# The Space of Measurement Outcomes as a Spectral Invariant for Non-Commutative Algebras

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**Abstract** The recently developed technique of Bohrification associates to a (unital) C*-algebra $A$

1. the Kripke model, a presheaf topos, of its classical contexts;
2. in this Kripke model a commutative C*-algebra, called the Bohrification of $A$;
3. the spectrum of the Bohrification as a locale internal in the Kripke model.

We propose this locale, the 'state space', as a (*n* intuitionistic) logic of the physical system whose observable algebra is $A$.

We compute a site which externally captures this locale and find that externally its points may be identified with *partial* measurement outcomes. This prompts us to compare Scott-continuity on the poset of contexts and continuity with respect to the C*-algebra as two ways to mathematically identify measurement outcomes with the same physical interpretation. Finally, we consider the not-not-sheafification of the Kripke model on classical contexts and obtain a space of measurement outcomes which for commutative C*-algebras coincides with the spectrum. The construction is functorial on the category of C*-algebras with commutativity reflecting maps.

**Keywords** Bohrification · Sheaves · Measurement · Boolean valued models

## 1 Introduction

By combining Bohr's philosophy of quantum mechanics, algebraic quantum theory [9, 11, 20], constructive Gelfand duality [1, 2, 5, 7], and inspiration from Butterfield, Isham and Hamilton's spectral presheaf [12], we proposed Bohrification as

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an intuitionistic, and spatial, logic for quantum theory [13, 14]. Given a unital C\*-algebra $A$ modeling a quantum system, consider the poset motivated by Bohr's classical concepts

$$\mathcal{C}(A) := \{C \mid C \text{ is a commutative unital C*-subalgebra of } A\}$$

partially ordered by inclusion. In the functor topos $\mathbf{Sets}^{\mathcal{C}(A)}$ we consider the *Bohrification* $\underline{A}$: the trivial functor $C \mapsto C$. This is an internal[1] C\*-algebra, of which we can compute the spectrum as an internal locale $\underline{\Sigma}$ in the topos $\mathbf{Sets}^{\mathcal{C}(A)}$. This locale is our proposal for an intuitionistic quantum logic associated to $A$ [13, 14]. When $A$ is a von Neumann algebra, in particular if it is a matrix algebra, this locale associates a Heyting algebra to the orthomodular lattice of projections. Heyting algebras satisfy the distributive law: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ which allows us to interpret these operations as disjunction and conjunction. The assignment $p \mapsto (C(p), p)$, to be defined below in Theorem 7, injects traditional orthomodular quantum logic of projections into our intuitionistic quantum logic. A similar embedding can be found in [8, Sect. 5.5] and [14, Sect. 6.6.]. See [26, Sect. 3.3] for a comparison.

Bohrification fits seamlessly in the methodology of Kripke-Joyal semantics. More precisely, it fits the motivation for Kripke models for intuitionistic logic [19], the posetal case of the general Kripke-Joyal semantics for Grothendieck toposes. The order $D \leq C$ on $\mathcal{C}(A)^{\mathrm{op}}$ is $D \supseteq C$: in the context $D$ we can measure more observables than in the context $C$. The domain of discourse increases when the Bohrian context increases while preserving the information we have about the domain. Hence the corresponding Kripke model is $\mathbf{Sets}^{\mathcal{C}(A)^{\mathrm{op\ op}}} = \mathbf{Sets}^{\mathcal{C}(A)}$. To this Kripke model we apply standard topos theoretic methods, like internal reasoning. In Sect. 3, we see how this methodology extends to iterated forcing, or iterating a sheaf construction. In this light, the Döring and Isham model may be seen as the co-Kripke model, the topos $\mathbf{Sets}^{\mathcal{C}(A)^{\mathrm{op}}}$. However, this not the viewpoint they use. See Wolters [26] for a comparison between the two approaches.

In Sect. 3 we give an external description of $\underline{\Sigma}$. It can be described as the space of partial measurement outcomes: its points are pairs consisting of a commutative C\*-subalgebra together with a point of its spectrum. This construction raises two natural questions:

- Can we restrict to the maximal commutative subalgebras, i.e. to the total measurement settings?
- Are we allowed to use classical logic internally?

In Sect. 4 we will see that, in a suitable sense, the answers to both of these questions are positive. The collection of maximal commutative subalgebras covers the space in the dense topology and this dense, or double negation, topology forces the logic to be classical. By considering the $\neg\neg$-sheafification, our locale coincides with the spectrum in the commutative case. Moreover, our previous constructions [13] of the phase space $\underline{\Sigma}$ and the state space still apply essentially unchanged.

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[1]We often underline objects internal to the topos under consideration.

## 2 Preliminaries

An extensive introduction to the context of the present paper can be found in [13, 14] and the references therein. We assume basic familiarity with that material and we will just repeat the bare minimum of definitions.

**Definition 1** A non-empty subset of a partial order is *directed* if every pair of its elements has an upper bound in the set.

An *ideal* in a join-semilattice $L$ is a strict subset $I$ such that[2]

- $\perp \in I$;
- $I$ is a *lower set*: if $a \in I$ and $b \leq a$, then $b \leq I$;
- $I$ is *directed*: if $a, b \in I$, then $a \vee b \in I$.

A *filter* in a meet-semilattice $L$ is a strict subset $F$ such that

- $\top \in F$;
- $F$ is an *upper set*: if $a \in F$ and $a \leq b$, then $b \leq F$;
- $F$ is *directed*: if $a, b \in F$, then $a \wedge b \in F$.

We will now introduce the notion of a locale, a point-free analogue of a topological space. In the constructive logic internal to a topos, locales have better properties than topological spaces. Locales can be conveniently presented by a base, a so-called site over a meet-semilattice [15, 2.11].

**Definition 2** Let $L$ be a meet-semilattice. A *covering relation* on $L$ is a relation $\lhd \subseteq L \times P(L)$ satisfying:

1. if $x \in U$ then $x \lhd U$;
2. if $x \lhd U$ and $U \lhd V$ (*i.e.* $y \lhd V$ for all $y \in U$) then $x \lhd V$;
3. if $x \lhd U$ then $x \wedge y \lhd U$;
4. if $x \lhd U$ and $x \lhd V$, then $x \lhd U \wedge V$, where $U \wedge V = \{x \wedge y \mid x \in U, y \in V\}$.

Such a pair $(L, \lhd)$ is called a *posite*, site on a poset, or a *formal topology*. We write $U \lhd V$ if for all $u \in U$, $u \lhd V$.

When $L$ is only a poset, we can obtain a lattice by passing from the poset to its ideal completion. The *ideal completion*, $\mathrm{Idl}(P)$, of a poset $P$ is the poset of its ideals; see [15, 25].

A *frame* is a completely lattice satisfying $a \wedge \bigvee S = \bigvee_{s \in S} a \wedge s$. Frame maps preserve $\wedge$ and $\bigvee$. The category of locales is the opposite of the category of frames. Every formal topology generates a locale with frame the ideals of $L$ modulo the relation $U \sim V$ iff $U \lhd V$ and $V \lhd U$. A formal topology may be seen as the description of a locale in terms of generators and relations. Conversely every locale can be presented in such a way; see [15, 2.11]. Its covering relation is then defined by $a \lhd U$ iff $a \leq \bigvee U$. Consequently, we will not always distinguish between the posite/formal topology and the locale it generates.

Thinking of a locale as a lattice of opens of a topological space, we can try to reconstruct the points of the space from the locale.

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[2]We write $\perp$, $\top$ for the bottom and top element of the lattice $L$.

**Definition 3** Let $(L, \lhd)$ be a formal topology. A *point* is a filter $\alpha \subseteq L$ such that for each $a \in \alpha$:

$$\text{if } a \lhd U, \text{ then } U \cap \alpha \text{ is inhabited.} \tag{1}$$

In short, $\alpha$ is a *completely prime filter*.

Equivalently, a point of a locale $X$ is a locale map $1 \to X$; where 1 is the one-point locale. When a locale is homeomorphic to the locale of opens of its topological space of points, the locale is called *spatial*. Locally compact locales are spatial [15, VII.4]. This fact uses the axiom of choice, and hence does not hold in an arbitrary topos. The functor sending a locale to its topological space of points is part of an adjunction between topological spaces and locales.

*Remark 4* The spectrum $\Sigma$ of a C*-algebra $A$ can be described directly as a lattice $L(A)$ together with a covering relation $\lhd_A$; see [7]. The distributive lattice $L$ is freely generated by the formal symbols $D_a$, $a \in A_{\text{sa}}$ subject to the relations

$$D_1 = \top,$$
$$D_a \wedge D_{-a} = \bot,$$
$$D_{-b^2} = \bot,$$
$$D_{a+b} \leq D_a \vee D_b,$$
$$D_{ab} = (D_a \wedge D_b) \vee (D_{-a} \wedge D_{-b}).$$

The spectrum has the same generators and relations, but moreover satisfies $D(a) = \bigvee_{s>0} D(a - s)$.

This construction is valid in any topos. If $\underline{A}$ denotes an internal C*-algebra, then we write $\underline{L}$ for this internal lattice and $\underline{\Sigma}$ for its internal spectrum.

## 2.1 Measurements

In algebraic quantum theory [9, 11, 20], a measurement context (the set-up of an apparatus) is modeled by a (maximal) Boolean subalgebra of the set of projections of a von Neumann algebra; see [17, 18]. The outcome of a measurement corresponds to the consistent assignment of either 0 or 1 to each element (test, proposition) of the Boolean algebra: i.e. the outcome is an element of its Stone spectrum.[3] Unlike von Neumann-algebras, C*-algebras need not have enough projections. However, one may replace the Boolean algebra by a commutative C*-subalgebra and the Stone spectrum by the Gelfand spectrum. With the previous motivation in mind we make the following definition.

**Definition 5** A *measurement outcome* on a C*-algebra $A$ is a point in the spectrum of a maximal commutative subalgebra of $A$.

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[3] Stone duality, between the categories of Boolean algebras and Stone spaces, assigns to a Boolean algebra its *Stone spectrum*; see [15].

In the special case $A = M_n(\mathbb{C})$, the $n$-dimensional matrix algebra, a maximal commutative subalgebra, a Bohrian context, corresponds to maximal set of simultaneously measurable observables, i.e. commuting self-adjoint matrices, and hence by the spectral theorem to a choice of a basis. A point of the spectrum is a particular base vector $e$, an eigenvector of all the observables. The outcome of measuring the observable $a$ is then $\langle e, ae \rangle$.

## 3 The Space of Partial Measurement Outcomes

The theory of sites and locales may be developed constructively, and hence can be interpreted in any topos. There is an equivalence between locales $\underline{Y}$ internal to sheaves over a locale $X$ and locale maps $Y \to X$ in **Sets**. To compute the *external* locale $Y$ corresponding to an *internal* locale $\underline{Y}$ we use iterated topos constructions as studied by Moerdijk [22, 16, C.2.5]. To wit, let $\mathcal{S}$ be the ambient topos. One may think of the topos **Sets**, but we envision applications where a different choice for $\mathcal{S}$ is appropriate. For example, as suggested in [13], algebraic quantum field theory defines a single C*-algebra in the topos of presheaves of opens of Minkowski space-time. Recently, Nuiten [23] obtained interesting results in this direction.

Let $\mathbb{C}$ be a site in a topos $\mathcal{S}$. Then we write $\mathcal{S}[\mathbb{C}]$, or $\mathrm{Sh}_{\mathcal{S}}[\mathbb{C}]$, for the topos of sheaves over $\mathbb{C}$.

**Theorem 6** (Moerdijk) *Let $\mathbb{C}$ be a site in $\mathcal{S}$ and $\mathbb{D}$ a site in $\mathcal{S}[\mathbb{C}]$. Then there is a site[4] $\mathbb{C} \ltimes \mathbb{D}$ in $\mathcal{S}$ such that*

$$(\mathcal{S}[\mathbb{C}])[\mathbb{D}] \cong \mathcal{S}[\mathbb{C} \ltimes \mathbb{D}].$$

Instead of repeating the construction in full generality, we will specialize to sites on a poset, as in Definition 2, and without further ado focus on our main example. We fix a unital C*-algebra $A$ for the rest of the paper. By [13, Cor. 28], the lattice $\underline{L}$ and the covering relation are computed 'pointwise', i.e. $C \Vdash u \lhd V$ iff $u \lhd_C V(C)$. The forcing relation $C \Vdash \phi$ denotes that the formula $\phi$ in the internal language holds at $C$ in the Kripke model $\mathbf{Sets}^{\mathcal{C}(A)^{\mathrm{op}\,\mathrm{op}}}$; see [21, p. 318].

**Theorem 7** *Consider the poset of pairs $(C, u)$, where $C$ in $\mathcal{C}(A)$ and $u$ in $\underline{L}(C)$ with the order $(D, v) \le (C, u)$ as $D \supseteq C$ and $v \le u$ in $\underline{L}(C)$. Equipped with the covering relation $(C, u) \lhd (D_i, v_i)$ iff $u \lhd_C \{v_i \mid D_i = C\}$, this posite presents the external locale which corresponds to $\underline{\Sigma}$.*

*Proof* To apply Theorem 6, let $\mathbb{C} := \mathcal{C}(A)^{\mathrm{op}}$ and let $\mathbb{D} = \underline{\Sigma}$ be the spectrum of the Bohrification, $\underline{A}$, of $A$. We claim that the posite $\mathbb{C} \ltimes \mathbb{D} = \mathcal{C}(A)^{\mathrm{op}} \ltimes \underline{\Sigma}$ is precisely the one stated in the present theorem. Specializing the covering relation from Theorem 6

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[4]The notation $\ltimes$ is motivated by the special case where $\mathbb{C}$ is a group $G$ considered to be a category with one object and $\mathbb{D}$ is a group $H$ in $\mathbf{Sets}^G$. Then $\mathbb{C} \ltimes \mathbb{D}$ is indeed the semi-direct product $H \rtimes G$. Where $H$ is considered as a group in $\mathbf{Sets}$ with an action of the group $G$.

to the present case, we obtain: $(C, u) \lhd (D_i, v_i)$ iff for all $i$, $C \subseteq D_i$ and $C \Vdash u \lhd V$, where $V$ is the pre-sheaf generated by[5] the conditions $D_i \Vdash v_i \in V$. By [13, Cor. 28] $C \Vdash u \lhd V$ iff $u \lhd_C V(C)$. Finally, the pre-sheaf $V$ generated by the conditions $D_i \Vdash v_i \in V$ is $V(D) := \{v_i \mid D \supseteq D_i\}$, as this is a pre-sheaf and clearly the smallest one satisfying the conditions.                                                                        □

When $A$ is a von Neumann algebra, in particular if it is a matrix algebra, the assignment $p \mapsto (C(p), p)$ injects traditional orthomodular quantum logic of projections into our intuitionistic quantum logic. Here $C(p)$ is the C*-algebra generated by $p$ and 1.

The posite in Theorem 7 naturally fits with the discussion in Sect. 2.1 as we will show in Theorem 9.

**Definition 8** A *partial measurement outcome* for $A$ is a point in the spectrum of a unital commutative subalgebra of $A$.

A *consistent ideal of partial measurement outcomes* is a family $(C_i, \sigma_i)$ of partial measurement outcomes such that the set of $C_i$s is an ideal in $\mathcal{C}(A)^{\mathrm{op}}$ and if $C_i \subseteq C_j$, then $\sigma_i = \sigma_j|_{C_i}$.

**Theorem 9** *The points of $\mathcal{C}(A)^{\mathrm{op}} \ltimes \underline{\Sigma}$ are consistent ideals of partial measurement outcomes.*

*Proof* Let $\tau$ be a point of $\mathcal{C}(A)^{\mathrm{op}} \ltimes \underline{\Sigma}$—that is, a completely prime filter; see Definition 3. We claim that

$$\{C \mid \exists u, (C, u) \in \tau\}$$

is an ideal $I_\tau$ in $\mathcal{C}(A)$. When both $(C, u)$ and $(C', u')$ are in $\tau$, then, by directedness of the filter $\tau$, there exists $(D, v)$ in $\tau$ such that $C, C' \subseteq D$ and $v \subseteq u, u'$. Moreover, $(C, u) \in \tau$ implies $(C, \top) \in \tau$, because $\tau$ is upward closed. Hence the set $\{C \mid (C, \top) \in \tau\}$ is directed and down-closed, i.e. an ideal.

We claim that for each $D \in I_\tau$, the set $\{u \mid (D, u) \in \tau\}$ defines a point of $\underline{\Sigma}(D)$. Suppose that $(D, u) \in \tau$ and $u \lhd V$ in $\underline{\Sigma}(D)$. Then $(D, u) \lhd \{(D, v) \mid v \in V\}$, so $(D, v) \in \tau$ for some $v$.

The point in the previous paragraph is defined consistently: If $C \subset D$ and $u \in L(C) \subseteq L(D)$, then $(D, u) \leq (C, u)$. Hence if $(D, u) \in \tau$, so is $(C, u)$ and the point in $\underline{\Sigma}(D)$ defines a point in $\underline{\Sigma}(C)$. The basic opens $u$ in $\Sigma(C)$ contain all the points $\{\sigma \mid \sigma(a) > 0\}$, for some $a$ in $C$, and hence together determine the value of the functional on $C$. Since, these constraints are preserved by the inclusion, the map $\underline{\Sigma}(D) \to \underline{\Sigma}(C)$ is the restriction of functionals.

Conversely, let $\mathcal{I} := \{(C_i, \sigma_i)\}$ be a consistent ideal of partial measurement outcomes. Then

$$\mathcal{F} := \{(C_i, u) \mid \sigma_i \in u\},$$

where $u \in \Sigma(C_i)$ and $\sigma_i$ is a point in the open $u$, defines a filter in $\mathcal{C}(A)^{\mathrm{op}} \ltimes \underline{\Sigma}$.

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[5]I.e. the smallest pre-sheaf satisfying the conditions; see [22].

- $\mathcal{F}$ is up-closed: if $(C_i, u) \in \mathcal{F}$ and $(C_i, u) \leq (D, v)$, then $D = C_j$, since we have an ideal. Moreover, $u \leq v$ in $\Sigma(C_i)$ and hence $\sigma_i \in u \leq v$.
- $\mathcal{F}$ is lower-directed: if $(C_i, u), (C_j, v) \in \mathcal{F}$, then there exists $(C_k, \sigma_k) \in \mathcal{I}$ such that $C_i, C_j \subset C_k$, since $\mathcal{I}$ is an ideal. Both $\sigma_i, \sigma_j$ are restrictions of $\sigma_k$, so $\sigma_k \in u \cap v$.

We claim that $\mathcal{F}$ is completely prime. Fix $(C, \sigma)$ and suppose that $(C, u) \in \mathcal{F}$ and $(C, u) \lhd (D_j, v_j)$, that is $u \lhd \{v_j \mid D_j = C\}$. Then some $(D_j, v_j) \in \mathcal{F}$, because $\sigma$ is a point/completely prime filter.

The two constructions constitute a bijection between points and consistent ideals of partial measurement outcomes, as is easy to check.                                              □

**Definition 10** We write $pMO$ for the locale $\mathcal{C}(A)^{\mathrm{op}} \ltimes \underline{\Sigma}$ of (consistent ideals of) partial Measurement Outcomes.

It is tempting to identify the ideal of partial measurement outcomes with its limit. However, the ideal and its limit define different points as we will see at the end of Sect. 3.1. The points of the latter will be identified in Sect. 4.

For commutative C*-algebras $pMO$ is similar, but not equal, to the Gelfand spectrum, as we see by a direct unfolding of the lemma.

**Corollary 11** *For a compact regular locale $X$, the points of $pMO(C(X))$ are consistent ideals of points in the spectra of C\*-subalgebras of $C(X)$.*

An alternative, but equivalent, external description of the locale $pMO(A)$ may be found in [14, 4.15]. The present computation gives a simpler description, using generators and relations, which makes it easier to compute the points.

### 3.1 Scott Continuity

It seems natural to identify consistent ideals of partial measurement outcomes with their limits. Hence we investigate the use of Scott continuity. A *Scott continuous* map is one that preserves directed joins.

**Definition 12** A *dcpo* is directed complete partial order: each of the directed subsets of the poset has a supremum.

**Lemma 13** *For any $A$, the poset $\mathcal{C}(A)$ is a dcpo.*

*Proof* A directed subset $\mathcal{D} \subset \mathcal{C}(A)$ is one such that any pair $D, D'$ of its elements have a common upper bound $E \supset D, D'$, hence $D, D'$ commute. Consequently, $\overline{\{a \in D \mid D \in \mathcal{D}\}}$ is a commutative C*-algebra, the supremum of $\mathcal{D}$.           □

A first attempt to include Scott continuity starts with the observation that the dcpo $\mathcal{C}(A)$ *seems* similar to the canonical example of an *algebraic* dcpo [15, p. 252]: the dcpo of subalgebras of an algebra with the finitely generated algebras as finite elements. An element $a$ of a dcpo $L$ is *finite* (or *compact*) if for all directed $S$ such that $\bigvee S \geq a$, there exists a finite subset $T \subset S$ such that $a \leq \bigvee T$. A *algebraic poset* is

one where every element is the supremum the finite elements below it. An algebraic dcpo is equivalent to $\mathrm{Idl}(P)$ for some poset $P$. For such $P$, the functor topos $\mathbf{Sets}^P$ is localic, it is equivalent to the topos of sheaves over $\mathrm{Idl}(P)$ with the Scott topology. Those, in turn, are equivalent to functors from $\mathrm{Idl}(P)$ to $\mathbf{Sets}$ that transform directed joins to colimits; e.g. [21, VII.7 Theorem 2][25, Lemma 59]. We can thus hope that $\mathbf{Sets}^{\mathcal{C}(A)}$ is equivalent to sheaves over $\mathrm{Idl}(P)$, where $P$ is the poset of finitely generated commutative subalgebras. If this would work, a point of the topos $\mathbf{Sets}^P$ would indeed be an ideal in $P$, equivalently an element of $\mathcal{C}(A)$.

This sounds promising *if* $\mathcal{C}(A)$ would be an algebraic dcpo. Unfortunately, $\mathcal{C}(A)$ need not be an *algebraic* dcpo. We do know that finite elements of the dcpo $\mathcal{C}(A)$ are finitely generated: for each $C$,

$$C = \bigcup \{C(a) \mid a \in C\},$$

where $C(a)$ is the C*-algebra generated by $a$ and 1. Hence if $C$ is finite in the dcpo, then a finite subset of $\{C(a) \mid a \in C\}$ covers $C$. However, the converse does not hold in general.

*Example 14* The finitely generated C*-algebra $C[0,1] = C(\mathrm{id})$ is not finite in the dcpo $\mathcal{C}(C[0,1])$.

*Proof* The C*-algebra $C := C[0,1]$ is generated by the identity function (by Stone-Weierstrass). However, it is not finite in the dcpo. Consider the set $U := \{C_n \mid n \in \mathbb{N}\}$, where

$$C_n := \left\{ a : C([0,1], \mathbb{C}) \mid a \text{ constant on } \left[0, \frac{1}{n}\right] \right\}.$$

Then $U$ is directed, and $C = \sup U$, but $C$ is not contained in any $C_n$.                    □

**Proposition 15** $C \in \mathcal{C}(A)$ *is finite iff* $C$ *is finite dimensional.*

*Proof* A commutative C*-algebra is finite dimensional if and only if it has a (finite and) discrete spectrum. We prove the equivalence between having a discrete spectrum and being finite in the dcpo.

Suppose that $C$ has a discrete spectrum and $C \subset \overline{\bigcup}^{\uparrow} C_i$, then by the finite dimensionality of $C$, $C \subset C_i$ for some $i$, i.e. $C$ is a finite element of the dcpo.

Conversely, suppose that $C$ is finite in the dcpo. By Gelfand duality, we may assume $C$ to be a C*-algebra of functions. Choose a point $\sigma$ in its spectrum. Let $U_\lambda$ be a directed net of opens converging to $\sigma$. Consider

$$C_\lambda := \{a \in C \mid \hat{a} \text{ constant on } U_\lambda\},$$

where $\hat{a}$ is the Gelfand transform of $a$. Since any uniformly continuous function on the spectrum can be approximated by one that is constant near $\sigma$, $C = \bigvee C_\lambda$. Hence by finiteness, some $C_\lambda$ contains $C$. By regularity any two points in the spectrum of $C$ can be separated by a function in $C$. Since all functions in $C$ are constant on $U_\lambda$, so $\sigma$ is the *only* point in $U_\lambda$. We conclude that for each point there is an open neighborhood that contains only that point—that is, the spectrum is discrete.                    □

**Corollary 16** *The dcpo $\mathcal{C}(C[0, 1])$ is not algebraic.*

*Proof* Let $C \subset C[0, 1]$ be finite, and hence finite dimensional. Then by Gelfand duality we have a continuous surjection $[0, 1] \rightarrow \{1, \ldots, n\}$, where $n$ is the dimension of $C$. By connectedness of $[0, 1]$, this function is constant. Hence, $C$ consists of the constant functions. It is clear that $C[0, 1]$ cannot be approximated by constant functions. □

Consequently, if $A$ is finite dimensional, then $\mathcal{C}(A)$ is an algebraic dcpo. Let $K$ be the Cantor Discontinuum.[6] By total disconnectedness, any continuous function from $K$ to $C$ can be approximated by a locally constant function on finitely many components which cover $K$. Hence, $\mathcal{C}(C(K, \mathbb{C}))$ is an algebraic dcpo. Obviously, the C*-algebra $C(K, \mathbb{C})$ is not finite dimensional.

The failure of algebraicity of the dcpo is caused by the completeness of C*-algebras. It can be avoided by using a more algebraic structure. For concreteness, we consider pre-C*-algebras. We write $\mathcal{C}'(A)$ for the poset of commutative pre-C*-subalgebras. This is an algebraic dcpo; its finite points are the finitely generated commutative subalgebras. Moreover, when we replace $\mathcal{C}(A)$ by $\mathcal{C}'(A)$, the theory we have developed in the Bohrification program will continue to work with only minor modifications. In particular, we obtain an internal commutative *pre*-C*-algebra of which we can compute the spectrum. When considering sheaves over this algebraic dcpo, we obtain as points pairs $(C, \sigma)$ with $C$ a pre-C*-subalgebra. We have now identified consistent ideals with their limit. This seems like an improvement. However, we have added new points which one may consider to carry the same physical information as the existing points. To wit, the point $(C, \sigma)$ is different from the point $(\overline{C}, \sigma)$, as their collection of opens differ. Here $\overline{C}$ is the completion of $C$. We conclude that we can choose to have as points either:

1. $(C, \sigma)$ with $C$ a pre-C*-algebra;
2. Consistent ideals of $(C, \sigma)$ with $C$ complete.

We could try to find a common refinement of the two approaches above. However, we will not pursue this in the present article.

## 4 Maximal Commutative Subalgebras, Classical Logic and the Spectrum

As stated in the introduction, we address the following questions:

- Can we restrict to the *maximal* commutative subalgebras?
- Are we allowed to use classical logic internally?

In a sense, the answers to both of these questions are positive. The collection of maximal commutative subalgebras covers the space $\mathcal{C}(A)$ in the dense topology and this dense, or double negation, topology forces the logic to be classical.

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[6]The Cantor Discontinuum, or Cantor comb, is the compact subset of [0,1] which is obtained by removing the middle third of the interval and then recursively applying this process to the two remaining intervals.

Sheaves for the dense topology may be used to present classical set theoretic forcing or, equivalently, Boolean valued models. In these non-standard models of set theory, one generalizes the usual two-valued logic to a Boolean valued logic. Common choices for the Boolean algebra of propositions include the measurable sets of a measure space or the projections in a commutative von Neumann algebra. As such these models have been used in probability theory and the foundations of physics; see [3] for an overview. Topos theoretically, one considers $\mathrm{Sh}(P, \neg\neg)$ [21, p. 277]. The dense topology on a poset $P$ is defined as $p \lhd D$ iff $D$ is dense below $p$: for all $q \leq p$, there exists a $d \in D$ such that $d \leq q$.[7] The locale presented by this site is a Boolean algebra, the topos is a Boolean valued model [21, VI.1].

This topos of $\neg\neg$-sheaves satisfies the axiom of choice [21, VI.2.9] when our base topos does. The associated sheaf functor sends the presheaf topos $\mathbf{Sets}^{P^{\mathrm{op}}}$ to the topos $\mathrm{Sh}(P, \neg\neg)$. The $\neg\neg$-sheafification can be described explicitly [21, p. 273] for $V \rightarrowtail W$:

$$V_{\neg\neg}(p) = \{x \in W(p) \mid \text{for all } q \leq p \text{ there exists } r \leq q \text{ such that } x \in V(r)\}.$$

We apply this to the poset $\mathcal{C}(A)$. We write $\Delta A$ for the constant functor $C \mapsto A$. Then $\underline{A} \subset \Delta A$ in $\mathbf{Sets}^{\mathcal{C}(A)}$.

For commutative $A$, $\mathcal{C}(A)^{\mathrm{op}}$ has $A$ as bottom element. Hence, for all $C$, $\underline{A}_{\neg\neg}(C) = A$. Here $\underline{A}_{\neg\neg}$ is the $\neg\neg$-sheafification of the Bohrification.

For the general case, we observe that each $C$ is covered by the collection of all its super-C*-algebras in $\mathcal{C}(A)$. By Zorn,[8] each commutative subalgebra is contained in a maximal commutative one. Hence the collection of maximal commutative subalgebras is dense. So, $\underline{A}_{\neg\neg}(C)$ is the intersection of all maximal commutative subalgebras containing $C$.

The covering relation for $(\mathcal{C}(A), \neg\neg) \ltimes \underline{\Sigma}$ is $(C, u) \lhd (D_i, v_i)$ iff $C \subseteq D_i$ and $C \Vdash u \lhd V_{\neg\neg}$, where $V_{\neg\neg}$ is the sheafification of the presheaf $V$ generated by the conditions $D_i \Vdash v_i \in V$. We computed this presheaf in the proof of Theorem 7, we proceed to compute the $\neg\neg$-sheaf. Since $V$ is an internal subset of the spectral lattice $L$ of the *presheaf $\underline{A}$*,

$$V_{\neg\neg}(C) = \{u \in L(C) \mid \forall D \leq C \, \exists E \leq D . u \in V(E)\}.$$

We conclude, $(C, u) \lhd (D_i, v_i)$ iff

$$\forall D \leq C \, \exists D_i \leq D . u \lhd V(D_i). \tag{2}$$

The following theorem should be compared to Theorem 9.

**Theorem 17** (Using Zorn's Lemma) *The points of $MO := (\mathcal{C}(A), \neg\neg) \ltimes \underline{\Sigma}$ are the measurement outcomes.*

---

[7]Constructively, this also defines a topology [6]. However, we need classical logic to prove that it coincides with the double negation topology.

[8]Here we use classical meta-logic.

*Proof* The covering relation of $MO$ contains that of $pMO$, hence $MO$ is a sublocale. We only need to prove the restriction to maximal elements.

Suppose we have a completely prime filter $\tau$. Then $(C, \top) \in \tau$ for some $C$. By Zorn's Lemma, the subalgebra $C$ is covered by all the maximal commutative subalgebras containing it. By directedness, one maximal subalgebra $M$ already covers $C$. We conclude that $(M, \top) \in \tau$. The proof of Theorem 9 shows that $\tau$ defines a measurement outcome.

Conversely, by Theorem 9, a measurement outcome $(M, \sigma)$ defines a completely prime filter

$$\{(C, u) \mid C \subseteq M, \sigma \in u\}$$

for $pMO$. We need to prove that it is also a point of $MO$. Suppose that $\sigma \in u$ and $(C, u) \lhd (D_i, v_i)$. Since $(M, u) \le (C, u)$, by Definition 2.3, $(M, u) \lhd (D_i, v_i)$. By (2), $\forall D \le M \, \exists D_i \le D. u \lhd V(D_i)$. By maximality of $M$, $u \lhd V(M)$. Since $\sigma \in u$, by (1), $\sigma \in v$ for some $v \in V(M)$. We have thus constructed a completely prime filter for $MO$.

Finally, the two operations above constitute a bijection between points and measurement outcomes. This follows from the restriction of the bijection in Theorem 9 and the uniqueness of the maximal element constructed above.                                      □

The following Corollary shows that the $MO$ construction is a non-commutative generalization of the spectrum. In this sense it behaves better than $pMO$; compare Corollary 11. Unfortunately, $MO$ is not functorial; see Sect. 5.

**Corollary 18** *For a compact regular locale* $X$, $X \cong MO(C(X))$.

*Proof* $C(X)$ is the only maximal commutative subalgebra of $C(X)$.                □

**Theorem 19** *Kochen-Specker*: *Let* $H$ *be a Hilbert space with* $\dim H > 2$ *and let* $A = B(H)$. *Then the* $\neg\neg$-*sheaf* $\underline{\Sigma}$ *does not allow a global section.*

*Proof* The Kochen-Specker theorem can be reformulated as the non-existence of certain global sections [4, 8, 14]. This connection carries over essentially unchanged to the present situation. A global section of a sheaf is (by definition) also a global section of the sheaf considered as a presheaf.                                                                    □

By considering the double negation we may use classical logic internally in our Boolean valued model. Moreover, the axiom of choice holds internally, so $\Sigma$ is a compact Hausdorff space. Still, the spectrum $\Sigma$ does not have a *global* point and the algebra does not have a global element.

## 5 Conclusions and Further Research

We have presented a generalization of the spectrum to non-commutative algebras. This was motivated by physical considerations elaborated on in [14]. The double negation allows us to restrict to maximal subalgebras.

We present another way to restrict to maximal subalgebras, while preserving the possibility to compute a unique functional from a global section. Let $A = M_n(\mathbb{C})$ be a matrix algebra. Consider the weakest topology on the set $\mathcal{M}$ of maximal subalgebras which makes the action of the unitary group continuous. Then $M \mapsto \underline{\Sigma}(M)$ defines a fiber bundle. Suppose that $\sigma$ is a continuous section. Let $M$ be a maximal subalgebra and let $p$ be a projection in $M$. Then $\sigma_M(p) \in \{0, 1\}$; say it is 0. If $M'$ is another maximal subalgebra which contains $p$ and $u$ a unitary transformation from $M$ to $M'$, i.e. $M' = uMu^*$, that leaves $p$ fixed. Since the unitary group is connected and acts transitively on the maximal subalgebras, $\sigma_{M'}(p) = \sigma_M(p) = 0$. We see that $\sigma_{M'}(p) = 0$ for all such $M'$. In particular, $\sigma_M(p)$ is defined independent of the choice of maximal subalgebra $M$. By linearity and density, this definition extends from projections to general elements: $\sigma$ may be uniquely defined on all elements, *independent* of the choice of the subalgebra. We conclude that, at least for matrix algebras, the independence of the functional from the subalgebra guaranteed by the order structure of $\mathcal{C}(A)$ may also be guaranteed by the group action of the unitary group. We leave the extension to general C*-algebras as an open question.

The $pMO$ construction, is not functorial when we equip C*-algebras with their usual morphisms. The construction *is* functorial on the category of C*-algebras with commutativity reflecting maps [24]. A map $\phi$ is *commutativity reflecting* if $[\phi(a), \phi(b)] = 0$ implies $[a, b] = 0$. More work seems to be needed for the $MO$ construction. Let $I_2$ be the subalgebra of constant functions of $\mathbb{C}^2$. Then $(I_2, \top) \lhd (\mathbb{C}^2, \{0, 1\})$. Here we write $\{0, 1\}$ for the spectrum of $\mathbb{C}^2$. However, this covering is not preserved when we embed $\mathbb{C}^2$ into $M_2$. In short, $\neg\neg$-covers need not be preserved under natural notions of morphism.

Bohrification may be described as a (co)limit [24]. While technically different from the double-negation, the intuitive meaning is similar: we are only interested in what happens 'eventually'.

As in [13, 14], we treat $\mathcal{C}(A)$ as a mere poset. At least in the finite dimensional case, this poset has an interesting manifold structure [14]. Escardo [10] provides a construction of the support of a locale which often coincides with its maximal points. It may be possible to use this construction to refine the present results by maintaining the topological structure.

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