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Values for Restricted Games with Externalities

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Abstract
We study cooperative games in which some of the coalitions are not viable and in addition, there are externalities among the feasible coalitions. These games are called here restricted partition function form games. For this class of games, two extensions of the Shapley value are proposed and characterized.

Keywords Restricted games · Games with externalities · Shapley value

1 Introduction

When studying transferable utility (TU) games, it is customarily assumed that the worth of all coalitions is known. Nevertheless, in real world applications there are coalitions whose worth is unknown or impossible to determine. These situations are modeled through restricted games (Faigle 1989), in short *R*-games, in which not all coalitions are assigned a worth.

Among the most recognized solutions for classical TU games is the Shapley value (1953), which is supported by three well known axioms: Carrier, Anonimity and Additivity. Regarding *R*-games, various extensions of the Shapley value have been proposed using different approaches beginning with Willson (1993). More recently, Aguilera et al. (2010) and Calvo and Gutiérrez (2015) proposed independently the same extension of the Shapley value for *R*-games from different points of view. This extended value is referred to herein as the *R*-value and was characterized in Albizuri et al. (2022) with three axioms: Carrier, Symmetric-Partnership and Additivity. That is, the same axioms employed by Shapley (1953)

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where Anonymity is replaced by Symmetric-Partnership, which is a weaker requirement in the context of classical TU games when all coalitions are feasible.

On the other hand, there are situations that present externalities between coalitions. Since traditional TU games are not suitable enough for analysis, Lucas and Thrall (1963) introduced partition function form games (PFF games). These games are based on the notion of embedded coalition, that is, a pair formed by a coalition and a partition of the set of players, the coalition being a member of the partition. Each embedded coalition is assigned a real number representing the worth of the coalition when players are arranged according to the partition.

Due to the richness of PFF games, a plethora of solutions have been proposed to extend the Shapley value to these games. For instance, Myerson (1977a) proposed and characterized an extension by means of the same three axioms as in Shapley (1953) adapted to PFF games: Carrier, Anonimity and Additivity. In Albizuri et al. (2005) an alternative extension is proposed and characterized with five axioms: Efficiency, Oligarchy, Anonymity, Embedded-Coalition Anonymity, and Additivity. The first two axioms, Efficiency and Oligarchy, are weaker than Myerson's Carrier axiom, and Embedded-Coalition Anonymity is an anonymity property related to embedded coalitions with the same fixed coalition. Other scholars have extended the Shapley value by considering different properties of this value. To name a few, Bolger (1989), de Clippel and Serrano (2008), McQuillin (2009), Hu and Yang (2010), Grabisch and Funaki (2012), Macho-Stadler et al. (2006, 2007, 2018), and Alonso-Meijide et al. (2019). These works use different approaches, and some of them are related to the Dummy Player property that is implicit in the original Shapley's Carrier axiom for TU games, but not in Myerson's Carrier axiom for PFF games.

In this paper, we combine R-games and PFF games by modifying the formulation of a game, and consider restricted partition function form games (R-PFF games), that is games in partition function form in which not all embedded coalitions are feasible. In addition, we propose and characterize two solutions for these games. The first one extends both the R-value and the (Myerson 1977a) value of PFF games. We offer a characterization of this value by substituting Axiom 1 (anonymity) by the Symmetric-Partnership axiom in Myerson's axiom system when adapted to R-PFF games. The second one is simultaneously an extension of the R-value and the value of Albizuri et al. (2005) of PFF games. This extension is characterized by substituting Oligarchy and Anonymity in the axiom system used by these authors adapted to R-PFF games. The first axiom is substituted by a very weak dummy player axiom and the second one by Partners-Symmetry.

A lot of research has been done on different types of restricted cooperative games. These are some of them. Myerson considered graphs (1977b) and conference structures (1980), Algaba et al. (2004) antimatroids, van den Brink and Gilles (1996) permission structures, Grabisch and Sudhölter (2016, 2018), precedence constraints, and Algaba et al. (2018) network structures with hierarchy and communication.

The contents of the paper are organized as follows. Section 2 is divided into two subsections. Section 2.1 contains a brief review of R-games, and Sect. 2.2 is devoted to PFF games and R-PFF games. Sections 3 and 4 deal respectively with the two extensions of the Shapley value mentioned above. In Sect. 5, concluding remarks are given and future research directions are suggested.

2 Restricted cooperative games

Throughout this paper, $N = \{1, 2, \dots, n\}$ is a fixed set of *players*. A coalition is any subset of N (possibly empty).

2.1 *R*-games

Traditionally, when studying characteristic function form games it is assumed that all coalitions can be formed. However, in this paper we will consider the case in which some coalitions are not viable.

Denote CL the set of all coalitions of N. Every family $\mathcal{K} \subseteq \mathrm{CL}$ such that $\emptyset, N \in \mathcal{K}$ is called a *set system* of feasible coalitions. Thus the grand coalition is assumed to be always viable.

A *restricted game* (in short, R-game) on a set system $\mathcal{K}$, is a mapping v which assigns each coalition $S \in \mathcal{K}$ to a real number $v(S)$, such that $v(\emptyset) = 0$. The real number $v(S)$ is the worth of coalition S, and symbolizes what members in S can guarantee themselves without the cooperation of the other players.

If $\mathcal{K}$ is a set system denote $G^{\mathcal{K}}$ the vector space of the R-games on $\mathcal{K}$. The addition of two R-games v and w is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \in \mathcal{K}$. The product of an R-game and a scalar $\lambda \in \mathbb{R}$ is defined by $(\lambda v)(S) = \lambda v(S)$ for all $S \in \mathcal{K}$.

When $\mathcal{K} = \mathrm{CL}$ we refer to such R-games as *full* games.

Given $T \in \mathcal{K}, T \neq \emptyset$, the *unanimity R-game* $u_T^{\mathcal{K}}$ on $\mathcal{K}$ is defined by

$$u_T^{\mathcal{K}}(S) = \begin{cases} 1, & \text{if } T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

It is easy to prove that the collection $\{u_T^{\mathcal{K}} : T \in \mathcal{K}, T \neq \emptyset\}$ forms a basis of $G^{\mathcal{K}}$. Given $v \in G^{\mathcal{K}}$, then

$$v = \sum_{\substack{T \in \mathcal{K}: \\ T \neq \emptyset}} \lambda_T u_T^{\mathcal{K}}, \tag{2}$$

where λ_T can be obtained recursively as follows:

$$\lambda_T = v(T) - \sum_{\substack{S \in \mathcal{K}: \\ T \supsetneq S}} \lambda_S. \tag{3}$$

Supposing the grand coalition N forms in the end, the question is how to divide the amount $v(N)$ among the players. Given a set system $\mathcal{K}$, define a *value* to be any mapping $\varphi : G^{\mathcal{K}} \longrightarrow \mathbb{R}^N$. The real number $\varphi_i(v)$ represents an evaluation of player i of her "*prospect that will arise as a result of a play*" (Shapley 1953).

A well-known value was developed by Shapley (1953) for full games. Aguilera et al. (2010) and Calvo and Gutiérrez (2015) proposed independently the same extension of the Shapley value for R-games, that later on was characterized in Albizuri et al. (2022). This extension, called here the R-value, is the linear map $\psi^{\mathcal{K}}$ from $G^{\mathcal{K}}$ to $\mathbb{R}^N$ defined for every unanimity game by

$$\psi_i^{\mathcal{K}}\left(u_T^{\mathcal{K}}\right) = \begin{cases} 1/|T|, & \text{if } i \in T; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

2.2 Restricted games in partition function form

There are situations where the worth of a coalition depends on how the rest of the players are arranged. As a way of mantaining information about externalities among the coalitions, Lucas and Thrall (1963) introduced partition function form games.

Before presenting the formal definitions some notation is needed.

Denote $\mathcal{P}$ the set of partitions of N, so $\Pi = \left\{S_1, \ldots, S_\ell\right\} \in \mathcal{P}$ if and only if

$$\bigcup_{i=1}^{\ell} S_i = N,\ \forall i\ S_i \neq \emptyset,\ \forall j\ S_i \cap S_j = \emptyset \text{ if } j \neq i. \tag{5}$$

Given $S \subseteq N, S \neq \emptyset$, and $\Pi \in \mathcal{P}$, denote

$$\Pi_S = \{S\} \cup \{S' \backslash S : S' \in \Pi,\ S' \backslash S \neq \emptyset\}, \tag{6}$$

that is, the partition that results if agents in S get together, leaving their corresponding coalition in Π. If $S = \emptyset$, denote $\Pi_\emptyset = \Pi$.

Let Π, Π' be two partitions. Define the partition $\Pi \wedge \Pi'$ as follows:

$$\Pi \wedge \Pi' = \left\{S \cap S' \ : S \in \Pi,\ S' \in \Pi',\ S \cap S' \neq \emptyset\right\} \tag{7}$$

An *embedded coalition* is any pair (S, Π), such that $\Pi \in \mathcal{P}$ and $S \in \Pi$ or $S = \emptyset$.[1]

Lucas and Thrall (1963) considered that all embedded coalitions (S, Π), where $S \in \Pi \cup \{\emptyset\}$ and $\Pi \in \mathcal{P}$ are feasible. Thus, players can form any partition of the grand coalition N. In this work we consider that not all coalitions of N are feasible and therefore, not any partition of N can be formed. In order to represent this situation, we consider a set system of feasible coalitions $\mathcal{K} \subseteq \mathrm{CL}$ such that $\emptyset, N \in \mathcal{K}$ and $\{i\} \in \mathcal{K}$ for all $i \in N$. That is, the singleton coalitions are feasible and the union of all these singleton coalitions is also feasible.

An embedded coalition (S, Π) will be *feasible* if $\Pi \subseteq \mathcal{K}$.

$\mathcal{E}(\mathcal{K})$ will denote the family of feasible embedded coalitions associated with $\mathcal{K}$. Note that for all $S \in \mathcal{K}$ there exists at least one feasible embedded coalition. Indeed, if $S = N$, then $\Pi = \{N\}$, and the grand coalition forms. If $S = \emptyset$, then Π can

[1] For ease of exposition we assume that $(\emptyset, \Pi)$ is an embedded coalition for every partition Π.

be any partition of N such that $\Pi \subseteq \mathcal{K}$. And if $S \notin \{\emptyset, N\}$, Π can be, for example, $\{S\} \cup \big\{\{i\} : i \in N \backslash S\big\}$, players outside S are alone.

Throughout the paper $\mathcal{K}$ is fixed, and therefore, we write $\mathcal{E}$ instead of $\mathcal{E}(\mathcal{K})$.

A *restricted game in partition function form* (in short, R-PFF game) on a family of feasible embedded coalitions $\mathcal{E}$, is any function $v : \mathcal{E} \rightarrow \mathbb{R}$ that assigns a real number to each $(S, \Pi) \in \mathcal{E}$, such that $v(\emptyset, \Pi) = 0$ for every $(\emptyset, \Pi) \in \mathcal{E}$.

The real number $v(S, \Pi)$ symbolizes the wealth which coalition S could divide among its members if all the players were aligned into the coalitions of partition Π.

The vector space of all R-PFF games on $\mathcal{E}$ will be denoted $\Gamma^{\mathcal{E}}$. The addition of two R-PFF games v and w is defined by $(v + w)(S, \Pi) = v(S, \Pi) + w(S, \Pi)$ for all $(S, \Pi) \in \mathcal{E}$. The product of an R-PFF game and a scalar $\lambda \in \mathbb{R}$ is defined by $(\lambda v)(S, \Pi) = \lambda v(S, \Pi)$ for all $(S, \Pi) \in \mathcal{E}$.

Occasionally, when $\mathcal{E}$ contains all embedded coalitions we will refer to such R-PFF games as a *full* R-PFF game.

Remark 1 Given a game $v \in \Gamma^{\mathcal{E}}$, if for every $S \in \mathcal{K}$ the real number $v(S, \Pi)$ does not depend on Π, then this game can be considered as an R-game on $\mathcal{K}$. In this way every R-game can be considered as an R-PFF game as well.

As in the case of R-games, define a *value* on $\Gamma^{\mathcal{E}}$ to be any mapping $\Phi : \Gamma^{\mathcal{E}} \longrightarrow \mathbb{R}^N$. The real number $\Phi_i(v)$ represents the evaluation of player i of her reward as a result of playing game v.

There have been proposed several values for full R-PFF games. We will focus here on the values proposed by Myerson (1977a) and Albizuri et al. (2005). Our goal in the next two sections is to propose and characterize the respective extension of these two values to general R-PFF games, which in turn can also be considered as an extension of the R-value by Remark 1.

As mentioned before, throughout the rest of the paper $\mathcal{E}$ will be a fixed family of feasible embedded coalitions.

3 An extension of the *R*-value to *R*-PFF games

If (S, Π) and (S', Π') are two feasible embedded coalitions of $\mathcal{E}$, following Myerson (1977a), we write:

$$(S, \Pi) \gg (S', \Pi') \quad \text{if and only if} \quad S \supseteq S' \text{ and } \Pi \wedge \Pi' = \Pi', \tag{8}$$

and read (S, Π) *covers* (S', Π').

Given $(T, \Sigma) \in \mathcal{E}, T \neq \emptyset$ define the *unanimity* R-PFF game $u_{(T,\Sigma)}$ on $\mathcal{E}$ by

$$u_{(T,\Sigma)}(S, \Pi) = \begin{cases} 1, & \text{if } (S, \Pi) \gg (T, \Sigma); \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Note that $u_{(T,\Sigma)}\big(N, \{N\}\big) = 1$ for every $(T, \Sigma) \in \mathcal{E}, T \neq \emptyset$.

A similar proof of the following proposition can be found in Myerson's (1977a) proof of Theorem 1.

Proposition 1 *The family* $\{u_{(T,\Sigma)}\ (T,\Sigma) \in \mathcal{E},\ T \neq \emptyset\}$ *is a basis of the vector space* $\Gamma^{\mathcal{E}}$.

From Proposition 1, every arbitrary game $v \in \Gamma^{\mathcal{E}}$ can be uniquely represented by means of a function $d_v : \mathcal{E} \longrightarrow \mathbb{R}$ as follows

$$v = \sum_{(T,\Sigma)\in\mathcal{E}} d_v(T,\Sigma) u_{(T,\Sigma)}. \tag{10}$$

The real numbers $d_v(T,\Sigma)$ can be interpreted as an extension of Harsanyi (1963) dividends to R-PFF games. From (9) to (10) we obtain the following recursive formula for the dividends $d_v(T,\Sigma)$:

$$d_v(T,\Sigma) = v(T,\Sigma) - \sum_{\substack{(S,\Sigma)\in\mathcal{E}: \\ (T,\Sigma)\gg(S,\Sigma) \\ (S,\Sigma)\neq(T,\Sigma)}} d_v(S,\Sigma). \tag{11}$$

For full R-PFF games the explicit expression for the dividends is given by Grabisch (2010) (Proposition 8).

Define the value $\Psi^{\mathcal{E}}$ for every R-PFF game $v \in \Gamma^{\mathcal{E}}$ as follows:

$$\Psi_i^{\mathcal{E}}(v) = \sum_{\substack{(S,\Pi)\in\mathcal{E}: \\ i\in S}} \frac{d_v(S,\Pi)}{|S|}. \tag{12}$$

Example 1 Let $N = \{1,2,3,4\}$, $\mathcal{K} = \big\{\{1\},\{2\},\{3\},\{4\},N,\{1,2\},\{3,4\}\big\}$ and $v \in \Gamma^{\mathcal{E}}$ defined by

$$v(T,\Sigma) = \begin{cases} 3, & \text{if } (T,\Sigma) = \big(N,\{N\}\big); \\ 1, & \text{if } (T,\Sigma) = \big(\{1,2\},\big\{\{1,2\},\{3,4\}\big\}\big); \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Let us calculate the dividends $d_v(T,\Sigma)$ for every $(T,\Sigma) \in \mathcal{E}$.

If $|T| = 1$, then it is straightforward that (11) implies $d_v(T,\Sigma) = 0$.

If $|T| = 2$, there are two cases. If $(T,\Sigma) = \big(\{1,2\},\big\{\{1,2\},\{3,4\}\big\}\big)$, then applying expression (11)

$$d_v(T,\Sigma) = v(\{1,2\},\{\{1,2\},\{3,4\}\}) - \sum_{\substack{(S,\Sigma)\in\mathcal{E}: \\ (T,\Sigma)\gg(S,\Sigma) \\ (S,\Sigma)\neq(T,\Sigma)}} d_v(S,\Sigma), \tag{14}$$

and therefore $d_v(T,\Sigma) = 1$. If $(T,\Sigma) \neq \big(\{1,2\},\big\{\{1,2\},\{3,4\}\big\}\big)$, expression (11) implies $d_v(T,\Sigma) = 0$.

If $T = N$, then

$$d_v(N,\{N\}) = v(N,\{N\}) - \sum_{\substack{(S,\Sigma)\in\mathcal{E}: \\ (N,\{N\})\gg(S,\Sigma) \\ (S,\Sigma)\neq(N,\{N\})}} d_v(S,\Sigma), \tag{15}$$

and hence, $d_v\big(N,\{N\}\big) = 3 - 1 = 2$.

Therefore, by the definition of $\Psi^{\mathcal{E}}$

$$\Psi^{\mathcal{E}}_1(v) = \frac{d_v\big(\{1,2\},\big\{\{1,2\},\{3,4\}\big\}\big)}{2} + \frac{d_v\big(N,\{N\}\big)}{4} = 1. \tag{16}$$

Similarly, $\Psi^{\mathcal{E}}_2(v) = 1$. And, again, by the definition of $\Psi^{\mathcal{E}}$

$$\Psi^{\mathcal{E}}_3(v) = \frac{d_v\big(N,\{N\}\big)}{4} = \frac{1}{2}, \tag{17}$$

and similarly, $\Psi^{\mathcal{E}}_4(v) = 1$.

If $\mathcal{E}$ is the set of all embedded coalitons, that is, if $\mathcal{K} = \mathrm{CL}$, this value is precisely the one proposed by Myerson (1977a) for full R-PFF games. Moreover, by Remark 1 this solution can also be considered as an extension of the R -value.

Next we are going to characterize the value $\Psi^{\mathcal{E}}$ with three axioms: Carrier, Symmetric-Partnership, and Additivity. Some definitions are needed in advance.

An embedded coalition $(S,\Pi) \in \mathcal{E}$ is said to be a *zero-embedded coalition* in $v \in \Gamma^{\mathcal{E}}$ if for every $(S',\Pi') \in \mathcal{E}$ it holds

$$(S,\Pi) \gg (S',\Pi') \text{ implies } v(S',\Pi') = 0. \tag{18}$$

Thus $(S,\Pi) \in \mathcal{E}$ is a zero-embedded coalition if all the feasible embedded coalitions covered by (S,Π) have no influence in the game. Note that the feasible embedded coalition $\big(\{3,4\},\big\{\{1,2\},\{3,4\}\big\}\big)$ of example 1 is a zero-embedded coalition. Note also that the associated dividend is zero.

A coalition M is a *carrier* of $v \in \Gamma^{\mathcal{E}}$ if for every $(S,\Pi) \in \mathcal{E}$ it holds:

$$1.\ \text{If } (S \cap M, \Pi_{S\cap M}) \in \mathcal{E}, \text{ then } v(S,\Pi) = v(S \cap M, \Pi_{S\cap M}), \tag{19}$$

$$2.\ \text{If } (S \cap M, \Pi_{S\cap M}) \notin \mathcal{E}, \text{ then } (S,\Pi) \text{ is a zero-embedded coalition.} \tag{20}$$

Players outside a carrier M cannot modify the worth of any feasible coalition. In the case that $(S \cap M, \Pi_{S\cap M}) \notin \mathcal{E}$, we require (S,Π) to be a zero-embedded coalition, and hence, no player modifies the zero worth of any embedded coalition covered by (S,Π). Note that N is always a carrier since for every $(S,\Pi) \in \mathcal{E}$ we have $(S,\Pi) = (S \cap N, \Pi_{S\cap N})$, and thus $v(S,\Pi) = v(S \cap N, \Pi_{S\cap N})$. Furthermore, in the case of full R-PFF games, since $(S \cap M, \Pi_{S\cap M}) \in \mathcal{E}$, we are never in second condition above (20) and this definition matches the definition of carrier given in Myerson (1977a).

Example 2 In Example 1 only N is a carrier of v. If $v\big(N,\{N\}\big) = 1$ (instead of 3), then $\{1,2\}$ would also be a carrier of v. The coalition $\{1,2,3\}$ would not be a carrier of v because condition 2 would not hold when $(S,\Pi) = \big(N,\{N\}\big)$. Indeed, $\big(\{1,2,3\},\big\{\{1,2,3\},\{4\}\big\}\big) \notin \mathcal{E}$ and $\big(N,\{N\}\big)$ is not a zero-embedded coalition.

If $N = \{1,2,3,4,5\}$, $\mathcal{K} = \mathrm{CL}\backslash\big\{\{2,3\},\{4,5\}\big\}$, and $v = u_{(T,\Sigma)}$, where $T = \{1,2,3\}$ and $\Sigma = \big\{\{1,2,3\},\{4\},\{5\}\big\}$, then $\{1,2,3\}$ is a carrier of v. Note that, for example, when $(S,\Pi) = \big(\{2,3,4\},\big\{\{1\},\{2,3,4\},\{5\}\big\}\big)$ condition 2 holds. In this case $\big(\{1,2,3\}\cap\{2,3,4\},\big\{\{1\},\{2,3\},\{4\},\{5\}\big\}\big) \notin \mathcal{E}$ and $\big(\{2,3,4\},\big\{\{1\},\{2,3,4\},\{5\}\big\}\big)$ is a zero-embedded coalition.

Axiom 1 (Carrier) Let Φ be a value on $\Gamma^{\mathcal{E}}$. If M is a carrier of $v \in \Gamma^{\mathcal{E}}$, then

$$\sum_{i\in M} \Phi_i(v) = v\big(N,\{N\}\big). \tag{21}$$

The Carrier axiom requires that players in a carrier allocate the full income of the grand coalition between them. This is due to the lack of influence of players outside M.

A coalition $T \subseteq N$, $T \neq \emptyset$, is said to be a *p-type coalition* in $v \in \Gamma^{\mathcal{E}}$ if for all $(S,\Pi) \in \mathcal{E}$ such that $T \nsubseteq S$ the following holds:

$$1.\text{ If } (S\backslash T,\Pi_{S\backslash T}) \in \mathcal{E},\text{ then } v(S,\Pi) = v(S\backslash T,\Pi_{S\backslash T}); \tag{22}$$

$$2.\text{ If } (S\backslash T,\Pi_{S\backslash T}) \notin \mathcal{E},\text{ then } (S,\Pi) \text{ is a zero-embedded coalition.} \tag{23}$$

Requirement 1 says that the presence in S of players of T do not affect the worth of any feasible embedded coalition (S,Π). This is true when $T \nsubseteq S$, that is, when players of T are not together. Requirement 1 implies that players of T are indistinguishable. We also require 2 because it may happen that $(S\backslash T,\Pi_{S\backslash T}) \notin \mathcal{E}$, and therefore, the equality in requirement 1 would be meaningless. But in this case we require (S,Π) to be a zero-embedded coalition, and therefore (S,Π) and the other feasible embedded coalitions covered by (S,Π) get zero worth. In this case, no players of S, in particular those of T, affect this zero worth. And again, players of T are indistinguishable.

Note that condition (22) above coincides with the definition of p-type coalition introduced by Kalai and Samet (1988) to characterize the weighted Shapley values.

Example 3 In Example 1 the coalition $\{1,2\}$ is a p-type coalition. Indeed, if $(S,\Pi) \in \mathcal{E}$ and $\{1,2\} \nsubseteq S$, then $v(S,\Pi) = 0$. Moreover, $\big(S\backslash\{1,2\},\Pi_{S\backslash\{1,2\}}\big) \in \mathcal{E}$ and $v\big(S\backslash\{1,2\},\Pi_{S\backslash\{1,2\}}\big) = 0$. We are not ever in condition 2.

As for the game $u_{(T,\Sigma)}$ in Example 2, $\{1,2,3\}$ is a p-type coalition. Note that, for example, if $(S,\Pi) = \big(\{2,3,4,5\},\big\{\{1\},\{2,3,4,5\}\big\}\big)$, then $(S\backslash T,\Pi_{S\backslash T}) = \big(\{4,5\},\big\{\{1\},\{2,3\},\{4,5\}\big\}\big) \notin \mathcal{E}$ and (S,Π) is a zero-embedded coalition.

Axiom 2 (Symmetric-Partnership) Let Φ be a value on $\Gamma^{\mathcal{E}}$. If T is a p-type coalition in $v \in \Gamma^{\mathcal{E}}$, then

$$\Phi_i(v) = \Phi_j(v) \quad \text{for all } i,j \in T. \tag{24}$$

Symmetric-Partnership requires the value of the players in a p-type coalition to be the same since they are indistinguishable. Note that in full R-PFF games this axiom is weaker than Axiom 1 (anonymity) in Myerson (1977a). Actually a requirement like Myerson's Axiom 1 is of little use when dealing with general R-PFF games, since a set system of feasible embedded coalitions might be not invariant under a permutation of N.

The last axiom is the conventional Additivity axiom.

Axiom 3 (Additivity) Let Φ be a value on $\Gamma^{\mathcal{E}}$. If $v, w \in \Gamma^{\mathcal{E}}$, then

$$\Phi(v+w) = \Phi(v) + \Phi(w). \tag{25}$$

Theorem 1 *There is one and only one solution on* $\Gamma^{\mathcal{E}}$ *satisfying Carrier, Symmetric-Partnership, and Additivity, and it is* $\Psi^{\mathcal{E}}$.

The proof is based on some lemmas and propositions.

Lemma 1 *Let* $v \in \Gamma^{\mathcal{E}}$, *then* $\sum_{i\in N} \Psi_i(v) = v\big(N, \{N\}\big)$.

Proof Let $v \in \Gamma^{\mathcal{E}}$. Then we have:

$$\begin{aligned}\sum_{i\in N} \Psi_i(v) &= \sum_{i\in N} \sum_{\substack{(S,\Pi)\in\mathcal{E}:\\ i\in S}} \frac{d_v(S,\Pi)}{|S|} = \sum_{(S,\Pi)\in\mathcal{E}} d_v(S,\Pi) \\ &= \sum_{(S,\Pi)\in\mathcal{E}} d_v(S,\Pi) u_{(S,\Pi)}(N,\{N\})\end{aligned} \tag{26}$$

Bearing in mind that $\big(N, \{N\}\big) \gg (S,\Pi)$ for every $(S,\Pi) \in \mathcal{E}$ we get

$$\sum_{i\in N} \Psi_i(v) = v\big(N, \{N\}\big). \tag{27}$$

□

Lemma 2 *Let* $v \in \Gamma^{\mathcal{E}}$ *and* $(S,\Pi) \in \mathcal{E}$.

(*i*) *If* (S,Π) *is a zero-embedded coalition in* v *then* $d_v(S,\Pi) = 0$.

(*ii*) *If* M *is a carrier of* v *and* $S \cap M = \emptyset$, *then* $d_v(S,\Pi) = 0$.

(*iii*) *If* M *is a carrier of* v *and* $S \backslash M \neq \emptyset$, *then* $d_v(S,\Pi) = 0$.

Proof (i) It is straightforward.

(ii) It is consequence of the definition of carrier.

(iii) Assume M is a carrier of v, $(S,\Pi) \in \mathcal{E}$, and $S \backslash M \neq \emptyset$. Then we consider two possibilities:

1. If $(S \cap M, \Pi_{S \cap M}) \notin \mathcal{E}$: By condition (20) we get that (S, Π) is a zero-embedded coalition, and the result follows from part (i) of this lemma.

2. If $(S \cap M, \Pi_{S \cap M}) \in \mathcal{E}$: Consider the set $\mathcal{A} = \{(S', \Pi') \in \mathcal{E}(S, \Pi) \gg (S', \Pi')$ $\mathcal{A} = \{(S', \Pi') \in \mathcal{E}(S, \Pi) \gg (S', \Pi') \neq (S, \Pi)\}$. We will show that $d_v(S, \Pi) = 0$ by induction on the cardinality of $\mathcal{A}$. First note that $(S \cap M, \Pi_{S \cap M}) \in \mathcal{A}$.

So assume first that $|\mathcal{A}| = 1$. In this case $\mathcal{A} = \{(S \cap M, \Pi_{S \cap M})\}$. Using expression (11) we get

$$\begin{aligned} v(S, \Pi) &= d_v(S, \Pi) + \sum_{\substack{(S',\Pi') \in \mathcal{E}: \\ (S,\Pi) \gg (S',\Pi') \neq (S,\Pi)}} d_v(S', \Pi') \\ &= d_v(S, \Pi) + d_v(S \cap M, \Pi_{S \cap M}), \end{aligned} \tag{28}$$

where the last equality follows since $|\mathcal{A}| = 1$. By the definition of a carrier it holds $v(S \cap M, \Pi_{S \cap M}) = v(S, \Pi)$, and hence $d_v(S, \Pi) = 0$, as desired.

Now assume that $|\mathcal{A}| \geq 2$. From expression (11) we have:

$$v(S, \Pi) = d_v(S, \Pi) + \sum_{\substack{(S',\Pi') \in \mathcal{E}: \\ (S,\Pi) \gg (S',\Pi') \neq (S,\Pi)}} d_v(S', \Pi') \tag{29}$$

$$= d_v(S, \Pi) + \sum_{\substack{(S',\Pi') \in \mathcal{E}: \\ (S,\Pi) \gg (S',\Pi') \neq (S,\Pi) \\ S' \subseteq S \cap M}} d_v(S', \Pi') + \sum_{\substack{(S',\Pi') \in \mathcal{E}: \\ (S,\Pi) \gg (S',\Pi') \neq (S,\Pi) \\ S' \setminus M \neq \emptyset}} d_v\big(S', \Pi'\big) \tag{30}$$

By the induction hypothesis the third term in this summation vanishes. Hence we get

$$v(S, \Pi) = d_v(S, \Pi) + \sum_{\substack{(S',\Pi') \in \mathcal{E}: \\ (S \cap M, \Pi_{S \cap M}) \gg (S',\Pi')}} d_v(S', \Pi') = d_v(S, \Pi) + v(S \cap M, \Pi_{S \cap M}), \tag{31}$$

where the last equality follows since $(S \cap M, \Pi_{S \cap M}) \in \mathcal{E}$ and expression (11).

Since M is a carrier of v, then $v(S \cap M, \Pi_{S \cap M}) = v(S, \Pi)$, and hence $d_v(S, \Pi) = 0$, as desired. □

Proposition 2 *The value* $\Psi^{\mathcal{E}}$ *satisfies the Carrier axiom on* $\Gamma^{\mathcal{E}}$.

Proof Let M be a carrier of $v \in \Gamma^{\mathcal{E}}$, and $i \notin M$. If $(S, \Pi) \in \mathcal{E}$ is such that $i \in S$, by parts (ii) and (iii) of Lemma 2, it must be $d_v(S, \Pi) = 0$. So ${\Psi^{\mathcal{E}}}_i(v) = 0$ for all $i \notin M$.

Consequently, by Lemma 1 we get

$$\sum_{i \in M} {\Psi^{\mathcal{E}}}_i(v) = \sum_{i \in N} {\Psi^{\mathcal{E}}}_i(v) = v\big(N, \{N\}\big), \tag{32}$$

and the proof is complete. □

Lemma 3 *Let T be a p-type coalition of $v \in \Gamma^{\mathcal{E}}$, and $(S,\Pi) \in \mathcal{E}$. If $T \cap S \neq \emptyset$ and $T \nsubseteq S$, then $d_v(S,\Pi) = 0$.*

Proof By induction on $|S|$. If $|S| = 1$, then $S \backslash T = \emptyset$. Since T is a p-type coalition $v(S,\Pi) = v(S\backslash T, \Pi_{S\backslash T}) = v(\emptyset, \Pi_{\emptyset}) = 0$, and hence $d_v(S,\Pi) = 0$.

Now assume that $|S| \geq 2$. By expression (11) we have:

$$v(S,\Pi) = d_v(S,\Pi) + \sum_{\substack{(S',\Pi')\in\mathcal{E}: \\ (S,\Pi)\gg(S',\Pi')\neq(S,\Pi) \\ S'\subseteq S\backslash T\neq\emptyset}} d_v(S',\Pi') + \sum_{\substack{(S',\Pi')\in\mathcal{E}: \\ (S,\Pi)\gg(S',\Pi')\neq(S,\Pi) \\ S'\cap T\neq\emptyset}} d_v(S',\Pi') \tag{33}$$

By the induction hypothesis the third term in the summation vanishes. So

$$v(S,\Pi) = d_v(S,\Pi) + \sum_{\substack{(S',\Pi')\in\mathcal{E}: \\ (S,\Pi)\gg(S',\Pi')\neq(S,\Pi) \\ S'\subseteq S\backslash T}} d_v(S',\Pi') \tag{34}$$

Consider two cases:

1. If $(S\backslash T, \Pi_{S\backslash T}) \in \mathcal{E}$: Taking into account expression (11), equality (34) above becomes

$$v(S,\Pi) = d_v(S,\Pi) + v(S\backslash T, \Pi_{S\backslash T}). \tag{35}$$

Since T is a p-type coaliton it holds $v(S,\Pi) = v(S\backslash T, \Pi_{S\backslash T})$, and hence $d_v(S,\Pi) = 0$.

2. If $(S\backslash T, \Pi_{S\backslash T}) \notin \mathcal{E}$: Since T is a p-type coalition, then (S,Π) is a zero-embedded coalition. From Lemma 2 we get $d_v(S,\Pi) = 0$, as desired. □

Proposition 3 *The value $\Psi^{\mathcal{E}}$ satisfies the Symmetric-Partnership axiom on $\Gamma^{\mathcal{E}}$.*

Proof Let T be a p-type coalition in $v \in \Gamma^{\mathcal{E}}$, and $i \in T$. If $(S,\Pi) \in \mathcal{E}$, $i \in S$ and $T \nsubseteq S$, from Lemma 3 it holds $d_v(S,\Pi) = 0$. Consequently

$$\Psi_i^{\mathcal{E}}(v) = \sum_{\substack{(S,\Pi)\in\mathcal{E}: \\ i\in S}} \frac{d_v(S,\Pi)}{|S|} = \sum_{\substack{(S,\Pi)\in\mathcal{E}: \\ T\subseteq S}} \frac{d_v(S,\Pi)}{|S|}. \tag{36}$$

But the last term in this expression is the same for every $i \in T$, so the proof is complete. □

Proposition 4 *The value $\Psi^{\mathcal{E}}$ satisfies the Additivity axiom on $\Gamma^{\mathcal{E}}$.*

Proof It is enough to observe that for all $(S,\Pi) \in \mathcal{E}$ it holds: $d_{v+w}(S,\Pi) = d_v(S,\Pi) + d_w(S,\Pi)$. □

Proposition 5 *Let $\Phi : \Gamma^{\mathcal{E}} \to \mathbb{R}^N$ be a solution satisfying Carrier and Symmetric-Partnership. Let also $c \in \mathbb{R}$, and $(T,\Sigma) \in \mathcal{E}$, then*

$$\Phi_i(c \cdot u_{(T,\Sigma)}) = \begin{cases} c/|T|, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases} \tag{37}$$

Proof Note first that T is a carrier and a p-type coalition in $c \cdot u_{(T,\Sigma)}$. By the Carrier axiom it follows $\sum_{i\in T} \Phi_i(c \cdot u_{(T,\Sigma)}) = c$. On the other hand, T is a p-type coalition in $c \cdot u_{(T,\Sigma)}$, so from the Symmetric-Partnership axiom it holds $\Phi_i(c \cdot u_{(T,\Sigma)}) = \Phi_j(c \cdot u_{(T,\Sigma)})$ for every $i,j \in T$. Combining these two facts we conclude that $\Phi_i(c \cdot u_{(T,\Sigma)}) = c/|T|$ for every $i \in T$.

Furthermore, N is a carrier of $c \cdot u_{(T,\Sigma)}$, and since $\sum_{i\in T} \Phi_i(c \cdot u_{(T,\Sigma)}) = c$ it follows $\sum_{i\notin T} \Phi_i(c \cdot u_{(T,\Sigma)}) = 0$. In addition $N \backslash T$ is a p-type coalition of $c \cdot u_{(T,\Sigma)}$, hence from Symmetric-Partnership we have that for every $i,j \in N\backslash T$ it holds $\Phi_i(c \cdot u_{(T,\Sigma)}) = \Phi_j(c \cdot u_{(T,\Sigma)})$. Therefore $\Phi_i(c \cdot u_{(T,\Sigma)}) = 0$ for every $i \in N\backslash T$ and the proof is complete. □

Proof of Theorem 1 In propositions 2, 3 and 4 it is shown that ψ satisfies the axioms. From propositions 1 and 5 together with the Additivity axiom, we can conclude that ψ is the only solution that satisfies them. □

4 An alternative extension of the *R*-value for *R*-PFF games

In Albizuri et al. (2005) a solution for full R-PFF games is proposed that is an extension of the Shapley value different from that of Myerson (1977a). This section is devoted to characterize its natural extension to R-PFF games.

Given $v \in \Gamma^{\mathcal{E}}$, define an R-game on $\mathcal{K}$ as follows

$$\tilde{v}(S) = \frac{1}{|\mathcal{E}(S)|} \sum_{(S,\Pi)\in\mathcal{E}(S)} v(S,\Pi), \tag{38}$$

where $\mathcal{E}(S) = \{(S,\Pi) \in \mathcal{E}\}$. That is, $\tilde{v}(S)$ is the average of the worth of the embedded coalitions of the form (S,Π).

Then define the value $\Lambda^{\mathcal{E}}$ on $\Gamma^{\mathcal{E}}$ by

$$\Lambda^{\mathcal{E}}(v) = \psi^{\mathcal{K}}(\tilde{v}), \tag{39}$$

where $\psi^{\mathcal{K}}$ is the R-value on $G^{\mathcal{K}}$.

Next we are going to characterize the value $\Lambda^{\mathcal{E}}$ using some new axioms.

Axiom 4 (Efficiency) Let Φ be a value on $\Gamma^{\mathcal{E}}$, then

$$\sum_{i\in N} \Phi_i(v) = v\big(N, \{N\}\big). \tag{40}$$

Since N is always a carrier of the game, this axiom is weaker than the Carrier axiom.

A coalition T such that $\mathcal{E}(T) \neq \emptyset$ is said to be an *oligarchy* of $v \in \Gamma^{\mathcal{E}}$ if for every $(S, \Pi) \in \mathcal{E}$ it holds:

$$1. \text{ If } T \subseteq S, \text{ then } v(S, \Pi) = v(N, \{N\}); \tag{41}$$

$$2. \text{ Otherwise } (S, \Pi) \text{ is a zero-embedded coalition.} \tag{42}$$

Note that if T is an oligarchy in $v \in \Gamma^{\mathcal{E}}$, then it is also a carrier. But a carrier is not necessarily an oligarchy.

Axiom 5 (Oligarchy Dummy Player) Let Φ be a value on $\Gamma^{\mathcal{E}}$. If T is an oligarchy then

$$\Phi_i(v) = 0 \quad \text{for all } i \in N \backslash T. \tag{43}$$

Example 4 The game in Example 1 has only one oligarchy, N. If $v\big(N, \{N\}\big) = 1$ instead of 3, then $\{1, 2\}$ would be a carrier of the game but N remains the only oligarchy. Since $v\big(\{1, 2\}, \big\{\{1, 2\}, \{3\}, \{4\}\big\}\big) = 0 \neq v\big(N, \{N\}\big)$, then $\{1, 2\}$ would not be an oligarchy. If $v\big(N, \{N\}\big) = 1$ and $v\big(\{1, 2\}, \Pi\big) = 1$ for every $\big(\{1, 2\}, \Pi\big) \in \mathcal{E}$, then $\{1, 2\}$ would be an oligarchy. Note that if $\{1, 2\} \nsubseteq S$, then (S, Π) would be a zero-embedded coalition, and thus requirement 2 would be satisfied. Oligarchy Dummy Player requires $\Phi_3(v) = 0$ and $\Phi_4(v) = 0$.

Remark 2 This axiom is weaker than several axioms existing in the literature of full R-PFF games about dummy players.

De Clippel and Serrano (2008) define a player $i \in N$ to be a weak null player in v if $v(S, \Pi) = v\big(S \backslash \{i\}, \Pi_{\{i\}}\big)$ for all (S, Π) such that $i \in S$. So i leaves S and remains alone. The weak null player axiom requires $\Phi_i(v) = 0$. Note that if T is an oligarchy and $i \in N \backslash T$, the worth $v(S, \Pi)$ does not change when i leaves S. And hence, i is a weak null player in v.

De Clippel and Serrano (2008), as well as Macho-Stadler et al. (2006) and (2007), define $i \in N$ to be a strong null player in v if $v(S, \Pi) = v\big(S \backslash \{i\}, \Pi'\big)$ for all (S, Π) and (S, Π') such that $\Pi_{\{i\}} = \Pi'_{\{i\}}$, that is, i leaves his or her coalition in Π and remains alone or joins any other coalition in Π. As a result, Π' is formed. The strong null player axiom requires $\Phi_i(v) = 0$. This axiom is weaker than the previous one. Note also that if T is an oligarchy and $i \in N \backslash T$, $v(S, \Pi)$ and $v\big(S \backslash \{i\}, \Pi'\big)$ coincide, they are zero or one. And therefore, i is a strong null player in v.

Alonso-Meijide et al. (2019) introduced a weaker axiom than the previous two. In this case, in addition it is considered a third type of movement. A player $i \in N$ is a complete null player in v if $v(S, \Pi) = v\big(S \backslash \{i\}, \Pi'\big)$ for all (S, Π) and (S, Π') such that $\Pi_{\{i\}} = \Pi'_{\{i\}}$ or

$$i \notin S \text{ and } \Pi' = \big(\Pi \backslash \{T_1, T_2\}\big) \cup \{T_1 \cup T_2\}, \tag{44}$$

where $T_1, T_2 \in \Pi \backslash \{S\}$ and $i \in T_1 \cup T_2$. So T_1 and T_2, where $i \in T_1 \cup T_2$, can join. The complete null player axiom requires $\Phi_i(v) = 0$. As Alonso-Meijide et al. (2019)

(Proposition 5) proved, the existence of a complete null player implies that v is a game without externalities. Again if T is an oligarchy and $i \in N \backslash T$, $v(S, \Pi)$ and $v\big(S\backslash\{i\}, \Pi'\big)$ coincide, and i is a complete null player in v.

Therefore, as written above, for full R-PFF games, Oligarchy Dummy Player is weaker than the previous three null player axioms.

We prove that for full R-PFF games $\Lambda^{\mathcal{E}}$ does not satisfy the weak and strong null player axioms. For this we consider two counterexamples.

Let $N = \{1, 2, 3\}$, $\mathcal{K} = \mathrm{CL}$, and $v \in \Gamma^{\mathcal{E}}$ such that

$$v(S, \Pi) = \begin{cases} 0 & \text{if } S = \emptyset \text{ or } S = \{2\}; \\ 0 & \text{if } (S, \Pi) = \big(\{1\}, \big\{\{1\}, \{2, 3\}\big\}\big); \\ 2 & \text{otherwise.} \end{cases} \tag{45}$$

Player 2 is clearly a weak null player. From expression (38), we get $\tilde{v}\big(\{1\}\big) = 1$, $\tilde{v}\big(\{2\}\big) = 0$ and $\tilde{v}(S) = 2$ if $\emptyset \neq S \notin \big\{\{1\}, \{2\}\big\}$. Hence, from (39) we have $\Lambda_2^{\mathcal{E}}(v) = Sh_2(\tilde{v}) = 1/6$, where Sh denotes the Shapley value for classical TU games. Consequently, $\Lambda^{\mathcal{E}}$ does not satisfy the weak null player axiom. Note that player 2 is not a strong player because $v\big(\{1, 2\}, \big\{\{1, 2\}, \{3\}\big\}\big) \neq v\big(\{1\}, \big\{\{1\}, \{2, 3\}\big\}\big)$.

Let $N = \{1, 2, 3, 4\}$, $\mathcal{K} = \mathrm{CL}$, and $v \in \Gamma^{\mathcal{E}}$ such that

$$v(S, \Pi) = \begin{cases} 0 & \text{if } S = \emptyset \text{ or } S = \{2\}; \\ 0 & \text{if } S = \{4\} \text{ and } \Pi \notin \big\{\big\{\{1, 3\}, \{2\}\big\}, \big\{\{1, 3, 2\}\big\}\big\}; \\ 2 & \text{if } S = \{4\} \text{ and } \Pi \in \big\{\big\{\{1, 3\}, \{2\}\big\}, \big\{\{1, 3, 2\}\big\}\big\}; \\ 2 & \text{otherwise.} \end{cases} \tag{46}$$

Player 2 is a strong null player but not a complete null player since $v\big(\{4\}, \big\{\{1, 2\}, \{3\}, \{4\}\big\}\big) \neq v\big(\{4\}, \big\{\{1, 2, 3\}, \{4\}\big\}\big)$. Taking into account expression (38), we have $\tilde{v}\big(\{2\}\big) = 0$, $\tilde{v}\big(\{4\}\big) = 4/5$, $\tilde{v}\big(\{2, 4\}\big) = 1$ and $\tilde{v}(S) = 2$ if $\emptyset \neq S \notin \big\{\{2\}, \{4\}, \{2, 4\}\big\}$. Therefore, (39) implies $\Lambda_2^{\mathcal{E}}(v) = Sh_2(\tilde{v}) = 1/60$. Hence, the strong null player axiom is not satisfied by $\Lambda^{\mathcal{E}}$.

As for the complete null player axiom, it is satisfied by $\Lambda^{\mathcal{E}}$ for full R-PFF games. As written before, if there exists a complete null player i, then v is a game without externalities. Moreover, i becomes a null player, and (39) implies $\Lambda_i^{\mathcal{E}}(v) = Sh_i(v) = 0$.

Next a pair of players $i, j \in N$ are called *partners* if for all $(S, \Pi) \in \mathcal{E}$ it holds

$$|\{i, j\} \backslash S| = 1 \quad \text{implies} \quad (S, \Pi) \text{ is a zero-embedded coalition.} \tag{47}$$

If two players i and j are partners in v, then each of them individually makes no contribution to any coalition to which neither of them belongs. Note also that if i and j are partners then $\{i, j\}$ is a p-type coalition in v, but not vice versa.

Example 5 In Example 1 players 1 and 2 are partners. To prove that players in a p-type coalition $\{i, j\}$ are not always partners, consider $N = \{1, 2, 3\}$ and $\mathcal{K} = \mathrm{CL}$. Let $v \in \Gamma^{\mathcal{E}}$ such that $v(S, \Pi) = 1$ if $3 \in S$ and $v(S, \Pi) = 0$ otherwise. Then $S = \{1, 2\}$

is a p-type coalition but 1 and 2 are not partners. Note that $|\{1,2\}\backslash\{1,3\}| = 1$ and $v(\{1,3\},\{\{1,3\},\{2\}\}) \neq 0$.

Axiom 6 (Partners-Symmetry) Let Φ be a value on $\Gamma^{\mathcal{E}}$. If i and j are partners in $v \in \Gamma^{\mathcal{E}}$, then

$$\Phi_i(v) = \Phi_j(v). \tag{48}$$

Since two partners make no contribution individually, the two players play the same role. Therefore, Partners-Symmetry requires that they receive the same value. Moreover, since a couple of partners forms a p-type coalition this axiom is weaker than Symmetric-Partnership, and hence weaker than Axiom 1 in Myerson (1977a) in the case of full R-PFF games as well.

Axiom 7 (Embedded-Coalition Anonymity) Let $v, w \in \Gamma^{\mathcal{E}}$ and $(S,\Pi), (S,\Pi') \in \mathcal{E}$. If

(a) $v(S,\Pi) = w(S,\Pi')$,
(b) $v(S,\Pi') = w(S,\Pi)$, and
(c) $v(T,\Sigma) = w(T,\Sigma)$ for every $(T,\Sigma) \in \mathcal{E}\backslash\{(S,\Pi),(S,\Pi')\}$

then

$$\sum_{i\in S} \Phi_i(v) = \sum_{i\in S} \Phi_i(w). \tag{49}$$

Assume that (S,Π) and (S,Π') are two feasible embedded coalitions, that only differ in the way in which the players out of S are rearranged. This axiom requires that in the game that results from exchanging the worth of these two embedded coalitions, all else being the same, the solution assigns to the players in S the same total payoff as in the original one. Alternatively, one can think that this axiom requires invariance of the total payoff assigned to S, under permutations of the set of embedded coalitions in which S is the first component of the pair. Thus this axiom is a very weak version of an anonymity property among embedded coalitions relative to coalition S.

Theorem 2 *There is a unique solution on $\Gamma^{\mathcal{E}}$ that satisfies Efficiency, Oligarchy Dummy Player, Partners-Symmetry, Embedded-Coalition Anonimity and Additivity, and it is $\Lambda^{\mathcal{E}}$.*

This theorem follows from Propositions 6 to 7.

Proposition 6 *The value $\Lambda^{\mathcal{E}}$ satisfies Efficiency, Oligarchy Dummy Player, Partners-Symmetry, Embedded-Coalition Anonimity and Additivity.*

Proof Since the R-value $\sum_{i\in N} \psi_i^{\mathcal{K}}(N) = \tilde{v}(N)$, it is straightforward to see that $\Lambda^{\mathcal{E}}$ satisfies Efficiency. Similarly, since $\psi^{\mathcal{K}}$ is linear, then $\Lambda^{\mathcal{E}}$ satisfies Additivity as well.

Let us show that $\Lambda^{\mathcal{E}}$ satisfies Oligarchy Dummy Player. Let T be an oligarchy in $v \in \Gamma^{\mathcal{E}}$. The corresponding R-game $\tilde{v}$ on $\mathcal{K}$ of expression (38) is

$$\tilde{v}(S) = \begin{cases} v\big(N, \{N\}\big), & \text{if} T \subseteq S; \\ 0 & \text{otherwise.} \end{cases} \tag{50}$$

Since $\mathcal{E}(T) \neq \emptyset$ then $\tilde{v} = u_T$, i.e. $\tilde{v}$ is the unanimity game $u_T \in G^{\mathcal{K}}$. So by the definition of $\psi^{\mathcal{K}}$ in expression (4), we get $0 = \psi_i^{\mathcal{K}}(u_T) = \psi_i^{\mathcal{K}}(\tilde{v}) = \Lambda_i^{\mathcal{E}}(v)$ for every $i \in N \backslash T$ as desired.

Now let us show that $\Lambda^{\mathcal{E}}$ satisfies Partners-Symmetry. Let $i, j \in N$ partners in v. Considering the decomposition of v in unanimity games given in (10) we get

$$\Lambda_i^{\mathcal{E}}(v) = \psi_i^{\mathcal{K}}(\tilde{v}) = \sum_{\substack{(T,\Sigma)\in\mathcal{E}: \\ i\in T}} d_v(T, \Sigma)\psi_i^{\mathcal{K}}(u_{(T,\Sigma)}) \tag{51}$$

Since i, j are partners $d_v(T, \Sigma) = 0$ whenever $i \in T$ and $j \notin T$. So

$$\Lambda_i^{\mathcal{E}}(v) = \sum_{\substack{(T,\Sigma)\in\mathcal{E}: \\ i,j\in T}} d_v(T, \Sigma)\psi_i^{\mathcal{K}}(u_{(T,\Sigma)}). \tag{52}$$

Analogously we can deduce that

$$\Lambda_j^{\mathcal{E}}(v) = \sum_{\substack{(T,\Sigma)\in\mathcal{E}: \\ i,j\in T}} d_v(T, \Sigma)\psi_j^{\mathcal{K}}(u_{(T,\Sigma)}). \tag{53}$$

Taking into account that since $i, j \in T$ it holds $\psi_i^{\mathcal{K}}(u_{(T,\Sigma)}) = \psi_j^{\mathcal{K}}(u_{(T,\Sigma)})$, we get $\Lambda_i^{\mathcal{E}}(v) = \Lambda_j^{\mathcal{E}}(v)$ as desired.

Finally we turn to show that $\Lambda^{\mathcal{E}}$ satisfies Embedded-Coalition Anonimity. Let $v, w \in \Gamma^{\mathcal{E}}$ and $(S, \Pi), (S, \Pi') \in \mathcal{E}$ such that $v(S, \Pi) = w(S, \Pi')$, $v(S, \Pi') = w(S, \Pi)$, and $v(T, \Sigma) = v(T, \Sigma)$ for $(T, \Sigma) \in \mathcal{E} \backslash \{(S, \Pi), (S, \Pi')\}$. Then for the corresponding R-games of expression (38), it holds $\tilde{v} = \tilde{w}$. So $\psi_i^{\mathcal{K}}(\tilde{v}) = \psi_i^{\mathcal{K}}(\tilde{w})$ for every $i \in T$, and hence $\sum_{i\in T} \Lambda_i(\tilde{v}) = \sum_{i\in T} \Lambda_i(\tilde{w})$, and the proof is concluded. □

The following games are useful for the proof of the next proposition.

Given $(T, \Sigma) \in \mathcal{E}$, define the game $u^*_{(T,\Sigma)} \in \Gamma^{\mathcal{E}}$ by

$$u^*_{(T,\Sigma)}(S, \Pi) = \begin{cases} 1, & \text{if}(S, \Pi) = (T, \Sigma); \\ 0, & \text{otherwise.} \end{cases} \tag{54}$$

It is clear that these games form a basis for $\Gamma^{\mathcal{E}}$.

Proposition 7 *There exists at least one value that satisfies Efficiency, Oligarchy Dummy Player, Embedded-Coalition Anonimity, Partners-Symmetry and Additivity.*

Proof Let Φ be a value that satisfies the axioms above. By Additivity it is sufficient to show that $\Phi\big(c \cdot u^*_{(T,\Sigma)}\big)$ is determined for every $(T,\Sigma) \in \mathcal{E}$ and all $c \in \mathbb{R}$. We will use backward induction on $|T|$.

If $|T| = n$, then $(T,\Sigma) = \big(N, \{N\}\big)$. But any couple of players are partners in $c \cdot u^*_{(N,\{N\})}$, and by Efficiency and Partners-Symmetry we get $\Phi\Big(c \cdot u^*_{(N,\{N\})}\Big) = c/n$ for all $i \in N$.

Let us suppose that $\Phi(c \cdot u^*_{(T,\Sigma)})$ is determined whenever $|T| > k$. So fix T such that $|T| = k$ and consider the game $c \cdot u^*_{(T,\Sigma)} \in \Gamma^{\mathcal{E}}$. We consider two cases.

(1) First, we will show $\Phi_i(c \cdot u^*_{(T,\Sigma)})$ is determined for all $i \in T$. For that, consider the game

$$v = \sum_{\substack{(S,\Pi)\in\mathcal{E}:\\ T\subseteq S}} c \cdot u^*_{(S,\Pi)} \tag{55}$$

Note that T is an oligarchy in v, so $\Phi_i(v) = 0$ for every $i \in N \backslash T$. By Efficiency we have

$$\sum_{i\in T} \Phi_i(v) = v\big(N, \{N\}\big). \tag{56}$$

And by Additivity

$$\sum_{i\in T} \Phi_i(v) = \sum_{\substack{(S,\Pi)\in\mathcal{E}:\\ T\subseteq S,\, T\neq S}} \sum_{i\in T} \Phi_i(c \cdot u^*_{(S,\Pi)}) + \sum_{(T,\Pi)\in\mathcal{E}} \sum_{i\in T} \Phi_i\Big(u^*_{(T,\Pi)}\Big). \tag{57}$$

Moreover if $(T,\Pi), (T,\Pi') \in \mathcal{E}$, by Embedded-Coalition Anonimity it holds $\sum_{i\in T} \Phi_i(u^*_{(T,\Pi)}) = \sum_{i\in T} \Phi_i(u^*_{(T,\Pi')})$. So by combining expressions (56) and (57) we get

$$v(N,\{N\}) - \sum_{\substack{(S,\Pi)\in\mathcal{E}:\\ T\subseteq S,\, T\neq S}} \sum_{i\in T} \Phi_i\Big(c \cdot u^*_{(S,\Pi)}\Big) = |\mathcal{E}(T)| \cdot \sum_{i\in T} \Phi_i\Big(c \cdot u^*_{(T,\Sigma)}\Big), \tag{58}$$

where $\mathcal{E}(T) = \big\{(T,\Pi) \in \mathcal{E}\big\}$. Now by the induction hypothesis the second term in the right side of the equality above is determined. Hence $\sum_{i\in T} \Phi_i(c \cdot u^*_{(T,\Sigma)})$ is also determined.

Furthermore, every pair of players of T are partners in $c \cdot u^*_{(T,\Sigma)}$, and hence by the Partners-Symmetry it holds $\Phi_i(c \cdot u^*_{(T,\Sigma)}) = \Phi_j(c \cdot u^*_{(T,\Sigma)})$ for all $i,j \in T$. Combining this with the fact that $\sum_{i\in T} \Phi_i(c \cdot u^*_{(T,\Sigma)})$ is determined, we can conclude that $\Phi_i(u^*_{(T,\Sigma)})$ is fully determined for every $i \in T$.

(2) Now we will show that $\Phi(c \cdot u^*_{(T,\Sigma)})$ is also determined for all $i \in N \backslash T$. Note that if $i, j \in N \backslash T$ then they are partners in $c \cdot u^*_{(T,\Sigma)}$. Then by Efficiency and Partners-Symmetry axioms and the fact $\Phi_i(c \cdot u^*_{(T,\Sigma)})$ is also determined for every $i \in T$, we can conclude the proof. □

5 Concluding remarks

In this paper we have considered restricted games in partition function form. These games are particularly suitable to study situations in which some coalitions may not be able to form, and in addition there are externalities. Two extensions of the Shapley value for these games have been proposed and characterized by means of standard axioms.

These two extensions of the Shapley value, are in turn the corresponding extensions of the Myerson (1977a) and Albizuri et al. (2005) values for PFF games. An open problem is to extend other values for PFF games mentioned in the Introduction to *R*-PFF games, using alternative approaches.

In our model of restricted games, the players form a partition of *N*, as in full *R*-PFF games. This is possible because all individual coalitions are feasible. It would also have been possible to define the restricted games with externalities without requiring that a partition forms. In this case, we would not require individual coalitions to be feasible. We have chosen the first model to maintain the formation of a partition and because we think that it makes sense that individual coalitions are feasible. Also mention that it might also have been possible to model restricted games considering a general family $\mathcal{E}$ of embedded coalitions such that $\left(N, \{N\}\right) \in \mathcal{E}$ and $(\emptyset, \Pi) \in \mathcal{E}$ for every $\Pi \in \mathcal{P}$.

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Declarations

Conflict of interest The authors declare no competing interests.

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