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RESEARCH ARTICLE

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Kaluza-Klein bubble like structure and celestial sphere in the inflationary universe

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Abstract We consider five dimensional deSitter spacetimes with a deficit angle due to the presence of a closed 2-brane and identify one dimension as an extra dimension. From the four dimensional viewpoint we can see that the spacetime has a structure similar to a Kaluza-Klein bubble of nothing, that is, four dimensional spacetime ends at the 2-brane. Since a spatial section of the full deSitter spacetime has the topology of a sphere, the boundary surface surrounds the remaining four dimensional spacetime, and can be considered as the celestial sphere. After the spacetime is created from nothing via an instanton which we describe, some four dimensional observers in it see the celestial sphere falling down, and will be in contact with a 2-brane attached on it.

Keywords Kaluza-Klein bubble · De Sitter spacetime

It is well known that the Kaluza-Klein vacuum is unstable against creation of bubbles of nothing in general [1]. The first solution found by Witten describes the Kaluza-Klein bubble in asymptotically flat spacetimes with a compactified space

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dimension. Recently, extension to asymptotically adS spacetimes was considered from the aspect of the adS/CFT correspondence or dynamical spacetimes [2]. The regular, dynamical and non-trivial geometries might give us good examination for superstring theory. (See also Ref. [3] for similar solution in the context of the Randall-Sundrum brane world.) On the other hand, a Kaluza-Klein bubble solution in de Sitter spacetime has not been found.

Consistent de Sitter vacua in superstring theory were just recently found [4] and, thus, we now have all kinds of maximally symmetric spacetimes in superstring theory: Minkowski, de Sitter and anti-de Sitter spacetimes. Among these three types of spcetimes, a Kaluza-Klein bubble solution has not been found in and only in de Sitter spacetime. Therefore, it seems natural to seek a Kaluza-Klein bubble solution in de Sitter spacetime.

This paper can be considered as a first step towards this direction. While we shall not find a Kaluza-Klein bubble solution itself in de Sitter spacetime, we examine properties of a spacetime with non-trivial topology, which has structures quite similar to those of a Kaluza-Klein bubble. First we shall discuss the five dimensional de Sitter spacetime with a deficit angle due to the presence of a closed 2-brane and then find a spacetime structure similar to a Kaluza-Klein bubble. That is, the four dimensional spacetime ends at the 2-brane if we identify one dimension with an extra dimension. The construction of full spacetime here is a straightforward extension of Ref. [5] in four dimension to five dimensions. Non-trivial issue presented here is analysis from the four dimensional point of view.

We begin with the five dimensional deSitter spacetime in the static chart

$$ds^2 = -(1 - H^2r^2)dt^2 + \frac{1}{1 - H^2r^2}dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\Omega_2^2\big). \tag{1}$$

Through the double Wick rotation

$$t \to i\chi/H \ \text{ and } \ \theta \to i\tau + \frac{\pi}{2}, \tag{2}$$

we obtain a new metric

$${}^{(5)}g = (H^{-2} - r^2)d\chi^2 + \frac{1}{1 - H^2r^2}dr^2 + r^2\big(-d\tau^2 + \cosh^2\tau d\Omega_2^2\big), \tag{3}$$

where $0 \le r \le H^{-1}$. If we require the spacetime to be regular at $r = H^{-1}$, the coordinate χ must have the period $\chi_p = 2\pi$. As seen later, the spacetime is exactly five dimensional deSitter spacetime. On the other hand, if we add a 2-brane at $r = H^{-1}$ then this condition is relaxed and the period can be any value smaller than this value:

$$0 < \chi_p \le 2\pi. \tag{4}$$

The equality holds if and only if the 2-brane tension vanishes or there is no 2-brane. In the following we consider a 2-brane with non-zero tension at $r = H^{-1}$ to make the period χ_p small enough and may regard χ as a coordinate of a compact dimension. Again, as seen later, the spacetime is five dimensional deSitter spacetime with a deficit angle.

For $\chi_p < 2\pi$, the 2-dimensional subspace spanned by (χ, r) and extended to the negative r region is like an American football as shown in Fig. 1. (For $\chi_p = 2\pi$

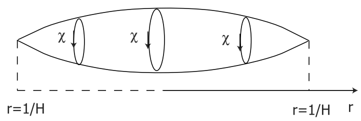

Fig. 1 The 2-dimensional subspace spanned by (χ, r). A 2-sphere is attached on each point and 2-spheres at $(\chi, \pm r)$ are identified

it is a round sphere.) At each point (χ, r) on the surface of the football, a 2-sphere with radius $r\cosh\tau$ is attached and it is identified with the 2-sphere at $(\chi, -r)$. Hence, if the longitudinal angular coordinate of each 2-sphere is suppressed, a $\tau = $ const. surface has a pancake-like (or convex lens-like) shape. The thickness of the pancake corresponds to the size of the compact dimension $\sqrt{H^{-2} - r^2}\chi_p$, and the edge of the pancake is at $r = H^{-1}$, where a 2-brane with the topology of 2-sphere is attached. While the thickness is independent of the time τ, the physical radius of the pancake changes as $H^{-1}\cosh\tau$.

Let us focus on the geometrical structure of timelike hypersurfaces $\chi = $ const. where we live. The induced metric is

$$^{(4)}g = \frac{1}{1 - H^2 r^2} dr^2 + r^2\big(-d\tau^2 + \cosh^2\tau d\Omega_2^2\big). \tag{5}$$

The Riemann tensor of the induced metric becomes

$$^{(4)}R_{\mu\nu\alpha\beta} = H^2\big({}^{(4)}g_{\mu\alpha}{}^{(4)}g_{\nu\beta} - {}^{(4)}g_{\mu\beta}{}^{(4)}g_{\nu\alpha}\big). \tag{6}$$

Since the induced metric has a constant curvature, the spacetime locally has maximal symmetry and is locally isometric to a four dimensional deSitter (dS_4) spacetime. To investigate the global structure we introduce a new coordinate σ defined by $r = H^{-1}\sin\sigma$. Then the induced metric is written as

$$H^{2\,(4)}g = d\sigma^2 + \sin^2\sigma\big(-d\tau^2 + \cosh^2\tau d\Omega_2^2\big). \tag{7}$$

Next we introduce the following coordinate (T, θ_1) via the coordinate transformation

$$\begin{aligned} \sinh T &= \sin\sigma \sinh\tau, \\ \cosh T \cos\theta_1 &= \cos\sigma. \end{aligned} \tag{8}$$

Finally the induced metric becomes deSitter one in the complete chart

$$H^{2\,(4)}g = -dT^2 + \cosh^2 T\left(d\theta_1^2 + \sin^2\theta_1 d\Omega_2^2\right). \tag{9}$$

To see where (τ, σ) chart covers the deSitter spacetime, it is convenient to embed it in a five dimensional Minkowski spacetime as

$$\begin{aligned} Y_0 &= \sinh T = \sin\sigma\sinh\tau, \\ Y_\sigma &= \cosh T\cos\theta_1 = \cos\sigma, \\ Y_1 &= \cosh T\sin\theta_1\cos\theta_2 = \sin\sigma\cosh\tau\cos\theta_2, \\ Y_2 &= \cosh T\sin\theta_1\sin\theta_2\cos\theta_3 = \sin\sigma\cosh\tau\sin\theta_2\cos\theta_3, \\ Y_3 &= \cosh T\sin\theta_1\sin\theta_2\sin\theta_3 = \sin\sigma\cosh\tau\sin\theta_2\sin\theta_3. \end{aligned}$$

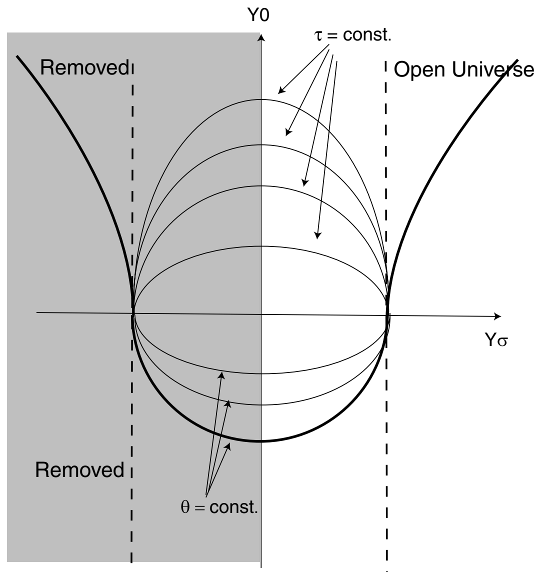

Fig. 2 The global structure of the four dimensional induced geometry. The figure shows the projection onto Y_0-Y_σ plane. The region of $Y_\sigma < 0$ is removed

The deSitter metric is

$$H^{2(4)}g = -dY_0^2 + dY_1^2 + dY_2^2 + dY_3^2 + Y_\sigma^2. \tag{10}$$

As shown in Fig. 2, the $\tau =$ const. slices are cross sections of the deSitter hyperboloid $-Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_\sigma^2 = 1$ with surfaces defined by

$$\left(\frac{Y_0}{\sinh\tau}\right)^2 + Y_\sigma^2 = 1. \tag{11}$$

The $\sigma =$ const. slices are the cross sections of the deSitter hyperboloid with $Y_\sigma =$ const. surfaces. Hence, the coordinate system (τ, r) covers the region $0 \le Y_\sigma \le 1$. The right boundary $Y_\sigma = 1$ is just a coordinate artifact and the five dimensional spacetime can be analytically continued beyond it so that the four dimensional projection of the full five dimensional spacetime covers the region $Y_\sigma > 1$ as well, where the deSitter open chart is fitted. On the other hand, we can see that the five dimensional spacetime cannot be extended beyond the other boundary $Y_\sigma = 0$. Actually, the five dimensional spacetime is geodesically complete

Therefore, as depicted in Fig. 2, the regions $Y_\sigma < 0$ and $Y_\sigma > 0$ represent a similar one with the Kaluza-Klein bubble of nothing and the four dimensional world, respectively. The boundary $Y_\sigma = 0$ (or $r = H^{-1}$) has the topology of a 2-sphere and the area radius $H^{-1}\cosh\tau$, and surrounds both regions. Hence, from four dimensional observers' viewpoints, it is interpreted that the four dimensional spacetime ends there.

In order to analyze the global structure we introduce null coordinates $u_\pm$ defined by

$$u_\pm = \tau \pm \int \frac{dr}{r\sqrt{1-H^2r^2}} = \tau \mp \frac{1}{2}\log\frac{1+\sqrt{1-H^2r^2}}{1-\sqrt{1-H^2r^2}}. \tag{12}$$

In this null coordinate system the induced metric becomes

$$^{(4)}g = H^{-2}\left[-\frac{1}{\cosh^2\frac{u_+-u_-}{2}}du_+du_- + R^2(u_+,u_-)d\Omega_2^2\right] \tag{13}$$

where

$$R(u_+,u_-) = \frac{\cosh\frac{u_++u_-}{2}}{\cosh\frac{u_+-u_-}{2}}. \tag{14}$$

The expansion rate of out/in-going null geodesic congruence is

$$\theta_\pm := \frac{\partial \ln R}{\partial u_\pm} = \frac{1}{2}\left[\frac{\sinh\frac{u_++u_-}{2}}{\cosh\frac{u_++u_-}{2}} \mp \frac{\sinh\frac{u_+-u_-}{2}}{\cosh\frac{u_+-u_-}{2}}\right]. \tag{15}$$

Since the spacetime is locally a four dimensional deSitter spacetime, the structure in terms of the expansion rates is simple as seen in Fig. 3. Each expansion $\theta_\pm$ vanishes on the null surface $u_\mp = 0$, is positive in the region $u_\mp > 0$ and is negative in the region $u_\mp < 0$. We note that $Hr = 1/\cosh\tau$ and $Y_1^2+Y_2^2+Y_3^2 = 1$ are satisfied on the $\theta_\pm = 0$ surfaces.

From Fig. 3, one can easily see that, from the view point of observers sitting at the center $R = 0$ (i.e. $x = -\infty$) of the spherical symmetry, the boundary $x = 0$ is always beyond the cosmological horizons $u_\pm = 0$. On the other hand, freely-falling observers off the center can be affected by the boundary. For example, let us consider comoving observers in the flat-chart covering the region $u_- > 0$.

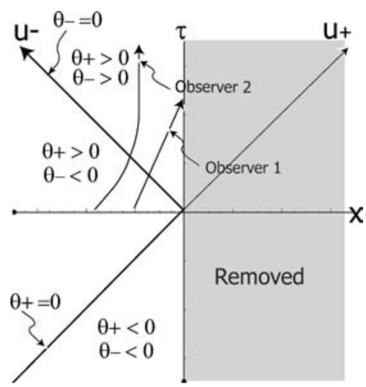

Fig. 3 The signature distribution of θ_+ and θ_- in τ-x plane. Only the region $x = \frac{1}{2}(u_+ - u_-) < 0$ is considered since the other region $x > 0$ is removed. The surface $\theta_\pm = 0$ corresponds to $\tau = \pm x < 0$ and separates the spacetime into two regions, that is, the region $\theta_\pm > 0$ (or $\tau > \pm x, x < 0$) and the region $\theta_\pm < 0$ (or $\tau < \pm x, x < 0$)

As easily seen from Fig. 3, some observers (the observer 1 in Fig. 3) will reach the edge $x = 0$ in finite affine time if the deviation from the center of spherical symmetry is large enough. For a small but non-zero deviation, the observer (the observer 2 in Fig. 3) will not reach the edge but will be in the causal future of the bubble in Lorentzian region after some time.

So far, we investigated the spacetime structure from the four dimensional viewpoint. Let us now consider the extra dimension and the five dimensional structure. The size of the extra dimension is given by $\sqrt{H^{-2} - r^2}\chi_p$ in the region covered by the coordinate system adopted in (3). This expression is equivalent to $Y_\sigma \chi_p$ and, thus, can be extended to the open universe region $Y_\sigma > 1$. Hence, the size of the extra dimension is not bounded from above in the open universe region. It is, thus, natural to ask what the ratio of the size of extra dimension to the size of the four dimensional universe is. We shall see below that the ratio is roughly $\chi_p/4$, which can be made arbitrarily small by fine-tuning the 2-brane tension.

The metric of the full five-dimensional spacetime, which is locally isometric to the deSitter spacetime, is written as

$$
\begin{aligned}
H^{2\,(5)}g &= -dT^2 + \cosh^2 T \left(d\theta_1^2 + \cos^2\theta_1 d\chi^2 + \sin^2\theta_1 d\Omega_2^2\right) \\
&= -dT^2 + \cosh^2 T\, ds_{S^4}^2.
\end{aligned} \tag{16}
$$

Here, $ds_{S^4}^2$ is the metric of S^4 and is transformed to the standard form by

$$
\begin{aligned}
\cos\tilde{\theta}_1 &= \cos\theta_1 \sin\chi, \\
\sin\tilde{\theta}_1 \cos\tilde{\theta}_2 &= \cos\theta_1 \cos\chi, \\
\sin\tilde{\theta}_1 \sin\tilde{\theta}_2 &= \sin\theta_1,
\end{aligned} \tag{17}
$$

so that $d\theta_1^2 + \cos^2\theta_1 d\chi^2 + \sin^2\theta_1 d\Omega_2^2 = d\tilde{\theta}_1^2 + \sin^2\tilde{\theta}_1 (d\tilde{\theta}_2^2 + \sin^2\tilde{\theta}_2 d\Omega_2^2)$. Hence, without the deficit angle due to the 2-brane, the five dimensional spacetime would be the maximally extended de Sitter spacetime not only locally but also globally. On the other hand, we shall see below that the presence of the 2-brane and the deficit angle introduces an intriguing structure. The 2-brane is located at $\theta_1 = \pi/2$, where the extra dimension of S^1 shrinks.

In order to visualize the five dimensional spacetime structure in the presence of the closed 2-brane and the deficit angle ($2\pi - \chi_p > 0$), it is convenient to go to a flat chart. Note, however, that the same structure looks differently for different choices of the origin of the flat chart since the presence of the 2-brane breaks the global translational invariance. Here, we choose the origin of the flat chart $(t_1, x_1, x_2, x_3, x_4)$ as

$$
\begin{aligned}
t_1 &= \ln(\sinh T + \cosh T \cos\tilde{\theta}_1), \\
x_1 &= \frac{\cosh T \sin\tilde{\theta}_1 \cos\tilde{\theta}_2}{\sinh T + \cosh T \cos\tilde{\theta}_1}, \\
x_2 &= \frac{\cosh T \sin\tilde{\theta}_1 \sin\tilde{\theta}_2 \cos\tilde{\theta}_3}{\sinh T + \cosh T \cos\tilde{\theta}_1},
\end{aligned}
$$

$$x_3 = \frac{\cosh T\, \sin\tilde{\theta}_1 \sin\tilde{\theta}_2 \sin\tilde{\theta}_3 \cos\tilde{\theta}_4}{\sinh T + \cosh T\, \cos\tilde{\theta}_1},$$
$$x_4 = \frac{\cosh T\, \sin\tilde{\theta}_1 \sin\tilde{\theta}_2 \sin\tilde{\theta}_3 \sin\tilde{\theta}_4}{\sinh T + \cosh T\, \cos\tilde{\theta}_1}. \tag{18}$$

In this flat chart, the 2-brane is located at the intersection of $x_1 = 0$ and $x_2^2 + x_3^2 + x_4^2 = 1 + e^{-2Ht_1}$. For a deficit angle larger than π or, equivalently, $\chi_p < \pi$, the spacetime region is restricted by [1]

$$|x_1| \leq \frac{\sqrt{1 + \left(1 + e^{-2Ht_1} - x_2^2 - x_3^2 - x_4^2\right)\tan^2(\chi_p/2)} - 1}{\tan(\chi_p/2)}$$
$$= \frac{\chi_p}{4}\left(1 + e^{-2Ht_1} - x_2^2 - x_3^2 - x_4^2\right) + O\left(\chi_p^2\right), \tag{19}$$

where we have chosen the origin of χ so that $-\chi_p/2 \leq \chi - \pi/2 \leq \chi_p/2$. Therefore, the 5-dimensional spacetime has a pancake-like (or convex lens-like) shape as already discussed. The expression (19) also makes it evident that the ratio of the size of the extra dimension (the thickness of the pancake) to the size of the four dimensional universe (the radius of the pancake) is roughly $\chi_p/4$.

The spacetime discussed here can be created from nothing via an instanton. The instanton is obtained simply by analytic continuation on the $\tau = 0$ surface. In the Euclidean regime ($\tau \to -i(\tau_E - \pi/2), Y_0 \to iY_E, T \to i(T_E - \pi/2)$), the five dimensional Euclidean geometry is locally isometric to the 5-sphere S^5 as

$$H^{2(5)}g_E = dT_E^2 + \sin^2 T_E ds_{S^4}^2 = ds_{S^5}^2, \tag{20}$$

while Eq. (11) becomes

$$\left(\frac{Y_E}{\cos\tau_E}\right)^2 + Y_\sigma^2 = 1. \tag{21}$$

Hence $\theta =$ const surfaces are cross sections of the four dimensional unit-sphere $Y_E^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_\sigma^2 = 1$ with the surface defined by Eq. (21) (See Fig. 4). The creation of the spacetime with 2-brane at $\tau = 0$ because $\tau = 0$ surface is momentary static. Note that the path depicted in Fig. 4 corresponds to a quantum creation of the whole universe with a closed 2-brane. As we have already mentioned, the region $Y_\sigma \geq 1$ can be regarded as an open universe. So the current path may be interpreted as creation of the open inflationary universe with a 2-brane. (See Ref. [6] for creation of open universes via a singular instanton. In Ref. [7, 8] a regular instanton similar to our instanton was proposed.)

[1] In the above argument we have assumed that the deficit angle is larger than π. If the deficit angle were smaller than π (and thus $\chi_p > \pi$) then it would be convenient to change the origin of χ so that $-\chi_p/2 \leq \chi + \pi/2 \leq \chi_p/2$. This would correspond to the change of the origin of the flat chart to the opposite point on the S^4. With this change, the inequality in (19) would be opposite and χ_p in the right hand side would be replaced by the deficit angle $2\pi - \chi_p$. Therefore, in the case of a large deficit angle ($\chi_p < \pi$) the spacetime is the outside of a pancake-shaped region but not the inside. In the limit of zero deficit angle ($\chi_p \to 2\pi$) the maximally extended de Sitter spacetime is recovered as the thickness of the pancake-shaped region vanishes.

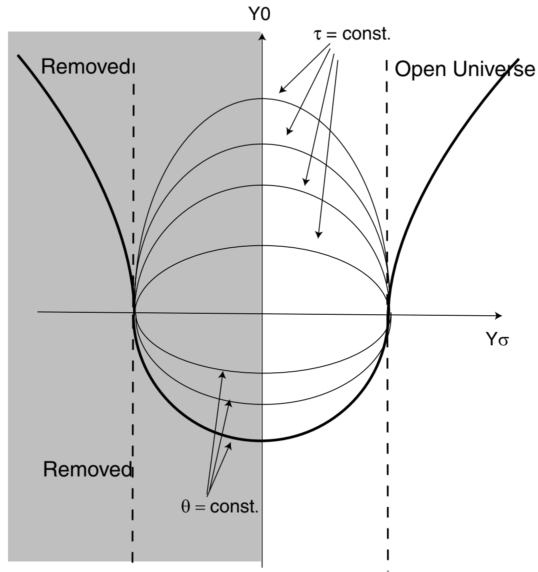

Fig. 4 Creation of the universe

Let us summarize our study. In a five dimensional theory admitting a positive cosmological constant, we considered a spacetime with a closed 2-brane. Despite the fact that it is locally a de Sitter spacetime, a large enough deficit angle due to the presence of the 2-brane introduces an intriguing structure. Depending on observers, the spacetime can be effectively four dimensional at low energy. Indeed, while the size of the extra-dimension is not bounded from above in general, its ratio to the size of the four dimensional universe can be made as small as one likes by fine-tuning the 2-brane tension. If the deficit angle exceeds π then the spacetime has a pancake-like (or convex lens-like) shape with the upper and lower surfaces identified. All observers are confined inside the pancake-shaped region surrounded by the closed 2-brane.

The edge of the pancake mimics the surface of the Kaluza-Klein bubble of nothing and all observers in the pancake are outside the “bubble.” Hence, it is the boundary of the four dimensional spacetime.

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References

1. Witten, E.: Nucl. Phys. **B195**, 481(1982)
2. Balasubramanian, V., Larjo, K., Simon, J.: hep-th/0502111; Dine, M., Fox, P.J., Gorbatov, E.: JHEP **0409**, 037 (2004); Birminghan, D., Rinaldi, M.: Phys. Lett. B **544**, 316 (2002); Balasubramanian, V., Simon, Ross, F.: Phys. Rev. D **66**, 086002 (2002); Aharony, O., Fabinger, M., Horowitz, G.T., Silverstein, E.: JHEP **0207**, 007 (2002); Fabinger, M., Horava, P.: Nucl. Phys. B **580**, 243 (2000); Horowitz, G.T., Maeda, K.: Class. Quant. Grav. **19**, 5543 (2002); Elvang, H., Horowitz, G.T.: Phys. Rev. D **67**, 044015 (2003)

3. Ida, D., Shiromizu, T., Ochiai, H.: Phys. Rev. D **65**, 023504 (2002); Ochiai, H., Ida, D., Shiromizu, T.: Prog. Theor. Phys. **107**, 703 (2002)
4. Kachru, S., Kallosh, R.: A. Linde and S. P. Trivedi, Phys. Rev. D **68**, 046005 (2003)
5. Basu, R., Guth, A.H., Vilenkin, A.: Phys. Rev. **44**, 340 (1991)
6. Hawking, S.W., Turok, N.: Phys. Lett. B **425**, 25 (1998)
7. Garriga, J.: hep-th/9804106
8. Bousso, R., Chamblin, A.: Phys. Rev. D **59**, 063504 (1999)