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Invariance of finiteness of K-area under surgery

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Abstract K-area is an invariant for Riemannian manifolds introduced by Gromov as an obstruction to the existence of positive scalar curvature. However in general it is difficult to determine whether K-area is finite or not in spite of its natural definition. In this paper, we study how the invariant changes under surgery.

1 Introduction

The notion of K-area was introduced by Gromov [3]. It is an invariant for Riemannian manifolds with values in $(0, +\infty]$. Roughly, K-area(M) measures how small C^0 curvature norms can be achieved for "non-trivial" vector bundles over a Riemannian manifold M. Here a "non-trivial" vector bundle E means a vector bundle with non-zero Chern numbers. Finiteness of K-area has a deep relationship with the existence of positive scalar curvature. The following theorem was proved by Gromov using the relative index theorem [2].

Theorem 1.1 [3] *Let M be an even dimensional complete spin Riemannian manifold. If the scalar curvature* Sc *of M satisfies* inf Sc $> \varepsilon^2$, *then K-area$(M) \leq c\varepsilon^{-2}$ where c is a constant depending on the dimension of M.*

In particular even dimensional spin manifold with K-area$(M) = \infty$ does not admit complete Riemannian metrics of uniformly positive scalar curvature.

Even though both notions of scalar curvature and K-area require Riemannian metrics, finiteness of K-area on a compact manifold depends only on its homotopy type. Hence infiniteness of K-area is a homotopical obstruction to the existence of positive scalar curvature on compact spin manifolds.

In this paper we verify the following.

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Theorem 1.2 *Let M be an oriented even dimensional Riemannian manifold with $p + q = \dim(M)$. Let $M^\sharp$ be a manifold obtained by p-surgery for $q \neq 2$. If K-area$(M) < \infty$, then K-area$(M^\sharp) < \infty$.*

As a special case:

Corollary 1.3 *Let M_1 and M_2 be oriented even dimensional Riemannian manifolds of the same dimension. Let $M_1 \sharp M_2$ denote the connected sum of M_1 and M_2. If both K-area(M_1) and K-area(M_2) are finite, then K-area$(M_1 \sharp M_2)$ is also finite.*

On the other hand the converse is easy to verify. Of course, the following Lemma 1.4 follows also from Theorem 1.2, but it can be verified without it.

Lemma 1.4 *Let M_1, M_2, and $M_1 \sharp M_2$ be as above. If either M_1 or M_2 has infinite K-area, then K-area$(M_1 \sharp M_2) = \infty$.*

In fact, if M_1 has infinite K-area then there exists a "non-trivial" vector bundle E over M_1 with small C^0 curvature norm. Then we can construct another vector bundle over $M_1 \sharp M_2$ by extending E trivially onto M_2.

We remark that the main theorem is analogous to the following.

Proposition 1.5 [1] *Let M be a compact manifold which carries a Riemannian metric of positive scalar curvature. Then any manifold obtained by surgeries in codimension ≥ 3 also carries a metric of positive scalar curvature.*

The proof of our main theorem is rather different. The idea of the above proposition is that $S^{q-1} \times N$ admits a Riemannian metric of positive scalar curvature for $q \geq 3$. On the other hand, we use a property that the cartesian product of spheres at the connecting region is simply connected. Any almost flat vector bundles over compact simply connected manifolds are trivial, which will be used to compute finiteness of K-area.

In [5] M. Listing studies so called "homology classes of finite K-area" and remarks that the homology of finite K-area in the dimensions lower than the largest one behave in the same way as the ordinary homology when taking the connected sums.

In [4] B. Hanke extends the concept of K-area by admitting Hilbert-A-module bundles of small or vanishing curvature. He defines the notion of infiniteness (and finiteness) of K-area of K-homology classes $h \in K_0(M) \otimes \mathbb{Q}$ for closed smooth manifolds M. It is shown that the K-area of the homological fundamental classes of area-enlargeable manifolds in the sense of [2] are infinite. Moreover he shows that oriented manifolds with fundamental classes of infinite K-area are essential. Manifolds are said to be essential if the classifying maps of universal covers map the homological fundamental classes to non-zero classes in the homology of the fundamental groups.

2 Definition and a fundamental lemma

Let $E \to M$ be a Hermitian vector bundle over a Riemannian manifold M, and let A be a section of $\bigwedge^* TM \otimes \operatorname{End}(E)$. Let us define

$$\|A\| := \sup_{\substack{\xi \in \bigwedge^*(TM) \\ \|\xi\|=1}} |A(\xi)|_{op} \tag{2.1}$$

where $|A(\xi)|_{op}$ denote the operator norm of $A(\xi) \in \operatorname{End}(E)$.

Let $K^{\times}(M)$ denote the isomorphism classes of Hermitian vector bundles equipped with compatible connections $E = (E, \nabla)$ over M, which satisfy the following conditions.

(i) (E, ∇) are isomorphic to the trivial bundles $\mathbb{C}^r$ equipped with flat connections outside compact subsets of M.

(ii) (E, ∇) have a non-zero Chern number. i.e. there exists a (multivariable) polynomial p such that

$$\int_M p(c_1(E), c_2(E), \cdots) \neq 0 \tag{2.2}$$

where $c_k(E) \in H_c^*(M)$ are the Chern classes of $E = (E, \nabla)$.

Definition 2.1 ([3]) *Let M be an even dimensional Riemannian manifold and let $R = R^E = R^{E,\nabla}$ denote the curvature tensor of (E, ∇). Then K-area of M is defined by*

$$K\text{-}area(M) := \sup_{(E,\nabla) \in K^{\times}(M)} \frac{1}{\|R^{E,\nabla}\|} \tag{2.3}$$

K-area$(M) = \infty$ if and only if for any $\varepsilon > 0$, there exists a vector bundle $(E, \nabla) \in K^{\times}(M)$ with a small curvature $\|R\| < \varepsilon$.

The following fundamental lemma is useful.

Lemma 2.2 *Let M and M' be Riemannian manifolds and let $f : M \to M'$ be a smooth Lipschitz map of non-zero degree which is proper or constant outside a compact subset in M. Then K-area$(M) \geq c^{-2}K\text{-}area(M')$ where c is the Lipschitz constant of f.*

Lemma 2.2 implies that finiteness or infiniteness of K-area(M) is independent of the deformation of Riemannian metrics on compact subsets in M. In particular, the finiteness or infiniteness of K-area is a homotopy invariant of compact manifolds. This is stated in [3] without proof. We give a proof for convenience.

Proof Set K-area$(M') = \frac{1}{a}$. If K-area$(M') = \infty$, take $a = 0$. For any $\varepsilon > 0$, there exists $E = (E, \nabla) \in K^{\times}(M')$ with $\|R^E\| < a + \varepsilon$. Let p be a polynomial satisfying $\int_{M'} p(c_1(E), c_2(E), \cdots) \neq 0$. Consider the vector bundle $f^*E \to M$ equipped with the induced connection $f^*\nabla$. Since f is proper or constant outside a compact subset, f^*E is isomorphic to a flat bundle $\mathbb{C}^r$ outside a compact subset. Moreover,

$$\int_M p(c_1(f^*E), c_2(f^*E), \cdots) = \deg(f) \int_{M'} p(c_1(E), c_2(E), \cdots) \neq 0 \tag{2.4}$$

Hence, $(f^*E, f^*\nabla) \in K^{\times}(M)$. On the other hand

$$R^{f^*E}(u \wedge v) = R^E(f_*(u \wedge v)) \tag{2.5}$$

$$\|R^{f^*E}\| \leq \|f_*(u \wedge v)\| \|R^E\| \leq c^2(a + \varepsilon) \tag{2.6}$$

$$K\text{-}area(M) \geq \frac{1}{\|R^{f^*E}\|} \geq \frac{1}{c^2(a + \varepsilon)} \tag{2.7}$$

Therefore K-area$(M) \geq c^{-2}$K-area(M') $\square$

Here, we give some examples of K-area.

Example 2.3 (1) Let S^{2m} denote even dimensional spheres K-area$(S^{2m}) < \infty$, which follows from Theorem 1.1.

(2) If M be an oriented even dimensional closed simply connected manifold, then K-area$(M) < \infty$. Later in Lemma 3.2, every vector bundle (E, ∇) over a closed simply connected manifold with sufficiently small curvature $\|R^{E,\nabla}\| < \delta$ is topologically trivial, which implies that all Chern numbers of E are zero. Hence K-area$(M) < \frac{1}{\delta}$. K-area$(S^n) < \infty$ can be verified also from this.

(3) K-area$(T^{2m}) = \infty$ where T^{2m} denote even dimensional tori. It follows from Theorem 1.1 that T^{2m} and hence T^{2m-1} do not admit Riemannian metrics of positive scalar curvature.

Proof of (3) Generally let $M = (M, g)$ be a Riemannian manifold equipped with a metric g. Observe that K-area$(M, c^2g) = c^2$K-area(M, g) by the preceding Lemma 2.2.

On the other hand let $\pi : \tilde{M} \to M$ be a finite covering space of M which is trivial outside a compact subset. Then K-area$(\tilde{M}) = $ K-area(M). In fact for $E = (E, \nabla) \in K^{\times}(\tilde{M})$, we can take $\pi_! E \in K^{\times}$ whose fiber is

$$\pi_! E_x = \bigoplus_{\tilde{x} \in \pi^{-1}(x)} E_{\tilde{x}} \tag{2.8}$$

So we can verify that $\|R^E\| \geq \|R^{\pi_! E}\|$ and hence K-area$(M) \geq $ K-area$(\tilde{M})$. Conversely, $\pi : \tilde{M} \to M$ satisfies the hypothesis of the preceding Lemma 2.2 with Lipschitz constant $c = 1$ so K-area$(\tilde{M}) \geq $ K-area(M). Therefore K-area$(\tilde{M}) = $ K-area(M).

Now consider an $2m$-dimensional tori equipped with flat metrics g_0 which are induced by $T^{2m} = \mathbb{R}^{2m}/\mathbb{Z}^{2m}$. There exist 2^{2m}-fold coverings $\pi : (T^{2m}, 4g_0) \to (T^{2m}, g_0)$. Hence K-area$(T^{2m}, g_0) = $ K-area$(T^{2m}, 4g_0) = 4$K-area(T^{2m}, g_0), which implies K-area$(T^{2m}, g_0) = \infty$.   □

3 Surgery

Let M_1 and M_2 be Riemannian manifolds and let $M_1 \sharp M_2$ denote the connected sum of M_1 and M_2 equipped with a Riemannian metric which coincides with the original metric of $M_1 \sqcup M_2$ outside a compact neighborhood of the connecting region.

Example 3.1 Let M be a 2m dimensional closed spin manifold. Then $T^{2m} \sharp M$ does not admit a Riemannian metric of positive scalar curvature. In fact K-area$(T^{2m}) = \infty$ implies K-area$(T^{2m} \sharp M) = \infty$ and apply Theorem 1.1.

Proof of Lemma 1.4 Suppose that K-area$(M_1) = \infty$. Write $M_1 \sharp M_2$ as $(M_1 \backslash D^n) \cup (M_2 \backslash D^n)$. There exits a smooth map $f : (M_1 \sharp M_2) \to M_1$ which satisfies the followings;

  $f(M_2 \backslash D^n) = \{x\}$ where x is the center of $D^n \subset M_1$.

  $f = $ id outside a neighborhood of $D^n \subset M_1$.

  deg $f = 1$.

Although f does not necessarily satisfy the assumption of Lemma 2.2, we can see that $(f^*E, f^*\nabla) \in K^{\times}(M_1 \sharp M_2)$ if $(f, \nabla) \in K^{\times}(M_1)$ just like as in the Proof of Lemma 2.2. Hence it follows that K-area$(M_1 \sharp M_2) \geq c^{-2}$K-area$(M_1) = \infty$ where c is the Lipschitz constant of f.   □

However, the converse of Lemma 1.4 is not trivial. The following two lemmata are used to verify Theorem 1.2.

Lemma 3.2 *Let N be a compact simply connected Riemannian manifold and take $E = (E, \nabla) \in K^\times(N)$. For any $\varepsilon > 0$, there exist $\delta > 0$ such that if $\|R^E\| < \delta$, there exists a global orthonormal frame $\bar{e} = \{e^i\}_{i=1}^r$ for E satisfying $\|\omega\| < \varepsilon$ where ω is the connection 1-form of (E, ∇) with respect to $\bar{e}$.*

Proof Fix a finite good open covering $\{V_\alpha\}$ of N equipped with geodesic coordinates whose centers are p_α. So each finite intersection of $\{V_\alpha\}$ is contractible unless it is empty. Let E be a Hermitian vector bundle with $\|R^E\| < \delta$ for some $\delta > 0$. Fix an orthonormal basis $e_{\alpha_0} = \{e_{\alpha_0}^i\}_{i=1}^r$ for $E|_{p_{\alpha_0}}$. Let $e_\alpha = \{e_\alpha^i\}_{i=1}^r$ be an orthonormal basis for $E|_{p_\alpha}$ obtained by the parallel transportation of e_{α_0} along γ_0^α, one of the minimal geodesics connecting p_{α_0} and p_α. Extend e_α on each V_α by the parallel transportation along the geodesic $t \mapsto \exp_{p_\alpha}(tv)$ where v is an unit tangent vector at p_α.

Let ω_α be the connection 1-form with respect to e_α on V_α.

For $x \in V_\alpha$ let γ_α^x be the (unique) geodesic connecting p_α and x and for a piece-wise smooth curve γ let T_γ be the parallel transportation along γ. Take $x \in U_\alpha$ and $X \in T_x M$. By the definition of e_α, $e_\alpha(\exp_x(tX)) = T_{\gamma_\alpha^{\exp_x(tX)}} T_{\gamma_\alpha^x}^{-1} e_\alpha(x)$. Then,

$$\nabla_X e_\alpha(x) = \lim_{t \to 0} \frac{1}{t} \left(T_{\exp_x(tX)}^{-1} e_\alpha(\exp_x(tX)) - e_\alpha(x) \right)$$

$$= \lim_{t \to 0} \frac{1}{t} \left(T_{\exp_x(tX)}^{-1} T_{\gamma_\alpha^{\exp_x(tX)}} T_{\gamma_\alpha^x}^{-1} - \mathrm{id} \right) e_\alpha(x) \tag{3.1}$$

$$\|\nabla_X e_\alpha(x)\| \leq \lim_{t \to 0} \frac{1}{t} \int_{D_t} \|R\| \tag{3.2}$$

where D_t is a 2-dimensional disk whose boundary is the closed curve $\exp_x(tX)^{-1} \gamma_\alpha^{\exp_x(tX)} (\gamma_\alpha^x)^{-1}$. Since $\mathrm{area}(D_t) = O(t) \quad (t \to 0)$, we have

$$\|\nabla e_\alpha\| \leq c_1 \delta \quad \text{i.e.} \quad \|\omega_\alpha\| \leq c_1 \delta \tag{3.3}$$

where c_1 is a constant depending on $\{V_u a\}$. Keep in mind that constants $c_1, c_2, \cdots, c_6$ which will appear below are independent of the vector bundle E.

Let $\psi_{\beta\alpha} : V_\alpha \cap V_\beta \to U(r)$ denote the transition functions, i.e., $\psi_{\beta\alpha} e_\alpha = e_\beta$. By the definition of e_α, $\psi_{\beta\alpha}(x) = T_\gamma$ where $\gamma = \gamma_\beta^x \gamma_0^\beta \left(\gamma_\alpha^x \gamma_0^\alpha \right)^{-1}$. Since N is simply connected, There exists a 2-dimensional disk $D \subset N$ whose boundary is γ. By the compactness, we can take D so that $\mathrm{area}(D) < c_2$ where c_2 is a constant depending on N and $\{V_\alpha\}$. Then we have

$$\|\psi_{\beta\alpha} - \mathrm{id}\| \leq \int_D \|R\| < c_2 \delta \tag{3.4}$$

By $\psi_{\beta\alpha} e_\alpha = e_\beta$, we have $\mathrm{d}\psi_{\beta\alpha} \otimes e_\alpha + \psi_{\beta\alpha} \nabla e_\alpha = \nabla e_\beta$. Hence by (3.3)

$$\|\mathrm{d}\psi_{\beta\alpha}\| < 2c_1 \delta \tag{3.5}$$

Taking into account the estimate (3.4) we can set $\psi_{\beta\alpha} = \exp(v_{\beta\alpha})$ for some $v_{\beta\alpha} : V_\alpha \cap V_\beta \to \mathfrak{u}(r)$ using $\exp : \mathfrak{u}(r) \to U(r)$ if $\delta > 0$ is sufficiently small since $\exp$ is a diffeomorphism from a neighbourhood of $0 \in \mathfrak{u}(r)$ to a neighbourhood of $\mathrm{id} \in U(r)$. Remark that (3.4) and (3.5) implies

$$\|v_{\beta\alpha}\| < c_2' \delta \quad \text{and} \quad \|\mathrm{d}v_{\beta\alpha}\| < 2c_1 \delta \tag{3.6}$$

There exist open subsets W_α and compact subsets K_α such that $W_\alpha \subset K_\alpha \subset V_\alpha$ and $\bigcup_\alpha W_\alpha = N$. Note that these are independent of (E, ∇). Let $\{\rho_\alpha, \rho_\beta, 1 - \rho_\alpha - \rho_\beta\}$ be

a partition of unity associated to $\{V_\alpha, V_\beta, N \setminus (K_\alpha \cup K_\beta)\}$. $(\rho_\alpha + \rho_\beta \equiv 1 \text{ on } K_\alpha \cup K_\beta.)$ Construct an orthonormal frame $e_{(2)}$ on $K_\alpha \cup K_\beta$ as follows;

$$e_{(2)} = \begin{cases} (\exp(\rho_\beta v_{\beta\alpha}))e_\alpha, & \text{on } K_\alpha \\ (\exp(\rho_\alpha v_{\alpha\beta}))e_\beta, & \text{on } K_\beta \end{cases} \tag{3.7}$$

$e_{(2)}$ is well defined. In fact on $K_\alpha \cap K_\beta$,

$$\exp(\rho_\alpha v_{\alpha\beta})e_\beta = \exp(\rho_\alpha(-v_{\beta\alpha})) \exp(v_{\beta\alpha})e_\alpha$$
$$= \exp((1 - \rho_\beta)(-v_{\beta\alpha}) + v_{\beta\alpha})e_\alpha = \exp(\rho_\beta v_{\beta\alpha})e_\alpha \tag{3.8}$$

There is a constant $c_3 > 0$ such that $|d\rho_\alpha| < c_3$, $|d\rho_\beta| < c_3$. Hence by (3.3), (3.6), and (3.7),

$$\|\nabla e_{(2)}\| = \|d(\exp(\rho_\beta v_{\beta\alpha})) \otimes e_\alpha + \exp(\rho_\beta v_{\beta\alpha})\nabla e_\alpha\|$$
$$\leq |d\rho_\beta| \|v_{\beta\alpha}\| + \rho_\beta \|dv_{\beta\alpha}\| + \|\nabla e_\alpha\| \qquad < c_4\delta \tag{3.9}$$

This means the connection 1-form $\omega_{(2)}$ associated to $e_{(2)}$ satisfies $\|\omega_{(2)}\| < c_4\delta$.

Next, choose another open subset V_γ, set $V_{(2)} := V_\alpha \cup V_\beta$, $K_{(2)} := K_\alpha \cup K_\beta$ and let $\psi_{\gamma(2)} \colon K_\gamma \cap K_{(2)} \to U(r)$ denote the transition function i.e., $e_\gamma = \psi_{\gamma(2)}e_{(2)}$.

Remark that $\text{id} = \psi_{\gamma(2)} \exp(\rho_\beta v_{\beta\alpha})\psi_{\alpha\gamma}$ implies $\|\psi_{\gamma(2)} - \text{id}\| < c_5\delta$ and $\|d\psi_{\gamma(2)}\| < c_5\delta$. Therefore, we can write $\psi_{\gamma(2)} = \exp(v_{\gamma(2)})$ for some $v_{\gamma(2)}$ satisfying $\|v_{\gamma(2)}\| < c_5'\delta$ and $\|dv_{\gamma(2)}\| < c_5'\delta$.

We can employ the similar argument to construct an orthonormal frame $e_{(3)}$ on $K_\gamma \cup K_{(2)}$ satisfying $\|\nabla e_{(3)}\| < c_6\delta$. Namely, let $\{\rho_\gamma, \rho_{(2)}, 1 - \rho_\gamma - \rho_{(2)}\}$ be a partition of unity associated to $\{V_\gamma, V_{(2)}, N \setminus (K_\gamma \cup K_{(2)})\}$, and define

$$e_{(3)} = \begin{cases} (\exp(\rho_{(2)}v_{(2)\gamma}))e_\gamma, & \text{on } K_\gamma \\ (\exp(\rho_\gamma v_{\gamma(2)}))e_{(2)}, & \text{on } K_{(2)} \end{cases} \tag{3.10}$$

It satisfies $\|\nabla e_{(3)}\| < c_6\delta$.

Repeat the above argument to construct a global orthonormal frame $\bar{e}$ for E which satisfies $\|\nabla\bar{e}\| < c\delta$. It means $\|\omega\| < c\delta$ where ω is the connection 1-form with respect to $\bar{e}$. Though c depends on N, it does not depend on (E, ∇). $\qquad \square$

Remark 3.3 The proof of Lemma 3.2 also holds if N is not connected but each connected component is simply connected by applying the arguments on each connected component.

Lemma 3.4 *Let M be a Riemannian manifold with a simply connected boundary $N = \partial M$, and let $E_0 = (E_0, \nabla_0)$ be a Hermitian vector bundle over M equipped with a compatible connection. Suppose that a neighborhood of ∂M is equipped with a product metric of $(-2, 2] \times N$ and that the connection ∇_0 restricted to $(-2, 2] \times N$ is invariant under the translation.*

Let $M_{(-2,a]}$ denote $(M \setminus (-2, 2] \times N) \cup ((-2, a] \times N)$. For instance the original M can be denoted by $M_{(-2,2]}$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|R^{E_0}\| < \delta$, there exists a vector bundle (E, ∇) over $M_{(-2,6]}$ satisfying the following;

(i) $\|R^E\| < \varepsilon$.
(ii) The restriction of (E, ∇) to $M_{(-2,2]}$ is isomorphic to (E_0, ∇_0).
(iii) (E, ∇) is trivial and flat on $(4, 6] \times N$.

Proof Choose $\varepsilon_0 > 0$ sufficiently small. For $\{0\} \times N$ and ε_0, we can find $\delta = \delta(\varepsilon_0) > 0$ as in the preceding Lemma 3.2. Suppose that $\|R^{E_0}\| < \delta$. Then we obtain a global orthonormal

frame e for $E_0|_{\{0\} \times N}$ such that the connection 1-form ω_0 with respect to e satisfies $\|\omega_0\| < \varepsilon_0$. Let E be a trivial Hermitian vector bundle without a connection on $M_{(-2,6]}$ which is an extension of E_0. Extending e we obtain a orthonormal frame for E denoted also by e. Now compose a connection 1-form ω with respect to e on on $(-2,6] \times N$ by

$$\omega|_{(t,y)} = \chi(t)\omega_0|_y \tag{3.11}$$

where χ is a smooth function on $(-2,6]$ satisfying

$$\chi(t) \begin{cases} \equiv 1, & t < 2 \\ \equiv 0, & t > 4 \end{cases} \tag{3.12}$$

$$0 \le \frac{d\chi}{dt} \le 1 \tag{3.13}$$

Since $\omega|_{(t,y)} = \omega_0|_y$ on $(-2,2) \times N$, the new connection denoted by ∇ can be patched with ∇_0.

$$
\begin{aligned}
\|R^{E,\nabla}\| &= \|\omega \wedge \omega + d\omega\| \\
&= \|\chi(t)^2 \omega_0 \wedge \omega_0 + d\chi(t) \wedge \omega_0 + \chi(t) \wedge d\omega_0\| \\
&\le |\chi(t)| \|\omega_0 \wedge \omega_0 + d\omega_0\| + |\chi(t)^2 - \chi(t)| \|\omega_0 \wedge \omega_0\| + \|d\chi \wedge \omega_0\| \\
&\le \|R^{E_0,\nabla_0}\| + \|\omega_0\|^2 + \|\omega_0\| \\
&\le \delta + \varepsilon_0^2 + \varepsilon_0
\end{aligned} \tag{3.14}
$$

Hence taking ε_0 depending on ε and δ depending on ε_0 sufficiently small, we obtain $\|R^{E,\nabla}\| < \varepsilon$. Moreover $\omega|_{(t,y)} = 0$ for $t > 4$ means that (E, ∇) is flat on $(4,6] \times N$. $\square$

Definition 3.5 *Let M be a Riemannian manifold, and $n = p+q = \dim(M)$. Fix an inclusion $\varphi \colon S^p \times D^q \hookrightarrow M$. Define another (smooth) manifold $M^\sharp$ as follows;*

$$M^\sharp := (M \setminus \varphi(S^p \times D^q)) \cup_{\partial(\varphi(S^p \times D^q))} (D^{p+1} \times S^{q-1}) \tag{3.15}$$

Remark that $\partial(S^p \times D^q) \cong S^p \times S^{q-1} \cong \partial(D^{p+1} \times S^{q-1})$. $M^\sharp$ is called a manifold obtained by p-surgery, or surgery in codimension q, along $\varphi \colon S^p \times D^q \hookrightarrow M$.

We assume that $M^\sharp$ is equipped with a Riemannian metric which coincides with the original one outside a compact neighborhood of $(D^{p+1} \times S^{q-1}) \subset M^\sharp$.

Proof of Theorem 1.2 Since the finiteness of K-area is invariant under deformations of Riemannian metrics on compact subsets, we may assume that the "connecting region", the neighborhood of $\partial(D^{p+1} \times S^{q-1}) \subset M^\sharp$ is isometric to $S^p \times (-4,4) \times S^{q-1}$ equipped with a canonical Riemannan metric.

Let $E_0 = (E_0, \nabla_0)$ be a Hermitian vector bundle equipped with a compatible connection. It is sufficient to verify that for sufficiently small $\delta > 0$, $\|R^{E_0}\| < \delta$ implies that all Chern numbers of E_0 are zero.

Let $f \colon M^\sharp \to M^\sharp$ be a smooth Lipschitz map such that $f = \text{id}$ outside $S^p \times (-4,4) \times S^{q-1}$, $f(x,t,y) = (x,0,y)$ for $|t| < 2$, and $\|f_*\| < 2$. Consider f^*E_0, the pull-back of E_0 by f equipped with the induced connection $f^*\nabla_0$. Then $\|R^{f^*E_0}\| \le 2\delta$ and the connection is invariant under the translation near the cylindrical boundary. Since $\deg(f) = 1$ the Chern numbers of f^*E_0 are equal to those of E_0.

Cut $M^\sharp$ along $S^p \times \{0\} \times S^{q-1}$ and remove $D^{p+1} \times S^{q-1}$ component. Let the resulting manifold be denoted by M'.

In the case of $p \neq 1$, we can apply to $f^*E_0|_{M'}$ the preceding Lemma 3.4 to obtain a vector bundle $E = (E, \nabla)$ over $M'_{(-2,6]}$ with $\|R^E\| < \varepsilon$ which is trivial and flat on $S^p \times (4, 6] \times S^{q-1}$. Remark that $\partial M' = S^p \times \{0\} \times S^{q-1}$ is simply connected by the condition $q \neq 2$.

Even in the case of $p = 1$, we claim that there exist a such extension of the vector bundle. In fact, consider the two copies of removed region $D^2 \times S^{n-2}$ and the vector bundle $f^*E_0 \to (D^2 \times S^{n-2})$ and reverse the orientation of one of them. We can patch them together along the boundary for the invariance of the connection of f^*E_0 under the translation near the cylindrical boundary. Since the resulting manifold, the double of $D^2 \times S^{n-2}$, is homeomorphic to $S^2 \times S^{n-2}$, by Lemma 3.2 there exists a global orthonormal frame e for the resulting vector bundle over $S^2 \times S^{n-2}$ such that the connection 1-form ω_0 with respect to e satisfies $\|\omega_0\| < \varepsilon_0$. Hence there exists a such orthonormal frame for the restriction of f^*E_0 onto a neighborhood of $\partial M'$. Then we can construct a vector bundle $E = (E, \nabla)$ over $M'_{(-2,6]}$ with $\|R^E\| < \varepsilon$ which is trivial and flat on $S^p \times (4, 6] \times S^{q-1}$ in the same way as the Proof of Lemma 3.4.

In the following argument the condition $q \neq 2$ is not needed. Deform the metric of $S^p \times D^q$ to have a product metric near the boundary $S^p \times (-1, 1) \times S^{q-1}$ so that it can be patched with $M'_{(-2,6]}$. The resulting manifold is homeomorphic to M. Since (E, ∇) is trivial and flat on $S^p \times (4, 6] \times S^{q-1} \subset (M'_{(-2,6]} \cup S^p \times D^q)$, it can be extended on $S^p \times D^q$ trivially.

Let X be $M^\sharp \setminus M'$ and let Y be $M \setminus M'$. They are homeomorphic to $D^{p+1} \times S^{q-1}$ and $S^p \times D^q$ respectively. Glue X and $(-Y)$ together to compose a Riemannian manifold homeomorphic to S^n where $(-Y)$ is the orientation reversed Y. Remark that $(f^*E_0, f^*\nabla_0)$ on X and (E, ∇) on $(-Y)$ can be joined smoothly. Hence they define a Hermitian vector bundle equipped with a compatible connection (E, ∇) with a small curvature $\|R\| < \varepsilon$ on $X \cup (-Y)$.

Since K-area$(M) = $ K-area$(M' \cup Y) < \infty$ and K-area$(X \cup (-Y)) < \infty$, there exist $\varepsilon > 0$ such that for any polynomial p,

$$\int_{M' \cup Y} p(c_1(E), c_2(E), \cdots) = 0$$

$$\int_{X \cup (-Y)} p(c_1(E), c_2(E), \cdots) = 0 \qquad (3.16)$$

Therefore,

$$\int_{M^\sharp} p(c_1(E_0), c_2(E_0), \cdots)$$

$$= \int_{M' \cup X} p(c_1(f^*E_0), c_2(f^*E_0), \cdots)$$

$$= \int_{M' \cup Y} p(c_1(E), c_2(E), \cdots) + \int_{X \cup (-Y)} p(c_1(E), c_2(E), \cdots) = 0 \qquad (3.17)$$

which implies K-area$(M^\sharp) < \frac{1}{\delta} < \infty$. $\qquad \square$

Remark 3.6 Notice that surgery is an invertible operator. Let $M^\sharp$ be a Riemannian manifold obtained from M by p-surgery. Then M is obtained by performing $(q-1)$-surgery to $M^\sharp$.

$M = (M^\sharp \setminus D^{p+1} \times S^{q-1}) \cup (S^p \times D^q)$. So if both $p \neq 1$ and $q - 1 \neq 1$ are satisfied, then
K-area$(M^\sharp) = \infty$ iff K-area$(M) = \infty$.

Proof of Corollary 1.3 By Lemma 1.4, K-area$(M_1 \sharp M_2) < \infty$ implies K-area$(M_1) < \infty$ and
K-area$(M_2) < \infty$. Suppose that both K-area(M_1) and K-area(M_2) are finite. Remark that
the K-area of the disjoint union K-area$(M_1 \sqcup M_2)$ is equal to max$\{$K-area(M_1), K-area$(M_2)\}$.
Then we can apply the case of $p = 0$ of the preceding Theorem 1.2 to conclude
K-area$(M_1 \sharp M_2) < \infty$. $\square$

Notice that we did not assume that M is compact in Theorem 1.2 and so we can "localize"
K-area in the following sense.

Example 3.7 Let M_∞ is an oriented even dimensional Riemannian manifold with a cylin-
drical end $(0, \infty) \times S^{n-1}$ and suppose that $M_0 := M_\infty \setminus ((0, \infty) \times S^{n-1})$ is compact. Let
M be a compact manifold obtained by sewing a disk D^n on M_0. K-area$(M_\infty) = \infty$ if and
only if K-area$(M) = \infty$.

Proof This is a direct consequence of Corollary 1.3. In fact M_∞ can be written as $M \sharp M'$
where M' is a complete Riemannian manifold homeomorphic to $\mathbb{R}^n$ whose metric is a smooth-
ing of the n dimensional hemispherical metric attached along the canonical cylindrical metric
on $[0, \infty) \times S^{n-1}$ of the same radius. Since inf Sc$_{M'} > 0$ and M' is spin, K-area(M') is
finite. Therefore K-area$(M_\infty) = \infty$ if and only if K-area$(M) = \infty$. $\square$

Example 3.7 suggests that the cylindrical region $(0, \infty) \times S^{n-1}$ have no effect on finiteness
or infiniteness of K-area.

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