Deformations of permutation representations of Coxeter groups

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Abstract The permutation representation afforded by a Coxeter group W acting on the cosets of a standard parabolic subgroup inherits many nice properties from W such as a shellable Bruhat order and a flat deformation over $\mathbb{Z}[q]$ to a representation of the corresponding Hecke algebra. In this paper we define a larger class of "quasi-parabolic" subgroups (more generally, quasiparabolic W-sets), and show that they also inherit these properties. Our motivating example is the action of the symmetric group on fixed-point-free involutions by conjugation.

1 Introduction

The motivation for the machinery developed in this paper arose when the first author was investigating certain conjectures involving Macdonald polynomials generalizing classical identities of Schur functions related to the representation theory of symmetric spaces [12]. These Schur function identities are closely related to Littlewood identities and thus via invariant theory [1, §8] to the action of S_{2n} by conjugation on

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fixed-point-free involutions. Given the close connection between Macdonald polynomials and Hecke algebras, this suggested that to prove those conjectures, one should first deform that permutation representation.

There is a straightforward way of deforming any given representation of a finite Coxeter group to its corresponding Hecke algebra. Indeed, when the ground field is algebraically closed of characteristic 0 and the parameter q is specialized to something other than a nontrivial root of unity, the Hecke algebra is isomorphic to the group algebra, and thus every irreducible representation deforms. So to deform any representation, one needs simply express it as a direct sum of irreducibles and deform each irreducible. However, the resulting deformation is quite complicated: this process does not give rise to any particularly nice basis. In addition, this construction can fail in the case of infinite Coxeter groups, as even their finite-dimensional representations can fail to be completely reducible. One would thus like a version of this construction that works for arbitrary Coxeter groups and in particular yields a basis for which the structure constants are polynomials in q rather than having poles at roots of unity.

There is one special case of a permutation representation, apart of course from the regular representation, which has a particularly nice deformation. If $W_I \subset W$ is a parabolic subgroup,[1] then there is a classical construction of a module deforming W/W_I. To be precise, if we define $\ell^I(w)$ to be the length of the shortest element of the coset wW_I, then we can define an action of the corresponding Hecke algebra $H_W(q)$ on $\bigoplus_{w \in W^I} \mathbb{Z}[q] w W_I$ (where W^I denotes the set of minimal length left coset representatives) as follows:

$$T(s)wW_I = \begin{cases} swW_I, & \ell^I(sw) > \ell^I(w), \\ qwW_I, & \ell^I(sw) = \ell^I(w), \\ (q-1)wW_I + qswW_I, & \ell^I(sw) < \ell^I(w). \end{cases} \tag{1.1}$$

This representation can also be obtained by inducing the trivial representation from the corresponding parabolic subalgebra. For our other deformations of permutation representations, no such subalgebra exists. We do not consider the natural problem of deforming other induced representations.

If one conjugates a fixed-point-free involution by a simple transposition, then the length either stays the same or changes by 2. This suggests constructing a Hecke algebra representation along the same lines as in the parabolic case, replacing the length function of a coset by half the length of the fixed-point-free involution ι, which we refer to as its height. If one does this, it turns out that the result indeed satisfies the relations of the Hecke algebra. Similarly, if one instead uses the negated height $-\frac{\ell(\iota)}{2}$, one again obtains a well-defined module, albeit with a rather different looking action. (In fact, the two modules are indeed isomorphic over $\mathbb{Z}[q, q^{-1}]$.) In either case the module is generated by the unique minimal (maximal) length element and thus is determined by the ideal annihilating that element.

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[1] Unless specifically noted otherwise, parabolic means standard parabolic throughout the paper, whereas a conjugate of a standard parabolic subgroup will be called conjugate parabolic.

It then turned out [13] that one could use the most natural extensions of these annihilator ideals to the affine Hecke algebra to settle the original Macdonald polynomial conjectures of the first author. It turns out that the existence of a deformation of the above form is actually a fairly stringent condition on a subgroup of a Coxeter group. It remains conjectural that these extended ideals are annihilators of minimal elements in modules of the affine Hecke algebra coming from the present construction.

The present work came about in an attempt to systematically understand which properties of the set of fixed-point-free involutions permit such a nice deformation.

One other property of the quotient by a parabolic subgroup is particularly notable. Namely, the existence of a very well-behaved partial order induced by the Bruhat order on W. It was thus natural to look for a corresponding partial order in the case of fixed-point-free involutions as well as other permutation representations which admit natural deformations.

With only two exceptions, all of the representations we had found indeed admitted an analogue of Bruhat order. This led us to formulate Definition 2.3 below. Since parabolic subgroups were our prototypical example of such subgroups, we called our larger class of subgroups *quasiparabolic*.[2] Note that just as conjugates of standard parabolic subgroups do not give rise to nice Hecke algebra modules, we similarly find that conjugates of quasiparabolic subgroups are rarely quasiparabolic.

Although our main examples of permutation representations take the form W/H for various subgroups H, we take a slightly more general approach and formulate at least the definition for arbitrary sets with Coxeter group actions.

In Sect. 2 after giving the definitions, we show that quasiparabolic W-sets satisfy an analogue of the Strong Exchange Condition (in a slightly odd form because there is no notion of reduced word in a general quasiparabolic set) and consider other elementary consequences of our definition.

Section 3 gives a number of constructions of quasiparabolic sets, both from Coxeter groups themselves and from other quasiparabolic sets. In particular, we show that any parabolic subgroup is quasiparabolic (as a very special case of Corollary 3.10 below), as is the regular representation of $W \times W$ on W. More generally there are combinatorial analogues of the module-theoretic operations of restriction to parabolics, induction from parabolics, and tensor product.

One of these constructions is: given a quasiparabolic set with odd length stabilizers, there is a natural quasiparabolic double cover with only even length stabilizers. This enables us to reduce many of our arguments to the even case. In particular one finds that the even subgroups of quasiparabolic subgroups share many of the same properties.

Our final construction (in Sect. 4) generalizes the case of fixed-point-free involutions by giving fairly general conditions under which a conjugacy class of involutions becomes quasiparabolic with the obvious height function. (Unfortunately the resulting notion does not appear adequate to address involutions in infinite Coxeter groups.) We also refer the reader to [9–11] for their results on Weyl groups acting on involutions.

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[2]After having developed much of the theory presented here, we discovered that there was an unrelated notion of "quasi-parabolic" subgroup for classical Weyl groups already in the literature (J. Du, L. Scott, Trans. AMS **352**(9), 4325–4353), but by then we had grown far too attached to the name to change it!

Section 5 introduces the analogue of Bruhat order and again deals with basic properties. In Proposition 5.5 we show that quasiparabolicity is equivalent to the existence of a partial order that behaves like Bruhat order. Some readers may find this a more natural definition and wish to skip to Sect. 5. The main result of that section is Theorem 5.12 which shows that the Bruhat order is essentially unchanged under restriction to a parabolic subgroup. We also characterize how the double cover construction affects Bruhat order. Apart from these two results (and the straightforward fact that our Bruhat order on the actions of $W \times W$ on W and of W on W/W_I agree with the usual Bruhat order), we have not attempted to understand how our constructions affect Bruhat order in general.

Even without a full understanding of Bruhat order, we are still able to show that two of the more important topological properties of Bruhat order apply to a general quasiparabolic subgroup. The two main results of Sect. 6 are Theorem 6.1 and its Corollary 6.2 which give a formula for the Möbius function of the Bruhat order, generalizing results of Verma and Deodhar [6, 15]. Note that our proof is somewhat simplified by the ability to reduce to the even stabilizer case when the poset is Eulerian. Our other topological result generalizes a theorem of Björner and Wachs by showing that for any interval in a (bounded) quasiparabolic set, the corresponding order complex is shellable and in fact homeomorphic to a sphere or to a cell (depending on the Euler characteristic).

In Sect. 7 we show that the construction suggested above indeed gives well-defined modules over the Hecke algebra. We also show that these modules admit a natural bilinear form induced by the height function. We also give an algorithm for finding small sets of generators of the annihilators of minimal elements of these modules and in particular show that the ideal associated to a finite quasiparabolic set is finitely generated even when the Coxeter group that is acting has infinite order.

In Sect. 8 we study Poincaré series, i.e., the generating function of height, of quasiparabolic sets over finite Coxeter groups and show these generating functions can be controlled by Hecke algebra modules. We find both that such Poincaré series are always palindromes (despite the fact that Bruhat order rarely admits an order-reversing automorphism) and that the Poincaré series of a quasiparabolic W-set always divides the Poincaré series of W, despite the absence of any combinatorial interpretation of the quotient. The palindrome property suggests the existence of order-reversing automorphisms of the Hecke algebra modules; we give an explicit conjecture along these lines. Note that automorphisms of the form prescribed by the conjecture would give rise to analogues of the R-polynomials of Kazhdan–Lusztig theory.

Finally, in Sect. 9, we discuss a number of examples of finite quasiparabolic sets, including several quasiparabolic subgroups of finite groups which do not come from any of our general constructions. We have fairly extensively explored the set of quasiparabolic subgroups of finite Coxeter groups (in rank ≤ 8), but space does not permit an exhaustive discussion. We thus focus primarily on the analogue of fixed-point-free involutions and cases that exhibit suggestive phenomena.

There are a number of open problems scattered throughout the paper. In the interest of timeliness we focused our efforts on developing enough of a theory to show that the class of quasiparabolic subgroups is a natural one. There are also some implicit open problems such as classifying all quasiparabolic subgroups of all finite

Coxeter groups. Though significant progress could most likely be made on this last problem without substantially new ideas, other problems such as the existence of R-polynomials appear to require new insight.

For basic results and notation on Coxeter groups, we refer the reader to [8]. As there, we will only consider Coxeter groups of finite rank. We denote by $R(W)$ the set of reflections (i.e., conjugates of the simple reflections $s \in S$) of a Coxeter system (W, S). We denote by $W^0 = \{w \in W \mid \ell(w) \equiv 0 \pmod 2\}$ the *even* subgroup of W, and more generally, for any $H \subset W$, let $H^0 = H \cap W^0$. For instance, in the case of the symmetric group S_n its even subgroup is the alternating group, denoted here Alt_n, to avoid confusion with A_n.

2 Definitions

Definition 2.1 Let (W, S) be a Coxeter system. A *scaled W-set* is a pair (X, ht) with X a W-set and $\text{ht} : X \to \mathbb{Z}$ a function such that $|\text{ht}(sx) - \text{ht}(x)| \leq 1$ for all $s \in S$. An element $x \in X$ is *W-minimal* if $\text{ht}(sx) \geq \text{ht}(x)$ for all $s \in S$, and similarly for *W-maximal*.

This is invariant under shifting and negation of heights; more precisely:

Proposition 2.2 *Let (X, ht) be a scaled W-set. Then for any $k \in \mathbb{Z}$, the new height functions $(k + \text{ht})(x) := k + \text{ht}(x)$ and $(k - \text{ht})(x) := k - \text{ht}(x)$ make $(X, k + \text{ht})$ and $(X, k - \text{ht})$ scaled W-sets.*

Definition 2.3 A *quasiparabolic W-set* is a scaled W-set X satisfying the following two properties:

(QP1) For all $r \in R(W)$, $x \in X$, if $\text{ht}(rx) = \text{ht}(x)$, then $rx = x$.
(QP2) For all $r \in R(W)$, $x \in X$, $s \in S$, if $\text{ht}(rx) > \text{ht}(x)$ and $\text{ht}(srx) < \text{ht}(sx)$, then $rx = sx$.

In Proposition 5.5 below, we show that quasiparabolicity is equivalent to the existence of a partial order that behaves like Bruhat order under the action of reflections.

The main motivating example of a quasiparabolic W-set is the homogeneous space W/W_I for a (standard) parabolic subgroup W_I, where the height of a coset is the length of its minimal representative; see Corollary 3.10 below. In general, as we will see, many of the well-known properties of these parabolic homogeneous spaces extend to general quasiparabolic W-sets. (This explains our choice of terminology.)

The case $I = \emptyset$ gives the *regular representation* of W, i.e., the action of W on itself by left multiplication, with height given by length. This extends to the action of $W \times W$ on W by left and right multiplication, see Theorem 3.1 below.

The situation of property QP2 is fairly rigid, as seen in the following lemma.

Lemma 2.4 *Let (X, ht) be a scaled W-set, and suppose $r \in R(W)$, $s \in S$, $x \in X$ are such that $\text{ht}(rx) > \text{ht}(x)$ and $\text{ht}(srx) < \text{ht}(sx)$. Then $\text{ht}(rx) = \text{ht}(sx) = \text{ht}(x) + 1 = \text{ht}(srx) + 1$.*

Proof Since (X, ht) is a scaled W-set, it follows that $|\mathrm{ht}(sx) - \mathrm{ht}(x)| \leq 1$ and $|\mathrm{ht}(rx) - \mathrm{ht}(srx)| \leq 1$. Since $\mathrm{ht}(rx) - \mathrm{ht}(x) \geq 1$ (by integrality) and similarly $\mathrm{ht}(sx) - \mathrm{ht}(srx) \geq 1$, the triangle inequality forces all differences to be 1 as claimed. □

Proposition 2.5 *If (X, ht) is quasiparabolic, then so are $(X, k + \mathrm{ht})$ and $(X, k - \mathrm{ht})$ for any $k \in \mathbb{Z}$.*

We omit several of the easier technical proofs of propositions and corollaries in this section and the next. The reader can refer to the longer arXiv article for full details.

Proposition 2.6 *If (X, ht) is a quasiparabolic W-set, then for any parabolic subgroup $W_I \subset W$, (X, ht) is a quasiparabolic W_I-set.*

Proof Follows immediately from the fact that $R(W_I) \subset R(W)$. □

Note that in the case of the regular representation of W, with $x = 1$, the following "strong exchange condition" becomes the usual strong exchange condition.

Theorem 2.7 (Strong Exchange) *Let (X, ht) be a quasiparabolic W-set, and let $x \in X$. Let $w = s_1 \cdots s_k \in W$ be an arbitrary word. If $\mathrm{ht}(rx) > \mathrm{ht}(x)$ and $\mathrm{ht}(wrx) < \mathrm{ht}(wx)$, then there exists $1 \leq i \leq k$ such that $wrx = s_1 \cdots s_{i-1} s_{i+1} \cdots s_k x$.*

Proof Let $x_i = s_{i+1} \cdots s_k x$ and $y_i = s_{i+1} \cdots s_k rx = r_i x_i$, where r_i is the reflection

$$r_i = s_{i+1} \cdots s_k r s_k \cdots s_{i+1}, \qquad (2.1)$$

and observe that $\mathrm{ht}(y_0) < \mathrm{ht}(x_0)$ and $\mathrm{ht}(y_k) > \mathrm{ht}(x_k)$, so there exists $1 \leq i \leq k$ such that $\mathrm{ht}(y_{i-1}) < \mathrm{ht}(x_{i-1})$ but $\mathrm{ht}(y_i) \geq \mathrm{ht}(x_i)$. If equality holds, then $y_i = r_i x_i = x_i$, so $y_{i-1} = s_{i-1} y_i = s_{i-1} x_i = x_{i-1}$, a contradiction. But then we may apply property QP2 to deduce that

$$y_i = r_i x_i = s_i x_i. \qquad (2.2)$$

The theorem follows. □

Theorem 2.8 *Let x be a W-minimal element of the quasiparabolic W-set (X, ht). Then for all $w \in W$,*

(i) $\mathrm{ht}(wx) \geq \mathrm{ht}(x)$, *with equality iff $wx = x$;*
(ii) *$\exists u \leq w$ in Bruhat order with $\ell(u) = \mathrm{ht}(wx) - \mathrm{ht}(x)$ such that $wx = ux$.*

Proof Suppose otherwise, and let $w = s_1 s_2 \cdots s_k$ be a counterexample of minimum length. In particular, $\mathrm{ht}(s_k x) = \mathrm{ht}(x) + 1$, since otherwise $s_k x = x$, and $w s_k$ is a counterexample of length $k - 1$. Let $1 \leq j < k$ thus be the largest index such that

$$\mathrm{ht}(s_j s_{j+1} \cdots s_k x) \leq \mathrm{ht}(s_{j+1} \cdots s_k x). \qquad (2.3)$$

If equality holds here, then again we can remove s_j from w to obtain a shorter counterexample, so

$$\mathrm{ht}(s_j s_{j+1} \cdots s_k x) = \mathrm{ht}(s_{j+1} \cdots s_k x) - 1. \tag{2.4}$$

Let r be the reflection

$$r = s_k \cdots s_{j+1} s_j s_{j+1} \cdots s_k. \tag{2.5}$$

Then by assumption

$$\mathrm{ht}(s_{j+1} \cdots s_k r x) = \mathrm{ht}(s_j s_{j+1} \cdots s_k x) = \mathrm{ht}(s_{j+1} \cdots s_k x) - 1, \tag{2.6}$$

while

$$\mathrm{ht}(rx) \geq \mathrm{ht}(s_j s_{j+1} \cdots s_k x) - (k - j) = \mathrm{ht}(x) + 1 > \mathrm{ht}(x), \tag{2.7}$$

where the second equality follows from maximality of j. We may thus apply strong exchange to conclude that

$$s_j s_{j+1} \cdots s_k x = s_{j+1} \cdots s_k r x = s_{j+1} \cdots s_{l-1} s_{l+1} \cdots s_k x \tag{2.8}$$

for some l. But then s_j and s_l can be removed from w without affecting wx, contradicting minimality of w.

To prove (2), note that $\mathrm{ht}(wx) - \mathrm{ht}(x) \leq \ell(w)$. If $\mathrm{ht}(wx) - \mathrm{ht}(x) < \ell(w)$, then there exists a j such that (2.3) holds, and we can repeat the argument above, replacing w with u such that $u = s_1 \cdots \widehat{s_j} \cdots s_k$ or $u = s_1 \cdots \widehat{s_j} \cdots \widehat{s_l} \cdots s_k$ and $ux = wx$. □

Remark 2.9 Of course, the analogous argument applies in the case of a W-maximal element, by symmetry.

Corollary 2.10 *Each orbit of a quasiparabolic W-set contains at most one W-minimal and at most one W-maximal element.*

Proof If x and wx are W-minimal elements, then $\mathrm{ht}(wx) \geq \mathrm{ht}(x)$, and so $\mathrm{ht}(x) = \mathrm{ht}(w^{-1}(wx)) \geq \mathrm{ht}(wx)$, so $\mathrm{ht}(wx) = \mathrm{ht}(x)$, and thus $wx = x$. □

Definition 2.11 Let (X, ht) be a quasiparabolic W-set with W-minimal element x_0. Suppose $x \in Wx_0 \subseteq X$ is in the W-orbit of x_0 and has height $\mathrm{ht}(x) = k + \mathrm{ht}(x_0)$. Then we call $s_1 s_2 \cdots s_k x_0$ a *reduced expression* for x if $x = s_1 s_2 \cdots s_k x_0$, $s_i \in S$. By abuse of notation, we also call wx_0 reduced where $w = s_1 s_2 \cdots s_k$ is the corresponding reduced word.

In the case that X has a W-minimal element, the strong exchange condition can be restated in the following way, which is more traditional in the parabolic case.

Corollary 2.12 *If x_0 is W-minimal in the quasiparabolic W-set (X, ht), and $r \in R(W)$, $w = s_1 \cdots s_k \in W$ are such that $\mathrm{ht}(wx_0) > \mathrm{ht}(rwx_0)$, then there exists $1 \le i \le k$ such that*

$$r w x_0 = s_1 \cdots s_{i-1} s_{i+1} \cdots s_k x_0. \tag{2.9}$$

If wx_0 is a reduced expression, then i is unique.

Proof Apply strong exchange to the reflection $w^{-1}rw$; by minimality, we have that $\mathrm{ht}(x_0) \le \mathrm{ht}(w^{-1}rwx_0)$, and equality would imply $wx_0 = rwx_0$. If i is not unique, so that

$$s_1 \cdots s_{i-1} s_{i+1} \cdots s_k x_0 = s_1 \cdots s_{j-1} s_{j+1} \cdots s_k x_0 \tag{2.10}$$

for some j, then

$$s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_k x_0 = wx_0, \tag{2.11}$$

and wx_0 is not reduced. $\qquad\square$

In the case that X is transitive, or equivalently is a homogeneous space, with a minimal element, the height function is essentially just given by the length of a minimal coset representative (as in the parabolic case). More precisely, we have the following.

Corollary 2.13 *Let x_0 be a W-minimal element of the transitive quasiparabolic W-set (X, ht). Then for all $y \in X$,*

$$\mathrm{ht}(y) = \mathrm{ht}(x_0) + \min_{wx_0=y}\big\{\ell(w)\big\}. \tag{2.12}$$

Remark 2.14 Note that the proof depended on quasiparabolicity only via the fact that x_0 is the unique W-minimal element of X.

In particular, a transitive quasiparabolic W-set with a minimal element is uniquely determined (up to an overall shift in height) by the stabilizer of that minimal element. More precisely, we have the following.

Proposition 2.15 *Let (X, ht) and (X', ht') be transitive quasiparabolic W-sets with minimal elements x_0 and x_0', respectively. If $\mathrm{Stab}_W(x_0) = \mathrm{Stab}_W(x_0')$ and $\mathrm{ht}(x_0) = \mathrm{ht}'(x_0')$, then X and X' are isomorphic scaled W-sets.*

Proof We construct the isomorphism $\phi : X \to X'$ as follows. Let $y \in X$, and let $g \in W$ be such that $y = gx_0$; then we take $\phi(gx_0) = gx_0'$. To see that this is well defined, observe that for fixed y, g is determined up to right multiplication by an element of the stabilizer of x_0, and this does not change gx_0', as x_0' has the same stabilizer. It thus remains only to show that this W-set isomorphism (since the analogous map $X' \to X$ is clearly inverse to ϕ) preserves height, which follows by the observation

$$\mathrm{ht}(y) = \mathrm{ht}(x_0) + \min_{wx_0=y}\{\ell(w)\} = \mathrm{ht}'(x_0') + \min_{wx_0'=\phi(y)}\{\ell(w)\} = \mathrm{ht}'(\phi(y)). \quad (2.13)$$

$$\square$$

Given a subgroup $H \subset W$, we can construct a scaled W-set W/H by defining

$$\mathrm{ht}(wH) = \min_{v\in wH}\{\ell(v)\}. \quad (2.14)$$

Note that W/H has a W-minimal element, namely the trivial coset H of height 0.

Definition 2.16 A subgroup $H \subset W$ is *quasiparabolic* if the scaled W-set W/H is quasiparabolic.

We remark that our motivating examples can be restated as saying that both standard parabolic subgroups and the diagonal subgroup of $W \times W$ are quasiparabolic.

It follows from the previous corollary that if X is a transitive quasiparabolic W-set with minimal element x_0 of height 0, then $X \cong W/\mathrm{Stab}_W(x_0)$, and thus $\mathrm{Stab}_W(x_0)$ is a quasiparabolic subgroup of W. Thus the quasiparabolic subgroups are precisely the stabilizers of W-minimal elements of quasiparabolic W-sets. (Note that via negation, every statement about W-minimal elements applies mutatis mutandum to W-maximal elements; e.g., the stabilizer of a W-maximal element is also quasiparabolic.)

It is important to note that the property of being a quasiparabolic subgroup is not invariant under conjugation. Indeed, the conjugate of a standard parabolic subgroup need not be quasiparabolic.

Example 2.17 For instance, the scaled W-set associated to the conjugate parabolic subgroup $\langle s_1s_2s_1 \rangle$ of S_3 cannot be quasiparabolic, since it has two distinct W-maximal elements.

If the height function of a quasiparabolic W-set is bounded from below, then each orbit has an element of minimum height, which is necessarily W-minimal, so unique in the orbit. In particular, the transitive quasiparabolic W-sets with height bounded from below (e.g., if $|X| < \infty$) are (up to isomorphism and shift of height) in one-to-one correspondence with the quasiparabolic subgroups of W. Many of our proofs require an assumption of boundedness (either from above or below), making this case particularly important. However, there are also interesting examples of unbounded quasiparabolic W-sets, so we have attempted to avoid as much as possible the assumption of existence of a W-minimal element. Note also that if X (transitive and quasiparabolic) is infinite, then it cannot have *both* a W-minimal and a W-maximal element. Indeed, if X (without loss of generality) has a W-minimal element (i.e., $X = W/H$ up to shifting), then its height function is unbounded from above, as there are only finitely many potential minimal coset representatives of any fixed length.

Proposition 2.18 *Suppose that the transitive scaled W-set (X, ht) has a unique W-minimal element and the stabilizer of that element is a quasiparabolic subgroup. Then X is quasiparabolic.*

Lemma 2.19 *Suppose that the quasiparabolic subgroup $H \subset W$ contains an element of odd length. Then it contains a simple reflection.*

Proof Let $w = s_1 \cdots s_k \in H$ have odd length, and consider the sequence

$$\text{ht}(H), \ \text{ht}(s_k H), \ \text{ht}(s_{k-1} s_k H), \dots, \ \text{ht}(wH) = \text{ht}(H). \tag{2.15}$$

Since k is odd, we find that there exists some j such that

$$\text{ht}(s_j \cdots s_k H) = \text{ht}(s_{j+1} \cdots s_k H), \tag{2.16}$$

and thus

$$s_j \cdots s_k H = s_{j+1} \cdots s_k H, \tag{2.17}$$

and so W/H contains an element fixed by a simple reflection. Let $gH \in W/H$ be an element of minimum height among all those fixed by a simple reflection s. If $g = 1$, we are done; otherwise, there exists $s' \in S$ such that $\text{ht}(s'gH) < \text{ht}(gH)$. Consider the orbit of gH under the action of the standard parabolic subgroup $\langle s, s' \rangle$. The heights in this orbit are bounded both above and below, so the orbit is finite, and the action of ss' on the orbit has finite order, say k. Now, consider the sequence

$$gH, \ s'gH, \ ss'gH, \ s'ss'gH, \dots, \ s'(ss')^{k-1}gH, \ (ss')^k gH. \tag{2.18}$$

By assumption, the last term in the sequence is gH, so the penultimate term is $sgH = gH$. Since the sequence has even length and the initial and final terms have the same height, it follows that there must be an even number of steps where the height stays the same, so at least one such step which is not the last. Each such step provides an element fixed by either s or s'; if fixed by s', the element is certainly not gH, while if fixed by s, it has the form $(ss')^l gH$ for $l < k$ and so by minimality of k is again not gH. In other words, the $\langle s, s' \rangle$-orbit contains an element different from gH which is also fixed by a simple reflection; since gH is $\langle s, s' \rangle$-maximal, this other element has strictly smaller height, a contradiction. $\qquad \square$

3 Constructions

Two trivial, but useful, examples of quasiparabolic subgroups of W are W itself and its even subgroup W^0. Less trivially, of our two motivating examples above, the parabolic case W_I has a natural generalization, so to avoid duplication of effort, we prove it as a corollary of that generalization. For the second example, we have the following proof.

Theorem 3.1 *Let (W, S) be a Coxeter system. The set W, together with the height function $\text{ht}(w) := \ell(w)$ and the $W \times W$-action*

$$(w', w'')w = w'w(w'')^{-1}, \tag{3.1}$$

is a quasiparabolic $W \times W$-set.

Proof Since $|\ell(sw) - \ell(w)| = |\ell(ws) - \ell(w)| = 1$, (W, ℓ) is indeed a scaled $W \times W$-set, and furthermore

$$\mathrm{ht}\big((w', w'')w\big) - \mathrm{ht}(w) \equiv \ell\big((w', w'')\big) \quad (\mathrm{mod}\ 2). \tag{3.2}$$

In particular, property QP1 is vacuous. For property QP2, there would normally be four cases (depending on which factor of $W \times W$ s and r belong to), but these reduce to two by symmetry and then to one via the observation that

$$rw = w\big(w^{-1}rw\big). \tag{3.3}$$

We thus reduce to showing that if $r \in R(W)$, $s \in S$, $w \in W$ are such that

$$\ell(wr) > \ell(w) \quad \text{and} \quad \ell(swr) < \ell(sw), \tag{3.4}$$

then $sw = wr$. Since $\ell(swr) < \ell(sw)$, the classical strong exchange condition tells us that, taking $w = s_1 s_2 \cdots s_k$, either $swr = w$ (and we are done), or

$$swr = s_1 \cdots s_{l-1} s_{l+1} \cdots s_k. \tag{3.5}$$

But then

$$wr = s s_1 \cdots s_{l-1} s_{l+1} \cdots s_k, \tag{3.6}$$

so $\ell(wr) = k - 1$, a contradiction. $\qquad\square$

Corollary 3.2 *The diagonal subgroup $\Delta(W) \subset W \times W$ is quasiparabolic.*

One important, if mostly trivial, construction for quasiparabolic W-sets is direct product.

Proposition 3.3 *Let (W, S) and (W', S') be Coxeter systems, let (X, ht) be a quasiparabolic W-set, and let (X', ht') be a quasiparabolic W'-set. The height function $(\mathrm{ht} \times \mathrm{ht}')(x, x') := \mathrm{ht}(x) + \mathrm{ht}'(x')$ on $X \times X'$ makes $X \times X'$ a quasiparabolic $W \times W'$-set.*

Proof We observe that any reflection in $W \times W'$ is either a reflection in W or a reflection in W', and the restriction of $X \times X'$ to W is the disjoint union of $|X'|$ copies of X (with appropriately shifted heights). Thus property QP1 is automatic, while property QP2 reduces immediately to the case $r \in R(W)$, $s \in S'$. But we find

$$\big(\mathrm{ht} \times \mathrm{ht}'\big)\big((r, 1)(1, s)(x, x')\big) - \big(\mathrm{ht} \times \mathrm{ht}'\big)\big((1, s)(x, x')\big)$$
$$= \mathrm{ht}(rx) - \mathrm{ht}(x) = \big(\mathrm{ht} \times \mathrm{ht}'\big)\big((r, 1)(x, x')\big) - \big(\mathrm{ht} \times \mathrm{ht}'\big)\big((x, x')\big), \tag{3.7}$$

and thus property QP2 is vacuous in this case. $\qquad\square$

If the scaled W-set (X, ht) satisfies $\mathrm{ht}(sx) \neq \mathrm{ht}(x)$ for all $s \in S$, $x \in X$, then the verification that X is quasiparabolic is noticeably simplified, as property QP1 is vacuous. Indeed, under this assumption,

$$\mathrm{ht}(wx) \equiv \mathrm{ht}(x) + \ell(w) \pmod 2 \tag{3.8}$$

for all $w \in W$, $x \in X$. This motivates the following definition.

Definition 3.4 The scaled W-set (X, ht) is *even* if for any pair $w \in W$, $x \in X$ s.t. $wx = x$, one has $\ell(w)$ even. Otherwise, we say that (X, ht) is *odd*.

It turns out that associated to any scaled W-set X is a canonically defined even scaled $W \times A_1$-set $\widetilde{X}$ (the *even double cover* of X) such that X is quasiparabolic iff $\widetilde{X}$ is quasiparabolic; thus in many proofs below, we can reduce our consideration to the even case.

The scaled $W \times A_1$-set $\widetilde{X}$ is defined as the set $\widetilde{X} = X \times \mathbb{F}_2$ with W-action

$$w(x, k) = \bigl(wx, k + \ell(w)\bigr) \tag{3.9}$$

and $A_1 = \langle s_0 \rangle$-action

$$s_0(x, k) = (x, k + 1). \tag{3.10}$$

The height function on $\widetilde{X}$ is defined by

$$\widetilde{\mathrm{ht}}(x, k) = \begin{cases} \mathrm{ht}(x), & \mathrm{ht}(x) \equiv k \pmod 2, \\ \mathrm{ht}(x) + 1, & \mathrm{ht}(x) \equiv k + 1 \pmod 2. \end{cases} \tag{3.11}$$

We also let $\widetilde{x}$ denote the preimage $(x, \mathrm{ht}(x)) \in \widetilde{X}$. Note that $\widetilde{x}$ is A_1-minimal whereas $s_0 \widetilde{x}$ is A_1-maximal.

Theorem 3.5 *The pair* $(\widetilde{X}, \widetilde{\mathrm{ht}})$ *is an even scaled* $W \times A_1$-set *and is quasiparabolic iff* (X, ht) *is quasiparabolic.*

Proof To see that $(\widetilde{X}, \widetilde{\mathrm{ht}})$ is even and scaled, we need simply observe that if $\mathrm{ht}(sx) = \mathrm{ht}(x) \pm 1$, then $\widetilde{\mathrm{ht}}(s(x, k)) = \widetilde{\mathrm{ht}}(x, k) \pm 1$, while if $\mathrm{ht}(sx) = \mathrm{ht}(x)$, then $\widetilde{\mathrm{ht}}(s(x, k)) - \widetilde{\mathrm{ht}}(x, k)$ is 1 or -1 depending on whether or not $\mathrm{ht}(x) \equiv k$ modulo 2, and similarly for $s_0(x, k)$.

Now, suppose that (X, ht) is quasiparabolic. We must show that for all $r \in R(W \times A_1)$, $s \in S \uplus \{s_0\}$, $(x, k) \in \widetilde{X}$, if $\widetilde{\mathrm{ht}}(r(x, k)) > \widetilde{\mathrm{ht}}((x, k))$ and $\widetilde{\mathrm{ht}}(sr(x, k)) < \widetilde{\mathrm{ht}}(s(x, k))$, then $r(x, k) = s(x, k)$. By parity considerations, $r(x, k) = (rx, k+1)$ and $s(x, k) = (sx, k+1)$, so it remains only to show that $rx = sx$. There are four cases, depending on whether r or s are in W or A_1. If both are in W, then we certainly have

$$\mathrm{ht}(rx) \geq \mathrm{ht}(x) \quad \text{and} \quad \mathrm{ht}(srx) \leq \mathrm{ht}(sx), \tag{3.12}$$

since $|\widetilde{ht}((x, k)) - ht(x)| \le 1$. But then by the quasiparabolicity of X, we either have $rx = x$ or $rx = sx$. In the latter case, we are done; in the former, we have

$$\widetilde{ht}((x, k+1)) > \widetilde{ht}((x, k)) \quad \text{and} \quad \widetilde{ht}((sx, k)) < \widetilde{ht}((sx, k+1)), \qquad (3.13)$$

so that $k \equiv ht(x) \pmod 2$ and $k \equiv ht(sx) \pmod 2$, implying $ht(sx) = ht(x)$ and $sx = x = rx$ as required. Similarly, if both are in A_1, the verification is trivial. If $r = s_0$ (the unique reflection in A_1) and $s \in S$, then r commutes with S and is a simple reflection, so this is a special case of $r \in R(W)$, $s = s_0$.

In that remaining case, we have

$$\widetilde{ht}((rx, k+1)) > \widetilde{ht}((x, k)) \quad \text{and} \quad \widetilde{ht}((rx, k)) < \widetilde{ht}((x, k+1)). \qquad (3.14)$$

The left- and right-hand sides of the two inequalities both differ by at most 1, so the only way to have opposite inequalities is to have

$$\widetilde{ht}((x, k+1)) = \widetilde{ht}((rx, k+1)) = \widetilde{ht}((x, k)) + 1 = \widetilde{ht}((rx, k)) + 1, \qquad (3.15)$$

implying $ht(rx) = ht(x)$, so $rx = x$ and $r(x, k) = (x, k+1) = s_0(x, k)$ as required.

Conversely, suppose that $(\widetilde{X}, \widetilde{ht})$ is quasiparabolic; we need to show that (X, ht) satisfies properties QP1 and QP2.

For QP1, suppose that $r \in R(W)$, $x \in X$ are such that $ht(rx) = ht(x)$, and observe that

$$\widetilde{ht}(r\widetilde{x}) > \widetilde{ht}(\widetilde{x}) \quad \text{and} \quad \widetilde{ht}(s_0 r\widetilde{x}) < \widetilde{ht}(s_0\widetilde{x}) \qquad (3.16)$$

so $r\widetilde{x} = s_0\widetilde{x}$ and thus $rx = x$ as required.

For QP2, suppose that $r \in R(W)$, $x \in X$, $s \in S$ are such that $ht(rx) > ht(x)$ and $ht(srx) < ht(sx)$. If $ht(rx) - ht(x)$ is odd, then $r\widetilde{x} = \widetilde{rx}$, so $\widetilde{ht}(r\widetilde{x}) > \widetilde{ht}(\widetilde{x})$. If $ht(rx) - ht(x)$ is even, then $r\widetilde{x} = s_0\widetilde{rx}$, but $ht(rx) - ht(x) \ge 2$, so again $\widetilde{ht}(r\widetilde{x}) > \widetilde{ht}(\widetilde{x})$. Similarly, $\widetilde{ht}(sr\widetilde{x}) < \widetilde{ht}(s\widetilde{x})$. It follows that $r\widetilde{x} = s\widetilde{x}$, so $rx = sx$ as required. □

Remark 3.6 Note that if X is already even, then $\widetilde{X} \cong X \times A_1$, where A_1 is the regular representation of A_1.

Corollary 3.7 *If $H \subset W$ is quasiparabolic, then so is its even subgroup $H \cap W^0$.*

Proof If $x_0 \in W/H$ is the unique minimal element, then the minimal element $\widetilde{x}_0 = (x_0, ht(x_0))$ of the even double cover has stabilizer $H \cap W^0$. □

The following construction is extremely powerful.

Theorem 3.8 *Let (X, ht) be a quasiparabolic $W \times W'$-set, and let $H \subset W'$ be a quasiparabolic subgroup such that every H-orbit in X has height bounded from below. Let X/H be the W-set of H-orbits in X, and define $ht'(Hx) = \min_{y \in Hx}\{ht(y)\}$. Then $(X/H, ht')$ is a quasiparabolic W-set.*

Proof Note that the $\Delta(W')$-orbit of $(x, w'H) \in X \times (W'/H)$ contains the representative $((1, h)(1, (w')^{-1})x, H)$, so that the H-orbits X/H are in bijection with the $\Delta(W')$-orbits $(X \times (W'/H))/\Delta(W')$. Since W commutes with W' and $\Delta(W')$ in $W \times W'$ and $W \times W' \times W'$, respectively, this bijection induces an isomorphism of scaled W-sets. Hence to check the quasiparabolicity of X/H, it suffices to check the quasiparabolicity of $(X \times (W'/H))/\Delta(W')$. In other words, we may reduce to the case that H is a diagonal subgroup. That is, (X, ht) is a quasiparabolic $W \times W' \times W'$-set, and we wish to show that $X/\Delta(W')$, with the induced height function, is quasiparabolic. This construction also commutes with taking the even double-cover, so we may assume that X is even.

We first need to verify that ht' is a height function. To see this, let x be a minimal representative of a $\Delta(W')$ orbit, and observe that

$$\mathrm{ht}'\big(\Delta(W')sx\big) \leq \mathrm{ht}(sx) \leq \mathrm{ht}(x) + 1 = \mathrm{ht}\big(\Delta(W')x\big) + 1. \qquad (3.17)$$

By symmetry, one also has

$$\mathrm{ht}'\big(\Delta(W')x\big) \leq \mathrm{ht}'\big(\Delta(W')sx\big) + 1, \qquad (3.18)$$

and thus $(X/\Delta(W'), \mathrm{ht}')$ is indeed a scaled W-set. Note furthermore that as we have assumed X even, and every element of $\Delta(W')$ has even length, $(X/\Delta(W'), \mathrm{ht}')$ is even. As a result, property QP1 is immediate, and it remains only to verify property QP2.

Consider the set of quadruples (x, s, r, w), $x \in X$, $s \in S$, $r \in R(W \times W' \times W')$, $w \in W'$ such that

1. $\mathrm{ht}'(\Delta(W')x) = \mathrm{ht}(x)$, i.e., x is minimal in its orbit;
2. $\mathrm{ht}'(\Delta(W')srx) = \mathrm{ht}((w, w)srx)$;
3. $\mathrm{ht}'(\Delta(W')sx) = \mathrm{ht}'(\Delta(W')rx) = \mathrm{ht}'(\Delta(W')srx) + 1 = \mathrm{ht}'(\Delta(W')x) + 1$;
4. and the orbits $\Delta(W')sx$, $\Delta(W')rx$ are distinct.

Note that by Lemma 2.4, to verify property QP2, it will suffice to show that no configuration of orbits as in the third and fourth conditions exists, with $r \in R(W)$. Since any such configuration extends to a quadruple as above (simply choose a minimal representative x and an appropriate minimizing w), if we can show that the full set quadruples is empty, this will imply property QP2.

Consider a quadruple which is minimal with respect to $\ell(w)$. We have the following two inequalities:

$$\mathrm{ht}(rx) \geq \mathrm{ht}'\big(\Delta(W')rx\big) = \mathrm{ht}'\big(\Delta(W')x\big) + 1 = \mathrm{ht}(x) + 1, \qquad (3.19)$$

$$\mathrm{ht}\big((w, w)sx\big) \geq \mathrm{ht}'\big(\Delta(W')sx\big) = \mathrm{ht}'\big(\Delta(W')srx\big) + 1 = \mathrm{ht}\big((w, w)srx\big) + 1. \quad (3.20)$$

We may thus apply the strong exchange condition for (X, ht) to conclude that $(w, w)srx$ can be obtained from $(w, w)sx$ by omitting a simple reflection from some reduced word. Since by assumption, $(w, w)srx \neq (w, w)x$ (since the two orbits are distinct), we conclude that there exists w' obtained from w by omitting a simple reflection such that

$$(w, w)srx \in \big\{(w, w')sx, (w', w)sx\big\}. \qquad (3.21)$$

Without loss of generality, $(w, w)srx = (w, w')sx$. Now, consider the new quadruple (x', s, r', w'), where

$$x' = (w, w)srx, \qquad r' = \left(w'w^{-1}, 1\right) \in R\left(W' \times W'\right). \qquad (3.22)$$

This new quadruple satisfies conditions 1 through 4, since one readily verifies that the four $\Delta(W')$ orbits have simply been pairwise swapped. Since $\ell(w') < \ell(w)$, this is a contradiction. □

Remark 3.9 Note that any minimal element of X maps to a minimal element of X/H. Also, observe that if X is a quasiparabolic $W \times W'$-set, and Y is a quasiparabolic $W' \times W''$-set, then we obtain a quasiparabolic $W \times W''$-set

$$X \times_{W'} Y := (X \times Y)/\Delta\left(W'\right); \qquad (3.23)$$

we then have

$$X/H = X \times_{W'} \left(W'/H\right). \qquad (3.24)$$

Corollary 3.10 *Let (W, S) be a Coxeter group, and let W_I be a parabolic subgroup of W. If $H \subset W_I$ is a quasiparabolic subgroup of W_I, then it is also quasiparabolic as a subgroup of W. In particular, W_I and its even subgroup are quasiparabolic in W.*

Proof The transitive quasiparabolic $W \times W$-set (W, ℓ) restricts to a quasiparabolic $W \times W_I$-set and thus induces by the theorem a transitive quasiparabolic W-set W/H. The stabilizer of the minimal element of this new set is precisely H, as required.

The remaining claim follows from the fact that for any Coxeter group W, both W and its even subgroup are quasiparabolic subgroups. □

Remark 3.11 It follows from this construction that any coset $wH \in W/H$ has a unique decomposition of the form $wH = uvH$ with $u \in W^I$ (i.e., $u \in W$ is (right) W_I-minimal) and $\mathrm{ht}(wH) = \ell(u) + \mathrm{ht}(vH)$. Existence follows by taking w to be a minimal H-coset representative and utilizing the standard decomposition $w = uv$ with $u \in W^I$, $v \in W_I$; one then has

$$\ell(u) + \ell(v) = \ell(w) = \mathrm{ht}(wH) \leq \ell(u) + \mathrm{ht}(vH) \leq \ell(u) + \ell(v), \qquad (3.25)$$

so $\mathrm{ht}(wH) = \ell(u) + \mathrm{ht}(vH)$ as required. To see uniqueness, observe that the above construction represents W/H as $W \times_{W_I} W_I/H$, or in other words as the quotient of the set of pairs (u, vH) by the action $w \cdot (u, vH) = (uw^{-1}, wvH)$ of W_I. In each such orbit, there is a unique choice of w such that $uw^{-1} \in W^I$, and thus a unique orbit representative of the desired form. It follows that the Poincaré series (see Sect. 7 below) of W/H can be written as

$$\mathrm{PS}_{W/H}(q) = \mathrm{PS}_{W/W_I}(q)\,\mathrm{PS}_{W_I/H}(q) = \frac{\mathrm{PS}_W(q)\,\mathrm{PS}_{W_I/H}(q)}{\mathrm{PS}_{W_I}(q)}. \qquad (3.26)$$

Equivalently, we have (compare Theorem 8.1 below)

$$\frac{\mathrm{PS}_W(q)}{\mathrm{PS}_{W/H}(q)} = \frac{\mathrm{PS}_{W_I}(q)}{\mathrm{PS}_{W_I/H}(q)}. \tag{3.27}$$

Remark 3.12 Experimentally (i.e., in every finite case which we have checked), there appears to be a partial converse to this statement, to wit that if quasiparabolic $H \subseteq W$ is contained in a conjugate parabolic subgroup $g W_I g^{-1}$, then it is a quasiparabolic subgroup of some standard parabolic subgroup. More precisely, we conjecture that the intersection of all conjugate parabolic subgroups containing H is standard parabolic, and H is a quasiparabolic subgroup of the intersection.

Definition 3.13 A *Coxeter homomorphism* $\phi : W \to W'$ is a group homomorphism such that $\phi(S) \subset S' \cup \{1\}$.

Remark 3.14 Note that if $\phi : W \to W'$ is a Coxeter homomorphism, then

$$\phi\big(R(W)\big) \subset R\big(W'\big) \cup \{1\}. \tag{3.28}$$

Corollary 3.15 *Let $\phi : W \to W'$ be a Coxeter homomorphism. If $H \subset W$ is quasiparabolic, then so is $\phi(H)$; if $H' \subset W'$ is quasiparabolic, then so is $\phi^{-1}(H')$.*

Proof If $H' \subset W'$ is quasiparabolic, then we have an inclusion $W/\phi^{-1}(H') \hookrightarrow W'/H'$ which is height-preserving, and Definition 2.3 is immediate.

For the other direction, we may assume that ϕ is surjective by Corollary 3.10, since $\phi(W) \subseteq W'$ is standard parabolic. We observe that

$$(1, \phi) : W' \times W \to W' \times W' \tag{3.29}$$

is also a Coxeter homomorphism, and thus the subgroup

$$\Delta_\phi\big(W'\big) := (1, \phi)^{-1}\big(\Delta\big(W'\big)\big) \subset W' \times W \tag{3.30}$$

is quasiparabolic. But then the set

$$\big(\big(W' \times W\big)/\Delta_\phi\big(W'\big)\big) \times_W W/H \tag{3.31}$$

is quasiparabolic and readily verified to be transitive such that $\phi(H)$ is the stabilizer of the minimal element. □

We have shown above that if H is an odd quasiparabolic subgroup of W, then it contains a simple reflection, and its even subgroup H^0 is quasiparabolic. To show the converse, namely that any subgroup of W containing a simple reflection and with quasiparabolic even subgroup is itself quasiparabolic, it suffices to construct a suitable action of A_1 on W/H^0 giving an isomorphism $W/H^0 \cong \widetilde{W/H}$ of scaled $W \times A_1$-sets and making W/H^0 quasiparabolic as a $W \times A_1$-set.

In fact, we have the following generalization of this fact.

Theorem 3.16 *Let $H \subset W$ be an even quasiparabolic subgroup, and let I be the set of all simple reflections normalizing H. Then the extension of W/H to a $W \times W_I$-set by*

$$(w, w') \cdot gH := wg(w')^{-1}H \tag{3.32}$$

preserves quasiparabolicity. In particular, HW_I is also a quasiparabolic subgroup of W.

Proof To see that W/H is a *scaled* $W \times W_I$ set, we need to show that $|\mathrm{ht}(gsH) - \mathrm{ht}(gH)| \leq 1$ for $s \in I$. If g has minimum length in its coset, then we find that

$$\mathrm{ht}(gsH) \leq \ell(gs) \leq \ell(g) + 1 \leq \mathrm{ht}(gH) + 1, \tag{3.33}$$

and thus the action of $(1, s)$ can never increase the height by more than 1; thus its inverse can never *decrease* the height by more than 1, and since $s^2 = 1$, we are done. Moreover, since H is even, we in fact find that W/H is an *even* scaled $W \times W_I$ set.

It thus remains to show that property QP2 holds. There are, as before, four cases to consider:

$$\mathrm{ht}(rgH) = \mathrm{ht}(sgH) = \mathrm{ht}(gH) + 1 = \mathrm{ht}(srgH) + 1, \quad r \in R(W),\ s \in S; \tag{3.34}$$

$$\mathrm{ht}(grH) = \mathrm{ht}(sgH) = \mathrm{ht}(gH) + 1 = \mathrm{ht}(sgrH) + 1, \quad r \in R(W_I),\ s \in S; \tag{3.35}$$

$$\mathrm{ht}(rgH) = \mathrm{ht}(gsH) = \mathrm{ht}(gH) + 1 = \mathrm{ht}(rgsH) + 1, \quad r \in R(W),\ s \in I; \tag{3.36}$$

$$\mathrm{ht}(grH) = \mathrm{ht}(gsH) = \mathrm{ht}(gH) + 1 = \mathrm{ht}(grsH) + 1, \quad r \in R(W_I),\ s \in I. \tag{3.37}$$

The second case is a special case of the first (replacing r by grg^{-1}), and similarly the fourth is a special case of the third. Moreover, the first case is just property QP2 as a W-set, so is immediate.

We are thus left with the case $r \in W$, $s \in I$. Now, suppose g is a shortest element of gH, and choose a reduced word $g = s_1 s_2 \cdots s_k$. Then

$$\mathrm{ht}(rs_1 s_2 \cdots s_k s H) < \mathrm{ht}(s_1 s_2 \cdots s_k s H), \tag{3.38}$$

so by Corollary 2.12, either

$$rs_1 s_2 \cdots s_k s H = s_1 s_2 \cdots s_k H \tag{3.39}$$

(and we are done), or there exists $1 \leq l \leq k$ such that

$$rs_1 s_2 \cdots s_k s H = s_1 s_2 s_{l-1} s_{l+1} \cdots s_k s H. \tag{3.40}$$

But then

$$rs_1 s_2 \cdots s_k H = s_1 s_2 s_{l-1} s_{l+1} \cdots s_k H, \tag{3.41}$$

since s normalizes H, so that

$$\mathrm{ht}(rs_1 s_2 \cdots s_k H) \leq k - 1 = \mathrm{ht}(gH) - 1, \tag{3.42}$$

a contradiction.

The final claim is a special case of the following corollary.   □

Corollary 3.17 *If H, I are as above, and K is any quasiparabolic subgroup of W_I, then HK is a quasiparabolic subgroup of W.*

Proof Consider the quasiparabolic W-set $W/H \times_{W_I} W_I/K$.   □

Corollary 3.18 *If the subgroup $H \subset W$ contains a simple reflection, then H is quasiparabolic iff $H \cap W^0$ is quasiparabolic.*

A given odd quasiparabolic subgroup $G \subset W$ can in general be obtained in more than one way via the construction of Theorem 3.16. The smaller the normal subgroup H of G being used in the construction, the more other odd subgroups it explains. We would thus in particular like to understand the minimal such subgroup (if it exists). There is, in fact, a natural candidate for this minimal normal subgroup, namely the subgroup generated by all elements that are forced to be in the stabilizer of the minimal element by property QP2 alone. Although we cannot as yet prove that this subgroup is quasiparabolic, we can at least prove the following.

Theorem 3.19 *Let H be a quasiparabolic subgroup of W, and let N be the subgroup of H generated by all elements of the form $w^{-1}srw$ for $w \in W$, $s \in S$, $r \in R(W)$ such that $\mathrm{ht}(rwH) \geq \mathrm{ht}(wH)$, $\mathrm{ht}(srwH) \leq \mathrm{ht}(swH)$. Then N is normal in H, and H/N is generated by simple reflections.*

Proof Since H is quasiparabolic, the conditions on w, s, r force $rwH = swH$, and thus $w^{-1}srw \in H$ as required. Moreover, for any $h \in H$, (wh, s, r) also satisfies the conditions; it follows that N is indeed a normal subgroup of H.

Now, let $s_1 s_2 \ldots s_k = h$ be an arbitrary reduced word of W multiplying to an element of H, and consider the sequence

$$0 = \mathrm{ht}(H),\ \mathrm{ht}(s_k H),\ \mathrm{ht}(s_{k-1} s_k H), \ldots,\ \mathrm{ht}(s_1 s_2 \cdots s_k H) = 0 \qquad (3.43)$$

of heights. For each j such that

$$\mathrm{ht}(s_j s_{j+1} \cdots s_k H) = \mathrm{ht}(s_{j+1} \cdots s_k H), \qquad (3.44)$$

choose a reduced expression

$$t_{j1} t_{j2} \cdots t_{jn_j} H = s_j s_{j+1} \cdots s_k H = s_{j+1} \cdots s_k H, \qquad (3.45)$$

and extend the given word by inserting

$$t_{j1} t_{j2} \cdots t_{jn_j} t_{jn_j} \cdots t_{j2} t_{j1} \qquad (3.46)$$

before and after s_j. If we then break the word between each pair $t_{jn_j} t_{jn_j}$, we obtain a factorization of h as a product $h = h_1 \cdots h_k$. Each $h_j = t_{j1} t_{j2} \cdots t_{jn_j} s_j t_{jn_j} \cdots t_{j2} t_{j1}$ is a product of $2n_j + 1$ simple reflections, and the sequence of heights given by the

suffixes of h_j changes at all $2n_j + 1$ steps but possibly one. In other words, there is at most one step in which the height does not change.

Let $s_1 s_2 \cdots s_k$ thus be such a word, and suppose first that at no step does the height remain unchanged. We claim that, in that case, $s_1 s_2 \cdots s_k \in N$. Suppose otherwise, and choose a counterexample of minimum length. Since the height must increase at the first step, and eventually decreases back to 0, there exists j such that

$$\mathrm{ht}(s_j s_{j+1} \cdots s_k H) = \mathrm{ht}(s_{j+1} \cdots s_k H) - 1. \tag{3.47}$$

Choose the largest such j. By Corollary 2.12, we find

$$s_j s_{j+1} \cdots s_k H = s_{j+1} \cdots s_{l-1} s_{l+1} \cdots s_k H \tag{3.48}$$

for some $l > j$, and by the definition of N it follows that

$$(s_{l+1} \cdots s_k)^{-1} s_l \cdots s_j \cdots s_{l-1} (s_{l+1} \cdots s_k) \in N. \tag{3.49}$$

But then

$$s_1 s_2 \cdots s_{j-1} s_{j+1} \cdots s_{l-1} s_{l+1} \cdots s_k \in hN \tag{3.50}$$

would be a shorter counterexample.

Similarly, let $s_1 s_2 \cdots s_k = h \in H$ be a word such that at precisely one step the height remains the same (necessarily of odd length, by parity considerations); we claim that

$$h \in sN \tag{3.51}$$

for some simple reflection s. Decompose the word for h as $h = vtw$, $v, w \in W$, $t \in S$, such that $twH = wH$. If $\mathrm{ht}(wH) = 0$, then $t \in H$ and $w \in H$ yields an expression such that at each step the height changes. So by the previous case $w \in N$ and hence $v \in N$, so we are done. Otherwise, there exists a simple reflection $u \in S$ such that $\mathrm{ht}(uwH) = \mathrm{ht}(wH) - 1$. Consider the $\langle t, u \rangle$-orbit generated by wH. By quasiparabolicity, this orbit has a unique minimal element $w'H$ such that $t'w'H = w'H$, $\mathrm{ht}(u'w'H) > \mathrm{ht}(w'H)$, with $\{t', u'\} = \{t, u\}$ (which depends on the parity of $\mathrm{ht}(wH) - \mathrm{ht}(w'H)$). We can freely replace v^{-1} and w by any words in the same coset such that the height changes at each step and may therefore assume that

$$v^{-1} = w = \begin{cases} (t'u')^{(\mathrm{ht}(wH)-\mathrm{ht}(H))/2} w', & \mathrm{ht}(wH) \equiv \mathrm{ht}(H) \pmod 2, \\ u'(t'u')^{(\mathrm{ht}(wH)-\mathrm{ht}(H)-1)/2} w', & \mathrm{ht}(wH) \equiv \mathrm{ht}(H) + 1 \pmod 2. \end{cases} \tag{3.52}$$

But then

$$vtw = (w')^{-1} u' (t'u')^{\mathrm{ht}(wH)-\mathrm{ht}(H)-1} w'. \tag{3.53}$$

Since

$$\mathrm{ht}(u'w'H) = \mathrm{ht}(u'(t'u')^{\mathrm{ht}(wH)-\mathrm{ht}(H)} w'H) = \mathrm{ht}(w'H) + 1$$
$$= \mathrm{ht}((t'u')^{\mathrm{ht}(wH)-\mathrm{ht}(H)} w'H) + 1,$$

we find

$$\left(w'\right)^{-1}\left(t'u'\right)^{\mathrm{ht}(wH)-\mathrm{ht}(H)}w' \in N, \tag{3.54}$$

so that

$$vtw \in \left(w'\right)^{-1}t'w'N, \tag{3.55}$$

giving us the desired result by induction (since $\mathrm{ht}(w'H) < \mathrm{ht}(wH)$). $\qquad\square$

Conjecture 3.20 *For any quasiparabolic subgroup $H \subset W$, the normal subgroup of Theorem 3.19 is also quasiparabolic.*

4 Perfect involutions

One of our original motivating examples of a quasiparabolic W-set is the set of fixed-point free involutions in S_{2n}, with height function $(\ell(\iota) - n)/2$. This generalizes considerably, to Theorem 4.6 below. We also refer the reader to results of Lusztig–Vogan on involutions in Weyl groups [9].

We first need to introduce some notation.

Let W be a Coxeter group, and let W^+ be the semidirect product of W by the group of permutations of S that induce Coxeter automorphisms of W; this inherits a length function from W by taking the length of such a Coxeter automorphism to be 0; equivalently, W^+ acts on the set of roots of W, and the length function counts (as usual) the number of positive roots taken to negative roots by the given element of W^+.

Lemma 4.1 *For any Coxeter group W, (W^+, ℓ) is a quasi-parabolic $W \times W$-set.*

Proof Indeed, each $W \times W$-orbit in W^+ has a unique minimal element, namely the associated Coxeter automorphism $\phi : W \to W$; the stabilizer of that minimal element is the quasiparabolic subgroup $\Delta_\phi(W)$. See (3.30). $\qquad\square$

Definition 4.2 An involution $\iota \in W^+$ is *perfect* if for all $r \in R(W)$, r commutes with $\iota r \iota$. We will denote by $\mathscr{I}$ the set of all perfect involutions.

Remark 4.3 Note that r commutes with $\iota r \iota$ iff $(r\iota)^4 = 1$. In the case $W = S_n$, any fixed-point-free involution is perfect, as follows easily from the fact that reflections are just 2-cycles. Similarly, any element conjugate to the diagram automorphism of S_{2n} is perfect for precisely the same reason. These two classes of perfect involutions will give rise to (quasiparabolic) scaled W-sets with negated heights.

Remark 4.4 This appears to be a too stringent condition when W is infinite; for instance, the obvious analogue in $\tilde{A}_{2n-1}$ of the case of fixed-point-free involutions in A_{2n-1} (i.e., one of the two conjugacy classes of preimages in $\tilde{A}_{2n-1}$) do not give perfect involutions, but are sufficiently well behaved that they very likely form a quasiparabolic set.

Lemma 4.5 *If $\iota \in W^+$ is perfect, then for all $r \in R(W)$, if*

$$\ell(\iota), \ell(r\iota r) < \ell(r\iota) = \ell(\iota r) \tag{4.1}$$

or

$$\ell(\iota), \ell(r\iota r) > \ell(r\iota) = \ell(\iota r), \tag{4.2}$$

then $\iota = r\iota r$.

Proof Note that since ι is an involution, $r\iota = (\iota r)^{-1}$, and thus the two elements have the same length as stated. Note also that the contrapositive of the lemma reads that if $r\iota$ has order 4, then $\ell(r\iota)$ is between $\ell(\iota)$ and $\ell(r\iota r)$, and this inclusion is strict by parity considerations.

Thus suppose that $r\iota$ has order 4. We can write $r = s_\alpha$ for some positive root α; the fact that $r\iota$ has order 4 implies that $\iota(\alpha)$ is orthogonal to α. But then

$$\ell(\iota) < \ell(\iota r) \iff \iota(\alpha) > 0 \tag{4.3}$$

$$\iff r\iota(\alpha) > 0 \tag{4.4}$$

$$\iff \ell(r\iota) < \ell(r\iota r), \tag{4.5}$$

as required, and similarly for the opposite inequalities. □

The set of perfect involutions is certainly acted on by W by conjugation; it very nearly becomes a scaled W-set by setting $\mathrm{ht}(\iota) = \ell(\iota)/2$; this fails only in that on some orbits the heights might lie in $1/2 + \mathbb{Z}$ rather than $\mathbb{Z}$, but this has no effect on the theory. (Indeed, for the above theory of quasiparabolic sets to work, we need only that on each orbit, the height function lies in a fixed coset of $\mathbb{Z}$.)

Theorem 4.6 *The set $\mathscr{I}$ of perfect involutions in W^+, together with the above height function, is a quasiparabolic W-set.*

Proof To show property QP1, let ι be a perfect involution, and let $r \in R(W)$ be such that $\mathrm{ht}(r \cdot \iota) = \mathrm{ht}(\iota)$; equivalently, $\ell(r\iota r) = \ell(\iota)$. But then Lemma 4.5 implies that $r\iota r = \iota$ as required.

It remains to show property QP2; let, therefore, ι be a perfect involution, and $r \in R(W)$, $s \in S$ such that (recalling Lemma 2.4)

$$\mathrm{ht}(r \cdot \iota) = \mathrm{ht}(s \cdot \iota) = \mathrm{ht}(\iota) + 1 = \mathrm{ht}(sr \cdot \iota) + 1, \tag{4.6}$$

or equivalently

$$\ell(r\iota r) = \ell(s\iota s) = \ell(\iota) + 2 = \ell(sr\iota rs) + 2. \tag{4.7}$$

We need to show that $r\iota r = s\iota s$.

We first observe that

$$\ell(r\iota) = \ell(\iota) + 1. \tag{4.8}$$

Indeed, by Lemma 4.5, $\ell(r\iota)$ is between $\ell(\iota)$ and $\ell(r\iota r)$, and thus (since the latter differ by 2), must be $\ell(\iota) + 1$ as required. Similarly,

$$\ell(s\iota) = \ell(\iota) + 1. \tag{4.9}$$

Moreover, since $sr\iota rs$ is also a perfect involution, and srs is a reflection, we find that

$$\ell(sr\iota rss) = \ell\big(sr\iota rs(srs)\big) = \ell(sr\iota rs) + 1 = \ell(\iota) + 1, \tag{4.10}$$

or in other words,

$$\ell(sr\iota r) = \ell(sr\iota s) = \ell(\iota) + 1. \tag{4.11}$$

Now, consider the element $sr\iota$. Since $\ell(r\iota) = \ell(\iota) + 1$ and s is a simple reflection, we find that $\ell(sr\iota) - \ell(\iota) \in \{0, 2\}$. We consider the two cases separately.

If $\ell(sr\iota) = \ell(\iota)$, then by the fact that W^+ is quasiparabolic as a $W \times W$-set and the fact that $\ell(s\iota) = \ell(r\iota) = \ell(\iota) + 1$, we conclude that $s\iota = r\iota$. But W^+ is a group, so we can cancel ι to find $s = r$, and thus in particular $s\iota s = r\iota r$ as required.

If $\ell(sr\iota) = \ell(\iota) + 2$, then on the one hand

$$\ell(sr\iota) = \ell(r\iota r) = \ell(r\iota) + 1 = \ell(sr\iota r) + 1, \tag{4.12}$$

so $sr\iota = r\iota r$, while on the other hand

$$\ell\big((srs)s\iota\big) = \ell(s\iota s) = \ell(s\iota) + 1 = \ell\big((srs)s\iota s\big) + 1, \tag{4.13}$$

so $sr\iota = (srs)s\iota = s\iota s$. Therefore $r\iota r = s\iota s$ as required. $\qquad\square$

Remark 4.7 In addition to the special case of fixed-point-free involutions in S_{2n}, there are two general instances of perfect involutions: the identity element is always perfect, as is the diagram automorphism of $W \times W$ that swaps the two factors. The latter quasiparabolic $W \times W$ set turns out to be naturally isomorphic to W.

Suppose ι is a W-minimal perfect involution, so that by the above result, the centralizer $Z_W(\iota)$ is quasiparabolic. Let $z_W(\iota)$ denote the normal subgroup attached to this group by Theorem 3.19. This has a particularly nice description directly in terms of ι. First a useful lemma.

Lemma 4.8 *Let ι be a perfect involution, and let r, r' be reflections such that $r\iota r = r'\iota r' \ne \iota$. Then $r' \in \{r, \iota r\iota\}$.*

Proof Rewrite the hypothesis as $(r\iota)^2 = (r'\iota)^2 \ne 1$ and work in the reflection representation of W^+. Let $r = r_\alpha$ for some root α normalized so that $\alpha \cdot \alpha = 2$. Since $(r\iota)^2 \ne 1$, it follows that $\iota(\alpha) \cdot \alpha = 0$, and thus

$$(r\iota)^2 = r_\alpha r_{\iota(\alpha)} = r_{\alpha_+} r_{\alpha_-}, \tag{4.14}$$

where $\alpha_\pm := (1 \pm \iota)(\alpha/2)$. Now, the vectors $\alpha_\pm$ are uniquely determined up to sign as unit vectors which are simultaneous eigenvectors of ι and $(r\iota)^2$, and thus if $r' = r_{\alpha'}$, we find

$$\alpha'_+ = \pm\alpha_+ \qquad \alpha'_- = \pm\alpha_-. \tag{4.15}$$

Solving for α', we find

$$\alpha' \in \{\pm\alpha, \pm\iota(\alpha)\}. \tag{4.16}$$

The claim follows. □

Theorem 4.9 *Let $\iota \in \mathscr{I}$. The group $z_W(\iota)$ is the subgroup of W generated by elements of the form $(r\iota)^2$ for $r \in R(W)$.*

Proof We first observe that if $w \in W$, $s \in S$, $r \in R(W)$ are such that

$$\mathrm{ht}(sw \cdot \iota) = \mathrm{ht}(rw \cdot \iota) = \mathrm{ht}(w \cdot \iota) + 1 = \mathrm{ht}(srw \cdot \iota) + 1, \tag{4.17}$$

so that $w^{-1}srw \in z_W(\iota)$, then

$$w^{-1}sw \cdot \iota = w^{-1}rw \cdot \iota \neq \iota, \tag{4.18}$$

and thus (assuming $r \neq s$) $w^{-1}rw = \iota w^{-1}sw\iota$, so that

$$w^{-1}srw = (w^{-1}sw\iota)^2, \tag{4.19}$$

and thus by the lemma $z_W(\iota)$ is generated by elements of the desired form.

It remains to show that for any reflection, $(r\iota)^2 \in z_W(\iota)$. Write $r = w^{-1}sw$ and define $r' := w\iota r\iota w^{-1}$, so that $sw \cdot \iota = r'w \cdot \iota$. If $\mathrm{ht}(sw \cdot \iota) > \mathrm{ht}(w \cdot \iota)$, then

$$\mathrm{ht}(sw \cdot \iota) = \mathrm{ht}(r'w \cdot \iota) = \mathrm{ht}(w \cdot \iota) + 1 = \mathrm{ht}(sr'w \cdot \iota) + 1, \tag{4.20}$$

so that $(r\iota)^2 = w^{-1}r'sw \in z_W(\iota)$. If $\mathrm{ht}(sw \cdot \iota) < \mathrm{ht}(w \cdot \iota)$, then

$$\mathrm{ht}(s(r'w) \cdot \iota) = \mathrm{ht}(r'(r'w) \cdot \iota) = \mathrm{ht}(r'w \cdot \iota) + 1 = \mathrm{ht}(sr'(r'w) \cdot \iota) + 1, \tag{4.21}$$

so that $(r\iota)^2 = (\iota r)^2 = w^{-1}r'sr'(r'w) \in z_W(\iota)$. Finally, if $\mathrm{ht}(sw \cdot \iota) = \mathrm{ht}(w \cdot \iota)$, then

$$r \cdot \iota = \iota, \tag{4.22}$$

so that $(r\iota)^2 = 1 \in z_W(\iota)$. □

Naturally, Conjecture 3.20 would imply that the groups $z_W(\iota)$ are quasiparabolic. It is fairly straightforward to classify the pairs (W, ι) with W finite and ι perfect. These include a number of sporadic cases (for which we have verified quasiparabolicity via computer), as well as infinite families coming from fixed-point-free involutions in A_{2n-1} and a corresponding case in D_{2n}. For the former, $z_W(\iota)$ is simply the even subgroup of $Z_W(\iota)$, and therefore its quasiparabolicity is automatic. Thus only the D_{2n} case (which we have verified through D_{12}) remains open among finite cases.

5 Bruhat order

Definition 5.1 Let (X, ht) be a quasiparabolic W-set. The *Bruhat order* on X is the weakest partial order such that for $x \in X$, $r \in R(W)$, $x \le rx$ iff $\mathrm{ht}(x) \le \mathrm{ht}(rx)$.

Remark 5.2 Note that as with the usual Bruhat order, x and rx are always comparable, and $x \le y$ implies $\mathrm{ht}(x) \le \mathrm{ht}(y)$.

Property QP2 can be rephrased in terms of the Bruhat order:

Proposition 5.3 *If $x < y$ and $sy < sx$, then $sy = x$.*

Proof Since $x < y$ implies $\mathrm{ht}(x) < \mathrm{ht}(y)$, we conclude that $\mathrm{ht}(y) = \mathrm{ht}(x) + 1$. But then y must cover x in the Bruhat order, so that $y = rx$ for some reflection r. The proposition is then precisely QP2. □

Remark 5.4 We had originally developed a theory in which instead of QP1 and QP2, we insisted only that every orbit of a parabolic subgroup should have at most one minimal and at most one maximal element. One can show that this condition need only be checked in rank 2, where it is equivalent to quasiparabolicity. When we eventually began considering how to extend Bruhat order to such sets, we discovered that there were essentially only two instances (see Example 9.1 below) of "quasiparabolic" sets in which such an extension failed to exist, so we decided that a change in definition was in order. In looking at the proofs in [8] concerning Bruhat order, we found that most of the arguments relied only on the claim of this proposition. Rewriting in terms of reflections gave QP1 and QP2.

There is a related reformulation of quasiparabolicity.

Proposition 5.5 *Let (X, ht) be a scaled W-set. Then X is quasiparabolic iff there exists a partial ordering $\le$ on X such that*

(1) ht *is strictly increasing*: *if $x < y$, then $\mathrm{ht}(x) < \mathrm{ht}(y)$.*
(2) *For any $x \in X$, $r \in R(W)$, x and rx are comparable.*
(3) *For any $s \in S$, if $x < y$, $sy < sx$, then $x = sy$.*

Proof If X is quasiparabolic, we have shown that Bruhat order satisfies these three properties. Conversely, QP1 follows by observing that if $\mathrm{ht}(rx) = \mathrm{ht}(x)$, then the only way x and rx can be comparable is if they are equal, while QP2 follows by observing that the hypothesis of QP2 implies via comparability that $x < rx$, $sx > srx$, and thus $x = srx$ as required. □

Remark 5.6 Recall as in Remark 4.4 above that there are apparently quasiparabolic conjugacy classes of involutions in W^+ for W infinite, so that Theorem 4.6 does not apply. For such conjugacy classes, Proposition 5.5 gives an alternate strategy, namely showing that the restriction of the usual Bruhat order to the conjugacy class gives an order satisfying the hypotheses of Proposition 5.5. The first condition of

the proposition always holds, since length is strictly increasing in ordinary Bruhat order. The third condition follows by observing that since $|\ell(s\iota s) - \ell(\iota)| \leq 2$ and is 0 if and only if $s\iota = \iota s$, then the only way the ordering of ι and ι' can be reversed is if their lengths differ by 2 and conjugation by s swaps their lengths. But then $\iota < s\iota$, so by Lemma 5.7 (or, rather, the statement for ordinary Bruhat order that this lemma generalizes), $\iota \leq s\iota'$, and $\iota < \iota s$, so $\iota \leq s\iota's$. Since the lengths agree, $\iota = s\iota's$ as required. Thus only the second condition of the proposition (comparability of ι and $r\iota r$) need be shown. Note that it follows from Lemma 4.5 that ι and $r\iota r$ are comparable whenever $r\iota$ has order 4. Checking this for the classes of interest in $\tilde{A}_{2n-1}$ or $\tilde{C}_{2n}$ (where nice combinatorial characterizations of Bruhat order are known [2, Sect. 2.6]) appears tractable, but quite technical, in the absence of further ideas.

We will also need a slight variant of Proposition 5.3.

Lemma 5.7 *If $x \leq y$ for $x, y \in X$ and $s \in S$, then*

(1) *Either $sx \leq y$ or $sx \leq sy$;*
(2) *Either $x \leq sy$ or $sx \leq sy$.*

In particular, if $sy \leq y$, then $sx \leq y$; if $x \leq sx$, then $x \leq sy$.

Proof By symmetry, we need only prove (1); the corresponding special case is immediate. Following Humphreys [8, Proposition 5.9], we reduce to the case $y = rx$ for some $r \in R(W)$. If $r = s$ or $sx \leq x$, then the lemma is immediate. So we may assume that $x < sx$. Then sx and $sy = (srs)sx$ are comparable, and the previous proposition gives the desired dichotomy. $\qquad\square$

Note as a special case, one has $\min\{x, sx\} \leq \min\{y, sy\}$ whenever $x \leq y$. This in turn is a special case of the following proposition. If $I \subset S$, $x \in X$ are such that the orbit $W_I x$ is bounded from below, let

$$\pi_I(x) := \min_{w \in W_I} wx \tag{5.1}$$

denote the (unique) minimal element of that orbit.

Proposition 5.8 *Let (X, ht) be a quasiparabolic W-set. Suppose $I \subset S$ is such that every W_I-orbit is bounded from below. Then the map $\pi_I : X \to X$ is order-preserving.*

Proof We have already seen that this holds when $|I| = 1$. Thus if $s_1, \ldots, s_k$ are the simple reflections in I in some order, the composition

$$\pi_{\langle s_1 \rangle} \circ \cdots \circ \pi_{\langle s_k \rangle} \tag{5.2}$$

is order-preserving and height-decreasing. If we iterate this operation starting at $x \in X$, we will eventually arrive at a fixed point clearly equal to $\pi_I(x)$; the proposition follows. $\qquad\square$

In particular, for any set of parabolic subgroups $W_{I_1}, \ldots, W_{I_k}$, one has the implication

$$x \leq y \implies \pi_{I_i}(y) \leq \pi_{I_i}(y), \quad 1 \leq i \leq k. \tag{5.3}$$

In the case of W viewed as a quasiparabolic W-set, it is a well-known result of Deodhar that the converse holds whenever $\cap_i I_i = \emptyset$. This does not hold for general quasiparabolic W-sets, however. For instance, if one views W as a $W \times W$-set, then every maximal parabolic of $W \times W$ is transitive, so the projected orderings provide no information. (An even worse example is W acting on W/W^0, since then every nontrivial parabolic subgroup is transitive.) One must thus at the very least add the necessary condition that it should be possible to reconstruct x from the projections $\pi_{I_i}(x)$. We do not know of any examples in which this necessary condition fails to be sufficient.

Let $[x, \infty)$ denote the subset of X consisting of elements $y \geq x$, and similarly for $(-\infty, y]$.

Corollary 5.9 *Let $x, y \in X$, $s \in S$ be such that $x \leq sx$ and $sy \leq y$. Then the intervals $[x, \infty)$ and $(-\infty, y]$ are preserved by s. If we further have $x \leq y$, then $[x, y]$ is also preserved by s.*

Proof If $x \leq z$, then since $x \leq sx$, the lemma gives $x \leq sz$; similarly, if $z \leq y$, then since $sy \leq y$, the lemma gives $sz \leq y$.   □

It will be useful to be able to restrict our attention to *even* quasiparabolic W-sets, so we will need to know how the Bruhat orders on X and $\widetilde{X}$ are related.

Lemma 5.10 *If $x \leq y$, then $\widetilde{x} \leq \widetilde{y}$, $s_0\widetilde{x} \leq s_0\widetilde{y}$. Conversely, if $(x, k) \leq (y, l)$ for some $k, l \in \mathbb{F}_2$, then $x \leq y$.*

Proof By transitivity, we reduce to the case $y = rx$, $\mathrm{ht}(y) > \mathrm{ht}(x)$. But then $r\widetilde{x} \in \{\widetilde{y}, s_0\widetilde{y}\}$, so that

$$\widetilde{\mathrm{ht}}(r\widetilde{x}) \geq \widetilde{\mathrm{ht}}(\widetilde{y}) = \mathrm{ht}(y) > \mathrm{ht}(x) = \widetilde{\mathrm{ht}}(\widetilde{x}). \tag{5.4}$$

This implies that either $\widetilde{x} \leq \widetilde{y}$ or $\widetilde{x} \leq s_0\widetilde{y}$. Since $\widetilde{x} < s_0\widetilde{x}$, it follows by Lemma 5.7 that $\widetilde{x} \leq \widetilde{y}$, and since $\widetilde{y} < s_0\widetilde{y}$, that $s_0\widetilde{x} \leq s_0\widetilde{y}$.

For the converse, we may again reduce to the case $(y, l) = r(x, k) = (rx, k + 1)$ with $r \in R(W \times A_1)$ and $\mathrm{ht}(y, l) > \mathrm{ht}(x, k)$. If $r = s_0$, then $y = x$, and we are done, so we may assume that $r \in R(W)$. Since x and $y = rx$ are comparable, it remains only to rule out the possibility that $y < x$. But then $\widetilde{y} < \widetilde{x}$ by the previous case, and thus (since the ordering of the heights changes) we must have $(x, k) = s_0\widetilde{x}$ and $(y, l) = s_0\widetilde{y}$. In other words, we have both $\widetilde{y} < \widetilde{x}$ and $s_0\widetilde{x} < s_0\widetilde{y}$. By Proposition 5.3, it follows that $s_0\widetilde{x} = \widetilde{y}$, but this is impossible.   □

Determining if $s_0\widetilde{x} \leq \widetilde{y}$ is somewhat more subtle.

Lemma 5.11 *If $x < y$, then either $s_0\widetilde{x}$ and $\widetilde{y}$ are incomparable or $s_0\widetilde{x} < \widetilde{y}$, with the latter occurring iff there exists a chain*

$$x = x_0 < x_1 = r_1 x_0 < x_2 = r_2 x_1 < \cdots < x_k = y, \tag{5.5}$$

$r_i \in R(W)$, *which at some point increases the height by an even amount.*

Proof We first note that if $x < rx = y$ with $\mathrm{ht}(rx) - \mathrm{ht}(x)$ even, then $r\widetilde{x} = s_0\widetilde{y}$, so that $s_0\widetilde{x} = r\widetilde{y}$, and thus $s_0\widetilde{x}$ and $\widetilde{y}$ are comparable, and comparing heights shows that $s_0\widetilde{x} < \widetilde{y}$ as desired. But then given any chain as hypothesized, with say the kth step even, we may apply the previous lemma to conclude

$$s_0\widetilde{x} < s_0\widetilde{x_1} < s_0\widetilde{x_2} < \cdots < s_0\widetilde{x_{k-1}} < \widetilde{x_k} < \widetilde{x_{k+1}} < \cdots < \widetilde{y}. \tag{5.6}$$

Conversely, suppose $s_0\widetilde{x} < \widetilde{y}$ and consider a chain

$$s_0\widetilde{x} < r_1 s_0\widetilde{x} < \cdots < \widetilde{y}. \tag{5.7}$$

Each element in this chain is either A_1-minimal or A_1-maximal; since the chain begins with a maximal element and ends with a minimal element, there is at least one step going from maximal to minimal. Now, the image of the chain in $\widetilde{X}$ is a chain in X, except that some consecutive elements may agree. If we remove these elements, however, we obtain a valid chain. It therefore suffices to show that any step going from maximal to minimal maps to a step increasing the height by an even amount. Note that such a step goes from $s_0\widetilde{u}$ to $\widetilde{v}$ for elements u and v which are related by a reflection in W since s_0 decreases the height of $s_0\widetilde{u}$. Since the heights in $\widetilde{X}$ differ by an odd amount, the heights in X differ by an even amount, and $u \neq v$, since again the height must increase. □

Since the restriction $X|_{W_I}$ of a quasiparabolic W-set X to a parabolic subgroup W_I is quasiparabolic, it is natural to ask how the Bruhat orders compare. Clearly if two elements are in distinct W_I-orbits, then they are incomparable with respect to the Bruhat order on $X|_{W_I}$. Otherwise, we conjecture in general that the Bruhat order of the restriction agrees with the restriction of the Bruhat order to any given orbit. Unfortunately, to date we can only prove this with an additional boundedness assumption.

Theorem 5.12 *Let (X, ht) be a quasiparabolic W-set, and let $W_I \subset W$ be a parabolic subgroup. If $x, y \in X$ are in the same bounded W_I-orbit, then $x \leq y$ in X iff $x \leq y$ in $X|_{W_I}$.*

Proof Since $R(W_I) \subset R(W)$, one direction is obvious (even without assuming boundedness). Thus assume that $x \leq y$ in X. By symmetry, we may assume that $W_I x = W_I y$ has a minimal element x_0. Moreover, by transitivity, we may assume that $y = rx$ for some $r \in R(W)$, and it will suffice to show that $y = r'x$ for some $r' \in R(W_I)$. Let $y = wx_0$ be a reduced expression for y inside $X|_{W_I}$. Then we have

$$\mathrm{ht}(w^{-1}rwx_0) = \mathrm{ht}(w^{-1}x) \geq \mathrm{ht}(x_0), \tag{5.8}$$

while

$$\mathrm{ht}\big(w\big(w^{-1}rw\big)x_0\big) = \mathrm{ht}(x) < \mathrm{ht}(y) = \mathrm{ht}(wx_0). \tag{5.9}$$

It follows by the strong exchange condition that $rwx_0 = w'x_0$ where w' is obtained from a reduced word representing w by removing a single simple reflection (which must be in W_I since $w \in W_I$). It follows that

$$y = ww'^{-1}x, \tag{5.10}$$

and the result follows by observing that $ww'^{-1} \in R(W_I)$.   □

Remark 5.13 In particular, if X is transitive as a W_I-set, then restricting to W_I does not change the Bruhat order. As a further special case, the Bruhat order of W viewed as a $W \times W$-set agrees with the usual Bruhat order.

Remark 5.14 Restriction is a special case of the operation $\times_W$ described above; it would be desirable to understand how to construct the Bruhat order on such a product from the two original Bruhat orders.

The proofs of our remaining results on Bruhat order require a global assumption of boundedness, though we conjecture in each case that this assumption is unnecessary. In addition to enabling us to perform induction on heights, boundedness also gives us an alternate characterization of the Bruhat order, analogous to the interpretation of the usual Bruhat order in terms of subwords. As a result, many of the proofs in the literature carry over with only minor changes required.

Below, we will fix a quasiparabolic W-set (X, ht) and a W-minimal element $x_0 \in X$; to simplify notations, we will assume that X is transitive and x_0 has height 0. (Since elements in different orbits are incomparable, there is no loss in assuming transitivity.)

Theorem 5.15 *Let $y = s_1 \cdots s_k x_0$ be a reduced expression. Then $x \le y$ iff one can write*

$$x = s_{i_1} \cdots s_{i_j} x_0 \tag{5.11}$$

for some $1 \le i_1 < i_2 < \cdots < i_j \le k$.

Proof The proof of [8, Theorem 5.10] carries over mutatis mutandum, except that one must multiply by reflections on the left and substitute Lemma 5.7 for [8, Proposition 5.9]   □

Similarly, Proposition 5.10 of Humphreys carries over immediately.

Proposition 5.16 *If $x < y$ and there is no z such that $x < z < y$, then $\mathrm{ht}(y) = \mathrm{ht}(x) + 1$. In particular, any saturated chain*

$$x = x_0 < x_1 < x_2 < \cdots < x_k = y \tag{5.12}$$

has length $k = \mathrm{ht}(y) - \mathrm{ht}(x)$, and the poset $(X, \le)$ is graded.

In principle, there are two natural partial orderings on the set of perfect involutions in W^+, namely the Bruhat order $\leq_{\mathscr{I}}$ viewed as a quasiparabolic set and the restriction $\leq_{W^+}$ of Bruhat order from W^+. It follows easily from the subword characterization of Bruhat order that

$$x \leq_{\mathscr{I}} y \quad \Longrightarrow \quad x \leq_{W^+} y; \tag{5.13}$$

indeed, any reduced word for y in $\mathscr{I}$ can be extended to a reduced word for y in W^+ by choosing a reduced word for the minimal perfect involution in its orbit. The converse appears to be true by experiment, but we have only been able to prove the following special case.

Proposition 5.17 *Suppose the perfect involution $\iota_0 \in W^+$ is a diagram automorphism. Then for x, y in the conjugacy class of ι_0,*

$$x \leq_{\mathscr{I}} y \quad \Longleftrightarrow \quad x \leq_{W^+} y. \tag{5.14}$$

Proof Suppose $x \leq_{W^+} y$. If $y = \iota_0$ (note that $\ell(\iota_0) = 0$, so ι_0 is minimal), the claim is immediate. Otherwise, there exists a simple reflection s such that $s \cdot y <_{\mathscr{I}} y$, and in particular $sys <_{W^+} sy$, $ys <_{W^+} y$. If $x <_{W^+} sx$, xs, then by Lemma 5.7 (applied to W^+ as a $W \times W$-set), $x <_{W^+} y$ implies $x <_{W^+} sy$ implies $x <_{W^+} sys$. Thus by induction on $\mathrm{ht}(x) + \mathrm{ht}(y)$, $x <_{\mathscr{I}} sys <_{\mathscr{I}} y$. If $x >_{W^+} sx$, $xs >_{W^+} sxs$, then we similarly have $sxs <_{W^+} sys$, so that by the same induction, $s \cdot x <_{\mathscr{I}} s \cdot y$, and again Lemma 5.7 applied to $\mathscr{I}$ gives $x <_{\mathscr{I}} y$.

It remains to consider the case $x = sxs >_{W^+} sx$, xs. But this implies that the simple root α_s is taken to its negative by x. Conjugating x to ι_0 would then give a root α negated by ι_0. However, ι_0 preserves the set of positive roots, so no such α can exist. $\qquad\square$

Remark 5.18 Clearly, the first part of the proof holds in general, and thus one reduces to showing that if $x = sxs >_{W^+} sx$, xs, then the subset $[x, \infty)_{W^+} \cap \mathscr{I}$ is preserved by the action of s by conjugation. (Note that the analogous statement for $[x, \infty)_{\mathscr{I}}$ follows by Corollary 5.9.) Using this criterion, one may verify the equivalence of these partial orders for various finite Coxeter groups, including all sporadic groups, B_n, and D_n for $n \leq 8$. It also follows for A_{2n-1}, as there are in that case two nontrivial conjugacy classes of perfect involutions: in one, the minimal element is a diagram automorphism ι_0 such that $\iota_0 x \iota_0 = w_0 x w_0$ where w_0 is the longest element. Multiplying by the central element $\iota_0 w_0$ swaps the two conjugacy classes and reverses both Bruhat orders.

6 The topology of Bruhat order

One of the more important invariants of a poset is its Möbius function, or equivalently the Euler characteristics of the order complexes associated to intervals in the poset. In the case of the regular representation of W (or equivalently, W viewed as a quasiparabolic $W \times W$-set, since the orders are the same), there is a classical formula

for the Möbius function ([6, 15]), which extends to a formula for the Möbius function of W/W_I. The latter is somewhat more complicated, which as we will see is a consequence of the fact W/W_I is not even. Indeed, for any bounded even quasiparabolic W-set, we have a very simple formula for the Möbius function.

Theorem 6.1 *Let (X, ht) be an even bounded quasiparabolic W-set. Then μ the Möbius function of $(X, \leq)$ is given by*

$$\mu(x, y) = \begin{cases} (-1)^{\mathrm{ht}(y) - \mathrm{ht}(x)}, & x \leq y, \\ 0 & \text{otherwise}. \end{cases} \tag{6.1}$$

Equivalently, any interval $[x, y]$ with $x < y$ is balanced in the sense that it has equal numbers of elements with odd and even heights.

Proof The equivalence of the two statements follows immediately from the definition of the Möbius function: an interval is balanced iff

$$\sum_{x \leq z \leq y} (-1)^{\mathrm{ht}(y) - \mathrm{ht}(z)} = 0, \tag{6.2}$$

so if every nontrivial interval is balanced, the claimed formula for μ indeed gives an inverse in the incidence algebra of $(X, \leq)$ to the "zeta function" of $(X, \leq)$ (i.e., the function which is 1 precisely when $x \leq y$ and 0 otherwise).

We now follow the proof in [6], with some simplification arising from the fact that X is even. Note first that by Corollary 5.9, it follows that any interval $[x, y]$ for which there exists a simple reflection s with $x < sx$ and $sy < y$ is preserved by s and thus balanced. We will use such intervals as a base case for an induction on $\mathrm{ht}(x) + \mathrm{ht}(y)$.

Since $y > x$, y is not minimal, and thus there exists some simple reflection s such that $sy < y$. If we had $sx > x$, the interval $[x, y]$ would be balanced per the previous paragraph, so we may as well assume that $sx < x$. We then claim that one has the following identity of sets:

$$[sx, y] \setminus [x, y] = [sx, sy] \setminus [x, sy]. \tag{6.3}$$

Note that each interval other than $[x, y]$ that appears in this expression is balanced by induction; since clearly $[x, y] \subset [sx, y]$ and $[sx, sy] \subset [x, sy]$, it will follow that $[x, y]$ is balanced.

In other words, we wish to show that

$$\{z : z \in X | sx \leq z \leq y; x \not\leq z\} = \{z : z \in X | sx \leq z \leq sy; x \not\leq z\}. \tag{6.4}$$

The set on the right is clearly contained (since $sy < y$) in the set on the left, so it suffices to prove the opposite inclusion.

Thus suppose that $sx \leq z \leq y$ but $x \not\leq z$; we need to show $z \leq sy$. Since $sx \leq z$ but $x \not\leq z$, Lemma 5.7 shows that $x \leq sz$ and thus $sz > z$ (since $z > sz$ would lead to a contradiction). Since $z \leq y$ and $z < sz$, the same lemma shows $z \leq sy$ as required. □

We can then compute the Möbius function in general by using the Möbius function on $\widetilde{X}$.

Corollary 6.2 *Let* (X, ht) *be a bounded quasiparabolic W-set with odd stabilizers. Then the Möbius function of* $(X, \leq)$ *can be described as follows. If* $x \not\leq y$, *then* $\mu(x, y) = 0$. *Otherwise,* $\mu(x, y) = (-1)^{\mathrm{ht}(y) - \mathrm{ht}(x)}$, *unless there exists a chain*

$$x < r_1 x < r_2 r_1 x < \cdots < y, \tag{6.5}$$

$r_i \in R(W)$, *in which the height increases by an even amount at some step; in that case,* $\mu(x, y) = 0$.

Proof The map $x \mapsto \min\{x, s_0 x\}$ is a (dual) closure operation on $\widetilde{X}$, with quotient poset X; we may thus apply [5, Theorem 1] (stated only for lattices in the reference, but the proof works for any locally finite poset) to compute the Möbius function of X from the Möbius function of $\widetilde{X}$. To be precise,

$$\mu(x, y) = \mu(\widetilde{x}, \widetilde{y}) + \mu(s_0 \widetilde{x}, \widetilde{y}), \tag{6.6}$$

since the preimage of x in $\widetilde{X}$ is $\{\widetilde{x}, s_0 \widetilde{x}\}$. Since $\widetilde{x} \leq \widetilde{y}$, the first term is given by

$$\mu(\widetilde{x}, \widetilde{y}) = (-1)^{\widetilde{\mathrm{ht}}(\widetilde{y}) - \widetilde{\mathrm{ht}}(\widetilde{x})} = (-1)^{\mathrm{ht}(y) - \mathrm{ht}(x)}. \tag{6.7}$$

The second term is either 0 or the negative of this and is nonzero precisely when $s_0 \widetilde{x} \leq \widetilde{y}$. The corollary then follows by Lemma 5.11. □

Remark 6.3 More generally, if one chooses $I \subset S$ such that each W_I-orbit is bounded below, then by Proposition 5.8 the operation π_I is again a dual closure operation with quotient the subposet X_I of W_I-minimal elements, and one has

$$\mu_{X_I}(\pi_I x, \pi_I y) = \sum_{z \in W_I x} \mu_X(z, \pi_I y). \tag{6.8}$$

It is unclear whether there is a simpler expression for the right-hand side.

We also have a generalization of the results of Björner and Wachs [3] regarding the shellability (and thus Cohen–Macaulay-ness) of Bruhat order on Coxeter groups.

Theorem 6.4 *Let* (X, ht) *be a bounded quasiparabolic transitive W-set. Then for any pair $x \leq y$ in X, the interval $[x, y]$ is lexicographically shellable. In particular, if* $\mathrm{ht}(y) - \mathrm{ht}(x) = d + 2 \geq 2$, *then the corresponding order complex is homeomorphic to either a d-sphere or a d-cell, depending on whether the Euler characteristic (i.e.,* $\mu(x, y)$*) is ± 1 or 0.*

Proof As in [3], we fix a reduced word

$$y = s_1 s_2 \cdots s_k x_0, \tag{6.9}$$

where x_0 is the minimal element of X, and label a saturated chain

$$x = x_0 < x_1 < x_2 < \cdots < x_l = y \tag{6.10}$$

by the sequence $(i_l, i_{l-1}, \ldots, i_1)$, where for each $1 \le m \le l$, i_m records the location in the reduced expression for y of the simple reflection being removed when passing from x_m to x_{m-1}. We then need only show that the chain with lexicographically minimal label is the unique chain with increasing label. The reduction in [3] to the existence of increasing chains of length 2 carries over mutatis mutandum, but the proof there for said existence involves cancellation and thus applies only to Bruhat order in groups.

We thus need to prove the following. Suppose $x < y$ with $\mathrm{ht}(y) = \mathrm{ht}(x) + 2$, and let $y = s_1 \cdots s_k x_0$ as above. Then there exists (uniqueness follows as in [3]) a pair $i < j$ such that

$$x = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k x_0 < s_1 \cdots \widehat{s_i} \cdots s_k x_0 < y. \tag{6.11}$$

Let i be the minimum index such that

$$x < s_1 \cdots \widehat{s_i} \cdots s_k x_0 < y. \tag{6.12}$$

Then x is obtained by removing some other simple reflection s_j from this word. We claim that $j > i$ and prove this by induction on $\mathrm{ht}(y)$.

We first observe that j cannot be 1 (note that this addresses the base case $\mathrm{ht}(y) = 2$). Indeed, we then see that

$$x = s_2 \cdots \widehat{s_i} \cdots s_k x_0 < s_2 \cdots s_k x_0 < s_1 \cdots s_k x_0 = y \tag{6.13}$$

since the fact that the chosen expression for y was reduced implies that

$$\mathrm{ht}(s_2 \cdots s_k x_0) = \mathrm{ht}(y) - 1. \tag{6.14}$$

In other words, i was not minimal, as we could have take $i = 1$.

Now, apply s_1 to the chain

$$x = s_1 \cdots \widehat{s_j} \cdots \widehat{s_i} \cdots s_k x_0 < s_1 \cdots \widehat{s_i} \cdots s_k x_0 < s_1 \cdots s_k x_0 = y. \tag{6.15}$$

Each of these expressions is reduced, so removing the s_1's preserves the ordering, and in particular $s_1 x < s_1 y$ with $\mathrm{ht}(s_1 y) = \mathrm{ht}(s_1 x) + 2$. Thus by induction, if i' is the corresponding minimal choice for passing from $s_1 y$ to $s_1 x$ (retaining the labels from y, since only the relative ordering matters), then there is a unique $j' > i'$ such that we have the chain

$$s_1 x = s_2 \cdots \widehat{s_{i'}} \cdots \widehat{s_{j'}} \cdots s_k x_0 < s_2 \cdots \widehat{s_{i'}} \cdots s_k x_0 < s_2 \cdots s_k x_0 = s_1 y. \tag{6.16}$$

Since j' is unique, we cannot have $i' = i$, since that would force $j' = j < i = i'$. On the other hand, i would be a legitimate choice for i', and thus minimality implies $i' < i$. Thus by the original minimality of i, multiplying *this* chain by s_1 cannot preserve the ordering. Again the fact that our original expression for y was reduced

implies that $\mathrm{ht}(y) > \mathrm{ht}(s_1 y)$, so the only way to break this chain is to change the ordering of the first two elements. But this implies by Lemma 5.7 that s_1 must *swap* the first two elements. In other words, the middle element in the chain (6.16) gives an expression for x, and we may thus construct the alternate chain

$$x = s_2 \cdots \widehat{s_{i'}} \cdots s_k x_0 < s_2 \cdots s_k x_0 < s_1 \cdots s_k x_0 = y. \tag{6.17}$$

But then we could again have chosen $i = 1$ in the first place.

The remaining topological statements follow from the above together with the fact that by our previous calculation of the Möbius function (i.e., Euler characteristic), any interval of length 2 has either one or two intermediate elements. In other words, the complex is thin. $\qquad \square$

7 Hecke algebra modules

Recall that for a Coxeter system (W, S), the Hecke algebra $H_W(q)$ is the $\mathbb{Z}[q]$-algebra with generators $T(s)$ for $s \in S$ with relations

$$\big(T(s)T(t)\big)^{m(s,t)/2} = \big(T(t)T(s)\big)^{m(s,t)/2} \tag{7.1}$$

if $m(s, t)$ is even,

$$\big(T(s)T(t)\big)^{(m(s,t)-1)/2} T(s) = \big(T(t)T(s)\big)^{(m(s,t)-1)/2} T(t) \tag{7.2}$$

if $m(s, t)$ is odd, and

$$\big(T(s) - q\big)\big(T(s) + 1\big) = 0. \tag{7.3}$$

More generally, one can choose parameters $q(s)$ for each s with the proviso that $q(s) = q(t)$ whenever s and t are conjugate. However, we will only consider the case of equal parameters in the sequel. The Hecke algebra has a natural basis $T(w)$ for $w \in W$ given by taking a reduced expression $w = s_1 s_2 \cdots s_k$, $k = \ell(w)$, and defining

$$T(w) = T(s_1)T(s_2) \cdots T(s_k); \tag{7.4}$$

since the Hecke algebra satisfies the braid relations, this is well defined. Moreover, one finds that in this basis, the generators have a particularly simple action:

$$T(s)T(w) = \begin{cases} T(sw), & \ell(sw) > \ell(w), \\ (q-1)T(w) + qT(sw), & \ell(sw) < \ell(w), \end{cases} \tag{7.5}$$

and similarly

$$T(w)T(s) = \begin{cases} T(ws), & \ell(ws) > \ell(w), \\ (q-1)T(w) + qT(ws), & \ell(ws) < \ell(w). \end{cases} \tag{7.6}$$

For convenience, we either extend scalars to $\mathbb{C}(q) \otimes_{\mathbb{Z}[q]} H_W(q)$ and consider representations over the field $\mathbb{C}(q)$, or we specialize q to some $q \in \mathbb{C}^\times$ (generically q is not a root of unity) and consider representations over $\mathbb{C}$ as specified below. The Hecke algebra $H_W(q)$ has two one-dimensional representations: the trivial $\mathbf{1}_+$ and sign (or alternating) $\mathbf{1}_-$ on which each generator $T(s)$ acts as $\varepsilon_+ := q$, respectively as $\varepsilon_- := -1$. We will also refer to the restriction to any subalgebra as its trivial, respectively sign, representation.

When we specialize to $q = 1$, $H_W(1) \cong \mathbb{Z}[W]$, and thus this is indeed a deformation of the group algebra of W. More generally, if W is finite, then $H_W(q) \otimes_{\mathbb{Z}[q]} \mathbb{C} \cong \mathbb{C}[W]$ for generic $q \in \mathbb{C}$, as in that case both algebras are semisimple (since any deformation of a semisimple algebra is generically semisimple); however, unlike the isomorphism for $q = 1$, this isomorphism does not respect the natural basis.

Any Coxeter homomorphism $\phi : W \to W'$ induces two natural homomorphisms $\phi_\pm$ of the corresponding Hecke algebras, by taking

$$\phi_\pm\big(T(s)\big) = \begin{cases} T(\phi(s)), & \phi(s) \in S', \\ \varepsilon_\pm, & \phi(s) = 1. \end{cases} \tag{7.7}$$

Of course if $\phi(S) \subseteq S'$, then $\phi_+ = \phi_-$, and we may omit the subscript. It suffices to check that the braid relation of length $m' = m'(\phi(s), \phi(t))$ is satisfied, as this implies the braid relation of length $m(s, t)$. Indeed, if one splits the left-hand side of the braid relation of length $m(s, t)$ into subwords of length m' and applies the braid relation of length m' to each such subword, one obtains the right-hand side of the desired braid relation.

As mentioned in the introduction, our original motivation for introducing quasi-parabolic subgroups was to construct modules for the Hecke algebra of W naturally deforming permutation representations.

From one perspective, the deformation problem is trivial (at least for finite W), as for generic $q \in \mathbb{C}$, $H_W(q) \otimes_{\mathbb{Z}[q]} \mathbb{C}$ is semisimple, and thus each irreducible representation of $\mathbb{C}[W]$ deforms to an irreducible representation of $H_W(q) \otimes_{\mathbb{Z}[q]} \mathbb{C}$. It follows that any representation deforms: simply decompose it as a sum of irreducibles and deform the irreducibles independently. This deformation suffers from two significant drawbacks, however. Not only are the formulas for the action of the generators on such a representation quite complicated, but also the coefficients of the action tend to have denominators, in sharp contrast to the Hecke algebra itself.

Given a standard parabolic subgroup W_I of W, there is a natural parabolic subalgebra $H_I(q)$ of $H_W(q)$ generated by those $T(s)$, $s \in I$. In that case, it also makes sense to induce representations of $H_I(q)$ to $H_W(q)$, via the functor $\mathrm{Ind}_{H_I(q)}^{H_W(q)} - := H_W(q) \otimes_{H_I(q)} -$. The analogue of the permutation representation of W on W/W_I is the representation induced from the trivial representation $\mathbf{1}_+$ of $H_I(q)$. (Theorem 7.11 below gives a quasiparabolic interpretation of slightly more general induced modules.) This representation has a basis naturally indexed by cosets W/W_I, and the action of the generators on this basis has a combinatorial description analogous to that given by (7.5), except that there is an additional case in which the height does not change.

In general, not all subgroups of W deform to a subalgebra of $H_W(q)$, so this induction construction cannot be used to deform the corresponding permutation representation. The parabolic case suggests a natural combinatorial action of the generators $T(s)$ in the case of a scaled W-set, but there are consistency conditions that must be satisfied in order for this to give a homomorphism. When the scaled W-set is quasiparabolic, these conditions are indeed satisfied, and we have the following.

Theorem 7.1 *Let (X, ht) be a quasiparabolic W-set, and let $T(X)$ be the free $\mathbb{Z}[q]$-module generated by elements $T(x)$ for $x \in X$. For $s \in S$, define endomorphisms $T_{\pm}(s)$ of $T(X)$ by*

$$T_{\pm}(s)T(x) = \begin{cases} T(sx), & \mathrm{ht}(sx) > \mathrm{ht}(x), \\ \varepsilon_{\pm} T(x), & \mathrm{ht}(sx) = \mathrm{ht}(x) \ (\implies sx = x), \quad (7.8) \\ (q-1)T(x) + qT(sx), & \mathrm{ht}(sx) < \mathrm{ht}(x). \end{cases}$$

Then the map $T(s) \mapsto T_+(s)$ (respectively $T(s) \mapsto T_-(s)$) afford $T(X)$ the structure of a $H_W(q)$-module, denoted by $H_X^{\pm}(q)$.

Remark 7.2 If X has even stabilizers, then $H_X^+(q)$ and $H_X^-(q)$ are naturally isomorphic, so in this case we may omit the sign.

Proof We need simply prove that $T_{\pm}(s)$ satisfy the relations of the Hecke algebra. The quadratic relation is straightforward (simply check the three cases), so it remains to check the braid relations. Since a braid relation only involves two simple reflections (say s and t), we may restrict X to the corresponding rank 2 parabolic subgroup of W, or equivalently, may assume that W has rank 2. We may also assume that X is transitive, as $H_X^{\pm}(q)$ is clearly a direct sum over orbits.

If $|W| = \infty$, there is nothing to check (as there is no braid relation in that case). Otherwise X has a minimal element, and by Lemma 2.19, the stabilizer of that element has the form

$$\langle (st)^{m'} \rangle, \ \langle (st)^{m'}, s \rangle, \ \langle (st)^{m'}, t \rangle \qquad (7.9)$$

for some m' dividing $m(s, t)$. We may thus (replacing W by a quotient as necessary and using the fact that Coxeter homomorphisms extend to Hecke algebras) reduce to the case that the stabilizer of the minimal element is trivial or generated by a simple reflection.

If the stabilizer is trivial, this is simply the regular representation of $H_W(q)$, so the result is immediate. Otherwise, say the stabilizer is $\langle s \rangle$. Then there is a natural isomorphism

$$T(X) \cong H_W(q)\big(T(s) - \varepsilon_{\mp}\big) \cong \mathrm{Ind}_{H_{\langle s \rangle}(q)}^{H_W(q)} \mathbf{1}_{\pm}, \qquad (7.10)$$

taking

$$T(wx) \mapsto T(w)\big(T(s) - \varepsilon_{\mp}\big) \qquad (7.11)$$

whenever $\mathrm{ht}(wx) = \ell(w) + \mathrm{ht}(x)$, and this respects the action of $T_{\pm}(s)$. $\qquad \square$

Remark 7.3 Note that even if s and t are not conjugate, there can be orbits with m' odd, forcing the corresponding parameters $q(s)$, $q(t)$ to agree; this is the reason for our simplifying assumption $q(s) \equiv q$.

Remark 7.4 In fact, in order for this to extend to a homomorphism, it suffices that the restriction of X to any rank 2 subgroup of W be quasiparabolic. Experimentally, this weaker condition is very nearly the same as quasiparabolicity (see Example 9.1 below) but fails to be preserved by many of the constructions above, especially Theorem 3.8 and its corollaries.

Taking $q = 0$ gives the following combinatorial fact. Define the Hecke monoid of a Coxeter group W to be the quotient $M(W)$ of the free monoid on idempotent generators $M(s)$, $s \in S$ by the braid relations.

Corollary 7.5 *There is a natural action of the Hecke monoid $M(W)$ on X in which the generators act by*

$$M(s)x = \max\{x, sx\}. \tag{7.12}$$

Proof We observe that apart from some sign changes, this is precisely the action of the generators of $H_W(0)$ on $H_X^-(0)$.   □

Remark 7.6 This action is compatible with Bruhat order in the sense of Richardson and Springer [14].

The construction in the case of a dihedral group with odd stabilizers generalizes as follows.

Proposition 7.7 *Let $\widetilde{X}$ be the even double cover of X. Then*

$$H_X^{\pm}(q) \cong H_{\widetilde{X}}(q)\big(T(s_0) - \varepsilon_{\mp}\big), \tag{7.13}$$

where s_0 generates the factor A_1 acting on $\widetilde{X}$. Over $\mathbb{Z}[q, 1/(q+1)]$, there is a natural isomorphism of $H_q(W)$-modules

$$H_{\widetilde{X}}(q) \cong H_X^+(q) \oplus H_X^-(q). \tag{7.14}$$

Specializing to $q = 1$, we find that $H_X^+(1)$ is simply the permutation representation associated to X, while $H_X^-(1)$ is the tensor product of this with the sign representation of W.

Recall the following notation. If M is an $H_W(q)$-module and N is an $H_{W'}(q)$-module with both M and N free over $\mathbb{Z}[q]$, then their tensor product, which carries the obvious $H_W(q) \otimes H_{W'}(q)$-module structure, is denoted $M \boxtimes N$. Note furthermore that $H_W(q) \otimes_{\mathbb{Z}[q]} H_{W'}(q) \cong H_{W \times W'}(q)$.

Proposition 7.8 *If (X, ht) and (X', ht') are quasiparabolic W- and W'-sets respectively, then there are natural isomorphisms*

$$H_X^+(q) \boxtimes H_{X'}^+(q) \cong H_{X \times X'}^+(q), \tag{7.15}$$

$$H_X^-(q) \boxtimes H_{X'}^-(q) \cong H_{X \times X'}^-(q). \tag{7.16}$$

Proposition 7.9 *Let $(X, \mathrm{ht}) = (W, \ell)$, viewed as a $W \times W$-set. Then there is a natural $H_W(q) \otimes H_W(q)$-module isomorphism*

$$H_X(q) \cong H_W(q). \tag{7.17}$$

Remark 7.10 Here we identify $H_W(q)^{\mathrm{op}}$ with $H_W(q)$ via the involution $T(w) \mapsto T(w^{-1})$.

It is not clear how to interpret the construction of Theorem 3.8 in the Hecke algebra setting. However, the special case of a quasiparabolic subgroup of a parabolic subgroup is straightforward.

Theorem 7.11 *Let $H \subset W_I$ be a quasiparabolic subgroup of the parabolic subgroup $W_I \subset W$. Then there is a natural isomorphism*

$$\mathrm{Ind}_{H_I(q)}^{H_W(q)} H_{W_I/H}(q) \cong H_{W/H}(q). \tag{7.18}$$

Proof It follows from standard results on induction of Hecke algebra modules from parabolic subalgebras that the induced module has a basis over $\mathbb{Z}[q]$ of the form $T(u) \otimes T(vH)$ with (u, vH) ranging over $W^I \times W_I/H$. The desired isomorphism is then given by

$$T(u) \otimes T(vH) \mapsto T(uvH). \tag{7.19}$$

This is clearly an isomorphism of free $\mathbb{Z}[q]$-modules, so it remains only to show that it is a homomorphism of $H_W(q)$-modules. Observe that there is in general a homomorphism

$$T(w) \otimes T(vH) \mapsto T(w)T(vH) \tag{7.20}$$

obtained via the left adjointness of induction to restriction from the natural inclusion $H_{W_I/H}(q) \subset H_{W/H}(q)$ of $H_I(q)$-modules. When $w = u \in W^I$, this map takes

$$T(u) \otimes T(vH) \mapsto T(u)T(vH) = T(uvH) \tag{7.21}$$

as required. (For the second equality (i.e., the fact that $\mathrm{ht}(uvH) = \ell(u) = \mathrm{ht}(vH)$), see Remark 3.11 following Corollary 3.10.)   □

One nice property of our Hecke algebras modules is the existence of a natural symmetric bilinear form. Define a pairing on $T(X)$ by

$$\langle T(x), T(y) \rangle = \delta_{xy} q^{\mathrm{ht}(x)} \tag{7.22}$$

and extending linearly.

Proposition 7.12 *The linear transformations $T_\pm(s)$ are self-adjoint with respect to this pairing.*

Proof We simply need to verify that

$$\langle T_\pm(s)T(x), T(y)\rangle = \langle T(x), T_\pm(s)T(y)\rangle. \tag{7.23}$$

If $\{x, sx\} \neq \{y, sy\}$, then both sides are 0, and if $y = x$, the claim is obvious by symmetry. We thus reduce to the case $y = sx \neq x$. Moreover, we may assume that $\mathrm{ht}(sx) = \mathrm{ht}(x) + 1$, as we can otherwise exchange x and y. But then

$$\langle T(s)T(x), T(sx)\rangle = q^{\mathrm{ht}(x)+1}, \tag{7.24}$$

while

$$\langle T(x), T(s)T(sx)\rangle = \langle T(x), qT(x) + (q-1)T(sx)\rangle = q q^{\mathrm{ht}(x)}. \tag{7.25}$$
□

In particular, in the case $X = W$, this is just the usual invariant inner product on the Hecke algebra, namely $\langle T(w), T(w')\rangle = \delta_{ww'}q^{\ell(w)}$.

Theorem 7.13 *Let W be a finite Coxeter group, and let (X, ht) be a transitive quasi-parabolic W-set with minimal element x_0. Then over $\mathbb{Z}[q, q^{-1}]$, there is a natural injection*

$$H_X^\pm(q) \to H_W(q) \tag{7.26}$$

of Hecke algebra modules.

Proof Simply take the adjoint of the surjection

$$H_W(q) \to H_X^\pm(q) \tag{7.27}$$

given by

$$T(w) \mapsto T(w)T(x_0). \tag{7.28}$$
□

In the case X is transitive and bounded from below, the corresponding Hecke algebra modules are cyclic, generated by $T(x_0)$, where x_0 is the minimal element of X. One can thus express $H_X^\pm(q)$ as a quotient $H_W(q)/I_X$, for some left ideal I_X, the annihilator of $T(x_0)$. It is therefore natural to ask whether we can give a nice set of generators for I_X. By Proposition 7.7, one can essentially reduce to the case X even; the ideal I_X can be obtained from the ideal $I_{\tilde{X}} \subset H_W(q)$ by adding one generator of the form $T(s) - \varepsilon_\mp, s \in S$.

Lemma 7.14 *Suppose H is an even quasiparabolic subgroup of W. The ideal $I_{W/H}$ is generated by elements of the form $T(w) - T(w')$ where $wH = w'H$ and $\mathrm{ht}(wH) = \ell(w) = \ell(w')$.*

Proof Since for any minimal coset representative w, $T(w)T(H) = T(wH)$, we see that the above elements are indeed in the ideal, so it remains to show that they generate. Choose a map $\phi : W/H \to W$ with the property that $\phi(x)H = x$ and $\mathrm{ht}(x) = \ell(\phi(x))$ for all $x \in W/H$, i.e., ϕ chooses a minimal representative of each coset. The elements $T(\phi(x))T(H) = T(x)$, $x \in W/H$, form a basis of $T(W/H)$, so we need simply to show that modulo the possibly smaller ideal, every element $T(w)$ is congruent to a linear combination of elements $T(\phi(x))$.

By induction on $\ell(w)$, it suffices to show that this holds for an element of the form $T(s)T(\phi(x))$. If $\mathrm{ht}(sx) = \mathrm{ht}(x) + 1$, then $s\phi(x)$ is a minimal coset representative, and therefore

$$T\big(s\phi(x)\big) - T\big(\phi(sx)\big) \tag{7.29}$$

is one of our chosen generators. Otherwise, $\mathrm{ht}(sx) = \mathrm{ht}(x) - 1$, in which case $s\phi(sx)$ is a minimal coset representative,

$$T\big(\phi(x)\big) - T\big(s\phi(sx)\big) \tag{7.30}$$

is one of our chosen generators, and

$$T\big(s\phi(x)\big) - \big(T(s) - (q-1)\big)\big(T\big(\phi(x)\big) - T\big(s\phi(sx)\big)\big)$$
$$= (q-1)T\big(\phi(x)\big) + qT\big(\phi(sx)\big). \tag{7.31}$$

$\square$

This set of generators is highly redundant, however, so we would like to find a small subset that still generates I_X. In general, it is too much to hope for this subset to be finite, but we can still reduce the complexity considerably. Note first that if w, w' are minimal representatives of the same coset of $H \subseteq W$, and there exists a simple reflection s such that $\ell(sw) = \ell(sw') = \ell(w) - 1$, then we have

$$T(w) - T\big(w'\big) = T(s)\big(T(sw) - T\big(sw'\big)\big), \tag{7.32}$$

so any such generator is redundant. More generally, if we could find an element w'' such that $\ell(sw) = \ell(sw'') = \ell(w) - 1$ and $\ell(tw') = \ell(tw'') = \ell(w') - 1$, $s, t \in S$, then

$$T(w) - T\big(w'\big) = \big(T(w) - T\big(w''\big)\big) - \big(T\big(w'\big) - T\big(w''\big)\big) \tag{7.33}$$

is also redundant. In general, $T(w) - T(w')$ will be redundant so long as there is any path from w to w' along the above lines. There is, however, an implied condition on s and t, namely that wH is $\{s, t\}$-maximal and moreover that the $\langle s, t\rangle$-orbit of wH has size $2m(s, t) = |\langle s, t\rangle|$. After all, w'' must have a reduced expression beginning with one side of the braid relation between s and t, and removing that subword gives a reduced expression for the minimal element of the $\langle s, t\rangle$-orbit. This suggests looking at the following graph Γ_x for each element $x \in W/H$: the vertices of Γ_x are precisely the simple reflections such that $\mathrm{ht}(sx) = \mathrm{ht}(x) - 1$, while the edges are the pairs $\{s, t\}$ such that the $\langle s, t\rangle$-orbit of x has size $2m(s, t)$. The above considerations tell us that

if the graph is connected, then *all* relations of the form $T(w) - T(w')$ arising from x are redundant, i.e., can be expressed in terms of relations arising from elements of smaller height. If the graph is not connected, we need simply add enough relations so that the corresponding additional edges make the graph connected.

Given a pair $\{s, t\}$ of vertices of Γ_x which is *not* an edge, there is a particularly nice choice of generator, as follows. Consider the $\langle s, t \rangle$-orbit of x, of size $2k$, k strictly dividing $m(s, t)$, and let y be the minimal element of that orbit, of height $\mathrm{ht}(y) = \mathrm{ht}(x) - k$. Choose an expression $y = wH$ with $\ell(w) = \mathrm{ht}(y)$ and observe that

$$(st)^{k/2}w \quad \text{and} \quad (ts)^{k/2}w \tag{7.34}$$

both give rise to reduced expressions for x (with the obvious interpretation of the $k/2$ power when k is odd, as in the braid relation), and thus

$$T\big((st)^{k/2}w\big) - T\big((ts)^{k/2}w\big) \tag{7.35}$$

is in the annihilator ideal. We call such generators *dihedral* generators based at x. Since adding an edge connecting vertices in different components of Γ_x reduces the number of components by 1, we obtain the following result.

Theorem 7.15 *The ideal I_X has a generating set consisting of dihedral generators, in such a way that the number of dihedral generators based at x in the generating set is one less than the number of components of Γ_x.*

Remark 7.16 When X is finite, this set of generators is finite (even if W itself is not) and in practice is quite small. For instance, in Example 9.2 below, we mention a quasiparabolic $E_8 \times A_2 \times A_2$-set X of size 113400 for which the above generating set consists of only 8 elements, whereas the first generating set we gave is *significantly* larger. Also observe that this gives rise to a fairly small set of generators of the quasiparabolic subgroup H stabilizing the minimal element.

8 Poincaré series

Given a quasiparabolic subgroup $H \subset W$, one natural invariant is the Poincaré series

$$\mathrm{PS}_{W/H}(q) := \sum_{x \in W/H} q^{\mathrm{ht}(x)} \in \mathbb{Z}[[q]]. \tag{8.1}$$

In the case $H = W_I$, one has

$$\mathrm{PS}_{W/W_I}(q)\,\mathrm{PS}_{W_I}(q) = \mathrm{PS}_W(q), \tag{8.2}$$

where the other two Poincaré series are in terms of the regular representation (i.e., the usual Poincaré series of a Coxeter group). In particular, it follows that when W is finite, the zeroes of $\mathrm{PS}_{W/W_I}(q)$ are roots of unity, as this is true for $\mathrm{PS}_W(q)$. The proof of this fact relies heavily on the fact that the restriction of W to a W_I-set is a union of $[W : W_I]$ copies of W_I, with appropriately shifted heights. No such decomposition exists for a general quasiparabolic subgroup, and yet the following still holds.

Theorem 8.1 *Let H be a quasiparabolic subgroup of the finite Coxeter group W. Then $\mathrm{PS}_{W/H}(q)$ is a divisor of $\mathrm{PS}_W(q)$.*

Proof We compute the product

$$\left(\sum_{w \in W} T_+(w)\right)\left(\sum_{x \in W/H} T(x)\right) \tag{8.3}$$

in two different ways. On the one hand, for any $w \in W$,

$$T_+(w) \sum_{x \in W/H} T(x) = q^{\ell(w)} \sum_{x \in W/H} T(x), \tag{8.4}$$

by induction on $\ell(w)$, so the product (8.3) is

$$\mathrm{PS}_W(q) \sum_{x \in W/H} T(x). \tag{8.5}$$

On the other hand, for all $x \in W/H$,

$$\left(\sum_{w \in W} T_+(w)\right) T(x) = q^{\mathrm{ht}(x)}\left(\sum_{w \in W} T_+(w)\right) T(H), \tag{8.6}$$

by induction on $\mathrm{ht}(x)$. Therefore,

$$\mathrm{PS}_{W/H}(q)\left(\sum_{w \in W} T_+(w)\right) T(H) = \mathrm{PS}_W(q) \sum_{x \in W/H} T(x). \tag{8.7}$$

But the coefficients of

$$\left(\sum_{w \in W} T_+(w)\right) T(H) \tag{8.8}$$

lie in $\mathbb{Z}[q]$, and thus so does the ratio of Poincaré series.   □

Remark 8.2 We may thus define a Poincaré series of H as the ratio

$$\mathrm{PS}_H(q) = \frac{\mathrm{PS}_W(q)}{\mathrm{PS}_{W/H}(q)}. \tag{8.9}$$

As observed in Remark 3.11 following Corollary 3.10, this is preserved by induction from parabolic subgroups. Note that because the proof that $\mathrm{PS}_H(q)$ is a polynomial is not combinatorial in nature, we do not obtain an interpretation of $\mathrm{PS}_H(q)$ as a generating function for elements of H. There does, however, seem to be a surprising amount of structure in $\mathrm{PS}_H(q)$. For instance, Example 9.5 below discusses a quasi-parabolic subgroup $H \subset E_8$ with

$$\mathrm{PS}_H(q) = \frac{(1 - q^2)(1 - q^{12})(1 - q^{20})(1 - q^{30})}{(1 - q)^4} = \mathrm{PS}_{H_4}(q), \tag{8.10}$$

and indeed H is abstractly isomorphic to the Coxeter group H_4; similar agreement appears to occur whenever the quasiparabolic subgroup is abstractly isomorphic to a Coxeter group. Even more strikingly, there are several examples of Poincaré series of a similar form in which the degrees of invariants are replaced by degrees of invariants in characteristic 2, see Example 9.4 below.

Corollary 8.3 *With hypotheses as above,*

$$q^m \, \mathrm{PS}_{W/H}\big(q^{-1}\big) = \mathrm{PS}_{W/H}(q), \qquad (8.11)$$

where $m = \max_{x \in W/H} \mathrm{ht}(x)$.

Proof It is known that the zeros of $\mathrm{PS}_W(q)$ are all roots of unity, and thus the same is true for its divisor $\mathrm{PS}_{W/H}(q)$. Since $\mathrm{PS}_{W/H}(q)$ has integer coefficients, its roots are permuted by the absolute Galois group of $\mathbb{Q}$. In particular, for every root of $\mathrm{PS}_{W/H}(q)$, its complex conjugate is also a root, or in other words, the reciprocal of each root is a root. But then

$$q^m \, \mathrm{PS}_{W/H}\big(q^{-1}\big) \qquad (8.12)$$

has the same roots with multiplicities as $\mathrm{PS}_{W/H}(q)$ (note that $m = \deg(\mathrm{PS}_{W/H}(q))$). Since W/H has unique maximal and minimal elements, both polynomials are monic, with the same roots, so must agree. $\square$

If W is infinite, it is too much to hope for the ratio $\mathrm{PS}_W(q)/\mathrm{PS}_{W/H}(q)$ to be a polynomial. When W is affine, we suspect that an analogous statement should hold, to wit that the ratio is a rational function, with zeros and poles only at roots of unity. The above methods appear to be completely insufficient for this case, however.

Despite the symmetry of the above corollary, the scaled W-sets $(W/H, \mathrm{ht})$ and $(W/H, m - \mathrm{ht})$ are not in general isomorphic. In particular, in such a case, we obtain two different deformations of the same permutation representation. We conjecture that not only are these deformations isomorphic, but also that the isomorphism can be chosen to have a particularly nice form. If (X, ht) is a scaled W-set, let X^- denote the scaled W-set $(X, -\mathrm{ht})$.

Conjecture 8.4 *Let $H \subset W$ be a quasiparabolic subgroup. Then there is an isomorphism (with coefficients in $\mathbb{Z}[q, 1/q]$)*

$$H^{\pm}_{W/H}(q) \cong H^{\pm}_{(W/H)^-}(q) \qquad (8.13)$$

of $H_W(q)$-modules in which $T^{\pm}(H)$ maps to $T^{\pm}(H)$.

Recall that on the left, the Hecke algebra acts by

$$T(s)T_+(x) = \begin{cases} T_+(sx), & \mathrm{ht}(sx) > \mathrm{ht}(x), \\ \varepsilon_{\pm} T_+(x), & \mathrm{ht}(sx) = \mathrm{ht}(x), \\ (q-1)T_+(x) + qT_+(sx), & \mathrm{ht}(sx) < \mathrm{ht}(x), \end{cases} \qquad (8.14)$$

while on the right, the Hecke algebra acts by

$$T(s)T_-(x) = \begin{cases} T_-(sx), & \text{ht}(sx) < \text{ht}(x), \\ \varepsilon_{\pm}T_-(x), & \text{ht}(sx) = \text{ht}(x), \\ (q-1)T_-(x) + qT_-(sx), & \text{ht}(sx) > \text{ht}(x). \end{cases} \quad (8.15)$$

We may rewrite this in terms of $T'(s) = q - 1 - T(s)$, $T'_-(x) = (-q)^{\text{ht}(x)}T_-(x)$:

$$T'(s)T'_-(x) = \begin{cases} T'_-(sx), & \text{ht}(sx) > \text{ht}(x), \\ \varepsilon_{\mp}T'_-(x), & \text{ht}(sx) = \text{ht}(x), \\ (q-1)T'_-(x) + qT'_-(sx), & \text{ht}(sx) < \text{ht}(x). \end{cases} \quad (8.16)$$

It follows that the desired isomorphism exists iff the annihilator of $T(H)$ in $T^{\pm}(W/H)$ is taken to the annihilator of $T(H)$ in $T^{\mp}(W/H)$ by the automorphism $T(s) \mapsto T'(s)$.

Since in finite cases we can find relatively small generating sets for these annihilators, it is straightforward in most cases to verify the existence of these isomorphisms. In particular, this isomorphism exists in every finite case we have checked.

Note that the coefficients of such an isomorphism would give an analogue for quasiparabolic W-sets of the R-polynomials of Kazhdan–Lusztig theory. There is a formula for the latter polynomials due to Deodhar [7] expressed in terms of a generating function for "distinguished" subexpressions. It appears that there is no obstacle to constructing the corresponding generating function in the quasiparabolic setting, but Deodhar's proof makes essential use of the fact that the natural family of recurrences for R-polynomials are consistent. In other words, if we knew that our isomorphism existed, there would almost certainly be a formula for the relevant coefficients à la Deodhar. In the few cases where quasiparabolic W-sets carry the structure of a "diamond," [4] has constructed corresponding R-polynomials and Kazhdan–Lusztig polynomials.

In addition, since the theory of Kazhdan–Lusztig polynomials themselves has analogues for quotients by parabolic subgroups, we expect there to be a corresponding analogue for quotients by quasiparabolic subgroups.

9 Examples

Example 9.1 We begin with an example of a non-quasiparabolic W-set. Let B_3 be the hyperoctahedral group of signed permutations, with simple reflections (12), (23), and $(3)_-$, and consider the subgroup $H \subset B_3$ of order 8 generated by $(13)(2)_-$, $(1)_-$, and $(3)_-$, and the corresponding self-dual scaled B_3-set B_3/H. This, together with its even subgroup, is the only indecomposable example we know of a non-quasiparabolic subgroup such that all restrictions to rank 2 are quasiparabolic. We also note that this subgroup violates many of the conclusions of our theorems above; for instance, it fails to map to a quasiparabolic subgroup of $A_2 \times A_1$ (not even when restricted to rank 2) and does not induce a well-behaved Bruhat order.

Example 9.2 One of the more fruitful constructions is the fact that perfect involutions form quasiparabolic W-sets. As we mentioned above, it is relatively straightforward to classify perfect involutions in finite Coxeter groups. Note that if W is a product, then the relevant diagram automorphism permutes the factors of W, and if the action on factors is not transitive, then the corresponding sets of perfect involutions are just products of the sets of perfect involutions from each orbit. In addition, if $W = W' \times W'$ and the diagram automorphism swap the factors, then up to conjugation by a further diagram automorphism, the only perfect involutions come from the diagonal. Thus the only interesting cases arise from diagram automorphisms of simple Coxeter groups.

In type A, as we mentioned above, there are two noncentral conjugacy classes of perfect involutions, one with the trivial diagram automorphism and one with the nontrivial diagram automorphism. Indeed, it is easy to see that a perfect involution must act as the inner automorphism corresponding to a fixed-point-free involution. (If r is the reflection swapping a fixed point and a non-fixed point of the image of ι in S_n, then ιr would have order 3 or 6.) Note that although the two resulting quasiparabolic sets are dual (i.e., related by negating heights), the stabilizers of their respective minimal elements are qualitatively quite different. For instance, the centralizer of the diagram automorphism contains only one simple reflection, while the centralizer of the minimal fixed-point-free involution in S_{2n} contains n simple reflections. However, it follows by considering the corresponding ideals that the two Hecke algebra modules are isomorphic as in Conjecture 8.4 and the discussion following it.

In type B/C, a noncentral order 2 element of B_n is a perfect involution iff its image in S_n is the identity or a fixed-point-free involution. In the former case, the small subgroup $z_W(\iota)$ contains the kernel of the natural Coxeter homomorphism $B_n \to A_{n-1} \times A_1$, and the image is a product of the form $\mathrm{Alt}_j \times \mathrm{Alt}_{n-j}$, embedded in the natural way. In the latter case, $z_W(\iota)$ is just the even subgroup of $Z_W(\iota)$.

In type D, again the image in S_n of a perfect involution must be the identity or a fixed-point-free involution. The first case includes all perfect involutions that involve the nontrivial diagram automorphism and is analogous to the B case. The fixed-point-free involutions now come in two conjugacy classes (swapped by the diagram automorphism), and the corresponding quasiparabolic sets are dual (i.e., differ by reversing the heights). As mentioned above, this is the one case where we do not know whether $z_W(\iota)$ (an index 4 subgroup of the centralizer of the minimal ι) is quasiparabolic.

It remains to consider the sporadic cases. Other than B_2, no dihedral group has a noncentral perfect involution (even including diagram automorphisms), and similarly for H_3 and H_4. For F_4, the noncentral perfect involutions (none of which involve the nontrivial diagram automorphism) form a single conjugacy class generated by the longest element of the parabolic $B_2 \subset F_4$. In each of E_6, E_7, and E_8, the longest element of the parabolic D_4 is perfect; in E_6 and E_7, one also has another conjugacy class of perfect involutions giving a dual quasiparabolic set. (For E_6, take the conjugacy class of the nontrivial diagram automorphism; for E_7, the other minimal involution is the product of three commuting roots.)

Note that for E_8, the small group $z_W(\iota)$ has index $36 = |A_2 \times A_2|$ in the centralizer, so we obtain an action of $E_8 \times A_2 \times A_2$ on a set of order 113400. Explicit computation gives the eight dihedral generators mentioned above: six of length 2 and two of

length 8. For E_7, the small group has index $12 = |A_2 \times A_1|$ giving a quasiparabolic action of $E_7 \times A_2 \times A_1$ on a set of size 3780.

Given a transitive quasiparabolic W-set X, sometimes we can extend the action to a larger Coxeter group W' in which W is standard parabolic, while retaining quasiparabolicity. Note that by Theorem 5.12, the Bruhat order as a W'-set will be the same as the original Bruhat order. We can thus produce a relatively short list of candidates for the actions of simple reflections in W'. We find, at least when X has even stabilizers, that a simple reflection in W' must be a special matching in the sense of [4]. There is, of course, the additional requirement that if we adjoin a new simple reflection, that all resulting new reflections must only swap comparable elements. Note, however, that even when W' is infinite, if X is finite, the image of W' in $\mathrm{Sym}(X)$ is finite, so it is a finite computation to verify quasiparabolicity. Moreover, only the condition of comparability remains to be checked, via Proposition 5.5. Note that as a special case, if W is a Weyl group, and the reflection in one of its highest roots induces a special matching, we can always extend to a quasiparabolic action the corresponding affine Weyl group, as this will not enlarge the image of $R(W)$ inside $\mathrm{Sym}(X)$.

Example 9.3 Let ι be the diagram automorphism of E_6, which as we have observed is perfect. We can directly check that the normal subgroup $z_{E_6}(\iota)$ (of index $6 = |A_2|$ in the centralizer) is quasiparabolic and thus gives rise to a transitive quasiparabolic $E_6 \times A_2$-set of size 270. The stabilizer in $E_6 \times A_2$ of the minimal element is abstractly isomorphic to F_4, and the corresponding Poincaré series agree.

There are, it turns out, precisely 10 special matchings on this set, of which eight are accounted for by the simple reflections in $E_6 \times A_2$. If we adjoin the remaining two special matchings, we obtain an action of a Coxeter group O_{10} of rank 10 on X, with diagram as in Fig. 1.

(The image in $\mathrm{Sym}(X)$ is isomorphic to $E_8/Z(E_8)$, but this does not factor through a Coxeter homomorphism $O_{10} \to E_8$.) It is computationally straightforward to verify that this gives a quasiparabolic action of the quite large Coxeter group O_{10} on X; moreover, the diagram automorphisms of O_{10} extend to Bruhat-preserving automorphisms of X. Since O_{10} is simply-laced, it has only one conjugacy class of reflections, which correspond to the 120 reflections of E_8. Various parabolic subgroups of O_{10} act transitively: any subgroup containing one of the two parabolic subgroups of type E_6 is transitive, as are the four parabolic subgroups of type A_8.

Fig. 1 Coxeter diagram of O_{10}
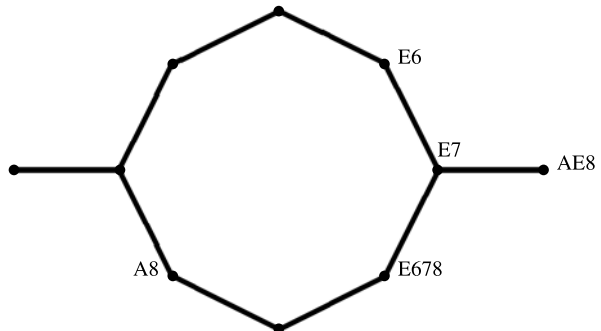
In particular, we obtain a quasiparabolic subgroup of E_7 abstractly isomorphic to $\mathbb{F}_2^7 \rtimes \text{PGL}_3(2)$, and a quasiparabolic subgroup of E_8 abstractly isomorphic to $2^{1+6} \rtimes \text{PGL}_4(2)$ (where 2^{1+6} denotes a 2-group of order 128 with center of order 2). We also obtain quasiparabolic actions of the corresponding affine groups, in which the additional reflection has the same action on X as the reflection in the highest root. The quasiparabolic subgroup of A_8 we obtain is actually contained in a maximal parabolic subgroup of type A_7 which we consider in the next example.

O_{10} contains various transitive parabolic subgroups which can be obtained in several ways and in particular by removing the $s \in S$ corresponding to marked nodes as in Fig. 1. One obtains A_8 removing the two nodes marked A8 and AE8; one obtains $E_6 \times A_2$ removing the two nodes marked E6 and E678; one obtains $\tilde{E}_8$ removing the node marked E678, and then E_8 by further removing AE8; one obtains $\tilde{E}_7 \times A_1$ removing the node marked E7, and then $E_7 \times A_1$ by further removing E678.

Example 9.4 The action of A_8 in the previous example is induced from an action of A_7 on a set of size 30, with quasiparabolic subgroup isomorphic to $\text{AGL}_3(2)$. The Poincaré series of this subgroup has the striking structure

$$\text{PS}_{\text{AGL}_3(2)}(q) = \frac{(1-q^4)(1-q^6)(1-q^7)(1-q^8)}{(1-q)^4}. \tag{9.1}$$

If $\text{AGL}_3(2)$ had been a Coxeter group, we would have concluded that its invariant ring was freely generated by elements of degrees 4, 6, 7, and 8. Surprisingly, there is indeed an action of $\text{AGL}_3(2)$ with such an invariant ring, but in characteristic 2. Indeed, there are two actions of $\text{AGL}_3(2)$ on $\bar{\mathbb{F}}_2^4$, one with an invariant subspace of dimension 1 and one with an invariant subspace of dimension 3. The latter has precisely the desired invariants.

We could also have obtained this set as in the previous example, beginning with either of the classes of perfect involutions in A_5. There is also a transitive action of A_4, induced from the action of A_3 on A_2 via the natural Coxeter homomorphism. In the other direction, the reflection in the highest root of A_7 is a special matching, and thus we obtain a quasiparabolic action of $\tilde{A}_7$ on this set.

This characteristic 2 invariant theory phenomenon also arises for two other quasiparabolic subgroups of symmetric groups. First, the transitive action of A_6 on the same set of size 30 has stabilizer $\text{PGL}_3(2) \cong \text{GL}_3(2)$. The invariant ring of $\text{GL}_3(2)$ in its three-dimensional characteristic 2 representation is freely generated by invariants of degrees 4, 6, and 7, which again agrees with the Poincaré series of the subgroup.

The other example comes from the transitive action of Alt_5 on a set of six elements. It turns out that one representative of the resulting conjugacy class of subgroups of $A_5 \cong S_6$ is actually quasiparabolic, of index 12. There is a three-dimensional representation of Alt_5 in characteristic 2, namely $\text{Alt}_5 \cong O_3^-(\mathbb{F}_4)$ (in the version with a two-dimensional invariant subspace), with invariant ring freely generated by elements of degrees 2, 5, and 6. Once more, the Poincaré series of this quasiparabolic Alt_5 agrees with the product suggested by the degrees of invariants.

It should be noted, however, that not all quasiparabolic subgroups have a Poincaré series of this form. For instance, the Poincaré series of the index 113400 quasi-

parabolic subgroup $H \subset E_8$ has the factorization

$$\mathrm{PS}_H(q) = \frac{(1+q^3)(1+q^6)(1+q^9)(1+q^5)(1+q^{10})(1+q^{15})(1-q^8)(1-q^{12})}{(1-q)^2}.$$

(9.2)

Note that this still has positive coefficients. There would seem to be no particular reason why such Poincaré series of nonparabolic quasiparabolic subgroups should have positive coefficients, but we do not know a counterexample.

Example 9.5 Inside E_8, apart from quasiparabolic subgroups of parabolic subgroups, the above index 270 example, and the index 113400 example coming from perfect involutions, there are essentially two more quasiparabolic subgroups (apart from those obtained via Theorem 3.16). The larger of the two (i.e., with the smaller E_8-set) corresponds to a subgroup of $E_8 \times A_1$ isomorphic to $\mathbb{F}_2^8 \rtimes \mathrm{AGL}_3(2)$, of index 4050. Again, the highest root induces a special matching, so we obtain an action of $\tilde{E}_8 \times A_1$. The other corresponds to an even subgroup of E_8 abstractly isomorphic to H_4, of index 48384; as in the $F_4 \subset E_6$ case above, again the Poincaré series of the subgroup is the same as its Poincaré series as a Coxeter group, despite the fact that it is far from being a reflection subgroup of E_8.

For E_7 and E_6, there are no quasiparabolic subgroups other than those already mentioned, or those they produce via Theorem 3.16.

Example 9.6 The case of fixed-point-free involutions in A_{2n-1} has a nice geometric interpretation due to Richardson and Springer [14]. Let k be an algebraically closed field of characteristic $\neq 2$, and consider the algebraic group $G = \mathrm{GL}_{2n}(k)$, with the Borel subgroup B of upper triangular matrices. It is classical that the double cosets $B \backslash G / B$ are in natural bijection with A_{2n-1}, with dimension essentially given by length. If we now consider the subgroup $H = \mathrm{Sp}_{2n}(k) \subset G$, we may consider instead the double cosets $B \backslash G / H$. It turns out that these are in natural bijection with fixed-point-free involutions in A_{2n-1}, or somewhat more naturally, with conjugates of the diagram automorphism. Moreover, the dimension of such a double coset is (up to an additive constant) given by height. The action of A_{2n-1} is somewhat tricky to reconstruct from the geometry, but the action of the corresponding Hecke monoid is straightforward: if one decomposes

$$Bs\,B\iota H$$

(9.3)

into double cosets (where $B\iota H$ denotes the double coset identified with ι), there will be a unique double coset of maximal dimension, which gives the image of ι under the monoid action of s. More generally, the Bruhat order on the given conjugacy class of involutions corresponds to inclusion of closures of double cosets.

One can also obtain the corresponding Hecke algebra module for q a prime power by considering the same double cosets, but now over the finite field $\mathbb{F}_q$. It appears that similarly the double cosets

$$B \backslash \mathrm{GL}_{2n}(\mathbb{F}_q) / \mathrm{GL}_n(\mathbb{F}_{q^2}),$$

(9.4)

where we map $\mathbb{F}_{q^2} \to \mathrm{Mat}_2(\mathbb{F}_q)$ in the obvious way, give rise to the Hecke algebra module corresponding to the dual quasiparabolic set.

The (extended) affine Weyl group $\tilde{A}_{2n-1}$ has a similar geometric interpretation in terms of double cosets of the Iwahori subgroup of $\mathrm{GL}_{2n}(K)$ where K is now a local field with residue field $\mathbb{F}_q$. One could then replace one of the Iwahori subgroups with $\mathrm{Sp}_{2n}(K)$ or $\mathrm{GL}_n(L)$ for either of the two quadratic extensions L/K and consider the resulting double cosets. In each case, a back of the envelope calculation suggests that the double cosets are classified by suitable conjugacy classes of involutions. This gives rise to three conjecturally quasiparabolic actions of $\tilde{A}_{2n-1}$. The corresponding (conjectural) ideals in the Hecke algebra were used in [13], along with two analogous ideals in $H_{\tilde{C}_{2n}}$.

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