Erkenn (2014) 79:1367–1372
DOI 10.1007/s10670-013-9572-y

INTRODUCTION

Basic Concepts of Structuralism

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Received: 27 September 2013 / Accepted: 27 September 2013 / Published online: 14 December 2013

Abstract Primarily addressed to readers unfamiliar with the structuralist approach in philosophy of science, we introduce the basic concepts that the contributions to this special issue presuppose. By means of examples, we briefly review set-theoretic structures and predicates, the potential and actual models of an empirical theory, intended applications, as well as links and specializations that are applied, among others, in reconstructing the empirical claim associated with a theory element.

Keywords Structuralist approach to scientific theories · Set-theoretic concepts · Logic of science

1 Set-Theoretic Structures and Predicates

There are three major accounts of the structuralist approach in the philosophy of science: (1) Josephs Sneed's *Logical Structure of Mathematical Physics* (1979), (2) Wolfgang Stegmüller's *The Structure and Dynamics of Theories* (1976), and (3) *An Architectonic for Science: The Structuralist Program* (1987) by Wolfgang Balzer, C. Ulises Moulines, and Joseph Sneed. In what follows, we briefly expound the basic meta-theoretical concepts that are foundational to all three accounts.

The core idea of structuralism is to represent empirical systems by means of sequences of sets, and to model the application of scientific theories by means of set-theoretic predicates. The systematic use of set-theoretic predicates for the representation of scientific knowledge, therefore, distinguishes the structuralist representation scheme from other formal accounts in the philosophy of science.

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Here is a simple example of a set-theoretic predicate from mathematics (Suppes 1957, p. 250):

Definition 1 *(Quasi-ordering)* $\mathfrak{A}$ is a quasi-ordering if and only if there is a set A and a binary relation R such that $\mathfrak{A} = \langle A, R \rangle$ and

(1) $R \subset A \times A$
(2) $\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$
(3) $\forall x R(x, x)$.

A set-theoretic predicate is simply one that applies to sequences of sets, which consist of a sub-sequence of base sets $D_1, \ldots, D_k$ and another sub-sequence of relations $R_1, \ldots, R_n$:

$$\langle D_1, \ldots, D_k, R_1, \ldots, R_n \rangle \tag{1}$$

Following, to some extent, the terminology of Bourbaki (1968) and common usage in model theory, sequences of this type are called *set-theoretic structures*. As structures in model theory of formal logic specify a domain of interpretation and an interpretation of the non-logical symbols, so are the base sets $D_1, \ldots, D_k$ to be understood as domains of interpretation and the relations $R_1, \ldots, R_n$ as interpretations of corresponding relation concepts. Hence, we can say that a structure of the type $\langle D_1, \ldots, D_k, R_1, \ldots, R_n \rangle$ specifies the interpretation of the relation symbols $\ulcorner R_1 \urcorner, \ldots, \ulcorner R_n \urcorner$, while noting that classical structuralism aims to avoid explicit references to the vocabulary of formal languages.

There are three types of set-theoretic concepts:

(1) models of **T**
(2) potential models of **T**
(3) intended applications of **T**.

The symbol **T** designates a *theory-element*, the basic unit of theory reconstruction in structuralism.

2 Potential and Actual Models

It has been observed, among others by Carnap (1958), that theory formation goes hand in hand with concept formation. That is, the advancement of a scientific theory comes with the introduction of concepts being specific to that theory. Such concepts are called **T**-theoretical in structuralism, where **T** stands for the theory or theory-element through which the concepts are introduced. Paradigmatic examples of **T**-theoretical concepts are mass and force in classical particle mechanics. Those concepts, by contrast, which are used to describe the empirical systems to which **T** is applied are called **T**-*non-theoretical*.

The distinction between **T**-theoretical and **T**-non-theoretical concepts gives rise to the following distinction between two kinds of set-theoretic entities:

$$\langle D_1, \ldots, D_k, N_1, \ldots, N_p \rangle \tag{2}$$

$$\langle D_1, \ldots, D_k, N_1, \ldots, N_p, T_1, \ldots, T_q \rangle \tag{3}$$

Structures of type (2) are intended to represent empirical systems that are the subject of the application of **T**, whereas structures of type (3) represent **T**-*theoretical extensions* of structures of type (2). The extension simply consists in an interpretation of the **T**-theoretical relation symbols. So the symbols $N_1, \ldots, N_p$ designate **T**-non-theoretical relations, whereas $T_1, \ldots, T_q$ designate **T**-theoretical ones. And the symbols $D_1, \ldots, D_k$ designate sets of empirical objects that make up the empirical system to which the theory **T** is applied.

If the theory involves some mathematical apparatus, such as functions to natural, rational, or real numbers, and operations on such functions, then symbols for sets of mathematical objects need to be introduced. This results in structures of the following types:

$$\langle D_1, \ldots, D_k, A_1, \ldots, A_m, N_1, \ldots, N_p \rangle \tag{4}$$

$$\langle D_1, \ldots D_k, A_1, \ldots, A_m, N_1, \ldots N_p, T_1, \ldots, T_q \rangle \tag{5}$$

$A_1, \ldots, A_m$ are sets of mathematical objects. Some or all of the **T**-non-theoretical and **T**-theoretical relations may be functions, i.e., binary many-to-one relations. If N_i (T_i) is required to be a function, we shall also write n_i (t_i) in place of N_i (T_i).

In physics, most quantities are introduced as functions taking empirical objects as arguments and having mathematical objects as values. Think of the concept of temperature, pressure, mass, force, electromagnetic field etc.

A simple and non-fundamental law of classical mechanics is the lever principle. The theory-element **LP** covers the case where the weights on either side of a lever are in equilibrium (Sneed 1979, p. 11):

Definition 2 *(Models of* **LP***)* x is a model of the lever principle $(x \in \mathbf{M}(\mathbf{LP}))$ if and only if there exist *D, n, t* such that

(1) $x = \langle D, \mathbb{R}, n, t \rangle$
(2) D is a finite, non-empty set
(3) $n : D \rightarrow \mathbb{R}$
(4) $s : D \rightarrow \mathbb{R}$
(5) $\sum_{y \in D} n(y) \cdot t(y) = 0.$

n has the intended meaning of the spatial distance function from the lever's centre of rotation, and t the intended meaning of the mass function. The particles on one arm of the lever have positive distance values, whereas particles of the opposite arm have negative distance values. Conditions (1)–(4) characterise the types of sets and relations that make up a model of **LP**, whereas (5) expresses a law concerning the descriptive concepts of **LP**.

The structuralist schema of representing knowledge has it that there is a one-to-one correspondence between *substantial laws* and theory-elements. Theory-elements are individuated by substantial laws, where there is some freedom of choice as to which axioms are grouped together to form the substantial law. In many cases, it is just one formal axiom that makes up a substantial law. In the above example, the lever principle being applied to the equilibrium case, individuates the theory-element **LP**. The structuralist analysis of a scientific theory usually results into a net of interrelated theory-elements.

Any set-theoretic definition of the models of a theory-element **T** consists of two parts: (1) conditions that a sequence of sets must meet to guarantee that the substantial law of **T** has a well defined truth-value when that sequence is taken to interpret the descriptive concepts of **T**, and (2) the substantial law of **T** itself. This division into conditions of applying the substantial law to a sequence of sets and the condition of satisfying this law leads to the distinction between potential and actual models of a theory-element. In the case of **LP** we have:

Definition 3 *(Potential models of* **LP***)* x is a potential model of the lever principle $(x \in \mathbf{M}_p(\mathbf{LP}))$ if and only if there exist D, n, t such that

(1) $x = \langle D, \mathbb{R}, n, t \rangle$
(2) D is a finite, non-empty set
(3) $n : D \to \mathbb{R}$
(4) $t : D \to \mathbb{R}$.

Definition 4 *(Models of* **LP***)* x is a model of the lever principle $(x \in \mathbf{M}(\mathbf{LP}))$ if and only if there exist D, n, t such that

(1) $x = \langle D, \mathbb{R}, n, t \rangle$
(2) $x \in \mathbf{M}_p(\mathbf{LP})$
(3) $\sum_{y \in D} n(y) \cdot t(y) = 0$.

Structures that satisfy the substantial law of **T** are called *models of* **T**, in line with well established conventions in model theory. *Potential models* of **T**, by contrast, are structures that meet the formal conditions of applying the substantial law of **T** but not necessarily satisfy that law itself.

3 Intended Applications

An intended application is a set-theoretic representation of an empirical system to which the substantial law of a theory-element **T** is applied or thought to be applicable. Formally, intended applications are structures of the following type:

$$\langle D_1, \ldots, D_k, A_1, \ldots, A_m, N_1 \ldots N_p \rangle \tag{6}$$

where $D_1, \ldots, D_k$ are empirical base sets, $A_1, \ldots, A_m$ mathematical base sets, and $N_1, \ldots, N_p$ **T**-non-theoretical relations. Any theory-element is associated with a set of intended applications, which thus encode the interpretation of a **T**-relativised observation language.

In less formal terms, intended applications are the particular phenomena to which the axioms of a scientific theory are applied, where the distinction between phenomenon and theory is relativised. A standard example of an intended application is the solar system to which Newton's equations and Newton's law of gravitation have been applied. The electromagnetic spectrum of the hydrogen atom likewise qualifies as an intended application of Bohr's theory of the atom and of quantum mechanics.

Applying an axiom of a scientific theory to an empirical system normally results in constraining the admissible interpretations of the **T**-theoretical relations of this system. In the case of **LP**, the interpretation of the mass function t is constrained by the axiom

$$\sum_{y \in D} n(y) \cdot t(y) = 0.$$

Any ordinary beam balance works on the basis of this axiom.

To formally capture the constraints upon the interpretations of the **T**-theoretical relations, the notion of a **T**-theoretical extension is introduced through a restriction function **r(T)**. This function "cuts off" the **T**-theoretical relations from a **T**-theoretical structure in order to obtain a **T**-non-theoretical structure:

Definition 5 *(Restriction function)* $\mathbf{r}(\mathbf{T})$ Let x be a structure of the type $\langle D_1 \ldots D_k, A_1, \ldots, A_m, N_1, \ldots, N_p, T_1 \ldots T_q \rangle$. $y = \mathbf{r}(\mathbf{T})(x)$ if and only if (i) y is a structure of the type $\langle D_1 \ldots D_k, A_1, \ldots, A_m, N_1, \ldots, N_p \rangle$ and (ii) for all $i, 1 \leq i \leq k + m + p$, $(x)_i = (y)_i$, where $(x)_i$ designates the i-th component of a structure x.

Definition 6 *(**T**-theoretical extension)* A structure x is a **T**-theoretical extension of a structure y if and only if $y = \mathbf{r}(\mathbf{T})(x)$.

For a **T**-theoretical extension x of an intended application y to be admissible, x must be a model of y.

4 Links and Specialisations

Intended applications of theory-elements are related to one another in various ways. One distinguishes between three kinds of relations:

1. internal links
2. external links
3. specialisations.

These relations concern intended applications with their **T**-theoretical extensions. More precisely, links further specify which theoretical extensions of an intended application are admissible, in addition to the requirement that any admissible theoretical extension must be a model of the respective theory-element. Internal links (also known as *constraints*) are relations between intended applications of one and the same theory-element, whereas external links relate intended applications of different theory-elements.

The motivation for introducing internal and external links derives from intended applications that overlap with regard to both their concepts and their empirical domains. As an example of an internal link, suppose one and the same particle a is placed subsequently together with certain other objects on a lever such that the lever is in equilibrium. Then, we have several intended applications that overlap with regard to the object a and with regard to their concepts. The admissible **LP**-theoretical extensions of these intended applications must agree on the value they assign to the mass of a, which expresses the assumption that the mass of a particle in classical mechanics remains constant.

External links are particularly important in accounting for the transfer of data between intended applications of different theory-elements. Such a transfer obtains when **T**-non-theoretical relations of a theory-element **T** are determined with the help of a measuring theory $\mathbf{T}'$. The general motivation for introducing external links is similar to that for internal ones: two intended applications of different theory-elements may overlap insofar as one and the same empirical object occurs as a member of the empirical base sets of different intended applications. External links are always binary.

Specialisation introduces another type of relation among theory-elements to account for the inner structure of theories. A large number of scientific theories, in the ordinary sense of the term, have been reconstructed in the form of a tree-like structure with a basic theory-element at the top and several branches of more special theory-elements. The underlying idea is that any intended application of any specialised theory-element **T** is also an intended application of the more basic theory-elements being higher up in the hierarchy. Through specialisation, the substantial laws of different theory-elements can be superimposed.

To give a simple example of specialisation from classical collision mechanics, both elastic and inelastic collisions must satisfy the law of conservation of momentum. Hence, both the theory-element of an elastic collision and the theory-element of an inelastic collision are specialisations of the theory-element that encodes the conservation of momentum for classical collisions (Moulines 2010).

In sum, structuralist theory representation consists to a large extent in specifying the admissible theoretical extensions of a given set of intended applications. The global empirical claim of a theory-element **T** is the proposition that there is a set $A(\mathbf{T})$ of structures such that, for all intended applications y of **T**, there is an $x \in A(\mathbf{T})$ such that (1) x is a model of **T**, (2) x is a **T**-theoretical extension of y, and (3) all members of $A(\mathbf{T})$ satisfy all internal and external links as well as the specialisations of **T**.

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