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A classification of smooth convex 3-polytopes with at most 16 lattice points

Anders Lundman

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Abstract We provide a complete classification up to isomorphism of all smooth convex lattice 3-polytopes with at most 16 lattice points. There exist in total 103 different polytopes meeting these criteria. Of these, 99 are strict Cayley polytopes and the remaining four are obtained as inverse stellar subdivisions of such polytopes. We derive a classification, up to isomorphism, of all smooth embeddings of toric threefolds in $\mathbb{P}^N$ where $N \leq 15$. Again we have in total 103 such embeddings. Of these, 99 are projective bundles embedded in $\mathbb{P}^N$ and the remaining four are blow-ups of such toric threefolds.

Keywords Smooth · Lattice polytopes · Toric varieties · Cayley polytopes · Toric fibrations

1 Introduction

There exists a fascinating correspondence between convex lattice polytopes and embeddings of toric varieties via complete linear series. In particular two embeddings of toric varieties are isomorphic if and only if the corresponding polytopes are isomorphic, i.e. if they differ by a lattice preserving affine isomorphism. Let $M \cong \mathbb{Z}^d$, recall that a d-dimensional convex lattice polytopes $P \subset M \otimes \mathbb{R}$ is called smooth if there are exactly d edges through every vertex of P and the edge-directions form a lattice basis for M. A d-dimensional toric variety embedded in $\mathbb{P}^k$ is smooth if and only if the corresponding d-dimensional convex lattice polytope is smooth (see [3] for details).

It has recently been proven in [1] that for any $d, k \in \mathbb{Z}^+$ there are, up to isomorphism, only finitely many smooth convex lattice d-dimensional polytopes $P \subset \mathbb{R}^d$

A. Lundman (✉)
Department of Mathematics, KTH, 100 44 Stockholm, Sweden
e-mail: alundman@math.kth.se

such that $|P \cap \mathbb{Z}^d| \le k$. By the correspondence mentioned above this implies that for a fixed choice of $d, k \in \mathbb{Z}^+$ there are, up to isomorphism, only finitely many embeddings of smooth toric varieties of dimension d into $\mathbb{P}^{k-1}$. An alternative proof for this theorem has also been given in [11]. From an elaboration of the proof given in [1] a complete classification of all smooth convex lattice d-dimensional polytopes $P \subset \mathbb{R}^d$ such that $|P \cap \mathbb{Z}^d| \le 12$ has been constructed by Lorenz in [10] for $d = 2$ and $d = 3$. The classification given by Lorenz relies on extensive calculations using the program Polymake.

In this paper we utilize Lorenz' classification of all smooth 2-dimensional convex lattice polytopes to obtain a classification of all smooth 3-dimensional convex lattice polytopes P such that $|P \cap \mathbb{Z}^3| \le 16$ as well as the corresponding toric embeddings. We prove the following.

Theorem 1 *Up to isomorphism there exist exactly* 103 *smooth* 3*-dimensional convex lattice polytopes* $P \subset \mathbb{R}^3$ *such that* $|P \cap \mathbb{Z}^3| \le 16$. *Equivalently there are, up to isomorphism,* 103 $\mathbb{P}^k$*-embeddings of smooth* 3*-dimensional toric varieties such that* $k \le 15$.

The smooth 3-dimensional convex lattice polytopes with at most 12 lattice points appearing in our classification coincide exactly with the 3-dimensional polytopes in the classification given in [10]. Our classification is obtained by analyzing the geometrical constraints imposed by the hypothesis. A key step in our approach is Lemma 25 in which we prove that any smooth 3-dimensional convex lattice polytope P such that $|P \cap \mathbb{Z}^3| \le 16$ has at most eight facets. This makes it possible to use the classification of triangulations of the 2-sphere given in [12, p. 59] to get the number of edges in the facets of any 3-dimensional convex lattice polytope meeting our restrictions.

The polytopes and embeddings appearing in our classification fall naturally into four categories; see Sect. 2. We will prove Theorem 1 in two steps. First we show in Proposition 16 that any smooth 3-dimensional convex lattice polytope P such that $|P \cap \mathbb{Z}^3| \le 16$ has to lie in one of the four categories. Sections 5 and 6 are then devoted to classifying all polytopes meeting our restrictions in each of the four categories. A complete list of polytopes and embeddings can be found in the Appendix.

This paper is based on the authors master thesis at the Department of Mathematics at KTH in Stockholm.

2 Notation and background

Let P be a d-dimensional convex lattice polytope in $\mathbb{R}^n$ and let Σ be the inner-normal fan of P. The polytope P defines an embedding of a d-dimensional toric variety X_Σ in $\mathbb{P}^{k-1}$ where $k = |P \cap \mathbb{Z}^d|$. Such embeddings will be called complete embeddings as they are defined by the complete linear system of the associated ample line bundle. For more details we refer to [3, 7]. In this paper we will call a convex lattice polytope of dimension d simply a *d-polytope*. Moreover a strongly convex rational polyhedral cone will be called simply a *cone* and a complete polyhedral fan is called simply a *fan*.

Definition 2 Let $N \cong \mathbb{Z}^d$ be a lattice and σ be a d-dimensional cone in $N \otimes_{\mathbb{R}} \mathbb{R}$. We call σ *unimodular* if there exist d lattice vectors $v_1, \dots, v_d \in N$ such that σ is the positive linear span of $v_1, \dots, v_d$ in $N \otimes_{\mathbb{R}} \mathbb{R}$ and $v_1, \dots, v_d$ form a lattice basis for N. A fan Σ is called *unimodular* if all cones in Σ are unimodular.

Smoothness of a toric variety can be defined in a strict algebraic geometry setting [3]. In fact the following statements are equivalent.

Proposition 3 [8, §2.1] *Let P be a full-dimensional polytope with inner-normal fan Σ and let X_Σ be the toric variety defined by Σ, then the following are equivalent*:

(i) *P is smooth*
(ii) *Σ is unimodular*
(iii) *X_Σ is smooth.*

2.1 Stellar subdivisions and blow-ups

For more details on blow-ups we refer to [7, §VI 7.]. Recall that the *relative interior* of a cone $\sigma \subset \mathbb{R}^n$ is the set of points $x \in \sigma$ such that there exists some ball $B \subset \sigma$ containing x. For a given cone $\sigma \subset \mathbb{R}^n$ we denote the relative interior of σ by $\operatorname{relint}(\sigma)$. Given two cones σ, τ we denote by $\operatorname{Cone}(\sigma \cup \tau)$ the cone spanned by the union of the spanning vectors in σ and τ.

Definition 4 Let σ be a cone and Σ a fan. Assume that $\operatorname{Span}(\sigma) \cap \operatorname{Span}(\sigma') = \{0\}$ for every $\sigma' \in \Sigma$ and that $\operatorname{relint}(\operatorname{Cone}(\sigma \cup \sigma')) \cap \operatorname{relint}(\operatorname{Cone}(\sigma \cup \sigma'')) = \emptyset$ for all $\sigma', \sigma'' \in \Sigma$ such that $\sigma' \neq \sigma''$. The *join* of σ and Σ is defined as

$$\sigma \cdot \Sigma := \bigl\{\operatorname{Cone}(\sigma \cup \tau) : \forall \tau \in \Sigma\bigr\}.$$

Let Σ be a fan. Then the *star* of a cone $\sigma \in \Sigma$ is the set $\operatorname{Star}_\Sigma(\sigma) := \{\tau \in \Sigma : \sigma \text{ is a face of } \tau\}$. The *closed star* of $\sigma \in \Sigma$ is the set $\overline{\operatorname{Star}_\Sigma(\sigma)} = \{\tau \in \Sigma : \tau \text{ is a face of } \tau' \in \operatorname{Star}_\Sigma(\sigma)\}$.

Definition 5 Let Σ be a fan, $\sigma \in \Sigma$ be a cone, $p \in \operatorname{relint}(\sigma)$ be a point and $\rho = \mathbb{R}_{\geq 0} p$ be the ray spanned by p. We call the set

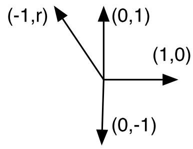
$$s(\Sigma; \rho) := \bigl(\Sigma \setminus \operatorname{Star}_\Sigma(\sigma)\bigr) \cup \rho \cdot \bigl(\overline{\operatorname{Star}_\Sigma(\sigma)} \setminus \operatorname{Star}_\Sigma(\sigma)\bigr)$$

the *stellar subdivision* of Σ in direction p. The fan Σ is called the *inverse stellar subdivision* of $s(\Sigma; \rho)$.

By $\mathbb{F}_r$ we denote the *Hirzebruch surface* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$. Recall that the defining fan of $\mathbb{F}_r$ is $\Sigma_r := \{\{0\}, \operatorname{Cone}((1,0)), \operatorname{Cone}((0,1)), \operatorname{Cone}((0,-1)), \operatorname{Cone}((-1,r)), \operatorname{Cone}((1,0),(0,1)), \operatorname{Cone}((1,0),(0,-1)), \operatorname{Cone}((0,-1),(-1,r)), \operatorname{Cone}((-1,r),(0,1))\}$ illustrated in Fig. 1.

Example 1 Let Σ be the fan of $\mathbb{P}^2$. The fan Σ_1 is the stellar subdivision of Σ in direction $p = (0,1)$.

Fig. 1 The fan of $\mathbb{F}_r$



The stellar subdivision of a unimodular fan in direction p is called *unimodular* if $s(\Sigma, \rho)$ in Definition 5 is unimodular. Remember our convention that fans are always complete.

Remark 1 For a fan $\Sigma \subset \mathbb{R}^n$ and a unimodular stellar subdivision $s(\Sigma, \rho) \subset \mathbb{R}^n$ the identity map $\mathbb{R}^n \to \mathbb{R}^n$ is a map of fans $s(\Sigma, \rho) \to \Sigma$ and therefore induce a toric morphism $X_{s(\Sigma,\rho)} \mapsto X_\Sigma$. Note that $X_{s(\Sigma,\rho)}$ is a blow-up of X_Σ.

All unimodular stellar subdivisions of a unimodular fan Σ are characterized by the following theorem.

Theorem 6 *Let Σ be a unimodular fan and assume that $\sigma = \mathrm{Cone}(x_1, \dots, x_r) \in \Sigma$ is a cone where $x_1, \dots, x_r$ are linearly independent lattice vectors that generate $\sigma \cap \mathbb{Z}^n$. Let $\rho = \mathbb{R}_{\geq 0} p$ where p generates $\rho \cap \mathbb{Z}^n$. Then $s(\rho; \sigma)$ is an unimodular stellar subdivision if and only if*

$$p = x_1 + \cdots + x_r.$$

Proof See [7, p. 179]. □

Let Σ be a unimodular fan and let $\sigma \in \Sigma$. As a consequence of Theorem 6 we write $s(\sigma) = s(\Sigma, \rho)$ for a unimodular stellar subdivision $s(\Sigma, \rho)$ of Σ. We refer to the blow-up associated to $s(\sigma)$ as the blow-up of X_Σ at $X_{\check{\sigma}}$.

Let Σ be the inner-normal fan of a smooth polytope P and let L_P be the associated ample line bundle whose global sections define the corresponding embedding, see [3, Chap. 6] for more details. Consider a unimodular stellar subdivision Σ' given by $s(\sigma)$ where $\sigma \in \Sigma$. Let F be the face of P associated to σ and $\pi : X_{\Sigma'} \to X_\Sigma$ be the induced blow-up map with exceptional divisor E. When the line bundle $\pi^* L_P - kE$ is ample for $k \geq 1$, it defines a polytope P' obtained by cutting off the face F at level k. We will denote the polytope P' by $\mathrm{Bl}_F^k(P)$. Consider for example $(X_P, L_P) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ and choose a fixed point p corresponding to the vertex v in P. Then the polytopes associated to the blow-ups of (X_P, L_P) at the fixed point p corresponding to a vertex v of P are as illustrated in Fig. 2.

Note that here $(X_P, L_P) \leftrightsquigarrow P$, $(\mathrm{Bl}_p(\mathbb{P}^3), \pi^*(\mathcal{O}_{\mathbb{P}^3}(3)) - E) \leftrightsquigarrow \mathrm{Bl}_v^{(1)}(P)$ and $(\mathrm{Bl}_p(\mathbb{P}^3), \pi^*(\mathcal{O}_{\mathbb{P}^3}(3)) - 2E) \leftrightsquigarrow \mathrm{Bl}_v^{(2)}(P)$. To shorten notation the family of polarized toric varieties obtained by consecutive blow-ups of (X_P, L_P) at n torus invariant subvarieties will be denoted $\mathrm{Bl}_n(X_P)$. The corresponding family of polytopes obtained via consecutive blow-ups of P at n faces will be denoted by $\mathrm{Bl}_n(P)$.

Definition 7 Let P be a smooth d-polytope. We call P *minimal* if it cannot be obtained as blow-up along a face of an other smooth d-polytope.

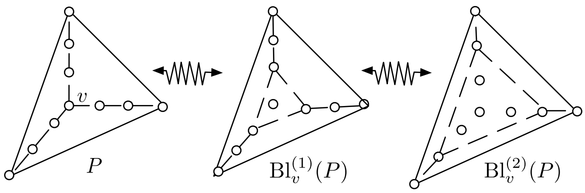


Fig. 2 The blow-ups of $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ at a fixed point p corresponding to a vertex v of P

Note that a polytope is minimal if and only if the corresponding embedded toric variety is minimal in the sense of equivariant blow-ups.

2.2 Toric fibrations and Cayley polytopes

Remember that two polytopes are *strictly isomorphic* if they have the same inner-normal fan.

Definition 8 Let $P_0, \dots, P_k \subset \mathbb{R}^n$ be strictly isomorphic polytopes with inner-normal fan Σ. Let $\{e_1, \dots, e_k\}$ be a basis for $\mathbb{Z}^k$ and let $e_0 = (0, \dots, 0) \in \mathbb{Z}^k$. Then a polytope P is called an *sth order strict Cayley polytope* associated to $P_0, \dots, P_k$ if it is isomorphic to

$$\mathrm{Cayley}^s_\Sigma(P_0, \dots, P_k) := \mathrm{Conv}\big((P_0, se_0), \dots, (P_k, se_k)\big) \subset \mathbb{R}^{n+k}$$

where $s \in \mathbb{Z}^+$ and $(P_i, se_i) := \{(p, se_i) : p \in P_i\}$.

Example 2 A 2-polytope $P \subset \mathbb{R}^2$ is strictly Cayley if and only if P is isomorphic to either $s\Delta_2$ or $\mathrm{Cayley}^s_\Sigma(P_0, P_1)$ where P_0 and P_1 are line segments and Σ is the fan associated to $\mathbb{P}^1$.

Example 3 A 3-polytope $P \subset \mathbb{R}^3$ is strictly Cayley if and only if it is of one of the following three types:

(i) $P \cong s\Delta_3$ where $s \in \mathbb{Z}^+$ and Δ_3 is the standard simplex.
(ii) $P \cong \mathrm{Cayley}^s_\Sigma(P_0, P_1, P_2)$ where P_0, P_1 and P_2 are line segments, and Σ is the fan associated to $\mathbb{P}^1$.
(iii) $P \cong \mathrm{Cayley}^s_\Sigma(P_0, P_1)$ where P_0 and P_1 are strictly isomorphic 2-polytopes with inner-normal fan Σ.

The following three lemmata follow directly from the definition of a smooth polytope.

Lemma 9 *Let $P \cong \mathrm{Cayley}^s_\Sigma(P_0, P_1)$ be a d-polytope, where P_0 and P_1 are strictly isomorphic and smooth $(d-1)$-polytopes. Then P is smooth if and only if there are exactly $s+1$ lattice points on every edge between $(P_0, 0)$ and (P_1, s).*

Lemma 10 *Let $P \subset \mathbb{R}^3$ be an sth order strict Cayley 3-polytope of the type $\mathrm{Cayley}^s_\Sigma(P_0, P_1, P_2)$ where $P_0 = [0, i]$, $P_1 = [0, j]$ and $P_2 = [0, k]$. Then P is smooth if and only if s divides $j - i$, $k - i$ and $k - j$. In particular every first order strict Cayley 3-polytope of this type is smooth.*

Lemma 11 *Let $P \cong \mathrm{Cayley}^s_{\Sigma}(P_0, P_1, P_2)$, where $P_0 \cong [0, i]$, $P_1 \cong [0, j]$ and $P_2 \cong [0, k]$ are line segments, be a 3-polytope such that $|P \cap \mathbb{Z}^3| \leq 16$. Then $s \leq 2$ and $i + j + k \leq 13$. Moreover up to isomorphism we may assume that $i \geq j \geq k$ and $k \leq 4$.*

For the following two definitions remember that by a *fan* we mean a complete polyhedral fan.

Definition 12 Let Σ and Σ' be fans and assume that the join $\sigma \cdot \Sigma'$ exists for all $\sigma \in \Sigma$ and that

$$\mathrm{relint}\big(\sigma \cdot \sigma'\big) \cap \mathrm{relint}\big(\tau \cdot \tau'\big) = \emptyset$$

for all $\sigma, \tau \in \Sigma$ and $\sigma', \tau' \in \Sigma'$ such that $\sigma \neq \tau$ and $\sigma' \neq \tau'$. Then we call the set

$$\Sigma \cdot \Sigma' := \big\{\sigma \cdot \sigma' : \sigma \in \Sigma, \sigma' \in \Sigma'\big\}$$

the join of Σ and Σ'.

Definition 13 Let $\Sigma \subseteq \mathbb{R}^n$ be a unimodular fan. Assume that Σ is the join of a unimodular fan Σ' which covers a lower-dimensional linear subspace U of $\mathbb{R}^n$ and a unimodular fan Σ''. The projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/U$ induces a map of fans $\pi : \Sigma'' \to \pi(\Sigma'') =: \Sigma_\pi$. We call X_Σ a *$X_{\Sigma'}$-fiber bundle over X_{Σ_π}* under the surjection $\overline{\pi} : X_\Sigma \to X_{\Sigma_\pi}$ induced by the projection $\pi : \Sigma \to \Sigma_\pi$.

An easy corollary of the results presented in [2, 5] is the following proposition, which is most useful for us.

Proposition 14 *Let P be the smooth polytope associated to an embedding of a toric variety X_P. Then $P \cong \mathrm{Cayley}^s_{\Sigma}(P_0, \ldots, P_k)$ where the strictly isomorphic polytopes $P_0, \ldots, P_k$ have inner-normal fan Σ if and only if X_P is a $\mathbb{P}^k$-fiber bundle over X_Σ.*

We are now ready to state Theorem 1 in full detail.

Theorem 15 *Up to isomorphism there exist 103 smooth 3-polytopes $P \subset \mathbb{R}^3$ such that $|P \cap \mathbb{Z}^3| \leq 16$. Equivalently there are, up to isomorphism, 103 complete embeddings of toric threefolds into $\mathbb{P}^k$ such that $k \leq 15$. All such pairs of 3-polytopes and embeddings may be arranged into the following four categories.*

(i) *$P \cong k\Delta_3$ where $k = 1, 2$ and $X_P \cong \mathbb{P}^3$ is embedded in either $\mathbb{P}^3$ or $\mathbb{P}^9$.*
(ii) *$P \cong \mathrm{Cayley}^s_{\Sigma}(P_0, P_1, P_2)$, where P_0, P_1 and P_2 are line segments and X_P is a $\mathbb{P}^2$-fiber bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^N$, $5 \leq N \leq 15$, $1 \leq s \leq 2$.*
(iii) *$P \cong \mathrm{Cayley}^s_{\Sigma}(P_0, P_1)$, where P_0 and P_1 are strictly isomorphic smooth 2-polytopes and X_P is a $\mathbb{P}^1$-fiber bundle either over $\mathbb{P}^2$, over $\mathrm{Bl}_2(\mathbb{P}^2)$, over $\mathrm{Bl}_3(\mathbb{P}^2)$ or over the Hirzebruch surface $\mathbb{F}_r$, with $0 \leq r \leq 4$, embedded in $\mathbb{P}^N$ where $5 \leq N \leq 15$, $1 \leq s \leq 3$.*

(iv) *P is the blow-up of a strict Cayley polytope at one, two or four vertices and does not lie in category* (i)–(iii). *The corresponding toric variety X_P is either the blow-up of $\mathbb{P}^3$ at four points embedded in $\mathbb{P}^{15}$ or the blow-up of a $\mathbb{P}^2$-fiber bundle over $\mathbb{P}^1$ at one or two points embedded in $\mathbb{P}^{13}$, $\mathbb{P}^{14}$ or $\mathbb{P}^{15}$.*

Up to isomorphism 99 *of the* 103 *pairs are in category* (i), (ii), *and* (iii). *In particular all complete embeddings of smooth toric varieties that are minimal in the sense of equivariant blow-ups lie in categories* (i), (ii), *and* (iii).

Note that since $\mathrm{Cayley}^s_\Sigma(k\Delta_2, k\Delta_2) \cong \mathrm{Cayley}^k(s\Delta, s\Delta, s\Delta)$ categories (ii) and (iii) of Theorem 15 are not mutually exclusive. Note moreover that Theorem 1 follows directly from Theorem 15.

3 Oda's classification

Our approach to prove Theorem 15 is to first prove the following proposition.

Proposition 16 *Let $P \subset \mathbb{R}^3$ be a smooth* 3*-polytope such that $|P \cap \mathbb{Z}^3| \leq 16$. If P is minimal then P is a strict Cayley* 3*-polytope.*

Recall that a minimal smooth toric surface is isomorphic to either $\mathbb{P}^2$ or to a Hirzebruch surface. Note that the toric surfaces $\mathbb{P}^2$ and $\mathbb{F}_r$ correspond exactly to the 2-dimensional strict Cayley polytopes. Theorem 15 states that a smooth 3-polytope with at most 16 lattice points is either strictly Cayley or the blow-up of a smooth strict Cayley 3-polytope. Hence our results reveal that, with the added bound to the number of lattice points, the set of all smooth 3-polytopes have an underlying Cayley structure analogous to smooth 2-polytopes.

Definition 17 Let $\Sigma \subset \mathbb{R}^3$ be a fan. The intersection $\Sigma \cap S^2$ of Σ with the unit sphere S^2 is called a *spherical cell complex*. If for every cone $\sigma \in \Sigma$, *the spherical cell* $\sigma \cap S^2$ is a triangle drawn on S^2, then we call $\Sigma \cap S^2$ a *triangulation* of the unit sphere.

Note that every full-dimensional cone in a unimodular fan is a simplex cone. Therefore, since we only consider complete fans, the spherical cell complex $\Sigma \cap S^2$ is a triangulation of S^2 for any unimodular fan Σ in $\mathbb{R}^3$.

Definition 18 Let T be a triangulation of S^2 and let v_m be the number of vertices in T with degree m. Then we associate to T the label

$$\prod_{m>0} m^{v_m},$$

as a word in the alphabet $\mathbb{Z}^+$. Note that the number of vertices in T can be read off as the sum $\sum_{m>0} v_m$.

The main reason we are interested in triangulations of S^2 is the following lemma which readily follows from Definition 18 (for details see [12, p. 52]).

Lemma 19 *Let Σ be the inner-normal fan of a simple 3-polytope P and assume that the triangulation T associated to Σ has the label $m_1^{v_1}\cdots m_k^{v_k}$. Then P has v_i facets with m_i edges for all $i\in\{1,\dots,k\}$.*

A complete classification of all combinatorial types of triangulation of the unit sphere containing at most eight spherical cells is presented in [12, p. 192]. The following theorem is a translation of Theorem 1.34 stated in [12, p. 59] into the language used in this paper.

Theorem 20 [12, p. 59] *Let P be a smooth and minimal 3-polytope with at most 8 facets. If Σ is the inner-normal fan of P, then the label of the triangulation $\Sigma\cap S^2$ will be one of the following.*

$$3^4,\quad 3^24^3,\quad 4^6,\quad 4^55^2,\quad 4^66^2,\quad 3^24^36^2,\quad 3^14^35^3,\quad 3^24^47^2$$
$$3^34^15^16^3,\quad 3^24^25^26^2,\quad 3^14^45^16^2,\quad 3^24^15^46^1,\quad 3^14^35^46^1,\quad 3^25^6,\quad 4^45^4$$

with weights as in [12]. Moreover, for the first five triangulations the associated toric varieties X_Σ are $\mathbb{P}^k$-fibrations as follows.

(i) *If $T=3^4$ then $X_\Sigma\cong\mathbb{P}^3$.*
(ii) *If $T=3^24^3$ then X_Σ is a $\mathbb{P}^1$-fiber bundle over $\mathbb{P}^2$ or a $\mathbb{P}^2$-fiber bundle over $\mathbb{P}^1$.*
(iii) *If $T=4^6$ then X_Σ is a $\mathbb{P}^1$-fiber bundle over the Hirzebruch surface $\mathbb{F}_a$ where $a>1$ or $a=0$.*
(iv) *If $T=4^55^2$ then X_Σ is a $\mathbb{P}^1$-fiber bundle over a smooth toric variety associated to a smooth pentagon.*
(v) *If $T=4^66^2$ then X_Σ is a $\mathbb{P}^1$-fiber bundle over a smooth toric variety associated to a smooth hexagon.*

From Proposition 14 we get the following corollary of Theorem 20.

Corollary 21 *Let P be a smooth and minimal 3-polytope with at most 8 facets. If the triangulation associated to P is $3^4, 3^24^3, 4^6, 4^55^2$ or 4^66^2 then P is strictly Cayley.*

Our next objective is to prove Proposition 16. We will do this in two steps. The first step is to prove that any smooth 3-polytope P, such that $|P\cap\mathbb{Z}^3|\le 16$, has at most 8 facets. The second step is to prove that if P is minimal, then the triangulation associated to P will be $3^4, 3^24^3, 4^6, 4^55^2$ or 4^66^2. Proposition 16 then follows directly from Corollary 21.

Lemma 22 *A simple 3-polytope with V vertices has $3V/2$ edges and $2+V/2$ facets.*

Proof See [9, §10.3]. □

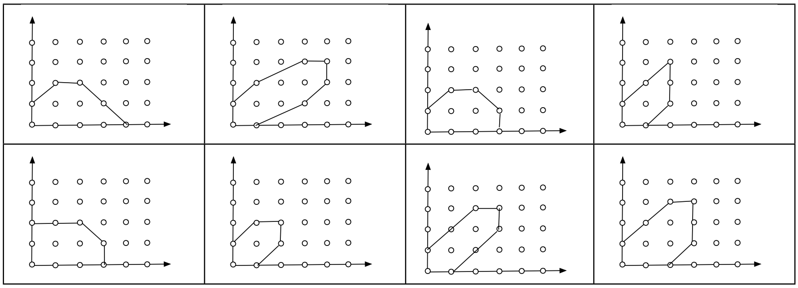


Fig. 3 All smooth 2-polytopes with at least 5 edges such that $|P \cap \mathbb{Z}^2| \leq 12$

In [10] Lorenz provides a complete list of all smooth 2-polytopes P such that $|P \cap \mathbb{Z}^2| \leq 12$. We are particularly interested in the n-gons of that classification with $n \geq 5$. For convenience of the reader we list these as a separate lemmata below.

Lemma 23 (See [10]) *Let P be a smooth 2-polytope with at least five edges such that $|P \cap \mathbb{Z}^2| \leq 12$. Then P is isomorphic to one of the following polytopes* (*see Fig*. 3).

Proof See [10]. □

We will need the following lemmata which follow readily by the definition of smoothness.

Lemma 24 *Let P be a n-dimensional smooth polytope*, *then every facet of P viewed as subset of the supporting hyperplane containing it*, *is smooth*.

Lemma 25 *Let P be a smooth 3-polytope such that $|P \cap \mathbb{Z}^3| \leq 16$, then P has at most eight facets and 12 vertices*.

Proof Let P be a smooth 3-polytope such that $|P \cap \mathbb{Z}^3| \leq 16$. Then by Lemma 22 we see that P has at most 10 facets. Let P be a smooth 3-polytope with nine facets and denote the facets by $F_0, \dots, F_8$. From Lemma 22 it follows that P has 14 vertices. Since every pair of facets share at most 1 edge, no facet of P may have more than eight edges. Therefore, there must be at least $14 - 8 = 6$ vertices which do not lie in a given facet. Hence if $|P \cap \mathbb{Z}^3| \leq 16$ then no facet of P contains more than 10 lattice points. Therefore by Lemmas 23 and 24 any facet of P that is not triangular or quadrilateral must be either a pentagon or a hexagon.

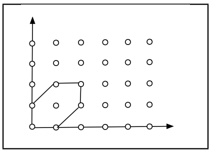
Let $V(F_i) \in [3, 6]$ be the number of vertices of the facet F_i. Since every vertex of P lies in exactly three facets $\sum_{i=0}^{8} V(F_i) = 3 \cdot 14 = 42$ holds. So by the pigeon hole principle there must be at least three facets such that $V(F_i) \geq 5$. Because every pentagonal or hexagonal facet has at least one interior lattice point this implies that $|P \cap \mathbb{Z}^3| \geq 17$. A completely analogous argument for a 3-polytope with 10 facets establishes the lemma. □

The last piece of information we need in order to prove Proposition 16 is Lemma 26.

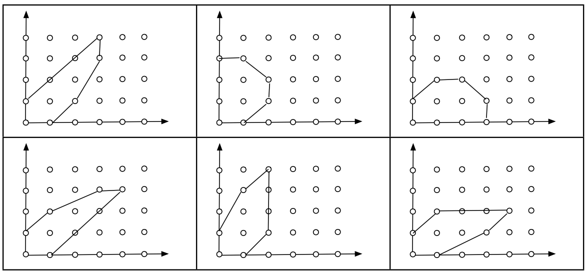
Lemma 26 *Up to isomorphism there exists exactly one smooth* 3*-polytope* P *that is associated to a triangulation with label* $3^24^36^2$ *such that* $|P \cap \mathbb{Z}^3| \leq 16$, *namely the blow-up of* $\text{Cayley}^2_\Sigma(3\Delta_1, \Delta_1, \Delta_1)$ *at two vertices. This polytope has* 7 *facets and* 10 *vertices.*

Lemma 26 implies that if P is a minimal smooth 3-polytope associated to a triangulation with label $3^24^36^2$ then $|P \cap \mathbb{Z}^3| > 16$. Note that a smooth 3-polytope with a triangulation associated to the label $3^24^36^2$ has seven facets and 10 vertices. These 10 vertices must all lie in a hexagonal facet since every pair of facets can share at most two vertices. The position of the hexagonal facets of P thus determines P. In order to prove Lemma 26 we first list the possible ways to align the smooth hexagons from the classification of Lorenz in the following lemmata. This allows us to exhaust all possible positions of the hexagonal facets of a smooth polytope P corresponding to a triangulation with label $3^24^36^2$ such that $|P \cap \mathbb{Z}^3| \leq 16$. The following lemmata follow by considering all possibilities.

Lemma 27 *The* 2*-polytope* $P = \text{Conv}((0,0), (1,0), (0,1), (1,2), (2,1), (2,2))$ *is invariant under any isomorphism that takes a vertex to the origin and each edge containing the vertex to a coordinate axis.*

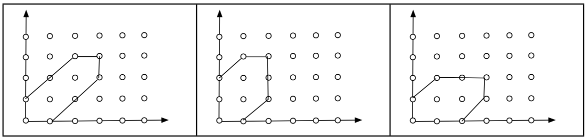


Lemma 28 *Consider an isomorphism* φ *that takes a vertex of the* 2*-polytope* $P = \text{Conv}((0,0), (1,0), (0,1), (2,1), (3,3), (3,4))$ *to the origin and each edge containing the vertex to a coordinate axis. The image of* P *under* φ *will be as illustrated in one of the six pictures below.*



Lemma 29 *Consider an isomorphism* φ *that takes a vertex of the* 2*-polytope* $P = \text{Conv}((0,0), (1,0), (0,1), (2,3), (3,2), (3,3))$ *to the origin and each edge contain-*

ing the vertex to a coordinate axis. The image of P under φ will be as illustrated in one of the pictures below.



Proof of Lemma 26 Let P be a smooth 3-polytope with $|P \cap \mathbb{Z}^3| \leq 16$ that is associated to a triangulation with label $3^2 4^3 6^2$ and let F_1 and F_2 be the hexagonal facets of P. Since P has 10 vertices F_1 and F_2 must share an edge. Moreover F_1 and F_2 must be smooth 2-polytopes with respect to the hyperplane containing them by Lemma 24. Therefore they must be isomorphic to some hexagonal 2-polytope given in the classification of [10]. These are exactly the 2-polytopes appearing in Lemmas 27, 28, and 29.

Note that because of smoothness we may, without loss of generality, assume that a shared vertex v of F_1 and F_2 is positioned at the origin and that the edges through v are aligned along the coordinate axes in the positive direction. By symmetry we may also assume that F_1 and F_2 lie in the xy- and xz-plane, respectively, with the shared edge of F_1 and F_2 along the x-axis. However for any such configuration the points $(1, 2, 0)$, $(2, 2, 0)$, $(1, 0, 2)$ and $(2, 0, 2)$ lie in P by Lemmas 27, 28, and 29. Then by convexity the points $(1, 1, 1)$ and $(2, 2, 2)$ also lie in P. Assume that there are m lattice points on the shared edge of F_1 and F_2. The position of every vertex of P is determined by the configuration of the hexagonal facets F_1 and F_2, so it must hold that $|P \cap \mathbb{Z}^3| \geq |F_1 \cap \mathbb{Z}^3| + |F_2 \cap \mathbb{Z}^3| + 2 - m$.

It is readily checked that since $m \leq 4$ the only choice which allows $|P \cap \mathbb{Z}^3| \leq 16$ is $F_1 \cong F_2 \cong \mathrm{Conv}((0, 0), (1, 0), (0, 1), (1, 2), (2, 1), (2, 2))$. Since every vertex of P is a vertex of F_1 or F_2 we see that P is the blow-up of $\mathrm{Cayley}^2_{\Sigma}(3\Delta_1, \Delta_1, \Delta_1)$ at two vertices via the isomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\varphi(x, y, z) = (2 - x, y, z)$. □

We are now in position to prove Proposition 16.

Proof of Proposition 16 We will prove the proposition by computing lower bounds for the number of lattice points of P for the labels of triangulations given in Theorem 20. Let P be a smooth 3-polytope. Then P has at most eight facets by Lemma 25. Since every edge of P is shared by exactly two facets no facet can have more than seven edges. However, if F is a heptagonal facet of P then P has eight facets and 12 vertices by Lemma 22. Thus exactly five vertices of P do not lie in F, so F contains at most 11 lattice points since $|P \cap \mathbb{Z}^3| \leq 16$. However, by Lemma 23 there exists no smooth heptagon with less than 13 lattice points. Hence every facet of P is a smooth n-gon with $n \leq 6$ and in particular P cannot have a triangulation with label $3^2 4^4 7^2$.

From Lemma 23 we see that any pentagonal or hexagonal facet of P has at least 1 interior point. Hence P cannot have a triangulation with any of the labels $3^2 4^1 5^4 6^1$, $3^1 4^3 5^4 6^1$ or $3^2 5^6$. Using Lemma 23 again we see that if P has exactly

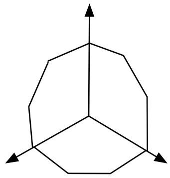


Fig. 4 Configuration of three pentagonal facets of P

one pentagonal facet, then that facet contains at least three lattice points apart from the vertices. So any smooth 3-polytope with a triangulation associated to the labels $3^3 4^1 5^1 6^3$ or $3^1 4^4 5^1 6^2$ contains at least 18 or 17 lattice points, respectively.

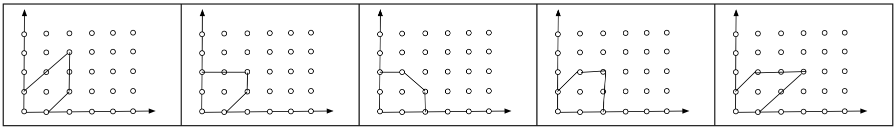
Assume that P contains exactly two pentagonal facets. Since the pentagonal facets can share their longest edge they contain at least five lattice points apart from the vertices. Hence P cannot have a triangulation with label $3^2 4^2 5^2 6^2$. If P has at least three pentagonal facets, then a priori three different configurations are possible: (i) At least two pentagonal facets do not share any edges. (ii) Three pentagonal facets share an edge pairwise but these three pentagonal facets do not meet at any vertex. (iii) Three pentagonal facets meet at a vertex. We will see that all three case are impossible if P is smooth and $|P \cap \mathbb{Z}^3| \leq 16$. Under these assumptions case (i) is impossible since each pair of pentagonal facets F and F' that do not share any edges must contain at least 16 lattice points by Lemma 23 and there is at least one vertex of P, coming from a third pentagonal facet, which lies neither in F nor F'. Case (ii) is impossible under the same assumptions, since a simple 3-polytope must have an even number of vertices by Lemma 22, so P has at least 10 vertices. Hence there must be at least five vertices which do not lie in any given pentagonal facet, so each such facet must be isomorphic to the one appearing in Lemma 23. Therefore in the configuration of case (ii) there is by exhaustion at least 8 lattice points apart from the vertices in P, which is a contradiction. For the final case note that three pentagonal facets contain a minimal number of lattice points if they are chosen and positioned in P as illustrated in Fig. 4.

In the configuration $(2,0,1)$ and $(0,2,1)$ lie in P so therefore also $(1,1,1) = \frac{1}{2}(2,0,1) + \frac{1}{2}(0,2,1)$ lies in P and is not a vertex. Therefore there are at least 7 lattice points of P which are not vertices in this case. We can conclude that with our assumptions P cannot contain three pentagonal facets, so in particular P cannot be associated to a triangulation with label $3^1 4^3 5^3$ or $4^4 5^4$. From the above and Theorem 20 we conclude that if P is minimal then it is associated to a triangulation with one of the labels $3^4, 3^2 4^3, 4^6, 4^5 5^2, 4^6 6^2$ or $3^2 4^3 6^2$. However, if P is minimal then the label of the triangulation associated to P cannot be $3^2 4^3 6^2$ by Lemma 26. Therefore it follows from Corollary 21 that if P is a minimal smooth 3-polytope with $|P \cap \mathbb{Z}^3| \leq 16$ then P is strictly Cayley. □

4 Some restrictions for the smoothness of strict Cayley 3-polytopes

Lemmas 10 and 11 provide us with enough restrictions to classify all smooth 3-polytopes P such that $|P \cap \mathbb{Z}^3| \leq 16$ and $P \cong \mathrm{Cayley}^s_{\Sigma}(P_0, P_1, P_2)$, where P_0, P_1 and P_2 are line segments. We will now establish similar restrictions for 3-polytopes of the type $P \cong \mathrm{Cayley}^s_\Sigma(P'_0, P'_1)$, where P'_0 and P'_1 are strictly isomorphic 2-polytopes. An obvious restriction is that we must have $|P'_0 \cap \mathbb{Z}^2| + |P'_1 \cap \mathbb{Z}^2| \leq 16$ which by the classification of smooth 2-polytopes in [10] implies that P'_0 and P'_1 must correspond to $\mathbb{P}^2$, $\mathbb{P}^2$ $\mathrm{Bl}_2(\mathbb{P}^2)$, $\mathrm{Bl}_3(\mathbb{P}^2)$ or the Hirzebruch surface $\mathbb{F}_r$ where $0 \leq r \leq 4$. All the following lemmata follow from the definition of strict Cayley polytopes.

Lemma 30 *Consider an isomorphism φ that takes a vertex of the 2-polytope $P :=$ $\mathrm{Conv}((0,0),(1,0),(0,1),(2,1),(2,3))$ to the origin and an edge to each coordinate axes. The image of P under φ will be as illustrated in one of the figures below.*



Lemma 31 *Let P_0 and P_1 be strictly isomorphic 2-polytopes with inner-normal fan Σ.*

1. *If Σ is the fan of $\mathbb{P}^2$, $\mathbb{F}_r$ with $r \in \{1,\dots\}$ or the inner-normal fan of one of the 2-polytopes in Lemmas 27, 28, 29 or 30, then there is exactly one polytope of the type $\mathrm{Cayley}^s_\Sigma(P_0, P_1)$ for every $s \in \mathbb{Z}^+ = \{1,\dots\}$.*
2. *If Σ is the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ then $P_0 \cong k_0\Delta_1 \times k'_0\Delta_1$ and $P_1 \cong k_1\Delta_1 \times k_1\Delta_1$. Moreover, if either $k_0 = k'_0$ or $k_1 = k'_1$ then there is exactly one 3-polytope of the type $\mathrm{Cayley}^s_\Sigma(P_0, P_1)$ for each $s \in \mathbb{Z}^+$. If instead $k_0 \neq k'_0$ and $k_1 \neq k'_1$ then there is exactly two polytopes of the type $\mathrm{Cayley}^s_\Sigma(P_0, P_1)$.*

Lemma 32 *Let $P \cong \mathrm{Cayley}^s_\Sigma(P_0, P_1)$ where P_0 and P_1 are strictly isomorphic 2-polytopes. If P is smooth and $|P \cap \mathbb{Z}^3| \leq 16$ then $s \leq 4$ and we may assume that $|P_0 \cap \mathbb{Z}^2| \geq |P_1 \cap \mathbb{Z}^2|$. Moreover P is not a simplex and is isomorphic to a 3-polytope of the type $\mathrm{Cayley}^s_{\Sigma'}(P'_0, P'_1, P'_2)$ if and only if $P_0 = P_1 = k\Delta_2$ for some $k \in \mathbb{Z}^+$.*

Proof Let P be oriented as in Lemma 31. Because P is convex, every lattice point of the form (i, j, k) where $i, j \in \{0, 1\}$ and $0 \leq k \leq s$ lies in P. This proves the first part. For the second part note that all simplices have 4 vertices, all strict Cayley 3-polytopes of the type $\mathrm{Cayley}^s_{\Sigma'}(P'_0, P'_1, P'_2)$ where P'_0, P'_1 and P'_2 are line segments have six vertices. If the number of vertices of P_0 and P_1 is k then the number of vertices of a 3-polytope of the type $\mathrm{Cayley}^s_\Sigma(P_0, P_1)$ is $2k$. Therefore a 3-polytope of the type $\mathrm{Cayley}^s_\Sigma(P_0, P_1)$ is never isomorphic to a simplex and may only be isomorphic to a 3-polytope of the type $\mathrm{Cayley}^s_{\Sigma'}(P'_0, P'_1, P'_2)$ if $P_0 \cong k_1\Delta_2$ and $P_1 \cong k_2\Delta_2$ for some $k_1, k_2 \in \mathbb{Z}^+$. If $k_1 \neq k_2$ then the 3-polytope $\mathrm{Cayley}^s_\Sigma(P_0, P_1)$ is a truncated pyramid. It is clear that such a 3-polytope does not contain three edges that are pairwise parallel and therefore cannot be isomorphic to a 3-polytope of the type $\mathrm{Cayley}^s_{\Sigma'}(P'_0, P'_1, P'_2)$. However, if $k_1 = k_2 = k$ then $\mathrm{Cayley}^s_\Sigma(k\Delta_2, k\Delta_2) \cong$ $\mathrm{Cayley}^k_{\Sigma'}(s\Delta_1, s\Delta_1, s\Delta_1)$. □

5 A classification of all smooth strict Cayley 3-polytopes with at most 16 lattice points

The following lemma lists all polytopes in category (i) of Theorem 15 and follows from the definition of $k\Delta_3$.

Lemma 33 *Δ_3 and $2\Delta_3$ are the only smooth 3-simplices such that* $|k\Delta_3 \cap \mathbb{Z}^3| \leq 16$.

The following lemma provides all polytopes in category (ii) of Theorem 15.

Lemma 34 *Up to isomorphism there are 69 smooth 3-polytopes P such that $P \cong \text{Cayley}^s_\Sigma(P_0, P_1, P_2)$ where $P_0 = [0, i]$, $P_1 = [0, j]$ and $P_2 = [0, k]$ are line segments and $|P \cap \mathbb{Z}^3| \leq 16$.*

Proof From Lemma 11 we know that $s \leq 2$. If $P \cong \text{Cayley}^2_\Sigma([0, i], [0, j], [0, k])$ then since we can assume $i \geq j \geq k$ it follows from Lemma 10 that we must have $i - j = 2m$ and $i - k = 2n$ where $m, n \in \mathbb{N}$. Observe that P is completely determined by the choice of i, m and n and that $i > 2n \geq 2m \geq 0$. In particular the first three choices of i, m and n give us the following smooth 3-polytopes for $s = 2$.

(i) $i = 1$ and $n = m = 0$ gives $P \cong \text{Cayley}^2_\Sigma(\Delta_1, \Delta_1, \Delta_1)$ and $|P \cap \mathbb{Z}^3| = 12$.
(ii) $i = 2$ and $n = m = 0$ gives $P \cong \text{Cayley}^2_\Sigma(2\Delta_1, 2\Delta_1, 2\Delta_1)$ and $|P \cap \mathbb{Z}^3| = 18$.
(iii) $i = 3$ and $n = m = 1$ gives $P \cong \text{Cayley}^2_\Sigma(3\Delta_1, \Delta_1, \Delta_1)$ and $|P \cap \mathbb{Z}^3| = 16$.

For $s = 2$ all remaining choices of i, m and n will clearly generate a smooth 3-polytope P with longer defining line segments than $\text{Cayley}^2_\Sigma(3\Delta_1, \Delta_1, \Delta_1)$. Therefore all remaining choices will give $|P \cap \mathbb{Z}^3| > 16$. Hence we may conclude that $\text{Cayley}^2_\Sigma(\Delta_1, \Delta_1, \Delta_1)$ and $\text{Cayley}^2_\Sigma(3\Delta_1, \Delta_1, \Delta_1)$ are up to isomorphism the only smooth 3-polytopes of the type $P \cong \text{Cayley}^2_\Sigma(P_0, P_1, P_2)$ such that $|P \cap \mathbb{Z}^3| \leq 16$.

If $P \cong \text{Cayley}^1_\Sigma([0, i], [0, j], [0, k])$ we know from Lemma 10 that P is smooth. Therefore by Lemma 11 the only restrictions are $i \geq j \geq k$, $i + j + k \leq 13$ and $k \leq 4$. As can be readily checked there are in total 67 choices of i, j and k meeting these restrictions, hence 67 associated smooth 3-polytopes. □

This lemma provides the polytopes in category (iii) of Theorem 15.

Lemma 35 *Up to isomorphism there are 33 smooth 3-polytopes P of the type $P \cong \text{Cayley}^s_\Sigma(P_0, P_1)$ where P_0 and P_1 are strictly isomorphic 2-polytopes such that $|P \cap \mathbb{Z}^3| \leq 16$. Of these five are isomorphic to a polytope counted in Lemma 34.*

Proof Consider the case when P_0 and P_1 are smooth 2-simplices. By Lemma 32 we know that $s \leq 4$. For a fixed choice of P_0, P_1 and s, Lemma 31 implies that there is up to isomorphism at most one smooth 3-polytope of the type $\text{Cayley}^s_\Sigma(P_0, P_1)$. In order for a 3-polytope P to fit our classification it must hold both that $|P_0 \cap \mathbb{Z}^2| + |P_1 \cap \mathbb{Z}^2| \leq |P \cap \mathbb{Z}^3| \leq 16$ and that every edge between $(P_0, 0)$ and (P_1, s) contains exactly $s + 1$ lattice points by Lemma 9. If $s = 4$ then the choice $P_0 \cong P_1 \cong \Delta_2$ gives the 3-polytope $P \cong \text{Cayley}^4_\Sigma(\Delta_2, \Delta_2)$, for which $|P \cap \mathbb{Z}^3| = 15$. Since $2\Delta_2$

contains three lattice points more than Δ_2 it follows that $\mathrm{Cayley}^4_\Sigma(\Delta_2,\Delta_2)$ must be the only smooth 3-polytope of the type $P\cong \mathrm{Cayley}^4_\Sigma(k_1\Delta_2,k_2\Delta_2)$. By considering all possible choices when $s=3,2$ and 1 in the same way we get seven more smooth 3-polytopes meeting our criteria. These are all 3-polytopes of the type $P\cong \mathrm{Cayley}^s_\Sigma(k_1\Delta_2,k_2\Delta_2)$ such that $|P\cap\mathbb{Z}^3|\le 16$. None of the eight smooth 3-polytopes considered so far in the proof are isomorphic to each other and using Lemma 32 we see that only those that are isomorphic to triangular prisms are isomorphic to a polytope covered in Lemmas 33 or 34.

In the same way one may utilize Lemma 31 to show that there are one, two, four, and nine smooth 3-polytopes of the form $\mathrm{Cayley}^s_\Sigma(P_0,P_1)$ where P_0 and P_1 have an inner-normal fan associated to $\mathbb{F}_4$, $\mathbb{F}_3$, $\mathbb{F}_2$ and $\mathbb{F}_1$, respectively. Similarly, one can check that there is one smooth 3-polytope P of the type $\mathrm{Cayley}^s_\Sigma(P_0,P_1)$ both when P_0 and P_1 have inner-normal fan associated to $\mathrm{Bl}_2(\mathbb{P}^2)$ and when the inner-normal fan of P_0 and P_1 is associated to $\mathrm{Bl}_3(\mathbb{P}^2)$. If the inner-normal fan of P_0 and P_1 is associated to $\mathbb{F}_0$ then $16\ge|P\cap\mathbb{Z}^3|\ge|P_0\cap\mathbb{Z}^2|+|P_1\cap\mathbb{Z}^2|$, $|P_0\cap\mathbb{Z}^2|\ge 4$ and $|P_1\cap\mathbb{Z}^2|\ge 4$. Therefore P_0 and P_1 contain at most 12 lattice points each and by the classification in [10] we have up to isomorphism 12 possible choices for P_0 and P_1. Lemma 31 implies that up to isomorphism for a choice such that neither P_0 nor P_1 is a square there are at most two smooth 3-polytopes of the type $\mathrm{Cayley}^s_\Sigma(P_0,P_1)$. Taking this into account one may proceed completely analogously to the case when P_0 and P_1 are simplices using Lemmas 31 and 9. Such a procedure gives up to isomorphism 16 smooth 3-polytopes of the type $\mathrm{Cayley}^s(P_0,P_1)$ where P_0 and P_1 have inner-normal fan associated to $\mathbb{F}_0$. These last 16 smooth 3-polytopes appear in the appendix as the nine polytopes corresponding to $\mathbb{P}^1$-bundles over $\mathbb{F}_0$ and 7 prisms which already have been considered as a $\mathbb{P}^1$-bundles over $\mathbb{F}_r$ for some $r>0$. The lemma now follows after we have excluded one polytope from every pair of isomorphic polytopes in our list. □

6 Classification of smooth 3-polytopes associated to blow-ups

Lemma 36 *Any smooth non-minimal* 3*-polytope* P *such that* $|P\cap\mathbb{Z}^3|\le 16$ *can be obtained by consecutive blow-ups of a polytope associated to a triangulation with one of the labels*: $3^4, 3^24^3, 4^6, 4^55^2, 3^24^36^2$ *or* $3^14^35^3$.

Proof By Lemma 25 any smooth 3-polytope P such that $|P\cap\mathbb{Z}^3|\le 16$ has at most 8 facets. Note that every blow-up of a 3-polytope P adds a facet to P. Therefore if P is a minimal smooth 3-polytope then a necessary condition for the nth consecutive blow-up of P to contain at most 16 lattice points is that P has at most $8-n$ facets. Moreover every non-minimal and smooth 3-polytope may be obtained by consecutive blow-ups of some minimal and smooth 3-polytope [7]. Theorem 20 now establishes the lemma. □

By definition we may construct a new polytope $\mathrm{Bl}^{(k)}_F(P)$ by blowing up a 3-polytope P either at a vertex or along an edge. Blowing up along an edge will add a quadrilateral facet to the 3-polytope and blowing up a vertex will add a triangular facet to the 3-polytope. When a vertex is blown-up the three facets meeting in that vertex will each gain one more edge. This implies that the label of $\mathrm{Bl}_v^{(k)}(P)$ may be obtained by adding a '3' and raising 3 base numbers in the label of P. Consider for example the case when P is associated to a triangulation with label $4^5 5^2$. If we blow-up P at a vertex where one pentagonal and two quadrilateral facets meet, then $\mathrm{Bl}_v^{(k)}(P)$ will have a triangulation with label $3^1 4^3 5^3 6^1$ since a triangular facet is added and each facet containing the vertex gets one more edge. Similarly if we blow-up along an edge e then the two facets that have the end-points of e as a vertex but do not contain the edge e will each gain one more edge. Hence the label of $\mathrm{Bl}_e^{(k)}(P)$ may be obtained from the label of P by adding a '4' and raising 2 base numbers.

Lemma 37 *There exists no smooth* 3*-polytope* P *such that*

(i) $|P \cap \mathbb{Z}^3| \leq 16$.
(ii) P *can be obtained by blow-ups starting from a* 3*-polytope associated to a triangulation with label* $3^1 4^3 5^3$, 4^6, $3^2 4^3 6^2$ *or* $4^5 5^2$.
(iii) P *has not been counted in Lemmas* 26, 33 *or* 34, 35.

Proof Recall that a smooth polytope P such that $|P \cap \mathbb{Z}^3| \leq 16$ has at most eight facets. Hence it is enough to consider polytopes obtained via a single blow-up of a polytope associated to a triangulation with label $3^1 4^3 5^3$, $3^2 4^3 6^2$ or $4^5 5^2$ and polytopes obtained via at most two blow-ups of a polytope associated to a triangulation with label 4^6. Given a 3-polytope P with label $3^1 4^3 5^3$ we list all the ways one can add 1 to the exponent of 3 in the label and raise three of the base numbers. This list includes the label associated to every possible blow-up $\mathrm{Bl}_v^{(k)}(P)$ at a vertex v of P. For each label we can check if it corresponds to a triangulation of S^2 listed in [12, A5] i.e. if it can be associated to a simple 3-polytope. Moreover we can compute a lower bound for $|\mathrm{Bl}_v^{(k)}(P) \cap \mathbb{Z}^3|$ by using Lemma 23 and considering the number of lattice points in the facets of $\mathrm{Bl}_v^{(k)}(P)$. Certainly any blow-up of P that fits our classification needs to be both associated to a simple 3-polytope and have at most 16 lattice points in its facets.

To compute a lower bound for the number of lattice points in the facets we use the same technique as in the proof of Proposition 16. Consider for example the case when P is associated to a triangulation with label $3^1 4^3 5^3$ and we blow-up P at a vertex v where the triangular facet F_3 and 2 of the quadrilateral facets F_4, F_4' meet. Then P will gain a triangular facet while F_3, F_4, F_4' all gain one more edge. Hence $\mathrm{Bl}_v^{(k)}(P)$ will be associated to a triangulation with label $3^1 4^2 5^5$. To get a lower bound for $|\mathrm{Bl}_v^{(k)}(P) \cap \mathbb{Z}^3|$ note that $\mathrm{Bl}_v^{(k)}(P)$ contains 5 pentagonal facets. Thus by the proof of Proposition 16 we see that $|\mathrm{Bl}_v^{(k)}(P) \cap \mathbb{Z}^3| > 16$. Next we note that the label $3^1 4^2 5^5$ does not appear in the classification in [12, A5]. Thus in this case $\mathrm{Bl}_v^{(k)}(P)$ fails both of our criteria. The results of such a calculation for all blow-ups at a vertex of a polytope P with label $3^1 4^3 5^3$ are given below.

Label of $\mathrm{Bl}_v^{(k)}(P)$	Associated to a triangulation	Lower bound
$3^1 4^2 5^5$	No	$\lvert\mathrm{Bl}_v^{(k)}(P)\cap\mathbb{Z}^3\rvert\ge 21$
$3^1 4^3 5^3 6^1$	Yes	$\lvert\mathrm{Bl}_v^{(k)}(P)\cap\mathbb{Z}^3\rvert\ge 19$
$3^1 4^4 5^1 6^2$	Yes	$\lvert\mathrm{Bl}_v^{(k)}(P)\cap\mathbb{Z}^3\rvert\ge 17$
$3^2 5^6$	No	$\lvert\mathrm{Bl}_v^{(k)}(P)\cap\mathbb{Z}^3\rvert\ge 22$
$3^2 4^1 5^4 6^1$	Yes	$\lvert\mathrm{Bl}_v^{(k)}(P)\cap\mathbb{Z}^3\rvert\ge 21$
$3^2 4^2 5^2 6^2$	Yes	$\lvert\mathrm{Bl}_v^{(k)}(P)\cap\mathbb{Z}^3\rvert\ge 19$
$3^2 4^3 6^3$	No	$\lvert\mathrm{Bl}_v^{(k)}(P)\cap\mathbb{Z}^3\rvert\ge 15$

From the above list it is clear that there exist no smooth polytopes of the type $\mathrm{Bl}_v^{(k)}(P)$, where P is a polytope with label $3^1 4^3 5^3$, such that $\lvert\mathrm{Bl}_v^{(k)}(P)\cap\mathbb{Z}^3\rvert\le 16$.

If we blow-up along an edge e of a 3-polytope P with label $3^1 4^3 5^3$ and then use the same technique as above a similar list including the label of every possible such blow-up can be created. From that list we then conclude that there exist no smooth polytopes of the type $\mathrm{Bl}_e^{(k)}(P)$, where P is a polytope with label $3^1 4^3 5^3$ such that $\lvert\mathrm{Bl}_e^{(k)}(P)\cap\mathbb{Z}^3\rvert\le 16$. Applying the same technique to blow-ups of 3-polytopes with the labels $3^2 4^3 6^2$, 4^6 and $4^5 5^2$ we see that there exist no blow-ups meeting all three stated criteria. In particular condition (iii) is used when we obtain blow-ups with the label 4^6 when blowing up a polytope with label $3^2 4^3 6^2$ along an edge. □

Lemma 38 *Up to isomorphism there are three smooth 3-polytopes* $\mathrm{Bl}(P)$, *such that* $\lvert\mathrm{Bl}(P)\cap\mathbb{Z}^3\rvert\le 16$ *which are not counted in Lemmas 26, 33, 34, 35, or 37. These can be obtained by a sequence of blow-ups of a 3-polytope P with one of the labels* 3^4 *or* $3^2 4^3$. *The three polytopes are the blow-up at a vertex of* $\mathrm{Cayley}^2_\Sigma(3\Delta_1,\Delta_1,\Delta_1)$, *the blow-up at four vertices of* $3\Delta_3$ *and the blow-up at two vertices of* $\mathrm{Cayley}^2_\Sigma(2\Delta_1,2\Delta_1,2\Delta_1)$.

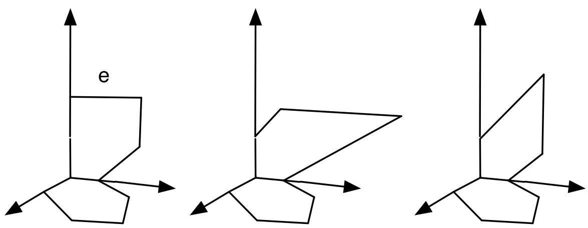
Proof For a smooth 3-polytope associated to the triangulation $3^2 4^3$ one triangular facet meets two quadrilateral facets at each vertex. Therefore if we blow-up along an edge joining the two triangular facets we get a 3-polytope with the label 4^6 i.e. already considered 3-polytopes. However if we blow-up a vertex or an edge of a triangular facet then we get a 3-polytope associated to the label $3^2 4^2 5^2$.

From the classification of all triangulations of S^2 given in [12] we see that two pentagonal facets of a 3-polytope P associated to the triangulation $3^2 4^2 5^2$ will share an edge. By the classification of smooth 2-polytopes given in [10] we see that the two pentagonal facets of P must be isomorphic to the 2-polytope $F := \mathrm{Conv}((0,0),(1,0),(0,1),(2,1),(2,3))$ since P has 8 vertices and there must be at least one interior lattice point in each pentagonal facet by smoothness and convexity. Note that every vertex of a 3-polytope associated to triangulation with label $3^2 4^2 5^2$ lies in a pentagonal facet. Hence all 3-polytopes with that label may be formed by taking the convex hull of every permissible configuration of the two pentagonal facets. To do this start by choosing 2 of the alignments of F given in Lemma 30 with the same edge length along the x-axis. Position one in the xy-plane and one in the xz-plane via the maps $(x, y) \mapsto (x, y, 0)$ and $(x, y) \mapsto (0, x, y)$. It is readily checked that exploring every such possibility gives us up to isomorphism exactly one smooth 3-polytope with at most 16 lattice points, namely the blow-up of $\mathrm{Cayley}^2_\Sigma(3\Delta_1, \Delta_1, \Delta_1)$ at a vertex.

Since a 3-polytope associated to a triangulation with label $3^2 4^2 5^2$ only has 6 facets it may be blown-up again. Therefore we apply the techniques of the proof of Lemma 37 to a polytope P associated to a triangulation with label $3^2 4^2 5^2$. This reveals that all blow-ups of P such that $|P \cap \mathbb{Z}^3| \leq 16$ have already been considered or have a triangulation with label $3^2 4^2 5^2 6^1$. If $\mathrm{Bl}^{(k)}_F(P)$ have a triangulation with label $3^2 4^2 5^2 6^1$ then by the proof of Proposition 16 there are at least 16 lattice points in the facets of $\mathrm{Bl}^{(k)}_F(P)$. When $\mathrm{Bl}^{(k)}_F(P)$ have exactly 16 lattice points the pentagonal and hexagonal facets must be the smallest possible and the pentagonal facets must share an edge of length 2. Since $\mathrm{Bl}^{(k)}_F(P)$ have 7 facets in total the hexagonal facet F_6 must share an edge with every other facet. Moreover if we fix a pentagonal facet F_5 then 9 of the 10 vertices must lie in either F_6 or F_5 while the remaining vertex lies in the other pentagonal facet F_5'. Thus we can list all polytopes $\mathrm{Bl}^{(k)}_F(P)$ associated to a triangulation with label $3^2 4^2 5^2 6^1$ meeting our criteria by finding all possible ways to position F_5, F_5' and F_6 so that the following holds.

1. F_6 lies in the xy-plane with a vertex at the origin and an edge along each coordinate axis.
2. F_5 lies in the xz-plane, sharing an edge with F_6 and has one edge along the z-axis.
3. F_5' shares one edge of length 2 with F_5 and one edge with F_6.
4. Let $\mathrm{Bl}^{(k)}_F(P)$ be the convex hull of the configuration; then $\mathrm{Bl}^{(k)}_F(P)$ is smooth and $|\mathrm{Bl}^{(k)}_F(P) \cap \mathbb{Z}^3| \leq 16$.

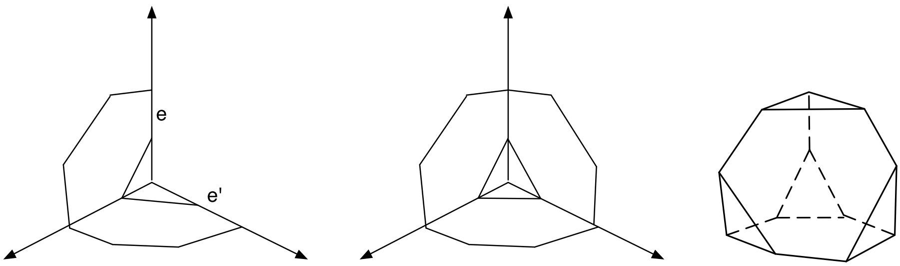
The three possible positions of F_5 and F_6 that meet requirement 1 and 2 are easily obtained from Lemma 30 and are illustrated below. Note that the convex hull of the



left configuration is not smooth at the vertex $(0, 1, 0)$. This reduces the possible ways to align F_5' into it either being contained in the yz-plane or sharing the edge e of length 2 with F_5 and having a vertex at $(0, 1, 1)$. The first configuration contains the point $(1, 1, 2) = \frac{1}{2}(2, 0, 2) + \frac{1}{2}(0, 2, 2)$ so in this case $|\mathrm{Bl}^{(k)}_F(P) \cap \mathbb{Z}^3| > 16$. The second configuration is $\mathrm{Cayley}^2_\Sigma(2\Delta_1, 2\Delta_1, 2\Delta_1)$ blown-up at two vertices. Note that the blow-up of $\mathrm{Cayley}^2_\Sigma(2\Delta_1, 2\Delta_1, 2\Delta_1)$ at two vertices is a smooth 3-polytope that contains exactly 16 lattice points. By the same argument we see that in the middle configuration $\mathrm{Bl}^{(k)}_F(P)$ must have a vertex at $(3, 1, 1)$. The resulting polytope in this case is easily seen to be isomorphic to $\mathrm{Cayley}^2_\Sigma(2\Delta_1, 2\Delta_1, 2\Delta_1)$ blown-up at two vertices. In the right configuration we see that $\mathrm{Bl}_F^{(k)}(P)$ must have a vertex at $(2,2,1)$ i.e. F_5' lies in the plane $x = 2$. However $\mathrm{Bl}_F^{(k)}(P)$ also contains the lattice point $(1,1,1) = \frac{1}{2}(1,0,2) + \frac{1}{2}(1,2,0)$ which is not a vertex so $|\mathrm{Bl}_F^{(k)}(P) \cap \mathbb{Z}^3| > 16$ in this case.

Next, we need to consider all smooth 3-polytopes that can be obtained via two blow-ups of a 3-polytope with a triangulation associated to the label $3^2 4^2 5^2$. This time the techniques of Lemma 37 reduce the blow-ups we need to consider to solely those associated with triangulation having the label $3^4 6^4$.

From the classification in [12, A5] each pair of the four hexagonal faces of a 3-polytope $\mathrm{Bl}(P)$ with label $3^4 6^4$ share exactly one edge. Moreover all hexagonal facets must be isomorphic to $F := \mathrm{Conv}((0,0), (1,0), (0,1), (1,2), (2,1), (2,2))$ since otherwise $\mathrm{Bl}(P)$ will contain more than 16 lattice points in its facets. Note that using Lemma 27 we can assume that two of the hexagonal facets lie in the xy- and yz-plane as illustrated in the figure to the left below. By the triangulation given in [12, A5] the facet in the xz-plane must be one of the other two hexagonal facets. Since every hexagonal facet has to be isomorphic to F this can only be done as illustrated in the second figure below. This procedure determines all 12 vertices of $\mathrm{Bl}(P)$. Therefore the polytope is obtained by taking the convex hull of the configuration, as indicated in the right figure. We note that $\mathrm{Bl}(P)$ is the blow-up of $3\Delta_3$ at four vertices,



that $|\mathrm{Bl}(P) \cap \mathbb{Z}^3| = 16$, and that this is the only 3-polytope with label $3^4 6^4$ meeting our requirements. Finally if we blow-up a simplex at a vertex or along an edge we get a 3-polytope with label $3^2 4^3$. Hence all blow-ups of simplices have already been considered. □

The above lemma covers the last cases to be considered by Lemma 36. Theorem 15 now follows and our classification is complete. A complete library of all polytopes and embeddings in our classification is given in the Appendix. Finally, we briefly mention how our results relate to two conjectures in toric geometry.

7 Smoothness and projective normality

The results of this paper relate to the following two conjectures in toric geometry.

Conjecture 1 Every smooth toric variety has a quadratic Gröbner basis.

Conjecture 2 Every smooth toric variety is projectively normal.

Recall that a projective toric variety X is projectively normal if the affine cone of X is a normal variety.

We have checked both these conjectures using the library toric.lib in Singular [4]. The reduced Gröbner basis with respect to the degree reverse lexicographical ordering consists entirely of quadratic binomials for every toric ideal $I_{P\cap\mathbb{Z}^3}$ corresponding to a polytope in the classification. This means that Conjecture 1 holds for all polytopes in our classification. To check Conjecture 2 we have used the following proposition.

Proposition 39 *Let $\mathcal{A} \subset \mathbb{Z}^d$ define a homogeneous toric ideal $I_{\mathcal{A}}$. Assume that $\prec$ is a term ordering on $\mathbb{C}[x_1, \dots, x_n]$ and that the initial ideal $in_{\prec}(I_{\mathcal{A}})$ with respect to $\prec$ is square free. Then the projective toric variety $X_{\mathcal{A}}$ associated to $I_{\mathcal{A}}$ is projectively normal.*

Proof See [14, p. 136]. □

For the ideals corresponding to the 3-polytopes in the classification the initial ideal $in_{\mathrm{lex}}(I_{\mathcal{A}})$ with respect to the lexicographical term ordering is square free. So in particular Conjecture 2 holds for all 3-polytopes in the classification. In combinatorial terms this means that the placing triangulation is a regular unimodular triangulation for all 3-polytopes in the classification (see [13, p. 256]). A hierarchic list of properties implying projective normality can be found in [6, p. 2313]. Having a regular unimodular triangulation is the strongest such property that holds for all polytopes in our classification. For more details on how these computations are done see: http://www.math.kth.se/~alundman.

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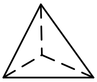
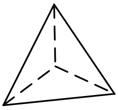
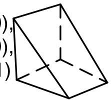
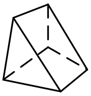
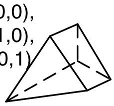
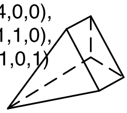
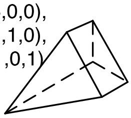
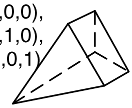
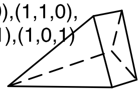
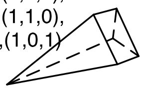
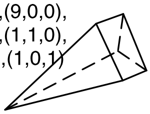
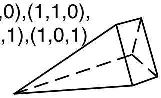
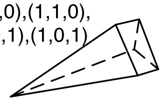
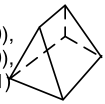
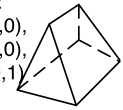
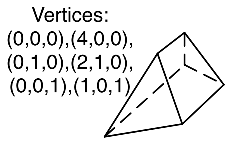
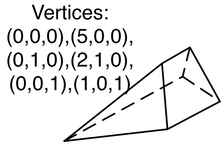
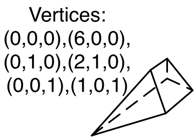
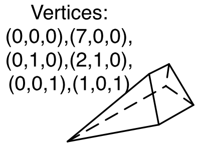
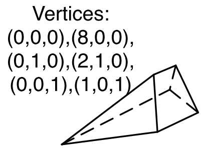
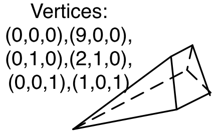
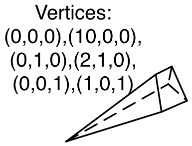
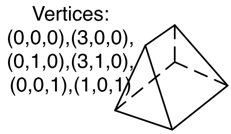
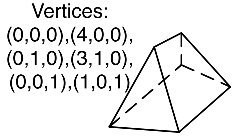
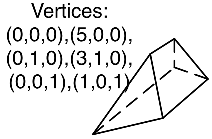
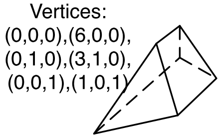
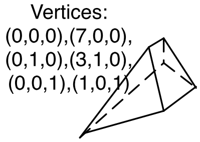
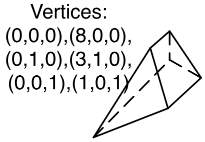
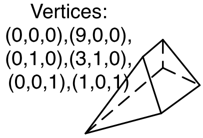
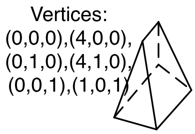
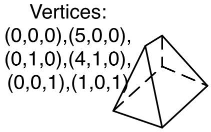
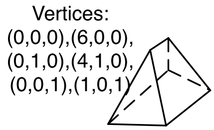
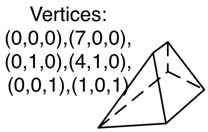
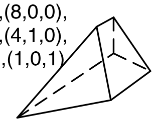
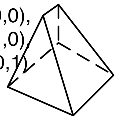
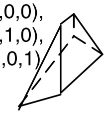
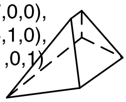
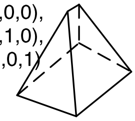
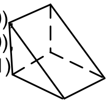
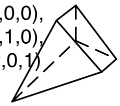
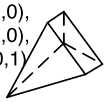
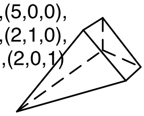
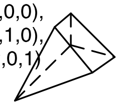
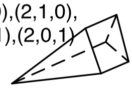
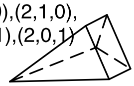
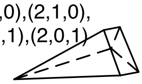
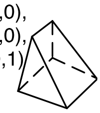
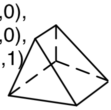
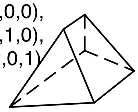
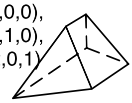
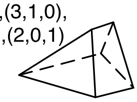
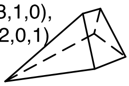
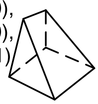
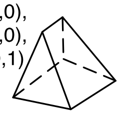
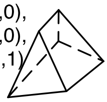
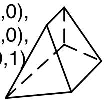
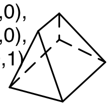
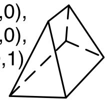
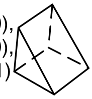
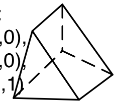
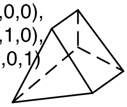
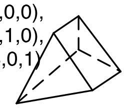
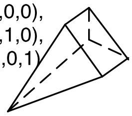
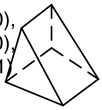
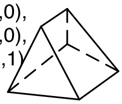
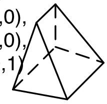
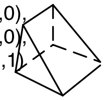
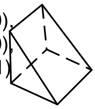
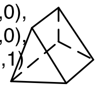
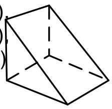
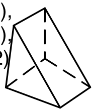
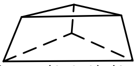
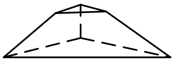
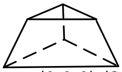
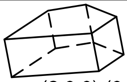
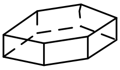
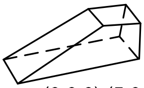
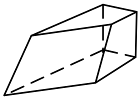
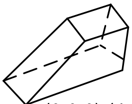
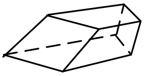
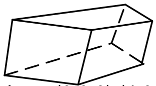
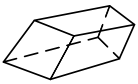
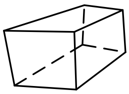
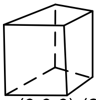
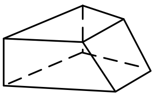
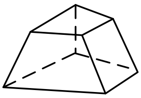
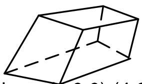
Appendix: A complete list of all smooth lattice 3-polytopes P such that $|P \cap \mathbb{Z}^3| \leq 16$

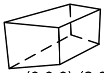
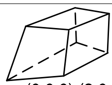
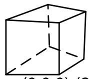
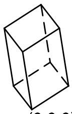
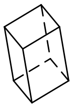
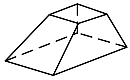
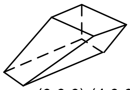
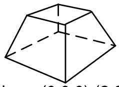
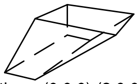
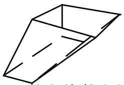
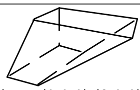
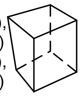
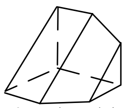
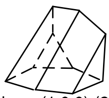
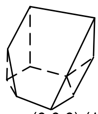
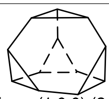
Vertices: (0,0,0), (1,0,0), (0,1,0), (0,0,1) 	Vertices: (0,0,0), (2,0,0), (0,2,0), (0,0,2) 	Vertices: (0,0,0),(1,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 
$\mathbb{P}^3$ embedded in $\mathbb{P}^3$	$\mathbb{P}^3$ embedded in $\mathbb{P}^9$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^5$
Vertices: (0,0,0),(2,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(3,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(4,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^6$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^7$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^8$
Vertices: (0,0,0),(5,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(6,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(7,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^9$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{10}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{11}$
Vertices: (0,0,0),(8,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(9,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(10,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$
Vertices: (0,0,0),(11,0,0), (0,1,0),(1,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(2,0,0), (0,1,0),(2,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(3,0,0), (0,1,0),(2,1,0), (0,0,1),(1,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^7$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^8$

		
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^9$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{10}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{11}$
		
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$
		
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^9$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{10}$
		
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{11}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$
		
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{11}$
		
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$

Vertices: (0,0,0),(8,0,0), (0,1,0),(4,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(5,0,0), (0,1,0),(5,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(6,0,0), (0,1,0),(5,1,0), (0,0,1),(1,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$
Vertices: (0,0,0),(7,0,0), (0,1,0),(5,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(6,0,0), (0,1,0),(6,1,0), (0,0,1),(1,0,1) 	Vertices: (0,0,0),(2,0,0), (0,1,0),(2,1,0), (0,0,1),(2,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{8}$
Vertices: (0,0,0),(3,0,0), (0,1,0),(2,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(4,0,0), (0,1,0),(2,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(5,0,0), (0,1,0),(2,1,0), (0,0,1),(2,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{9}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{10}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{11}$
Vertices: (0,0,0),(6,0,0), (0,1,0),(2,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(7,0,0), (0,1,0),(2,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(8,0,0), (0,1,0),(2,1,0), (0,0,1),(2,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$
Vertices: (0,0,0),(9,0,0), (0,1,0),(2,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(3,0,0), (0,1,0),(3,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(4,0,0), (0,1,0),(3,1,0), (0,0,1),(2,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{10}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{11}$
Vertices: (0,0,0),(5,0,0), (0,1,0),(3,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(6,0,0), (0,1,0),(3,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(7,0,0), (0,1,0),(3,1,0), (0,0,1),(2,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$

Vertices: (0,0,0),(8,0,0), (0,1,0),(3,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(4,0,0), (0,1,0),(4,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(5,0,0), (0,1,0),(4,1,0), (0,0,1),(2,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$
Vertices: (0,0,0),(6,0,0), (0,1,0),(4,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(7,0,0), (0,1,0),(4,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(5,0,0), (0,1,0),(5,1,0), (0,0,1),(2,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$
Vertices: (0,0,0),(6,0,0), (0,1,0),(5,1,0), (0,0,1),(2,0,1) 	Vertices: (0,0,0),(3,0,0), (0,1,0),(3,1,0), (0,0,1),(3,0,1) 	Vertices: (0,0,0),(4,0,0), (0,1,0),(3,1,0), (0,0,1),(3,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{11}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{12}$
Vertices: (0,0,0),(5,0,0), (0,1,0),(3,1,0), (0,0,1),(3,0,1) 	Vertices: (0,0,0),(6,0,0), (0,1,0),(3,1,0), (0,0,1),(3,0,1) 	Vertices: (0,0,0),(7,0,0), (0,1,0),(3,1,0), (0,0,1),(3,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$
Vertices: (0,0,0),(4,0,0), (0,1,0),(4,1,0), (0,0,1),(3,0,1) 	Vertices: (0,0,0),(5,0,0), (0,1,0),(4,1,0), (0,0,1),(3,0,1) 	Vertices: (0,0,0),(6,0,0), (0,1,0),(4,1,0), (0,0,1),(3,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$
Vertices: (0,0,0),(5,0,0), (0,1,0),(5,1,0), (0,0,1),(3,0,1) 	Vertices: (0,0,0),(4,0,0), (0,1,0),(4,1,0), (0,0,1),(4,0,1) 	Vertices: (0,0,0),(5,0,0), (0,1,0),(4,1,0), (0,0,1),(4,0,1) 
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$
Vertices: (0,0,0),(1,0,0), (0,2,0),(1,2,0), (0,0,2),(1,0,2) 	Vertices: (0,0,0),(3,0,0), (0,2,0),(1,2,0), (0,0,2),(1,0,2) 	 Vertices: (0,0,0),(3,0,0), (0,3,0),(0,0,1)(2,0,1), (0,2,1)
A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{11}$	A $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ embedded in $\mathbb{P}^{15}$

 Vertices: (0,0,0),(3,0,0),(0,3,0),(0,0,1)(1,0,1),(0,1,1)	 Vertices: (0,0,0),(2,0,0),(0,2,0),(0,0,1),(1,0,1),(0,1,1)	 Vertices: (2,0,0),(0,1,0),(1,0,0),(0,3,0)(2,1,0),(2,0,1),(0,1,1),(1,0,1),(0,3,1)(2,1,1)
A $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ embedded in $\mathbb{P}^8$	A $\mathbb{P}^1$-bundle over $\mathrm{Bl}_2(\mathbb{P}^2)$ embedded in $\mathbb{P}^{15}$
 Vertices: (2,0,0),(0,1,0),(1,0,0),(0,2,0),(1,2,0),(2,1,0),(2,0,1),(0,1,1),(1,0,1),(0,2,1),(2,1,1),(1,2,1)	 Vertices: (0,0,0),(5,0,0),(0,1,0),(5,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)	 Vertices: (0,0,0),(5,0,0),(0,1,0),(2,1,0),(0,0,1),(4,0,1),(0,1,1),(1,1,1)
A $\mathbb{P}^1$-bundle over $\mathrm{Bl}_3(\mathbb{P}^2)$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_4$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_3$ embedded in $\mathbb{P}^{15}$
 Vertices: (0,0,0),(4,0,0),(0,1,0),(4,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)	 Vertices: (0,0,0),(5,0,0),(0,1,0),(3,1,0),(0,0,1),(3,0,1),(0,1,1),(1,1,1)	 Vertices: (0,0,0),(4,0,0),(0,1,0),(2,1,0),(0,0,1),(4,0,1),(0,1,1),(2,1,1)
A $\mathbb{P}^1$-bundle over $\mathbb{F}_3$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_2$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_2$ embedded in $\mathbb{P}^{15}$
 Vertices: (0,0,0),(4,0,0),(0,1,0),(2,1,0),(0,0,1),(3,0,1),(0,1,1),(1,1,1)	 Vertices: (0,0,0),(3,0,0),(0,1,0),(1,1,0),(0,0,1),(3,0,1),(0,1,1),(1,1,1)	 Vertices: (0,0,0),(2,0,0),(0,1,0),(1,1,0),(0,0,2),(2,0,2),(0,1,2),(1,1,2)
A $\mathbb{P}^1$-bundle over $\mathbb{F}_2$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_2$ embedded in $\mathbb{P}^{11}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_1$ embedded in $\mathbb{P}^{14}$
 Vertices: (0,0,0),(3,0,0),(0,2,0),(1,2,0),(0,0,1),(3,0,1),(0,1,1),(2,1,1)	 Vertices: (0,0,0),(3,0,0),(0,2,0),(1,2,0),(0,0,1),(2,0,1),(0,1,1),(1,1,1)	 Vertices: (0,0,0),(4,0,0),(0,1,0),(3,1,0),(0,0,1),(3,0,1),(0,1,1),(2,1,1)
A $\mathbb{P}^1$-bundle over $\mathbb{F}_1$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_1$ embedded in $\mathbb{P}^{15}$

 Vertices: (0,0,0),(3,0,0), (0,1,0),(2,1,0),(0,0,1), (3,0,1),(0,1,1),(2,1,1)	 Vertices: (0,0,0),(3,0,0), (0,1,0),(2,1,0),(0,0,1), (2,0,1),(0,1,1),(1,1,1)	 Vertices: (0,0,0),(2,0,0), (0,1,0),(1,1,0),(0,0,1), (2,0,1),(0,1,1),(1,1,1)
A $\mathbb{P}^1$-bundle over $\mathbb{F}_1$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_1$ embedded in $\mathbb{P}^{11}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_1$ embedded in $\mathbb{P}^9$
 Vertices: (0,0,0),(1,0,0), (0,1,0),(1,1,0),(0,0,3), (1,0,3),(0,1,3),(1,1,3)	 Vertices: (0,0,0),(1,0,0), (0,1,0),(1,1,0),(0,0,2), (1,0,2),(0,1,2),(1,1,2)	 Vertices: (0,0,0),(3,0,0), (0,2,0),(3,2,0)(0,0,1), (1,0,1),(0,1,1),(1,1,1)
A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^{11}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^{15}$
 Vertices: (0,0,0),(4,0,0), (0,1,0),(4,1,0),(0,0,1), (1,0,1),(0,2,1),(1,2,1)	 Vertices: (0,0,0),(2,0,0), (0,2,0),(2,2,0)(0,0,1), (1,0,1),(0,1,1),(1,1,1)	 Vertices: (0,0,0),(3,0,0), (0,1,0),(3,1,0)(0,0,1), (1,0,1),(0,3,1),(1,3,1)
A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^{15}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^{12}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^{15}$
 Vertices: (0,0,0),(3,0,0), (0,1,0),(3,1,0)(0,0,1), (1,0,1),(0,2,1),(1,2,1)	 Vertices: (0,0,0),(2,0,0), (0,1,0),(2,1,0)(0,0,1), (1,0,1),(0,2,1),(1,2,1)	Vertices: (0,0,0),(1,0,0), (0,1,0),(1,1,0) (0,0,1),(1,0,1), (0,1,1),(1,1,1) 
A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^{13}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^{11}$	A $\mathbb{P}^1$-bundle over $\mathbb{F}_0$ embedded in $\mathbb{P}^7$
 Vertices: (0,0,0),(2,0,0), (0,2,0),(0,0,2)(2,1,0), (2,0,1),(1,2,0),(1,0,2)	 Vertices: (1,0,0),(2,0,0), (0,1,0),(0,2,0),(0,0,1), (0,0,2),(2,1,0),(2,0,1), (1,2,0),(1,0,2),	 Vertices: (0,0,0),(1,0,0), (0,1,0),(1,2,0),(2,1,0), (2,2,0),(0,0,2),(2,0,1), (2,0,2),(0,1,1)
The blow-up at one point of a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{14}$	The blow-up at two points of a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ embedded in $\mathbb{P}^{13}$	The blow-up at two points of a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ embedded in $\mathbb{P}^{15}$
	 Vertices: (1,0,0),(2,0,0), (0,1,0),(0,2,0),(0,0,1), (0,0,2)(2,1,0),(2,0,1), (1,2,0),(0,2,1),(1,0,2), (0,1,2)	
	The blow-up at four points of $\mathbb{P}^3$ embedded in $\mathbb{P}^{15}$	

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