

An analysis of the temperature field of the workpiece in dry continuous grinding

J. L. González–Santander · J. Pérez ·
P. Fernández de Córdoba · J. M. Isidro

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Abstract The recent model for heat transfer during intermittent grinding described in Skuratov, Ratis, Selezneva, Pérez, Fernández de Córdoba and Urchueguía (Appl Math Model 31:1039–1047, 2007) is considered. This model is particularized to the case of continuous dry grinding, where an alternative solution is obtained in the steady state. This alternative solution is analytically equivalent to the well-known formula of Jaeger (Proc R Soc NSW 76:204–224, 1942) for the steady-state temperature field created by an infinite moving source of heat and proves to be very useful for evaluating the maximum point of the temperature.

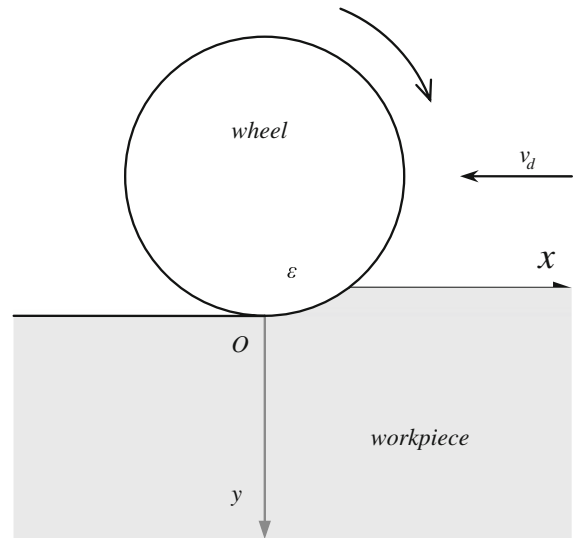
Keywords Dirac delta representation · Dry grinding process · Jaeger’s formula

1 Introduction

Two relevant contributions on mathematical modelling of the grinding problem [1,2] use coupled systems of two-dimensional partial differential equations to calculate the evolution of the temperature fields in the wheel, the workpiece and the grinding fluid. These models are nonlinear in that they allow for temperature-dependent contact heat source and heat-exchanger coefficients. These works extend an improved model allowing heat-flux variation along the grinding zone [3]. Recently, a simplified mathematical model was proposed in terms of a two-dimensional boundary-value problem where the interdependence among the grinding wheel, the workpiece and the coolant was described by two-variable functions in the boundary condition [4]. An analytical expression for the evolution of the workpiece temperature field during intermittent wet grinding was given. The heat partitioning to other heat sinks [5] is then fully decoupled. A numerical analysis of this solution was presented in [6]. In the present paper, we show that the solution of the mathematical problem [4] for the particular case of dry continuous grinding and in the steady-state is given by an expression that is analytically equivalent to the well-known integral formula of Jaeger [7]. The proof of the equivalence of both expressions (3.6) and (4.8), is derived from the uniqueness of the solution of the problem (see Sect. 5.2)

Our setup is depicted in Fig. 1. The workpiece moves at a constant speed v_d and is assumed to be infinite along Ox and Oz , and semiinfinite along Oy . The plane $y = 0$ is the surface being ground. The contact area between

J. L. González–Santander · J. Pérez · P. F. de Córdoba · J. M. Isidro (✉)
Instituto Universitario de Matemática Pura y Aplicada, IUMPA Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, Valencia 46022, Spain
e-mail: joissan@mat.upv.es

Fig. 1 Grinding setup

the wheel and the workpiece is an infinitely long strip of width ε located parallel to the Oz -axis and on the plane $y = 0$. Both the wheel and the workpiece are assumed rigid. Although our equations below allow for the case of wet and intermittent grinding, we will mostly consider the case of dry and continuous grinding.

This paper is organized as follows. Section 2 presents the differential equations governing our problem which are extracted from [4]. Section 3 gives a brief summary of the steady-state solution to these equations following [7]. The time-dependent thermal field is derived in Sect. 4 and analyzed in detail in Sect. 5. Emphasis is laid on determining the maximum of the temperature for the workpiece that takes place in the steady-state. Our conclusions are summarized in Sect. 6.

2 Problem formulation

Our problem is modelled by the heat equation in the presence of a convective term [4]

$$\partial_t T(t, x, y) = k [\partial_{xx} T(t, x, y) + \partial_{yy} T(t, x, y)] - v_d \partial_x T(t, x, y), \quad (2.1)$$

subject to the initial condition

$$T(0, x, y) = 0, \quad (2.2)$$

and to the boundary condition

$$k_0 \partial_y T(t, x, 0) = b(t, x) T(t, x, 0) + d(t, x), \quad (2.3)$$

where $-\infty < x < \infty$ and $t \geq 0, y \geq 0$. Above, $b(t, x)$ is the heat exchange coefficient between the workpiece and the grinding wheel, and $d(t, x)$ is the heat generated by friction between the two. Under the assumption of dry grinding, the workpiece is thermally insulated, and $b(t, x) = 0$. In this case we have that $d(t, x)$ equals the heat flux ϕ between the wheel and the workpiece. Now the heat flux ϕ across the plane $y = 0$ is

$$\phi = \rho c (k \nabla T + T \mathbf{v}_d) \cdot \mathbf{n}, \quad (2.4)$$

where \mathbf{n} is the unit normal to the plane $y = 0$, pointing in the direction of $y > 0$. Since $\mathbf{v}_d \cdot \mathbf{n} = 0$ we have

$$\phi = \rho c k \left. \frac{\partial T}{\partial y} \right|_{y=0}. \quad (2.5)$$

3 The steady-state solution

We will first review the steady-state solution to (2.1)–(2.3) as given in [7]. In the absence of lubrication fluid, the workpiece is assumed to be thermally isolated from the environment, so we can set

$$b(t, x) = 0. \tag{3.1}$$

Moreover, in the case of continuous grinding, we may assume that the physical contact between the wheel and the workpiece extends over an interval $x \in (0, \varepsilon)$, friction being zero outside. This can be modeled as

$$d(t, x') = -Q_s H(x') H(\varepsilon - x'), \tag{3.2}$$

where Q_s is the frictional heat-generation source term into the workpiece and $H(x)$ is a step function. For the details of parameter Q_s see references [6,8]. That is, the heat flux due to friction is localized exactly on the contact area (an infinitely long strip of width ε) between the wheel and the workpiece. This solution is constructed from the expression for the Green function corresponding to (2.1) when $v_d = 0$. For a point source of heat power Q_p located at (x', y', z') , this function is given by

$$T(t, x, y, z; x', y', z') = \frac{Q_p}{8(\pi kt)^{3/2}} \exp \left\{ -\frac{1}{4kt} \left[(x - x')^2 + (y - y')^2 + (z - z')^2 \right] \right\}. \tag{3.3}$$

The superposition principle is applied to (3.3). This is first done for $v_d = 0$. The temperature field $T(x, y)$ corresponding to an infinite linear source along the z -axis on the plane $y = 0$ can be obtained by superposition of point sources such as (3.3). Then the motion of the source along the Ox -axis, with a speed v_d , is modeled by changing coordinates to a moving reference frame. In the stationary state, when $t \rightarrow \infty$, one finds that an infinitely long, infinitely thin linear source causes a temperature field within the medium given by

$$T(x, y) = \frac{Q_l}{2\pi\rho ck} \exp\left(-\frac{v_d x}{2k}\right) K_0\left(\left|\frac{v_d}{2k}\right| \sqrt{x^2 + y^2}\right), \tag{3.4}$$

where Q_l is the heat power of the infinite linear source and $K_0(x)$ is the modified Bessel function of order zero [9, Sect. 9.6].

Further, applying the superposition principle to (3.4), one obtains the temperature field created by an infinitely long source of finite width ε moving exactly as above. If the band releases heat at a rate Q_s , then the steady-state temperature field is found to be given by

$$T(x, y)_{\text{infinite}} = \frac{Q_s}{\pi\rho cv_d} \int_{-v_d x/2k}^{-v_d(x-\varepsilon)/2k} e^u K_0\left(\sqrt{u^2 + \left(\frac{v_d y}{2k}\right)^2}\right) du. \tag{3.5}$$

Equation 3.5 gives the steady-state temperature field created by an infinitely long flat band of width ε located on the plane $y = 0$ along the z -axis, within an infinite medium along the axis Oy . This implies that the temperature field (3.5) does not solve the boundary-value problem (2.1)–(2.3). In order to obtain the temperature field created by grinding a semi-infinite workpiece (above called medium) in contact with a grinding wheel (above called source), without lubrication, we multiply the temperature field (3.5) by a factor of 2:

$$T(x, y)_{\text{semi-infinite}} = 2T(x, y)_{\text{infinite}} = \frac{2Q_s}{\pi\rho cv_d} \int_{-v_d x/2k}^{-v_d(x-\varepsilon)/2k} e^u K_0\left(\sqrt{u^2 + \left(\frac{v_d y}{2k}\right)^2}\right) du. \tag{3.6}$$

Now (3.6) does solve the boundary-value problem (2.1)–(2.3) along with (3.1)–(3.2). We can verify that the thermal flux across $y = 0$, Eq. 2.5, indeed is nonzero only on the contact area. In order to compute the thermal flux corresponding to (3.6), we first introduce the dimensionless variables $X := v_d x/2k$ and $Y := v_d y/2k$. We can also extend the integral (3.6) to the whole line if we include the appropriate Heaviside functions $H(u)$:

$$T(X, Y)_{\text{semi-infinite}} = \frac{2Q_s}{\pi\rho cv_d} \int_{-\infty}^{\infty} H(X - u) H(u - X + \Delta) e^{-u} K_0\left(\sqrt{u^2 + Y^2}\right) du, \tag{3.7}$$

where $\Delta = v_d \varepsilon / 2k$. This allows one to compute the derivative at $y = 0$ if one uses the representation of the Dirac δ -function given in (5.13), to be derived in Sect. 5.3. We find

$$\phi(x, y = 0) = -Q_s H(x) H(\varepsilon - x), \quad (3.8)$$

as expected.

4 The time-dependent solution

For the time-dependent solution to (2.1)–(2.3) we refer to [4]; what follows is a brief summary thereof. We apply a number of integral transformations on the variables x, y, t ; this will turn the differential equation (2.1) into an algebraic equation that can be readily solved. Transforming back into the original variables will yield the solution. Since $t \geq 0$ and $y \geq 0$, the natural transformation to apply on them will be Laplace's. On the other hand, x varies on the whole real line \mathbb{R} , so it will be Fourier-transformed. Details can be found in [4]. The result is

$$\begin{aligned} T(t, x, y) = & \frac{1}{4\pi} \int_0^t \frac{\exp\left(-\frac{y^2}{4ks}\right)}{s} \left(\int_{-\infty}^{\infty} \exp\left[-\frac{(x' - x - v_d s)^2}{4ks}\right] \right. \\ & \left. \times \left[\left(\frac{y}{2ks} - \frac{b(t-s, x')}{k_0} \right) T(t-s, x', 0) - \frac{d(t-s, x')}{k_0} \right] dx' \right) ds. \end{aligned} \quad (4.1)$$

Equation 4.1 solves the boundary-value problem (2.1–2.3) along with (3.1, 3.2) exactly. If the workpiece had an initial temperature T_0 , then T_0 is to be added as a constant to the above.

Equation 4.1 splits into two summands,

$$T(t, x, y) = T^{(0)}(t, x, y) + T^{(1)}(t, x, y), \quad (4.2)$$

where

$$T^{(0)}(t, x, y) = -\frac{1}{4\pi k_0} \int_0^t \frac{\exp\left(-\frac{y^2}{4ks}\right)}{s} \left(\int_{-\infty}^{\infty} d(t-s, x') \exp\left[-\frac{(x' - x - v_d s)^2}{4ks}\right] dx' \right) ds, \quad (4.3)$$

and

$$\begin{aligned} T^{(1)}(t, x, y) = & \frac{1}{4\pi} \int_0^t \frac{\exp\left(-\frac{y^2}{4ks}\right)}{s} \left(\int_{-\infty}^{\infty} T(t-s, x', 0) \right. \\ & \left. \times \left[\frac{y}{2ks} - \frac{b(t-s, x')}{k_0} \right] \exp\left[-\frac{(x' - x - v_d s)^2}{4ks}\right] dx' \right) ds. \end{aligned} \quad (4.4)$$

Neither $T^{(0)}$ nor $T^{(1)}$ separately is a solution to the boundary-value problem (2.1)–(2.3), but their sum is.

For convenience we will use the notations

$$T(x, y) = \lim_{t \rightarrow \infty} T(t, x, y), \quad (4.5)$$

$$T^{(0)}(x, y) = \lim_{t \rightarrow \infty} T^{(0)}(t, x, y), \quad (4.6)$$

$$T^{(1)}(x, y) = \lim_{t \rightarrow \infty} T^{(1)}(t, x, y) \quad (4.7)$$

for the temperature field in the steady state, so (4.2) simplifies to

$$T(x, y) = T^{(0)}(x, y) + T^{(1)}(x, y). \quad (4.8)$$

5 Analysis of the solution

Following [10, <http://hdl.handle.net/10251/4769>, Chap.5], next we analyze the solution (4.2)–(4.4). As compared with that found in Sect. 4, the solution (4.2)–(4.4) is more general: it is valid in the transient regime, when t is finite; it also allows for wet and/or pulsed grinding. However, for comparison purposes with the solution of Sect. 4, we will make some simplifying assumptions.

5.1 Continuous dry grinding in the steady state

The simplifying assumptions (3.1) and (3.2) reduce (4.3) and (4.4) to

$$T^{(0)}(t, x, y) = \frac{Q_s}{4\pi k_0} \int_0^t \frac{\exp\left(-\frac{y^2}{4ks}\right)}{s} \left(\int_0^\varepsilon \exp\left[-\frac{y^2 + (x' - x - v_d s)^2}{4ks}\right] dx' \right) ds, \tag{5.1}$$

and

$$T^{(1)}(t, x, y) = \frac{y}{8\pi k} \int_0^t \frac{\exp\left(-\frac{y^2}{4ks}\right)}{s^2} \left(\int_{-\infty}^\infty T(t - s, x', 0) \exp\left[-\frac{y^2 + (x' - x - v_d s)^2}{4ks}\right] dx' \right) ds. \tag{5.2}$$

It will be useful to re-express (5.1) as

$$T^{(0)}(t, x, y) = \frac{Q_s}{4\pi k_0} \int_0^t \frac{\exp\left(-\frac{y^2}{4ks}\right)}{s} F(x, s) ds, \tag{5.3}$$

with

$$F(x, s) = \int_0^\varepsilon \exp\left(-\frac{(x' - x - v_d s)^2}{4ks}\right) dx'. \tag{5.4}$$

In the steady state, some manipulations reduce (5.3) to

$$T^{(0)}(x, y) = \frac{Q_s}{\pi \rho c v_d} \int_{-v_d x/2k}^{-v_d(x-\varepsilon)/2k} e^u K_0 \left(\sqrt{u^2 + \left(\frac{v_d y}{2k}\right)^2} \right) du, \tag{5.5}$$

or, equivalently, to

$$T^{(0)}(x, y) = \frac{Q_s}{4k_0 \sqrt{\pi}} \int_0^\infty \exp\left(-\frac{y^2}{\sigma^2}\right) \left[\operatorname{erf}\left(\frac{x}{\sigma} + \frac{v_d \sigma}{4k}\right) - \operatorname{erf}\left(\frac{x - \varepsilon}{\sigma} + \frac{v_d \sigma}{4k}\right) \right] d\sigma. \tag{5.6}$$

Equation 5.5 is useful in showing the complete agreement with the solution (3.5) for a workpiece stretching along the whole axis Oy . Incidentally, this shows again that (3.5) cannot be an exact solution to the boundary-value problem (2.1)–(2.3), because the term $T^{(1)}(x, y)$ does not vanish identically.

Concerning $T^{(1)}(t, x, y)$, we can rewrite (5.2) as

$$T^{(1)}(t, x, y) = \frac{y}{8\pi k} \int_0^t \frac{\exp\left(-\frac{y^2}{4ks}\right)}{s^2} G(x, t, s) ds, \tag{5.7}$$

with

$$G(x, t, s) = \int_{-\infty}^\infty T(t - s, x', 0) \exp\left(-\frac{(x' - x - v_d s)^2}{4ks}\right) dx'. \tag{5.8}$$

Passing again to the steady state, one may reduce (5.7) to

$$T^{(1)}(x, y) = \frac{yv_d^2}{8\pi k^2} \int_{-\infty}^{\infty} T(x', 0) \exp\left(\frac{v_d(x' - x)}{2k}\right) K_1\left(\frac{|v_d|}{2k} \sqrt{y^2 + (x - x')^2}\right) \frac{2k/|v_d|}{\sqrt{y^2 + (x - x')^2}} dx'. \tag{5.9}$$

In the dimensionless variables $X' = v_d x' / 2k$, $X = |v_d| x / 2k$ and $Y = |v_d| y / 2k$ one finds

$$T^{(1)}(X, Y) = \frac{Y}{2\pi} \int_{-\infty}^{\infty} T(X', 0) e^{X' - X} \frac{K_1\left(\sqrt{Y^2 + (X' - X)^2}\right)}{\sqrt{Y^2 + (X' - X)^2}} dX'. \tag{5.10}$$

The above integral contains the kernel function

$$N(X - X', Y) := Y \frac{K_1\left(\sqrt{Y^2 + (X' - X)^2}\right)}{\sqrt{Y^2 + (X' - X)^2}}. \tag{5.11}$$

Below we prove that this kernel behaves like a Dirac δ -function on the workpiece surface,

$$\lim_{Y \rightarrow 0^+} N(X - X', Y) = \pi \delta(X - X'), \tag{5.12}$$

as one approaches $y = 0$ from above. Equivalently we can write (5.12) as

$$\lim_{y \rightarrow 0^\pm} \frac{y}{\pi} \frac{K_1\left(\sqrt{y^2 + u^2}\right)}{\sqrt{y^2 + u^2}} = \pm \delta(u). \tag{5.13}$$

Then by (5.12) in (5.10) we find

$$\begin{aligned} T^{(1)}(X, 0) &= \lim_{Y \rightarrow 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} T(X', 0) e^{X' - X} N(X - X', Y) dX' \\ &= \frac{1}{2} \int_{-\infty}^{\infty} T(X', 0) e^{X' - X} \delta(X - X') dX' = \frac{1}{2} T(X, 0). \end{aligned} \tag{5.14}$$

This is an important result: the surface temperature $T(X, 0)$ is twice the value of $T^{(1)}(X, 0)$,

$$T(X, 0) = 2T^{(1)}(X, 0). \tag{5.15}$$

Substitution in (4.8) leads to another important conclusion:

$$T^{(0)}(X, 0) = T^{(1)}(X, 0). \tag{5.16}$$

5.2 Uniqueness of the solution

We can generalize our result (5.16) if we impose further assumptions. In the case of continuous dry grinding, and in the steady state, the boundary-value problem posed in Sect. 2 has a unique solution.¹ By the method of Sect. 3, in the steady state our equation reads

$$\nabla^2 T(x, y) = 0, \tag{5.17}$$

subject to the boundary conditions

$$k_0 \partial_y T(x, 0) = -Q_s H(x) H(\varepsilon - x), \quad \lim_{x \rightarrow \pm\infty} T(x, 0) = 0. \tag{5.18}$$

Now non-constant harmonic functions can only attain their extrema on the boundary of their domain [11, Chap. X, Sect. 1], and a standard argument establishes that the solution is unique. We conclude that, under the assumption of continuous dry grinding, the steady-state solutions of Sects. 3 and 4 must be equal:

$$T(x, y)_{\text{semi-infinite}} = 2T^{(0)}(x, y). \tag{5.19}$$

This is stronger than our previous result (5.16), at the cost of imposing the additional assumption that $\lim_{x \rightarrow \pm\infty} T(x, 0) = 0$, which was not required to derive (5.16).

¹ Of course, the solution may also be unique under more general assumptions than continuous dry grinding in the steady state, but here we are interested in this case only.

5.3 Proof of (5.12)

In (5.11) we have, for $X' \neq X$,

$$\lim_{Y \rightarrow 0^+} N(X - X', Y) = 0. \tag{5.20}$$

However, when $X = X'$, we have

$$\lim_{Y \rightarrow 0^+} N(0, Y) = \lim_{Y \rightarrow 0^+} K_1(|Y|) = \infty, \tag{5.21}$$

because of the singularity of the Bessel function K_1 at the origin. This behavior is reminiscent of the Dirac δ -function. It remains to compute the integral of the kernel function (5.11) and to prove that it is finite, in order to conclude that N is indeed a multiple of the Dirac δ -function. Let us therefore consider the integral

$$I_N := \lim_{Y \rightarrow 0^+} \int_{-\infty}^{\infty} N(X - X', Y) dX'. \tag{5.22}$$

Performing the change of variables $\chi = X' - X$ and remembering that the integrand is even, we can write

$$I_N = 2 \lim_{Y \rightarrow 0^+} Y \int_0^{\infty} \frac{K_1(\sqrt{Y^2 + \chi^2})}{\sqrt{Y^2 + \chi^2}} d\chi. \tag{5.23}$$

Now the substitution $u = \sqrt{Y^2 + \chi^2}$ leads to

$$I_N = 2 \lim_{Y \rightarrow 0^+} Y \int_Y^{\infty} \frac{K_1(u)}{\sqrt{u^2 - Y^2}} du. \tag{5.24}$$

Finally, setting $u = Y \cosh z$, we find

$$I_N = 2 \lim_{Y \rightarrow 0^+} Y \int_0^{\infty} K_1(Y \cosh z) dz. \tag{5.25}$$

A useful expression for K_1 reads [12, Eq. 5.7.11]

$$K_1(x) = \frac{1}{x} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!} \left[2 \log\left(\frac{x}{2}\right) - \psi(k+1) - \psi(k+n+1) \right], \tag{5.26}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler Gamma function. Hence

$$\lim_{Y \rightarrow 0^+} x K_1(\lambda x) = \frac{1}{\lambda}. \tag{5.27}$$

Substitution of (5.27) in (5.25) gives

$$I_N = 2 \int_0^{\infty} \frac{dz}{\cosh z}. \tag{5.28}$$

This integral is readily evaluated by applying the change of variables $\zeta = \tanh z$, with the result that

$$I_N = \pi, \tag{5.29}$$

and (5.12) follows immediately.

5.4 The maximal temperature as a function of time

We claim that the maximal temperature as a function of the time is attained in the steady state. In order to prove it we observe that, since the integrand of (5.3) is positive and $Q_s > 0$, the integral (5.3) is a monotonically increasing function of the time. We have

$$T^{(0)}(t, x, y) < T^{(0)}(x, y), \quad \forall t \in (0, \infty), \quad (5.30)$$

and the maximum value of $T^{(0)}$ is attained in the steady state. Analogous arguments apply to (5.7), so

$$T^{(1)}(t, x, y) < T^{(1)}(x, y), \quad \forall t \in (0, \infty), \quad (5.31)$$

and again the maximum value of $T^{(1)}$ is reached in the steady state. Altogether, Eqs. 5.30 and 5.31 imply that

$$T(t, x, y) < T(x, y), \quad \forall t \in (0, \infty), \quad (5.32)$$

with $T(t, x, y)$ being a monotonically increasing function of the time. Hence the maximum temperature, as a function of time, is reached in the steady-state, as claimed.

5.5 The maximal temperature as a function of space

By the maximum principle for harmonic functions [11, Chap X, Sect. 1], the maximal temperature must be reached on the boundary, $y = 0$ in our case. Let us analyze the maximum of the function $T(x, 0)$. By (5.15), (5.16) and (5.19) we need to compute the derivative $\partial T^{(0)}(x, 0)/\partial x$. This can be done with the help of (5.6):

$$\frac{\partial T^{(0)}(x, 0)}{\partial x} = \frac{Q_s}{2\pi k_0} \int_0^\infty \left\{ \exp \left[- \left(\frac{x}{\sigma} + \frac{v_d \sigma}{4k} \right)^2 \right] - \exp \left[- \left(\frac{x - \varepsilon}{\sigma} + \frac{v_d \sigma}{4k} \right)^2 \right] \right\} \frac{d\sigma}{\sigma} \quad (5.33)$$

where the derivative of the error function has been used. To further manipulate (5.33) let us consider the function

$$R(\chi) := \int_0^\infty \exp \left[- \left(\frac{\chi}{\sigma} + \frac{v_d \sigma}{4k} \right)^2 \right] \frac{d\sigma}{\sigma}. \quad (5.34)$$

The change of variables $h = (v_d \sigma / 4k)^2$ reduces (5.34) to

$$R(\chi) = \frac{1}{2} \exp \left(- \frac{v_d \chi}{2k} \right) \int_0^\infty \exp \left(- \frac{z^2}{4h} - h \right) \frac{dh}{h}, \quad z := \left| \frac{v_d \chi}{2k} \right|, \quad (5.35)$$

and this can be expressed in terms of the Bessel function K_0 as [12, Eq. 5.10.25]

$$R(\chi) = \exp \left(- \frac{v_d \chi}{2k} \right) K_0 \left(\left| \frac{v_d \chi}{2k} \right| \right). \quad (5.36)$$

Therefore

$$\frac{\partial T^{(0)}(x, 0)}{\partial x} = \frac{Q_s}{2\pi k_0} [R(x) - R(x - \varepsilon)]. \quad (5.37)$$

Since $\lim_{x \rightarrow 0} K_0(x) = \infty$, we have

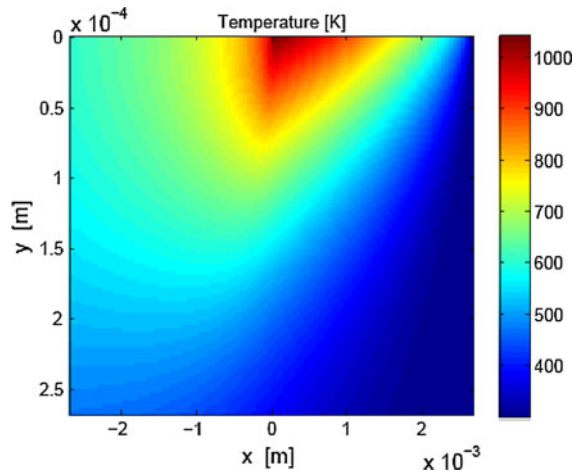
$$\lim_{x \rightarrow 0} \frac{\partial T^{(0)}(x, 0)}{\partial x} = \infty, \quad \lim_{x \rightarrow \varepsilon} \frac{\partial T^{(0)}(x, 0)}{\partial x} = -\infty, \quad (5.38)$$

and we can limit our search for the zeroes of $\partial T^{(0)}(x, 0)/\partial x$ to the interval $(0, \varepsilon)$. A detailed analysis, supplemented with Bolzano's theorem, establishes the following. When $v_d > 0$, there exists a unique value $c_+ \in (0, \varepsilon)$ at which $T^{(0)}(x, 0)$ attains a maximum. When $v_d < 0$, there also exists a unique value $c_- \in (0, \varepsilon)$ at which $T^{(0)}(x, 0)$ attains a maximum. Moreover, the fact that the error function is odd implies that $c_- = \varepsilon - c_+$.

Table 1 Input data for numerical simulations

ε (m)	2.663×10^{-3}
v_d (m s ⁻¹)	0.53
Q_s (W m ⁻²)	5.89×10^7
k_0 (W m ⁻¹ K ⁻¹)	13
k (m ² s ⁻¹)	4.23×10^{-6}
T_0 (K)	300

Fig. 2 $T_0 + 2T^{(0)}(x, y)$ for $(x, y) \in (-\varepsilon, \varepsilon) \times (0, \varepsilon/10)$



5.6 Numerical example

Very fast and simple numerical algorithms have been implemented in MATLAB to compute the maximum of the workpiece temperature in the steady-state, see Appendix E of reference [10, <http://hdl.handle.net/10251/4769>, Chap.5] for further details. These algorithms are based on formulas (5.5), (5.19) and (5.37) presented before. Assuming that the workpiece is a VT20 titanium alloy and using the values of the parameters tabulated in Table 1, see references [4,6], we show that the maximum point $c_+ = 0.0072 \varepsilon$ m and that the maximum value of the temperature, reached on the workpiece surface, $T_{\max} = T_0 + 2T^{(0)}(c_+, 0) = 1042.23$ K. These values agree with the temperature field shown in Fig. 2.

6 Conclusions

Our first conclusion is the equality between the (apparently different) solutions for the temperature field given in the literature (and summarized here in Sects. 3 and 4). Beyond summarizing the existing approaches to the boundary-value problem (2.1)–(2.3), we have presented a detailed analysis of its exact solution in the case of dry grinding. For this purpose, a new representation of the Dirac delta distribution has been developed, involving a modified Bessel function; this new representation of the Dirac delta has not been tabulated in the literature yet.

From an applied point of view, the computation of the maximum temperature is the principal goal. According to this, it has been proved that this maximum is reached in the stationary state within the contact zone on the workpiece surface. It is shown that the numerical computation of this maximum is quite simple from the approach given in the Samara–Valencia model.

The maximal surface temperature, and also the maximal temperature as a function of the time, have been computed theoretically and numerically.

Our analysis can be generalized to the case of wet grinding [13] again considering the assumption of constant heat-transfer coefficient on the workpiece surface. One can expect the results concerning the maximal temperature to be quantitatively different from those of dry grinding. We expect to report on these issues soon.

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