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Comparing the Lower Secondary Textbooks of Japan and England: a Praxeological Analysis of Symmetry and Transformations in Geometry

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Abstract

This study is aimed at comparing the content in mathematics textbooks from Japan and England, focusing on symmetry and transformations at the lower secondary level. We adopted the concept of *praxeology*, a main construct of the Anthropological Theory of the Didactic (ATD). Using the ATD framework, our approach regarded textbooks as an empirical source which can reveal knowledge to be taught in the didactic transposition process. The praxeological analysis results indicated that symmetry and transformations in the Japanese textbooks were strongly influenced by the teaching of geometric proofs, while transformations in the textbooks from England had many connections to other contexts or contents across domains. These findings suggested the ways in which the knowledge of symmetry and transformations is differently situated in the two countries' textbooks in terms of different praxeological organisations. We further discussed our findings for elaborating the theoretical and methodological aspects of this study, which can potentially contribute to future research on textbooks in mathematics education.

Keywords Comparative study · Praxeology · Symmetry · Textbooks · Transformations

Introduction

The study of mathematics textbooks has become widely recognised as a scientific research field in the international mathematics education research community (Fan,

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2013; Fan, Trouche, Qi, Rezat, & Visnovska, 2018; Jones, Bokhove, Howson, & Fan, 2014). According to Pepin, Gueudet, and Trouche (2013), "mathematics textbooks are widely used as the main resource for teaching, and they are perceived to reflect the views expressed in curricular documents" (p. 686). Although the same textbook does not necessarily provide the same classroom practice, it is still important to recognise textbooks as a "potentially implemented curriculum" (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002) and study them as such. According to reviews, a major research concern in this field is textbook comparison, which may be undertaken in various ways (national, international, and historical) and on different scales (large-scale survey or small-scale case study) (Fan, Zhu, & Miao, 2013; Howson, 2013; Kilpatrick, 2014). In comparative studies at the international level, much can be learned from textbooks used in other countries (Howson, 2013). However, to find common ground in this growing field, it is necessary for researchers to elaborate on both the theoretical and methodological aspects of research on mathematics textbooks (Fan et al., 2013).

For our study, we developed these theoretical and methodological aspects using the Anthropological Theory of the Didactic (ATD) (Chevallard, 2006, 2019; Chevallard & Sensevy, 2014). Certain ATD constructs can be used effectively for international comparative studies (Artigue et al., 2017; Artigue & Winsløw, 2010; Miyakawa, 2017). International comparative studies are often aimed at identifying and explaining the similarities and differences among two or more countries. According to ATD, these similarities and differences can be understood and characterised by the conditions and constraints of knowledge used by different educational systems (called *institutions*). ATD posits that certain mathematical knowledge exists in specific institutions, such as in a community of mathematicians, an educational system, a mathematics classroom, or a community of study (Bosch & Gascón, 2006, 2014; Chevallard, 1985). Further, specific curricula or textbooks are produced from a didactic transposition under the influence of the different cultural and historical elements of a given country (Kang & Kilpatrick, 1992). From this perspective, textbooks can be considered the main empirical source allowing us to understand that what constitutes a piece of knowledge to be taught varies from country to country. To contribute to the theoretical aspects of textbook research, we adopted *praxeology*, a main construct of ATD.

Our study investigated the knowledge to be taught regarding symmetry and transformations in two specific countries with different educational traditions: Japan and England. After the late-nineteenth-century educational reforms, Japanese geometry education in secondary schools began with the publication of a seminal textbook strongly influenced by geometry education in England (Cousin, 2018; Cousin & Miyakawa, 2017; Ueno, 2012). Hence, it is worthwhile to compare the mathematics textbooks of these two countries. In the current curricula of both countries, symmetry and transformations are included in the domain of geometry. As the textbooks from these two countries exhibit interesting and contrasting approaches to school geometry (Jones & Fujita, 2013), concepts would also be expected to be situated differently. While symmetry and transformations are central to mathematics (Coxeter, 1961; Weyl, 1952), research on the teaching and learning of these concepts is still relatively under-represented in mathematics education literature (Healy, 2003; Usiskin, 2014). Although Jones and Fujita (2013) provided an illustration of the geometrical contents of both the national curricula and textbooks for Grade 8 in England and Japan, our study offers a more detailed perspective by focusing on symmetry and transformations. In summary, by means of a praxeological analysis, this study examined how symmetry and transformations is differently situated in the domain of geometry in textbooks from England and Japan.

Theoretical Perspective

Literature Review: Research on Mathematics Textbooks

While textbook comparison is a common avenue for conducting mathematics education research, studies may focus on different topics and adopt various viewpoints. Many studies have focused on particular mathematical content or specific tasks in two or more countries' textbooks (e.g. Alajmi, 2012; Jones & Fujita, 2013; Miyakawa, 2017; Wang, Barmby, & Bolden, 2017). Some have investigated mathematical processes rather than content, such as problem-solving (Fan & Zhu, 2007), problem-posing (Cai & Jiang, 2017), and reasoning (Fujita & Jones, 2002, 2014). Other research has illustrated the more general or cultural features of textbooks (e.g. Dowling, 1996; Haggarty & Pepin, 2002; Park & Leung, 2006). Comparative studies between East Asian and Western countries have provided valuable insights into such cultural aspects (Leung, 2006).

Overall, textbook comparison research is multi-faceted and offers different scopes ranging from the particular to the general, such as whether a textbook is content-focused or process-focused or what cultural aspects are represented. Although our comparative study focused on a specific content, the theoretical frameworks used in this study can also help to conceptualise the different scopes and layers exhibited by different educational systems.

Theoretical Framework: Praxeology

In this study, we adopted the concept of praxeology, one of the main constructs of ATD. The basic tenet of ATD includes this description of praxeology: "ATD postulates that any activity related to the production, diffusion, or acquisition of knowledge should be interpreted as an ordinary human activity, and thus proposes a general model of human activities built on the key notion of praxeology" (Bosch & Gascón, 2014, p. 68). A praxeology comprises two blocks—praxis and logos—and each block comprises two elements, as shown in Table 1. The praxis block is usually explicit, but the logos block is often implicit and not easily identified and interpreted. As per this framework, the quadruplet [$\mathbf{T}/\tau/\theta/\Theta$] provides a basic model for praxeological analysis.

Table 1 A praxeology (descriptions adapted from Chevallard & Sensevy, 2014, p. 40)

Praxis block		Logos block	
Type of task ($\mathbf{T}$)	Technique (τ)	Technology (θ)	Theory (Θ)
Problems of a given type	A way of performing this type of task	A way of explaining and justifying (or designing) the technique	To explain, justify, or generate whatever part of the technology that may sound unclear or missing

Question ⇆ Theme ⇆ Sector ⇆ Domain ⇆ Discipline ⇆
Pedagogy ⇆ School ⇆ Society ⇆ Civilization

Fig. 1 Levels of co-determinacy (adapted from Bosch & Gascón, 2014, p. 73)

In a praxeological analysis, it is useful to consider different delimitations depending on the activities or knowledge at stake. To do so, "a distinction is made between 'point praxeology' (containing a single type of task), a 'local praxeology' (containing a set of types of tasks organised around a common technological discourse), and a 'regional praxeology' (containing all point and local praxeologies that share a common theory)" (Bosch & Gascón, 2014, p. 69). Using these notions, ATD focuses on the conditions and constraints shaping both what mathematical knowledge is taught (mathematical praxeologies) and how it is taught (didactic praxeologies) in different institutions (i.e. different educational systems in different cultural settings).

The different sizes of praxeological organisations (i.e. point, local, and regional) are explained in terms of another ATD construct: *the scale of didactic co-determinacy*. The nine institutional levels shown in Fig. 1 depict how conditions and constraints interact with each other (Artigue & Winsløw, 2010; Bosch & Gascón, 2014).[1]

As the notion of didactic co-determinacy highlights the institutional relativity of knowledge, it can be useful for international comparative studies (Artigue & Winsløw, 2010). From this perspective, the similarities and differences found at a certain level can be explained by the conditions and constraints that appear at a higher level. In the present study, our focus was on the mathematical praxeologies at the levels from *question* to *domain* because our primary data were sourced from textbooks from a specific country at a particular moment in time. Although it is also important to consider the higher levels (from *discipline* to *civilization*), exploring this matter further would involve many factors other than textbooks, going beyond the scope of the present study.

According to Artigue and Winsløw (2010) and Chevallard (2019), a domain can be created from a global praxeology (or a collection of regional praxeologies), a sector is characterised by one regional praxeology, a theme refers to one local praxeology, and a question is concentrated on a point praxeology. For the purposes of our study, geometry can be understood as a domain, symmetry and transformations can be seen as a sector (which should be specified in the chapters of the textbooks), reflective symmetry is an example of a theme, and questions are represented by the problems described in the textbook.

Research Question and the Scope of the Paper

Bearing all this in mind, the research question for this study is as follows: What are the similarities and differences between the knowledge regarding symmetry and transformations in the content of mathematics textbooks from England and those in mathematics textbooks from Japan? To answer this question, a praxeological analysis of the textbooks of both countries was performed. In doing so, we investigated the conditions

[1] In this paper, we use the English translations "co-determinacy" and "question" (as the first level of co-determinacy) from French according to the latest translations by Chevallard (2019), although these words have previously been translated as "co-determination" and "subject." It is also notable that Chevallard (2019) posits "*Humanity*" as the highest level of co-determinacy.

and constraints of the knowledge in textbooks and compared them in terms of the different sizes of their praxeological organisations. Although we attempted to clarify the different praxeological characteristics by focusing on the sub-disciplinary levels of co-determinacy (from *question* to *domain*), this does not imply that the present study ignored the cultural aspects which may affect the textbooks used in each country.

Methodology

Educational System Background

The different cultural backgrounds of Japan and England have likely led to their educational systems and textbooks showing some particularities in design. The educational system in Japan is outlined in a national curriculum called the *Course of Study* (Ministry of Education, Culture, Sports, Science and Technology [MEXT], 2008a), which is typically revised about every 10 years. Schooling is compulsory for 9 years in Japan, with primary education comprising Grades 1 to 6 (from age 6 to age 12), lower secondary comprising Grades 7 to 9 (from age 12 to age 15), and upper secondary comprising Grades 10 to 12 (from age 15 to age 18). The Japanese Ministry of Education authorises textbooks strictly based on the Course of Study. Currently, seven different mathematics textbook series are used for lower secondary schools, each published by a different company. The choice of textbook is made by local boards of education, not by schools or teachers. Each local board of education publishes its choice on its official website or in other documents, enabling us to determine which textbook series is used in each school.

The *National Curriculum* in England is provided by the Department for Education, with the latest edition implemented in September 2014 (Department for Education [DfE], 2013). The curriculum comprises four key stages (KS): primary education includes KS1 and KS2 (Grades 1 to 6, ages 5 to 11) and secondary education includes KS3 and KS4 (Grades 7 to 11, ages 11 to 16). Both primary and secondary education are compulsory, while further education after age 16, called "sixth form", is optional. Textbooks in England are provided by a wide range of publishers. Although no government agency officially approves textbooks, some secondary school textbooks have been published in association with the General Certificate of Secondary Education (GCSE), a school leaving qualification in England. The choice of textbooks is, in principle, made by each school, but "the degree of use range[s] widely, determined by teachers individually rather than by their school" (Siedel & Stylianides, 2018, p. 126). The wide range of publishers and the high degree of teacher autonomy make it difficult to determine which textbook series is actually used in each English school.

Selection of Textbooks

The primary data analysed were taken from lower secondary mathematics textbooks, and some data came from primary school textbooks. To examine Japanese textbooks, we chose Keirinkan's *Gateway to the future: Math* (Okamoto, Koseki, Morisugi, & Sasaki, 2012), among the most popular textbook series and publishers in Japan. Although the textbook has undergone changes before the time of writing, the revised version is used in the majority of the Japanese lower secondary schools. For England, we chose Cambridge University Press's *SMP Interact* series (School Mathematics Project, 2008a, b), which is edited and published in association with the GCSE. Although it is difficult to determine which textbook series is most commonly used in England, *SMP Interact* can be considered the most traditionally representative textbook series because it has been used as the source for a TIMSS survey on mathematics textbooks (Foxman, 1999), an international comparative research project on mathematics textbooks (Japan Textbook Research Center, 2012) and other research studies (e.g. Dowling, 1996; Hodgen, Küchemann, & Brown, 2010).

Although both countries' selected textbooks are based on the official curriculum content at the time (Department for Education and Employment [DfEE], 1999; MEXT, 2008a), they are not the newest editions due to recent curriculum reforms and are not entirely representative of the present teaching of these topics. However, each country's latest official curriculum documents (DfE, 2013; MEXT, 2017) still include symmetry and transformations in the geometry domain at the lower secondary level. Therefore, they were appropriate as the primary sources in this study.

The English textbook series has three different levels (namely, "foundation", "standard", and "higher") to accommodate variations in individuals' abilities, while the Japanese textbooks have only one type for lower secondary mathematics. In our study, we mainly used "standard" textbooks, but some information was taken from "higher" textbooks as well. For Japan, we also examined the primary mathematics textbook series *Fun with Math* from Keirinkan (Shimizu & Funakoshi, 2010), as the content focuses on symmetry, which is included in the English textbook for Grade 7, and is included at the primary level in Japan (see Table 2).

Table 2 presents basic information about the chapters on symmetry and transformations from the Japanese and English textbooks. The "chapter" refers to the name of the chapter or section excerpted from the textbooks. The "objectives" refer to what each chapter describes regarding the aim and content, but these have been lightly edited by the authors so that the main topics included in the chapter are more explicit. As can be seen from Table 2, the organisation of the chapter on symmetry and transformations from the Japanese textbook is different from that from the English textbook. In Japan, each theme has one chapter that covers all contents to be taught in a specific grade, while in England, each theme has two chapters with the same title that can be used across grades. It is also notable that point reflection (or point symmetry) is introduced in Grade 6 in Japan, and then in Grade 7, this content is considered a particular case of rotation symmetry referring to rotation symmetry restricted to "180° rotation".

Analytical Model: a Reference Epistemological Model

Our analytical model was designed to be what Bosch and Gascón (2006, 2014) termed a *reference epistemological model* (REM[2]), which constitutes the basic theoretical lens through which researchers analyse different mathematics-based knowledge in different institutions (Chevallard, 1985/1991). In our study, an REM was required to analyse knowledge pertaining to symmetry and transformations. For this model, it is useful to refer to four situations regarding symmetry, a concept based on an idea from Vergnaud

[2] The reference epistemological model is also called the *reference praxeological model* (Wijayanti & Winsløw, 2017) because it is used for praxeological analysis.

Table 2 Main topics focusing on symmetry and transformations in the textbooks from Japan and England

	Japan		England	
	Chapter (Grade, page number)	Objectives	Chapter (Grade, page number)	Objectives
Symmetry	Symmetric figures (6th grade, 18 pages)	To understand reflection symmetry and point reflection and their respective properties by observing and composing geometrical figures	Symmetry 1 (7th grade, 8 pages)	To make shapes that have reflection symmetry and to identify symmetrical shapes and their lines of reflection symmetry
			Symmetry 2 (7th grade, 7 pages)	To recognise rotation symmetry, to draw patterns with rotation symmetry, and to find all the different symmetries in a pattern
Transformations	Figure transformation (7th grade, 6 pages)	To learn transformations in geometry, such as translation, rotation, and reflection and their properties	Transformations (7th grade, 7 pages)	To follow instructions for transformations in geometry, such as translation, rotation, and reflection, and to describe these transformations clearly
			Transformations (9th grade, 10 pages)	To transform points and shapes using translations, reflections, rotations, and a combination of these and to describe clearly how points and shapes have been transformed

(2009). Vergnaud (2009) started with two situations considering the symmetrical shape of a given figure such as the one in Fig. 2.

By adding two other situations, Vergnaud (2009) clearly illustrated the following four situations (sentences) which are related to both symmetry and transformations (p.

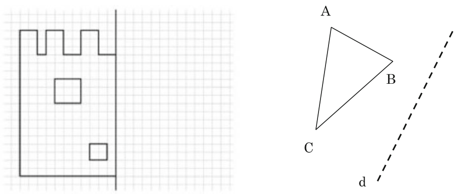

Fig. 2 Two situations of symmetry (Vergnaud, 2009, p. 90)

91) (while these four sentences were originally given in both French and English, we have only quoted the English sentences):

1. The fortress is symmetrical (the left side of Fig. 2)
2. Triangle $A'B'C'$ is symmetrical to triangle ABC in relation to line d' (the right side of Fig. 2)
3. Symmetry conserves lengths and angles
4. Symmetry is an isometry

Essential to this characterisation are the qualitative and epistemological jumps in the transitions, as "inevitably the succession of jumps in the operational and in the predicative forms of mathematical knowledge causes difficulties for students" (Vergnaud, 2009, p. 91). Below, we summarise what Vergnaud (2009) meant by these jumps.

- Transition between 1 and 2: the adjective "symmetrical" moves from the status of a one-element predicate to the status of a three-element predicate (*A* is symmetrical to *B* in relation to *C*)
- Transition between 2 and 3: the predicate "symmetrical" is transformed into an object of thought, "symmetry", which has its own properties; it conserves lengths and angles
- Transition between 3 and 4: the retention of lengths and angles then becomes an object of thought as "isometry"

Although the present study did not address the difficulties that students may have with learning these concepts, these four points (sentences) and the jumps between them can still provide a useful reference model for knowledge pertaining to symmetry and transformations. In our analysis, this model was specifically used for interpreting the implicit logos blocks of each praxeology.

Results of Analysis

The Praxis Block

In the present study, we analysed all the questions (problems) which are described in the main text pages of the textbooks as well as the additional questions ("exercises") which are usually listed at the end of the chapter. While some of these might be used during classroom lessons, others might not be carried out in the classroom but, rather, assigned as homework. For the purpose of this study, we included all questions in the analysis because we were not able to judge the availability or frequency of each question in an actual classroom situation.

The Task Types In this analysis, we first considered the praxis block (task types and technique) and then interpreted the logos block. Each task type was defined by classifying questions into sets. Labelling and coding the task types are listed as **T1** to **T8**; in Tables 3 and 4, we attempted to describe the general characteristics of a set of questions included in the same task type, as well as to show some examples

Table 3 Types and number of tasks for symmetry

	Japan ($n = 49$)	England ($n = 110$)
T1: to identify symmetrical shapes (e.g. "Which of the letters below have line symmetry?" [G6, JP]) (e.g. "How many lines of symmetry does each pattern have?" [G7, EN])	15 (36.6%)	56 (51%)
T2: to draw the symmetrical shape (e.g. "Draw a figure with point symmetry so that point O is the centre of symmetry." [G5, JP]) (e.g. "Copy and complete each pattern so it has rotation symmetry of order 4." [G7, EN])	6 (12.2%)	54 (49%)
T3: to find the corresponding points and lines (e.g. "Find out where point *B* is if point *B* corresponds to point *A*." [G6, JP])	22 (45%)	–
Other (e.g. "How does line *AK*, which connects the corresponding points *A* and *K*, intersect the axis of symmetry?" [G6, JP])	6 (12.2%)	–

G, grade; *EN*, English textbook; *JP*, Japanese textbook

from both countries' textbooks. Although the types of tasks provide basic data to be analysed in terms of point, local, and regional praxeologies, here, we briefly present the distributions of task types for symmetry and transformations according to each theme.

The number of identified task types for symmetry in the textbooks from both countries is shown in Table 3. For the Japanese textbook, 49 tasks were categorised into three different types ("Other" refers to tasks which do not fit the other categories). The textbook from England had 110 tasks that were categorised into two different types. From Table 3, it is evident that both countries' textbooks shared two types of tasks (**T1** and **T2**), while **T3** was only observed in the Japanese textbook.

The number of transformation-based task types identified and their descriptions are presented in Table 4. This table shows that both countries' textbooks share **T4** and **T5**, whereas **T6** is only identified in Japanese textbooks, and **T7** and **T8** are only observed in English ones.

Although there was a quantitative difference between the countries' textbooks in terms of the number of tasks, we did not pay much attention to this in our praxeological analysis. What we considered to be the relevant data from Tables 3 and 4 were the identified types rather than the individual tasks.

Techniques To identify a technique for each type of task, we first examined what solutions and approaches are required to solve questions included in each type of task, and then, we tried to create categories that described those solutions and approaches. As a result, we identified four different techniques: *perceptual*, *physical*, *operational*, and *algebraic*. These can be described as follows[3]: a perceptual technique mainly relies on

[3] Some readers might feel that there is a similarity between the four techniques and Duval's (1995) four geometric apprehensions: *perceptual*, *sequential*, *operational*, and *discursive*. Although in developing the categories we have considered the adaptability of Duval's idea, we do not intend to make a theoretical connection to the present praxeological analysis.

Table 4 Types and number of tasks for transformation

	Japan ($n = 21$)	England ($n = 181$)
T4: to draw the shape that is transformed (translated, rotated, and reflected) (e.g. "Construct the shape that rotated 180° from $\triangle ABC$ around the centre of rotation O." [G7, JP]) (e.g. "Draw a mirror image of the L-shape for each of the mirror lines." [G7, EN])	13 (62%)	89 (49%)
T5: to describe a (single or pair of) transformation(s) that will map one shape to another (e.g. "Which types of transformation does this movement combine?" [G7, JP]) (e.g. "Describe a rotation that maps shape A to G." [G9, EN])	5 (24%)	49 (27%)
T6: to identify a property of a given transformation (translation, rotation, and reflection) (e.g. "What relationship is there among the segments AP, BQ, and CR linking corresponding points and the axis of reflection?" [G7, JP])	3 (14%)	–
T7: to combine transformations (e.g. "What single transformation could replace a clockwise rotation of 90° about (0, 0) followed by a reflection in the x-axis?" [G9, EN]) (e.g. "Translating by $\begin{bmatrix} 4 \\ a \end{bmatrix}$ then $\begin{bmatrix} b \\ -2 \end{bmatrix}$ has the same effect as translating by $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$.Find the values of a and b." [G9, EN])	–	16 (9%)
T8: to represent a given translation as a column vector (e.g. "Use column vectors to describe the translation that maps G to D." [G9, EN])	–	27 (15%)

G, grade; *EN*, English textbook; *JP*, Japanese textbook

visual judgement based on the appearance of given shapes; a physical technique is performed using physical tools for drawing and measuring, such as a mirror, ruler, or compass; an operational technique is performed with shapes on a grid sheet (squared paper) for drawing or by using coordinates; and an algebraic technique is based on expressions by column vectors. As summarised and explained in Table 5, the possibility exists that one type of task may involve different kinds of techniques. In addition to the techniques identified in the corresponding row, other techniques could be used for the same type of task, but our analysis was primarily based on the contexts and content provided by the textbooks. In Table 5, we describe some features which represent typical approaches to perform the task rather than concrete solutions from the textbooks.

As shown in Tables 3, 4, and 5, the first three techniques (**τ1**, **τ2**, **τ3**) appeared in both countries' textbooks. In terms of frequency, the perceptual and physical techniques covered four types of tasks in the Japanese textbooks, and the operational technique covered four types of tasks in the English textbooks. The algebraic technique in **T7** and **T8** existed only in the English textbooks (see Table 4). Although a more detailed analysis of the praxis block might have clarified more characteristics of each question in each text, what was more informative for the praxeological analysis in our study was the praxis block's relationship to the two elements of the logos block.

Table 5 Techniques for each of type of task relating to symmetry and transformations

Type of task	Technique	Descriptions of techniques with examples
T1	**τ1**: perceptual **τ2**: physical	**τ1**: relying on visual judgement (to distinguish one set of shapes from another) **τ2**: using a mirror or holding (to find how many lines of symmetry a given shape has)
T2	**τ2**: physical **τ3**: operational	**τ2**, **τ3**: using a mirror or holding (to construct a reflected shape in relation to the line of symmetry)
T3	**τ1**: perceptual	**τ1**: relying on visual judgement (to find the corresponding points and lines in relation to the line of symmetry)
T4	**τ1**: perceptual **τ2**: physical **τ3**: operational	**τ1**, **τ2**, **τ3**: relying on visual judgement by using ruler, and with a grid sheet (to draw a reflected image of a cat in the mirror)
T5	**τ1**: perceptual **τ3**: operational	**τ1**, **τ3**: relying on visual judgement, and with a grid sheet (to describe a rotation that maps one shape to another)
T6	**τ2**: physical	**τ2**: using a ruler and protractor for measuring (describing properties among the segments linking corresponding points and the line of symmetry)
T7	**τ3**: operational **τ4**: algebraic	**τ3**: with a grid sheet or coordinate plane (to describe a single transformation that could replace one transformation followed by another transformation) **τ4**: using vector (to combine a pair of translations)
T8	**τ3**: operational **τ4**: algebraic	**τ3**, **τ4**: with a grid sheet and by using vector (to represent the given translation as a column vector)

The Logos Block

To identify the logos block, the REM, based on Vergnaud's (2009) idea, was used as an interpretative lens. In terms of local praxeology, the first three task types for symmetry (**T1/T2/T3**) can be organised by a common technological discourse such that "one shape is symmetrical (i.e. a one-element predicate)" (**θ1**). This technology can be supposed to justify each of the techniques (**τ1/τ2/τ3**) because it is based sometimes on visual evidence, sometimes on experiment or measurement, and sometimes on arithmetic properties of the grid sheet. Furthermore, this technological discourse is based on the property of isometry that "symmetry conserves length and angles," although it is often only implicit in the textbooks. On the contrary, the technological discourse "*A* is symmetrical to *B* in relation to *C* (i.e. a three-element predicate)" (**θ2**) seems to justify the two task types (**T4/T5**) and the set of their techniques (**τ1/τ2/τ3**) for transformations.

Regarding **T6** and its technique, (**τ2**) can be justified by a more property-oriented technology according to which "there are geometric relationships between the before and after shapes that are transformed" (**θ3**). While a bit general, this statement refers to properties of a particular transformation. For example, a praxis block of reflection (**T6**) can be explained as the particular technology (**θ3**), "the axis of reflection is the perpendicular bisector of the segments linking pairs of corresponding points." This property can underlie the physical approach of, say, measuring the length or angles of a given shape. In addition, **T7** and its techniques (**τ3/τ4**) share the same technological discourse (**θ2**) as **T4/T5**, although **T4/T5** refers to one kind of transformation and **T7** refers to another. Lastly, **T8** is organised by a technology in which "translation is interpreted as additions of column vectors" (**θ4**). This technology justifies the conversion from geometric representation (translation) to algebraic representation (additions of column vectors). In terms of our REM based on Vergnaud (2009), we can interpret this as meaning that the four different technologies share the same theory of "isometry" (**Θ**).

As a result, Tables 6 and 7 summarise praxeological organisations within the sectors of symmetry and transformations in each country. The two tables allow us to further compare and discuss the differences and similarities in terms of point, local, and regional praxeologies.

Discussion

In this section, we discuss the conditions and constraints that shape the knowledge in each country's textbooks by elaborating on the analysis of their similarities and differences. According to the results shown in Tables 6 and 7, the similarities between the two can be clarified by focusing on the sets of praxeological elements such as [**T1/τ/θ1/Θ**] and [**T2/τ/θ1/Θ**] within symmetry and [**T4/τ/θ2/Θ**] and [**T5/τ/θ2/Θ**] within transformations. These common types of tasks can be performed by perceptual, physical, or operational techniques. The four-point praxeologies share the same theory but rely on different technologies. In terms of the REM, these similarities appear natural because the four identified praxeological organisations are associated with Vergnaud's (2009) characterisation of the two situations of "symmetry" (Fig. 2). We can, therefore, recognise similar processes and results of didactic transposition in both countries' textbooks.

Table 6 A regional praxeology of symmetry and transformations in Japan

Theme	Type of Task	Technique	Technology	Theory
Symmetry	**T1**: to identify the symmetrical shape	**τ1**: perceptual **τ2**: physical	**θ1**: one shape is symmetrical (i.e. a one-element predicate)	**Θ**: isometry
	T2: to draw the symmetrical shape	**τ2**: physical **τ3**: operational		
	T3: to find the corresponding points and lines	**τ1**: perceptual		
Transformation	**T4**: to draw the shape that is transformed (translated, rotated, and reflected)	**τ1**: perceptual **τ2**: physical **τ3**: operational	**θ2**: *A* is symmetrical to *B* in relation to *C* (i.e. a three-element predicate)	
	T5: to describe a (single or pair of) transformation(s) that will map one shape onto another	**τ1**: perceptual **τ3**: operational		
	T6: to identify the property of a given transformation (translation, rotation, and reflection)	**τ2**: physical	**θ3**: there are geometric relationships between the before- and after shapes that are transformed	

Table 7 A regional praxeology of symmetry and transformations in England

Theme	Type of Task	Technique	Technology	Theory
Symmetry	**T1**: to identify the symmetrical shape	**τ1**: perceptual **τ2**: physical	**θ1**: one shape is symmetrical (i.e. a one-element predicate)	**Θ**: isometry
	T2: to draw the symmetrical shape	**τ2**: physical **τ3**: operational		
Transformation	**T4**: to draw the shape that is transformed (translated, rotated, and reflected)	**τ1**: perceptual **τ2**: physical **τ3**: operational	**θ2**: *A* is symmetrical to *B* in relation to *C* (i.e. a three-element predicate)	
	T5: to describe a (single or pair of) transformation(s) that will map one shape onto another	**τ1**: perceptual **τ3**: operational		
	T7: to combine transformations	**τ3**: operational **τ4**: algebraic		
	T8: to represent a given translation as a column vector		**θ4**: translation is interpreted as additions of column vectors	

In contrast, essential differences are identified as sets of elements such as [**T3/τ/θ1/Θ**] and [**T6/τ/θ3/Θ**] for the Japanese textbooks and [**T7/τ/θ2/Θ**] and [**T8/τ/θ4/Θ**] for the English textbooks (see Tables 6 and 7). Although these differences are evident in terms of point and local praxeologies, the types of tasks and technologies require special attention, as these elements differ between the two countries. The common ground between **θ1**, **θ2**, and **θ3** is that these discourses are recognised as geometric properties, although **θ3** is more specialised than **θ2**. This shapes the knowledge of symmetry and transformations in Japan, as shown by the **T6** question in Fig. 3 regarding identifying the property of a given reflection such that segments *AP*, *BQ*, and *CR* intersect the axis of reflection perpendicularly at a point that divides them into equal halves.

Considering a regional and global praxeology which covers the geometry domain in Japan, such differences are conceived as follows. It seems that **θ1**, **θ2**, and **θ3** can be understood as general statements (particularly considering that "the axis of reflection is the perpendicular bisector of the segments linking pairs of corresponding points") and that these statements have a close relationship with other sectors, such as "geometric construction" (for Grade 7) or "geometric proof" (for Grades 8 and 9), because the properties of symmetry and transformations are used to explain constructions and to link them with the statements to be proven in these sectors. In fact, an official guidebook for the Course of Study in Japan provides the following description:

> By connecting the study of transformations with the study of the content of geometrical constructions, students' understanding of plane figures may be deepened and the important connection to the study of congruence of geometrical

In this figure, $\triangle PQR$ is reflected from $\triangle ABC$ over the axis of reflection l.
What is the relationship between segments AP, BQ, and CR that links the corresponding points and the axis of reflection l?

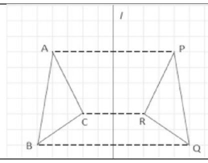

Fig. 3 Reflection in Grade 7 in Japan (adapted from Okamoto, Koseki, Morisugi, & Sasaki, 2012, p. 135)

figures in grade 2 of lower secondary school. (MEXT, 2008b, p. 81) (English translation cited from Isoda (Isoda, 2010, p. 133))

The congruence of geometrical figures is the content for introducing mathematical proofs in the Japanese curriculum. This is consistent with the predominance of geometric proofs in the geometry curriculum in Japan (Miyakawa, 2017; Shinno, et al. 2018), and hence, teaching geometric transformations is designed to enhance students' deductive reasoning (e.g. Miyakawa & Hatsugai, 2014; Okazaki, 2015). In contrast, there are no explicit descriptions of properties or general statements for symmetry or transformations in the English textbooks, probably because geometric proofs are underrepresented in the recent versions of the English curriculum (Hoyles, 1997).

The roles of technologies tell us that transformations are interrelated with a wide range of content in English textbooks. This summarises the condition of the knowledge of transformations in England, which may explain the specialised nature of **T7** and **T8**. Some **T7** tasks are situated in a functional setting. For example, the set of questions included in Fig. 4 was originally described together with a set of shapes (*A* to *H*) in functional graphs ($y = x$ and $y = -x$); although, the picture and some subsequent questions are omitted here due to a limited space. This context is obviously more or less connected to other chapters, such as "rules and coordinates" (for Grade 8), "straight-line graphs" (for Grade 9), and likely other chapters in different domains. In England, it seems, transformations are first introduced within the geometry domain but then situated to understand and analyse other mathematical concepts in different theoretical contexts, especially those involving coordinates.

Additionally, Fig. 5 shows a **T8** task which asks students to use vectors to explain a given translation. In the selected textbook series, column vectors were first introduced in the "transformations" chapter to describe a translation. The

- Reflect shape *A* in the y-axis and then rotate the image 180° about (0, 0). What shape is the final image of *A*?
- What single transformation could replace a reflection in the y-axis followed by a rotation of 180° about (0, 0)?
- What single transformation could replace a reflection in the line $y = -x$ followed by a clockwise rotation of 90° about (0, 0)?

Fig. 4 Combining transformations in Grade 9 in England (SMP, 2008b, p. 165)

You want to translate shape R on to shape S.
You can use three out of these four vectors.

$$\begin{bmatrix}4\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}$$

(a) Which three vectors do you need?
(b) Does it matter the order in which you use them?

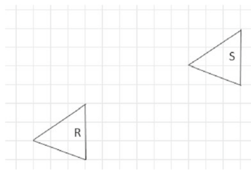

Fig. 5 Translation in Grade 9 in England (adapted from SMP, 2008a, p. 163)

vector additions were then placed in more algebraic contexts under the chapter "vectors."[4]

Even within the geometry domain, **θ2** appears such that "a shape and its image after a translation, reflection, or rotation are congruent to each other" in the English KS4 textbook, implying a technology that can be distinguished from another type of transformation: "enlargement" (which conserves angles but cannot conserve lengths). The "enlargement" theme is also included in the "transformations" chapter of the KS4 textbook. In Japan, the notion of enlargement is also situated in the textbook for Grade 9 but is set apart from the context of transformation. In this way, the roles of the technologies suggest that in England, transformations have many connections to other contexts or content across domains. In contrast, transformations in Japanese textbooks are relatively independent, even taking into consideration other domains and textbooks used in upper secondary schools. In Japan, symmetry and transformations are situated only within the domain of geometry, in which those contents are influenced by geometric proofs.

Finally, the regional praxeological organisations can be discussed. As mentioned above, our analysis shows that geometric proofs are predominant and influence many topics within the domain of geometry in Japan, but this is not the case in England. This is especially clear from the "Pythagorean theorem" chapters in each country. While the textbooks from both countries include the same types of tasks, such as "use the Pythagorean theorem to find the length of the missing side of the right triangle", in Japan, the theorem is introduced with various proofs, while in England, it is only given as a formula. This can be understood and explained by a strong versus a weak emphasis on Euclidean geometry, which can be considered a theory (**Θ**) in a collection of regional praxeologies (a global praxeology) at the domain level.

Conclusion

In this study, the different characteristics (conditions and constraints) of the knowledge regarding symmetry and transformations in lower secondary textbooks from Japan and England were examined using theoretical frameworks within ATD, specifically, a praxeological analysis with the REM. In this concluding section, we summarise the main findings of this study and then discuss the implications and limitations for further research and practice.

[4] The "vectors" chapter is only included in the higher-level textbook for Grade 9.

Summary

Concerning the praxis block, we revealed the commonalities and specialities of both countries' textbooks by considering all questions included in the relevant chapters. Of the eight types of tasks in symmetry and transformations identified, four of them (**T1**, **T2**, **T4**, and **T5**) were common to textbooks from both countries, while the other four (**T3**, **T6**, **T7**, and **T8**) were specific to one country or the other (as shown in Tables 3 and 4). Similar techniques (such as perceptual, physical, or operational) for most of these common types of tasks were found in both countries, while the algebraic technique for **T7** and **T8** was specific to England (see also Table 5). Considering the logos block, a praxeological analysis with the help of the REM allowed us to gain a deeper understanding of the content of both countries' textbooks. We used the REM, based on Vergnaud (2009), as an analytical lens for identifying technologies (θ), theories (**Θ**), and different praxeological organisations. In terms of point praxeology, a set of elements [**T**/**τ**/θ/**Θ**] can describe the relevant similarities and differences in each question or theme, and then, the local and regional praxeological organisations can clarify the characteristics in the sectors of symmetry and transformations by focusing on similarities and differences in technologies and theories. Explaining the regional praxeologies comprising different technologies (θ**1**, θ**2**, θ**3**, θ**4**) under the same theory ("isometry") can lead to discussions on a global praxeological organisation covering the domain level. One essential difference comes from the constraint that symmetry and transformations are strongly influenced by the teaching of geometric proofs in Japan, which is not the case in England, where transformations have many connections to other contexts or content across domains.

Implications and Limitations

This study suggests that a praxeological analysis using the REM is useful for identifying and explaining the characteristics (conditions and constraints) of the knowledge in textbooks. Analyses from point and local praxeologies to regional and global ones offer a deeper insight into the object of the study. Moreover, following ATD, our approach regarded textbooks as an empirical source which can reveal knowledge to be taught in the didactic transposition process; this differs from the approaches of previous textbook research studies. Although there are some existing studies using ATD for textbook analysis (González-Martîn, Giraldo, & Souto, 2013; Lundberg & Kilhamn, 2018; Wijayanti, 2015; Wijayanti & Winsløw, 2017), the innovation this paper provides is in the precision achieved at the logos level. The theoretical and methodological aspects of this study can contribute to future research on mathematics textbooks. In addition, in terms of the REM, the findings have implications for both the teaching and the learning of the concept in question. An epistemological jump occurs in the first transition between the technologies (θ**1** and θ**2**). This transition is explicitly situated and identified as the difference between two chapters ("symmetry" and "transformations") in both countries' textbooks. Therefore, to make a pedagogical connection between two different technologies might be essential when designing a lesson focusing on symmetry and transformations. On the other hand, it seems that the second transition, where the predicate "symmetrical" is changed to an object of thought, is more challenging to achieve in classroom practice because this shift is almost implicit in both countries' textbooks. Thus, this point may be an important pedagogical issue to be considered for the teaching and learning of the topic.

However, further research on the following three points, at a minimum, is required to substantiate the results of our study. First, a notable limitation is that some similarities and differences in transformations might be more explicit if we refer to other content included in other domains or in upper (or lower) grades' textbooks. The concept of transformations is related to other content such as algebraic matrices/coordinates and function (e.g. Usiskin, 2014). Second, it is important to extend this study to comparatively analyse the taught knowledge about symmetry and transformations in the classroom setting, which is another institutional production in the didactic transposition. This analysis may clarify some of the constraints caused by the differences in the types of tasks identified in this study. Third, it is important to consider higher levels of co-determinacy (from *discipline* to *civilisation*) because some of the findings in our study could be further explained by referring to the cultural and historical traditions of each country. For example, the geometry curricula in the given countries have undergone changes before and after the curriculum reforms of the New Math era. Although our analysis and the REM are limited in examining such aspects, further analyses of the conditions and constraints of the educational systems from a historical and cultural perspective are potential avenues for future research.

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