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ERRATUM

Erratum to: Weyl–Einstein structures on K-contact manifolds

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The main result, Theorem 3.2, of [1] is incorrect and must be replaced by the following statement:

Theorem 3.2 *Let (M, g, ξ) be a compact K-contact manifold of dimension $n = 2m+1$, $m \geq 1$, carrying a closed Weyl–Einstein structure D compatible with the conformal class $c = [g]$. Then g is Einstein and D is the Levi–Civita connection of an Einstein metric g_0 in c, which, up to scaling, is equal to g, except if (M, c) is the flat conformal sphere. In the latter case,* any *K-contact structure is isomorphic to the standard Sasaki–Einstein structure.*

Accordingly, the last sentence in the Introduction of [1] must be replaced by the following one:

We show—Theorem 3.2 and Corollary 3.1 below—that g is then Einstein and D is the Levi–Civita of an Einstein metric g_0, which is actually equal to g up to scaling, except if (M, c) is the flat conformal sphere; in all cases, the K-contact structure is Sasaki–Einstein.

In the proof of Theorem 3.2 the last sentence before **Case 1** must be replaced by:

Since ξ is conformal with respect to g_0, from Proposition 2.2, ξ is Killing with respect to g_0, or (M, c) is the flat conformal sphere.

The end of the proof of Theorem 3.2 starting with **Case 2** must be removed and replaced by:

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Case 2. (M, c) is the standard flat conformal sphere $(\mathbb{S}^{2m+1}, c_0)$. In this case we conclude the proof of Theorem 3.2 by using the following general result:

Lemma 1 *For any K-contact structure (g, ξ) on $\mathbb{S}^{2m+1}$ with $[g] = c_0$, the metric g has constant sectional curvature equal to 1 and the K-contact structure is then isomorphic to the standard Sasaki–Einstein structure.*

Proof We first recall that for K-contact manifold (M, g, ξ), we have

$$\mathrm{R}_{\xi,X}\xi = X - g(\xi, X)\xi, \tag{1}$$

for any vector field X. Indeed, since $\varphi = \nabla\xi$ and ξ is Killing with respect to g and of norm 1, we have

$$\nabla_X\varphi = \mathrm{R}_{\xi,X}, \tag{2}$$

where R denotes the curvature of g, cf. [2], so that $\mathrm{R}_{\xi,X}\xi = \nabla_X(\nabla_\xi\xi) - \nabla_{\nabla_X\xi}\xi = -\nabla_{\nabla_X\xi}\xi = -\varphi^2(X) = X - g(\xi, X)\,\xi$. In the current case, when the conformal structure $[g]$ is flat, the curvature of g is of the form

$$\mathrm{R}_{X,Y} = \mathrm{S}(X) \wedge Y + X \wedge \mathrm{S}(Y), \tag{3}$$

where, in general, for any n-dimensional Riemannian manifold (M, g), the *normalized Ricci tensor* (or *Schouten tensor*) S is defined by

$$\mathrm{S} := \frac{1}{(n-2)}\left(\mathrm{Ric} - \frac{\mathrm{Scal}}{2(n-1)}\right).$$

It then follows from (1), (3) and the identity

$$\mathrm{Ric}(\xi) = (n-1)\,\xi, \tag{4}$$

cf. Proposition 3.1 in [1], that:

$$\mathrm{S}(X) = \frac{1}{(n-2)}\left[\left(\frac{\mathrm{Scal}}{2(n-1)} - 1\right)X + \left(n - \frac{\mathrm{Scal}}{(n-1)}\right)g(\xi, X)\,\xi\right] \tag{5}$$

[in (4), (5) and in the sequel, Ric and S are regarded as endomorphisms of the tangent bundle via the metric g]. By using the contracted Bianchi identity $\delta\mathrm{S} + \frac{\mathrm{d\,Scal}}{2(n-1)} = 0$, we readily infer from (5) that Scal is constant, so that

$$(\nabla_X\mathrm{S})(Y) = \kappa\,\left(g(\nabla_X\xi, Y)\xi + g(\xi, Y)\,\nabla_X\xi\right), \tag{6}$$

for any vector fields X, Y, by setting $\kappa := \frac{1}{(n-2)}\left(n - \frac{\mathrm{Scal}}{(n-1)}\right)$. Since the conformal structure is flat, $(\nabla_X\mathrm{S})(Y)$ must be *symmetric* in X, Y, so in particular the expression $\kappa\, g(\nabla_X\xi, Y) = g((\nabla_X\mathrm{S})(Y), \xi)$ must be symmetric in X, Y. On the other hand, $g(\nabla_X\xi, Y)$ is clearly skew-symmetric in X, Y, thus showing that $\kappa = 0$; it follows that $\mathrm{Scal} = n(n-1)$, hence, by (5), that $S = \frac{1}{2}\mathrm{Id}$, so by (3), that g is a metric of sectional curvature equal to 1.

Finally, Eq. (2) shows that $\nabla_X\varphi = \xi \wedge X$ for every tangent vector X, meaning that the K-contact structure is Sasaki–Einstein, and it is well known that the isometry group of $\mathbb{S}^{2m+1}$ acts transitively on the set of Sasaki–Einstein structures on the sphere. □

References

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