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A Random Field Formulation of Hooke's Law in All Elasticity Classes

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Abstract For each of the 8 symmetry classes of elastic materials, we consider a homogeneous random field taking values in the fixed point set V of the corresponding class, that is isotropic with respect to the natural orthogonal representation of a group lying between the isotropy group of the class and its normaliser. We find the general form of the correlation tensors of orders 1 and 2 of such a field, and the field's spectral expansion.

Keywords Elasticity class · Random field · Spectral expansion

Mathematics Subject Classification 60G60 · 74A40

1 Introduction

Microstructural randomness is present in just about all solid materials. When dominant (macroscopic) length scales are large relative to microscales, one can safely work with deterministic homogeneous continuum models. However, when the separation of scales does not hold and spatial randomness needs to be accounted for, various concepts of continuum mechanics need to be re-examined and new methods developed. This involves: (1) being able to theoretically model and simulate any such randomness, and (2) using such results as input into stochastic field equations. In this paper, we work in the setting of linear elastic random media that are statistically wide-sense homogeneous and isotropic.

Regarding the modeling motivation (1), two basic issues are considered in this study: (i) type of anisotropy, and (ii) type of correlation structure. Now, with reference to Fig. 1

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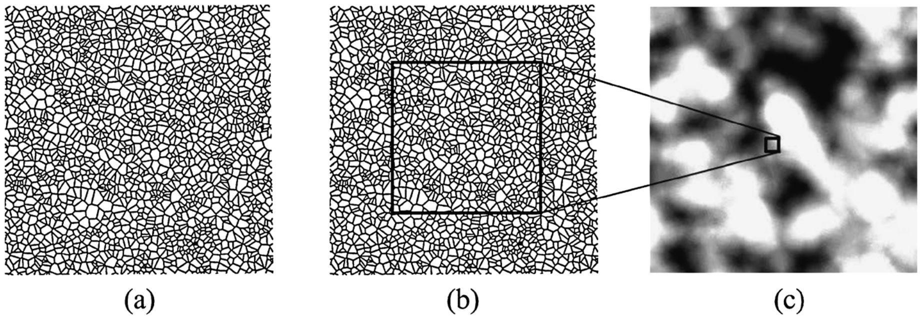

Fig. 1 (**a**) A realisation of a Voronoi tesselation (or mosaic); (**b**) placing a mesoscale window leads, via upscaling, to a mesoscale random continuum approximation in (**c**)

showing a planar Voronoi tessellation of E^2 which serves as a planar geometric model of a polycrystal (although the same arguments apply in E^3), each cell may be occupied by a differently oriented crystal, with all the crystals belonging to any specific crystal class. The latter include:

– transverse isotropy modelling, say, sedimentary rocks at long wavelengths;
– tetragonal modelling, say, wulfenite ($PbMoO_4$);
– trigonal modelling, say, dolomite ($CaMg(CO_3)_2$);
– orthotropic, modelling, say, wood;
– orthotropic modelling, say, orthoclase feldspar;
– triclinic, modelling, say, microcline feldspar.

Thus, we need to be able to model 4th-rank tensor random fields, pointwise taking values in any crystal class. While the crystal orientations from grain to grain are random, they are not spatially independent of each other—the assignment of crystal properties over the tessellation is not a white noise. This is precisely where the two-point characterisation of the random field of elasticity tensor is needed. While the simplest correlation structure to admit would be white-noise, a (much) more realistic model would account for any mathematically admissible correlation structures as dictated by the statistically wide-sense homogeneous and isotropic assumption. A specific correlation can then be fitted to physical measurements.

Regarding the modeling motivation (1), it may also be of interest to work with a mesoscale random continuum approximation defined by placing a mesoscale window at any spatial position as shown in Fig. 1(b). Clearly, the larger is the mesoscale window, the weaker are the random fluctuations in the mesoscale elasticity tensor: this is the trend to homogenise the material when upscaling from a statistical volume element (SVE) to a representative volume element (RVE), e.g., [27, 30]. A simple paradigm of this upscaling, albeit only in terms of a scalar random field, is the opacity of a sheet of paper held against light: the further away is the sheet from our eyes, the more homogeneous it appears. Similarly, in the case of upscaling of elastic properties (which are tensor in character), on any finite scale there is (almost surely) an anisotropy, and this anisotropy, with mesoscale increasing, tends to zero hand-in-hand with the fluctuations. It is in the infinite mesoscale limit (i.e., RVE) that material isotropy is obtained as a consequence of the statistical isotropy.

Regarding the motivation (2) of this study, i.e., input of elasticity random fields into stochastic field equations, there are two principal routes: stochastic partial differential equations (SPDE) and stochastic finite elements (SFE). The classical paradigm of SPDE [19] can be written in terms of the anti-plane elastostatics (with $u \equiv u_3$):

$$\nabla \cdot \big(C(\mathbf{x}, \omega)\nabla u\big) = 0, \quad \mathbf{x} \in E^2,\ \omega \in \Omega \tag{1}$$

with $C(\cdot, \omega)$ being spatial realisations of a scalar RF. In view of the foregoing discussion, (1) is well justified for a piecewise-constant description of realisations of a random medium such as a multiphase composite made of locally isotropic grains. However, in the case of a boundary value problem set up on coarser (i.e., mesoscales) scales, having continuous realisations of properties, a 2nd-rank tensor random field (TRF) of material properties would be much more appropriate, Fig. 1(c). The field equation should then read

$$\nabla \cdot \big(\mathsf{C}(\mathbf{x}, \omega) \cdot \nabla u\big) = 0, \quad \mathbf{x} \in E^2,\ \omega \in \Omega, \tag{2}$$

where C is the 2nd-rank tensor random field. Indeed, this type of upscaling is sorely needed in the stochastic finite element (SFE) method, where, instead of assuming the local isotropy of the elasticity tensor for each and every material volume (and, hence, the finite element), full triclinic-type anisotropy is needed [28].

Moving to the in-plane or 3d elasticity, if one assumes local material isotropy, a simple way to introduce material spatial randomness is to take the pair of Lamé constants (λ, μ) as a "vector" random field, resulting in

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \nabla\mu \cdot \big(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\big) + \nabla\lambda\nabla \cdot \mathbf{u} = \rho\ddot{\mathbf{u}} \tag{3}$$

as a generalisation of the classical Navier equation. However, just like in (1) above, the local isotropy is a crude approximation in view of micromechanics upscaling arguments, and (3) should be replaced by

$$\nabla \cdot (\mathsf{C} \cdot \nabla \mathbf{u}) = \rho\ddot{\mathbf{u}}, \tag{4}$$

where the stiffness C $(= C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)$ is a TRF. At any scale finitely larger than the microstructural scale, it is almost surely (a.s.) anisotropic, e.g. [27, 29]. Clearly, instead of (3), one should work with this SPDE (4) for $\mathbf{u}$.

While the mathematical theory of SPDEs with anisotropic realisations is not yet developed, one powerful way to numerically solve such equations is through stochastic finite elements (SFE). However, the SFE, just like the SPDE, require a general representation of the random field C, so it can be fitted to micromechanics upscaling studies, as well as its spectral expansion. Observe that each and every material volume (and, hence, the finite element) is an SVE of Fig. 1(c), so that a full triclinic-type anisotropy is needed: all the entries of the 4th-rank stiffness tensor C are non-zero with probability one. While a micromechanically consistent procedure for upscaling has been discussed in [33] and references cited there, general forms of the correlation tensors are sorely needed.

In this paper we develop second-order TRF models of linear hyperelastic media in each of the eight elasticity classes. That is, for each class, the fourth-rank elasticity tensor is taken as an isotropic and homogeneous random field in a three-dimensional Euclidean space, for which the one-point (mean) and two-point correlation functions need to be explicitly specified. The simplest case is that of an isotropic class, which implies that two Lamé constants are random fields. Next, we develop representations of seven higher crystal classes: cubic, transversely isotropic, trigonal, tetragonal, orthotropic, monoclinic, and triclinic. We also find the general form of field's spectral expansion for each of the eight isotropy classes.

2 The Formulation of the Problem

Let $E = E^3$ be a three-dimensional Euclidean point space, and let V be the translation space of E with an inner product $(\cdot, \cdot)$. Following [35], the elements A of E are called the *places* in E. The symbol $B - A$ is the vector in V that translates A into B.

Let $\mathscr{B} \subset E$ be a deformable body. The *strain tensor* $\varepsilon(A)$, $A \in \mathscr{B}$, is a *configuration variable* taking values in the symmetric tensor square $\mathsf{S}^2(V)$ of dimension 6. Following [25], we call this space a *state tensor space*.

The *stress tensor* $\sigma(A)$ also takes values in $\mathsf{S}^2(V)$. This is a *source variable*, it describes the source of a field [34].

We work with materials obeying *Hooke's law* linking the configuration variable $\varepsilon(A)$ with the source variable $\sigma(A)$ by

$$\sigma(A) = \mathsf{C}(A)\varepsilon(A), \quad A \in \mathscr{B}.$$

Here the *elastic modulus* C is a linear map $\mathsf{C}(A) \colon \mathsf{S}^2(V) \to \mathsf{S}^2(V)$. In linearised hyperelasticity, the map $\mathsf{C}(A)$ is symmetric, i.e., an element of a *constitutive tensor space* $\mathsf{V} = \mathsf{S}^2(\mathsf{S}^2(V))$ of dimension 21.

We assume that $\mathsf{C}(A)$ is a single realisation of a *random field*. In other words, denote by $\mathfrak{B}(\mathsf{V})$ the σ-field of Borel subsets of V. There is a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ and a mapping $\mathsf{C} \colon \mathscr{B} \times \Omega \to \mathsf{V}$ such that for any $A_0 \in \mathscr{B}$ the mapping $\mathsf{C}(A_0, \omega) \colon \Omega \to \mathsf{V}$ is $(\mathfrak{F}, \mathfrak{B}(\mathsf{V}))$-measurable.

Translate the whole body $\mathscr{B}$ by a vector $\mathbf{x} \in V$. The random fields $\mathsf{C}(A + \mathbf{x})$ and $\mathsf{C}(A)$ have the same finite-dimensional distributions. It is therefore convenient to assume that there is a random field defined *on all of* E such that its restriction to $\mathscr{B}$ is equal to $\mathsf{C}(A)$. For brevity, denote the new field by the same symbol $\mathsf{C}(A)$ (but this time $A \in E$). The random field $\mathsf{C}(A)$ is *strictly homogeneous*, that is, the random fields $\mathsf{C}(A + \mathbf{x})$ and $\mathsf{C}(A)$ have the same finite-dimensional distributions. In other words, for each positive integer n, for each $\mathbf{x} \in V$, and for all distinct places $A_1, \dots, A_n \in E$ the random elements $\mathsf{C}(A_1) \oplus \cdots \oplus \mathsf{C}(A_n)$ and $\mathsf{C}(A_1 + \mathbf{x}) \oplus \cdots \oplus \mathsf{C}(A_n + \mathbf{x})$ of the direct sum on n copies of the space V have the same probability distribution.

Let K be the material symmetry group of the body $\mathscr{B}$ acting in V. The group K is a subgroup of the orthogonal group $\mathrm{O}(V)$. Fix a place $O \in \mathscr{B}$ and identify E with V by the map f that maps $A \in E$ to $A - O \in V$. Then K acts in E and rotates the body $\mathscr{B}$ by

$$g \cdot A = f^{-1} g f A, \quad g \in K, \ A \in \mathscr{B}.$$

Let $A_0 \in \mathscr{B}$. Under the above action of K the point A_0 becomes $g \cdot A_0$. The random tensor $\mathsf{C}(A_0)$ becomes $\mathsf{S}^2(\mathsf{S}^2(g))\mathsf{C}(A_0)$. The random fields $\mathsf{C}(g \cdot A)$ and $\mathsf{S}^2(\mathsf{S}^2(g))\mathsf{C}(A)$ must have the same finite-dimensional distributions, because $g \cdot A_0$ is the same material point in a different place. Note that this property does not depend on a particular choice of the place O, because the field is strictly homogeneous.

To formalise the non-formal considerations of the above paragraph, note that the map $g \mapsto \mathsf{S}^2(\mathsf{S}^2(g))$ is an *orthogonal representation* of the group K, that is, a continuous map from K to the orthogonal group $\mathrm{O}(\mathsf{V})$ that respects the group operations:

$$\mathsf{S}^2\big(\mathsf{S}^2(g_1 g_2)\big) = \mathsf{S}^2\big(\mathsf{S}^2(g_1)\big)\mathsf{S}^2\big(\mathsf{S}^2(g_2)\big), \quad g_1, g_2 \in K.$$

Let U be an arbitrary orthogonal representation of the group K in a real finite-dimensional linear space V with an inner product $(\cdot, \cdot)$, and let O be a place in E. A V-valued field $\mathsf{C}(A)$

is called *strictly isotropic* with respect to O if for any $g \in K$ the random fields $\mathsf{C}(g \cdot A)$ and $U(g)\mathsf{C}(A)$ have the same finite-dimensional distributions. If in addition the random field $\mathsf{C}(A)$ is strictly homogeneous, then it is strictly isotropic with respect to any place.

Assume that the random field $\mathsf{C}(A)$ is *second-order*, that is

$$\mathsf{E}\big[\|\mathsf{C}(A)\|^2\big] < \infty, \quad A \in E.$$

Define the *one-point correlation tensor* of the field $\mathsf{C}(A)$ by

$$\big\langle \mathsf{C}(A) \big\rangle = \mathsf{E}\big[\mathsf{C}(A)\big]$$

and its *two-point correlation tensor* by

$$\big\langle \mathsf{C}(A), \mathsf{C}(B) \big\rangle = \mathsf{E}\big[\big(\mathsf{C}(A) - \big\langle \mathsf{C}(A) \big\rangle\big) \otimes \big(\mathsf{C}(B) - \big\langle \mathsf{C}(B) \big\rangle\big)\big].$$

Assume that the field $\mathsf{C}(A)$ is *mean-square continuous*, that is, its two-point correlation tensor $\langle \mathsf{C}(A), \mathsf{C}(B) \rangle \colon E \times E \to \mathsf{V} \otimes \mathsf{V}$ is a continuous function. If the field $\mathsf{C}(A)$ is strictly homogeneous, then its one-point correlation tensor is a constant tensor in V, while its two-point correlation tensor is a function of the vector $B - A$, i.e., a function on V. Call such a field *wide-sense homogeneous*.

Similarly, if the field $\mathsf{C}(A)$ is strictly isotropic, then we have

$$\begin{aligned} \big\langle \mathsf{C}(g \cdot A) \big\rangle &= U(g) \big\langle \mathsf{C}(A) \big\rangle, \\ \big\langle \mathsf{C}(g \cdot A), \mathsf{C}(g \cdot B) \big\rangle &= (U \otimes U)(g) \big\langle \mathsf{C}(A), \mathsf{C}(B) \big\rangle. \end{aligned}$$

Call such a field *wide-sense isotropic*. In what follows, we consider only wide-sense homogeneous and isotropic random fields and omit the words "wide-sense".

For simplicity, identify the field $\{\, \mathsf{C}(A) \colon A \in E \,\}$ defined on E with the field $\{\, \mathsf{C}'(\mathbf{x}) \colon \mathbf{x} \in V \,\}$ defined by $\mathsf{C}'(\mathbf{x}) = \mathsf{C}(O + \mathbf{x})$. Introduce the Cartesian coordinate system (x, y, z) in V. Use the introduced system to identify V with the coordinate space $\mathbb{R}^3$ and $\mathrm{O}(V)$ with $\mathrm{O}(3)$. The action of $\mathrm{O}(3)$ on $\mathbb{R}^3$ is the matrix-vector multiplication.

Forte and Vianello [7] proved the existence of 8 symmetry classes of elasticity tensors, or *elasticity classes*. In other words, consider the action

$$g \cdot \mathsf{C} = \mathsf{S}^2\big(\mathsf{S}^2(g)\big)\mathsf{C}$$

of the group $K = \mathrm{O}(3)$ in the space $\mathsf{V} = \mathsf{S}^2(\mathsf{S}^2(\mathbb{R}^3))$. The symmetry group of an elasticity tensor $\mathsf{C} \in \mathsf{V}$ is

$$K(\mathsf{C}) = \big\{\, g \in \mathrm{O}(V) \colon g \cdot \mathsf{C} = \mathsf{C} \,\big\}.$$

Note that the symmetry group $K(g \cdot \mathsf{C})$ is conjugate through g to $K(\mathsf{C})$:

$$K(g \cdot \mathsf{C}) = \big\{\, ghg^{-1} \colon h \in K(\mathsf{C}) \,\big\}. \tag{5}$$

Whenever two bodies can be rotated so that their symmetry groups coincide, they share the same symmetry class. Mathematically, two elasticity tensors C_1 and C_2 are equivalent if and only if there is $g \in \mathrm{O}(3)$ such that $K(\mathsf{C}_1) = K(g \cdot \mathsf{C}_2)$. In view of (5), C_1 and C_2 are equivalent if and only if their symmetry groups are conjugate. The equivalence classes of the above relation are called the *elasticity classes*.

The first column of Table 1 adapted from [2], contains the name of an elasticity class. The second column represents a collection of subgroups H of $\mathrm{O}(3)$ such that H is conjugate

Table 1 Elasticity classes

Elasticity class	H	$N(H)$
Triclinic	Z_2^c	$\mathrm{O}(3)$
Monoclinic	$Z_2 \times Z_2^c$	$\mathrm{O}(2) \times Z_2^c$
Orthotropic	$D_2 \times Z_2^c$	$\mathscr{O} \times Z_2^c$
Trigonal	$D_3 \times Z_2^c$	$D_6 \times Z_2^c$
Tetragonal	$D_4 \times Z_2^c$	$D_8 \times Z_2^c$
Transverse isotropic	$\mathrm{O}(2) \times Z_2^c$	$\mathrm{O}(2) \times Z_2^c$
Cubic	$\mathscr{O} \times Z_2^c$	$\mathscr{O} \times Z_2^c$
Isotropic	$\mathrm{O}(3)$	$\mathrm{O}(3)$

to a symmetry group of any elasticity tensor of the given class. In other words, the above symmetry group lies in the conjugacy class $[H]$ of the group H. The third column contains the notation for the normaliser $N(H)$:

$$N(H) = \left\{ g \in \mathrm{O}(3) \colon gHg^{-1} = H \right\}.$$

Here $Z_2^c = \{I, -I\}$, where I is the 3×3 identity matrix, Z_n is generated by the rotation about the z-axis with angle $2\pi/n$, $\mathrm{O}(2)$ is generated by rotations about the z-axis with angle θ, $0 \le \theta < 2\pi$ and the rotation about the x-axis with angle π, D_n is the *dihedral group* generated by Z_n and the rotation about the x-axis with angle π, and $\mathscr{O}$ is the *octahedral group* which fixes an octahedron. See also [26, Appendix B] for the correspondence between the above notation and notation of Hermann–Mauguin [16, 23] and Schönfließ [32].

The importance of the group $N(H)$ can be clarified as follows. Consider the *fixed point set* of H:

$$\mathsf{V}^H = \{ \mathsf{C} \in \mathsf{V} \colon g \cdot \mathsf{C} = \mathsf{C} \text{ for all } g \in H \}.$$

By [2, Lemma 3.1], if H is the symmetry group of some tensor $\mathsf{C} \in \mathsf{V}$, then $N(H)$ is the maximal subgroup of $\mathrm{O}(3)$ which leaves V^H invariant. In the language of the representation theory, V^H is an *invariant subspace* of the representation $g \mapsto \mathsf{S}^2(\mathsf{S}^2(g))$ of any group K that lies between H and $N(H)$, that is, $\mathsf{S}^2(\mathsf{S}^2(g))\mathsf{C} \in \mathsf{V}^H$ for all $g \in K$ and for all $\mathsf{C} \in \mathsf{V}^H$. Denote by $U(g)$ the restriction of the above representation to V^H.

The problem is formulated as follows. For each elasticity class $[H]$ and for each group K that lies between H and $N(H)$, consider an V^H-valued homogeneous random field $\mathsf{C}(\mathbf{x})$ on $\mathbb{R}^3$. Assume that $\mathsf{C}(\mathbf{x})$ is isotropic with respect to U:

$$\begin{aligned} \langle \mathsf{C}(g\mathbf{x}) \rangle &= U(g) \langle \mathsf{C}(\mathbf{x}) \rangle, \\ \langle \mathsf{C}(g\mathbf{x}), \mathsf{C}(g\mathbf{y}) \rangle &= (U \otimes U)(g) \langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y}) \rangle. \end{aligned} \tag{6}$$

We would like to *find the general form of the one- and two-point correlation tensors of such a field and the spectral expansion of the field itself in terms of stochastic integrals*.

To explain what we mean consider the simplest case when the answer is known. Put $K = H = \mathrm{O}(3)$, $\mathsf{V}^H = \mathbb{R}^1$, and $U(g) = 1$, the *trivial representation* of K. Recall that a measure Φ on the σ-field of Borel sets of a Hausdorff topological space X is called *tight* if for any Borel set B, $\Phi(B)$ is the supremum of $\Phi(K)$ over all compact subsets K of B. A measure Φ is called *locally finite* if every point of X has a neighbourhood U for which $\Phi(U)$ is finite. A measure Φ is called a *Radon measure* if it is tight and locally finite. In what follows we consider only Radon measures and call them just measures.

Schoenberg [31] proved that the equation

$$\langle\tau(\mathbf{x}),\tau(\mathbf{y})\rangle=\int_0^\infty\frac{\sin(\lambda\|\mathbf{y}-\mathbf{x}\|)}{\lambda\|\mathbf{y}-\mathbf{x}\|}\,\mathrm{d}\Phi(\lambda)$$

establishes a one-to-one correspondence between the class of two-point correlation tensors of homogeneous and isotropic random fields $\tau(\mathbf{x})$ and the class of finite measures on $[0,\infty)$.

Let $L^2_0(\Omega)$ be the Hilbert space of centred complex-valued random variables with finite variance. Let Z be a $L^2_0(\Omega)$-valued measure on the σ-field of Borel sets of a Hausdorff topological space X. A measure Φ is called the *control measure* for Z, if for any Borel sets B_1 and B_2 we have

$$\mathsf{E}\big[\overline{Z(B_1)}Z(B_2)\big]=\Phi(B_1\cap B_2).$$

Yaglom [37] and independently M.Ĭ. Yadrenko in his unpublished PhD thesis proved that the field $\tau(\mathbf{x})$ has the form

$$\tau(\rho,\theta,\varphi)=C+\pi\sqrt{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}S^m_\ell(\theta,\varphi)\int_0^\infty\frac{J_{\ell+1/2}(\lambda\rho)}{\sqrt{\lambda\rho}}\,\mathrm{d}Z^m_\ell(\lambda),$$

where $C=\langle\tau(\mathbf{x})\rangle\in\mathbb{R}^1$, (ρ,θ,φ) are spherical coordinates in $\mathbb{R}^3$, $S^m_\ell(\theta,\varphi)$ are real-valued spherical harmonics, $J_{\ell+1/2}(\lambda\rho)$ are the Bessel functions of the first kind of order $\ell+1/2$, and Z^m_ℓ is a sequence of centred uncorrelated real-valued orthogonal random measures on $[0,\infty)$ with the measure Φ as their common control measure.

Other known results include the case of $\mathsf{V}^H=\mathbb{R}^3$, and $U(g)=g$. Yaglom [36] found the general form of the two-point correlation tensor. Malyarenko and Ostoja-Starzewski [22] found the spectral expansion of the field. In the same paper, they found both the general form of the two-point correlation tensor and the spectral expansion of the field for the case of $\mathsf{V}^H=\mathsf{S}^2(\mathbb{R}^3)$, and $U(g)=\mathsf{S}^2(g)$. In [20] they solved one of the cases for two-dimensional elasticity, when $V=\mathbb{R}^2$, $K=\mathrm{O}(2)$, $\mathsf{V}^H=\mathsf{S}^2(\mathsf{S}^2(\mathbb{R}^2))$, and $U(g)=\mathsf{S}^2(\mathsf{S}^2(g))$.

Remark 1 Another approach to studying tensor-valued random fields was elaborated by Guilleminot and Soize [11–15]. Using a stochastic model alternative to our model, they constructed a generator for random fields that have prescribed symmetry properties, take values in the set of symmetric nonnegative-definite tensors, depend on a few real parameters, and may be easily simulated and calibrated. The question of constructing a generator with similar properties based on the stochastic model described below raises several interesting issues and will be considered in forthcoming publications.

3 A General Result

The idea of this Section is as follows. Let V be a finite-dimensional real linear space, let K be a closed subgroup of the group $\mathrm{O}(3)$, and let U be an orthogonal representation of the group K in the space V. Consider a homogeneous and isotropic random field $\mathsf{C}(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^3$, and solve the problem formulated in Section 2. In Section 5, apply general formulae to our cases. The resulting Theorems 1–16 are particular cases of general Theorem 0.

To obtain general formulae, we describe all homogeneous random fields taking values in V and throw away non-isotropic ones. The first obstacle here is as follows. The complete description of such fields is unknown. We use the following result instead.

Let $\mathsf{V}^{\mathbb{C}}$ be a *complex* finite-dimensional linear space with an inner product $(\cdot,\cdot)$ that is linear in the *second* argument, as is usual in physics. Let J be a *real structure* on $\mathsf{V}^{\mathbb{C}}$, that is, a map $J\colon \mathsf{V}^{\mathbb{C}} \to \mathsf{V}^{\mathbb{C}}$ satisfying the following conditions:

$$J(\alpha\mathsf{C}_1 + \beta\mathsf{C}_2) = \overline{\alpha}J(\mathsf{C}_1) + \overline{\beta}J(\mathsf{C}_2),$$
$$J\bigl(J(\mathsf{C})\bigr) = \mathsf{C}$$

for all α, $\beta \in \mathbb{C}$ and for all C_1, $\mathsf{C}_2 \in \mathsf{V}^{\mathbb{C}}$. In other words, J is a multidimensional and coordinate-free generalisation of complex conjugation. The set of all eigenvectors of J that correspond to eigenvalue 1, constitute a *real* linear space, denote it by V. Let H be the real linear space of Hermitian linear operators in $\mathsf{V}^{\mathbb{C}}$. The real structure J induces a linear operator J in H. For any $A \in \mathsf{H}$, the operator $\mathsf{J}A$ acts by

$$(\mathsf{J}A)\mathsf{C} = J(AJ\mathsf{C}), \quad \mathsf{C} \in \mathsf{V}^{\mathbb{C}}.$$

In coordinates, the operator J is just the transposition of a matrix.

The result by Cramér [5] in coordinate-free form is formulated as follows. Equation

$$\bigl\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y})\bigr\rangle = \int_{\hat{\mathbb{R}}^3} \mathrm{e}^{\mathrm{i}(\mathbf{p},\mathbf{y}-\mathbf{x})}\,\mathrm{d}F(\mathbf{p}) \tag{7}$$

establishes a one-to-one correspondence between the class of two-point correlation tensors of homogeneous mean-square continuous $\mathsf{V}^{\mathbb{C}}$-valued random fields $\mathsf{C}(\mathbf{x})$ and the class of measures on the σ-field of Borel sets of the *wavenumber domain* $\hat{\mathbb{R}}^3$ tasking values in the set of nonnegative-definite Hermitian linear operators in $\mathsf{V}^{\mathbb{C}}$. For V-valued random fields, there is only a *necessary condition*: if $\mathsf{C}(\mathbf{x})$ is V-valued, then the measure F satisfies

$$F(-B) = \mathsf{J}F(B), \quad B \in \mathfrak{B}\bigl(\hat{\mathbb{R}}^3\bigr),$$

where $-B = \{-\mathbf{p} \colon \mathbf{p} \in B\}$.

Introduce the *trace measure* μ by $\mu(B) = \operatorname{tr} F(B)$, $B \in \mathfrak{B}(\hat{\mathbb{R}}^3)$ and note that F is absolutely continuous with respect to μ. This means that (7) may be written as

$$\bigl\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y})\bigr\rangle = \int_{\hat{\mathbb{R}}^3} \mathrm{e}^{\mathrm{i}(\mathbf{p},\mathbf{y}-\mathbf{x})} f(\mathbf{p})\,\mathrm{d}\mu(\mathbf{p}),$$

where $f(\mathbf{p})$ is a measurable function on the wavenumber domain taking values in the set of all nonnegative-definite Hermitian linear operators in $\mathsf{V}^{\mathbb{C}}$ with unit trace, that satisfies the following condition

$$f(-\mathbf{p}) = Jf(\mathbf{p}). \tag{8}$$

Using representation theory, it is possible to prove the following. Let C_1, $\mathsf{C}_2 \in \mathsf{V}$. Let $L(\mathsf{C}_1 \otimes \mathsf{C}_2)$ be the operator in H acting on a tensor $\mathsf{C} \in \mathsf{V}^{\mathbb{C}}$ by

$$L(\mathsf{C}_1 \otimes \mathsf{C}_2)\mathsf{C} = (J\mathsf{C}_1, \mathsf{C})\mathsf{C}_2.$$

By linearity, this action may be extended to an isomorphism L between $\mathsf{V} \otimes \mathsf{V}$ and H. The orthogonal operators $LU \otimes U(g)L^{-1}$, $g \in K$, constitute an orthogonal representation of the group K in the space H, *equivalent* to the tensor square $U \otimes U$ of the representation U. The operator L is an *intertwining operator* between the spaces $\mathsf{V} \otimes \mathsf{V}$ and H where equivalent representations $U \otimes U$ and $LU \otimes UL^{-1}$ act. In what follows, we are working only with the latter representation, for simplicity denote it again by $U \otimes U$ and note that it acts in the space H by

$$(U \otimes U)(g)A = U(g)AU^{-1}(g), \quad A \in \mathsf{H}.$$

Denote $\mathsf{H}_+ = L\mathsf{S}^2(\mathsf{V})$. In coordinates, it is the subspace of Hermitian matrices with real-valued matrix entries. If $-I \in K$, then the second equation in (6) and (8) together are equivalent to the following conditions:

$$\mu(gB) = \mu(B), \quad B \in \mathfrak{B}\big(\hat{\mathbb{R}}^3\big) \tag{9}$$

and

$$f(\mathbf{p}) \in \mathsf{H}_+, \qquad f(g\mathbf{p}) = \mathsf{S}^2\big(U(g)\big)f(\mathbf{p}). \tag{10}$$

The description of all measures μ satisfying (9) is well known, see [3]. There are finitely many, say M, orbit types for the action of K in $\hat{\mathbb{R}}^3$ by

$$(g\mathbf{p}, \mathbf{x}) = \big(\mathbf{p}, g^{-1}\mathbf{x}\big).$$

Denote by $(\hat{\mathbb{R}}^3/K)_m$, $0 \le m \le M-1$ the set of all orbits of the mth type. It is known, see [2], that all the above sets are manifolds. Assume for simplicity of notation that there are charts $\boldsymbol{\lambda}_m$ such that the domain of $\boldsymbol{\lambda}_m$ is dense in $(\hat{\mathbb{R}}^3/K)_m$. The orbit of the mth type is the manifold K/H_m, where H_m is a stationary subgroup of a point on the orbit. Assume that the domain of a chart $\boldsymbol{\varphi}_m$ is a dense set in K/H_m, and let $\mathrm{d}\boldsymbol{\varphi}_m$ be the unique probabilistic K-invariant measure on the σ-field of Borel sets of K/H_m. There are the unique measures Φ_m on the σ-fields of Borel sets in $(\hat{\mathbb{R}}^3/K)_m$ such that

$$\int_{\hat{\mathbb{R}}^3} \mathrm{e}^{\mathrm{i}(\mathbf{p},\mathbf{y}-\mathbf{x})} f(\mathbf{p})\,\mathrm{d}\mu(\mathbf{p}) = \sum_{m=0}^{M-1} \int_{(\hat{\mathbb{R}}^3/K)_m} \int_{K/H_m} \mathrm{e}^{\mathrm{i}((\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m),\mathbf{y}-\mathbf{x})} f(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m)\,\mathrm{d}\boldsymbol{\varphi}_m\,\mathrm{d}\Phi_m(\boldsymbol{\lambda}_m).$$

To find all functions f satisfying (10), proceed as follows. Fix an orbit $\boldsymbol{\lambda}_m$ and denote by $\boldsymbol{\varphi}_m^0$ the coordinates of the intersection of the orbit $\boldsymbol{\lambda}_m$ with the set $(\hat{\mathbb{R}}^3/K)_m$. Let U^m be the restriction of the representation $\mathsf{S}^2(U)$ to the group H_m. We have $g(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0) = (\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0)$ for all $g \in H_m$, because H_m is the stationary subgroup of the point $(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0)$. For $g \in H_m$, (10) becomes

$$f\big(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0\big) = U^m(g) f\big(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0\big). \tag{11}$$

Any orthogonal representation of a compact topological group in a space H has at least two invariant subspaces: $\{0\}$ and H. The representation is called *irreducible* if no other invariant subspaces exist. The space of any finite-dimensional orthogonal representation of a compact topological group can be uniquely decomposed into a direct sum of *isotypic subspaces*. Each isotypic subspace is the direct sum of finitely many subspaces where the copies of the same irreducible representation act. Equation (11) means that the operator $f(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0)$ lies in the isotypic subspace H_m which corresponds to the trivial representation of the group H_m. The intersection of this subspace with the convex compact set of all nonnegative-definite operators in H_+ with unit trace is again a convex compact set, call it $\mathscr{C}_m$. As $\boldsymbol{\lambda}_m$ runs over $(\hat{\mathbb{R}}^3/K)_m$, $f(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0)$ becomes an arbitrary measurable function taking values in $\mathscr{C}_m$.

An irreducible orthogonal representation of the group K is called a *representation of class* 1 with respect to the group H_m if the restriction of this representation to H_m contains at least one copy of the trivial representation of H_m. Let $\mathsf{S}^2(U)_m$ be the restriction of the representation $\mathsf{S}^2(U)$ to the direct sum of the isotypic subspaces of the irreducible representation of class 1 with respect to H_m. Let $g_{\boldsymbol{\varphi}_m}$ be an arbitrary element of K such that $g_{\boldsymbol{\varphi}_m}(\boldsymbol{\varphi}_m^0)=\boldsymbol{\varphi}_m$. Two such elements differ by an element of H_m, therefore the second equation in (10) becomes

$$f(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m)=\mathsf{S}^2\big(U(g_{\boldsymbol{\varphi}_m})\big)_m f\big(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0\big).$$

The two-point correlation tensor of the field takes the form

$$\big\langle \mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\big\rangle=\sum_{m=0}^{M-1}\int_{(\hat{\mathbb{R}}^3/K)_m}\int_{K/H_m}\mathrm{e}^{\mathrm{i}(g_{\boldsymbol{\varphi}_m}(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0),\mathbf{y}-\mathbf{x})}\mathsf{S}^2\big(U(g_{\boldsymbol{\varphi}_m})\big)_m \times f\big(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0\big)\,\mathrm{d}\boldsymbol{\varphi}_m\,\mathrm{d}\Phi_m(\boldsymbol{\lambda}_m). \tag{12}$$

Choose an orthonormal basis $\mathsf{T}^1,\dots,\mathsf{T}^{\dim\mathsf{V}}$ in the space V. The tensor square $\mathsf{V}\otimes\mathsf{V}$ has several orthonormal bases. The *coupled basis* consists of tensor products $\mathsf{T}^i\otimes\mathsf{T}^j$, $1\le i$, $j\le\dim\mathsf{V}$. The mth *uncoupled basis* is build as follows. Let $U^{m,1},\dots,U^{m,k_m}$ be all non-equivalent irreducible orthogonal representations of the group K of class 1 with respect to H_m such that the representation $\mathsf{S}^2(U)$ contains isotypic subspaces where c_{mk} copies of the representation $U^{m,k}$ act, and let the restriction of the representation $U^{m,k}$ to H_m contains d_{mk} copies of the trivial representation of H_m. Let T^{mkln}, $1\le l\le d_{mk}$, $1\le n\le c_{mk}$ be an orthonormal basis in the space where the nth copy act. Complete the above basis to the basis T^{mkln}, $1\le l\le\dim U^{m,k}$ and call this basis the mth uncoupled basis. The vectors of the coupled basis are linear combinations of the vectors of the mth uncoupled basis:

$$\mathsf{T}^i\otimes\mathsf{T}^j=\sum_{k=1}^{k_m}\sum_{l=1}^{\dim U^{m,k}}\sum_{n=1}^{c_{mk}}c_{ij}^{mkln}\mathsf{T}^{mkln}+\cdots,$$

where dots denote the terms that include the tensors in the basis of the space $\mathsf{S}^2(\mathsf{V})\ominus\mathsf{S}^2(\mathsf{V})_m$. In the introduced coordinates, (12) takes the form

$$\big\langle \mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\big\rangle_{ij}=\sum_{m=0}^{M-1}\sum_{k=1}^{k_m}\sum_{l=1}^{\dim U^{m,k}}\sum_{l'=1}^{d_{mk}}\sum_{n=1}^{c_{mk}}c_{ij}^{mkln}\int_{(\hat{\mathbb{R}}^3/K)_m}\int_{K/H_m}\mathrm{e}^{\mathrm{i}(g_{\boldsymbol{\varphi}_m}(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0),\mathbf{y}-\mathbf{x})} \times U_{ll'}^{m,k}(\boldsymbol{\varphi}_m)f_{l'n}\big(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}_m^0\big)\,\mathrm{d}\boldsymbol{\varphi}_m\,\mathrm{d}\Phi_m(\boldsymbol{\lambda}_m). \tag{13}$$

The choice of bases inside the isotypic subspaces is not unique. One has to choose them in such a way that calculation of the transition coefficients c_{ij}^{mkln} is as easy as possible.

To calculate the inner integrals, we proceed as follows. Consider the action of K on $\mathbb{R}^3$ by matrix-vector multiplication. Let $(\mathbb{R}^3/K)_m$, $0\le m\le M-1$ be the set of all orbits of the mth type. Let $\boldsymbol{\rho}_m$ be such a chart that its domain is dense in $(\mathbb{R}^3/K)_m$. Let $\boldsymbol{\psi}_m$ be a chart in K/H_m with a dense domain, and let $\mathrm{d}\boldsymbol{\psi}_m$ be the unique probabilistic K-invariant measure on the σ-field of Borel sets of K/H_m. It is known that the sets of orbits of one of the types, say $(\hat{\mathbb{R}}^3/K)_{M-1}$ (resp. $(\mathbb{R}^3/K)_{M-1}$), are dense in $\hat{\mathbb{R}}^3$ (resp. $\mathbb{R}^3$). Write the plane wave $\mathrm{e}^{\mathrm{i}(g_{\boldsymbol{\varphi}_{M-1}}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\varphi}_{M-1}^0),\mathbf{y}-\mathbf{x})}$ as

$$\mathrm{e}^{\mathrm{i}(g_{\boldsymbol{\varphi}_{M-1}}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\varphi}_{M-1}^0),\mathbf{y}-\mathbf{x})}=\mathrm{e}^{\mathrm{i}(g_{\boldsymbol{\varphi}_{M-1}}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\varphi}_{M-1}^0),g_{\boldsymbol{\psi}_{M-1}}(\boldsymbol{\rho}_{M-1},\boldsymbol{\psi}_{M-1}^0))},$$

and consider the plane wave as a function of two variables $\boldsymbol{\varphi}_{M-1}$ and $\boldsymbol{\psi}_{M-1}$ with domain $(K/H_{M-1})^2$. This function is K-invariant:

$$\mathrm{e}^{\mathrm{i}(gg_{\boldsymbol{\varphi}_{M-1}}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\varphi}^0_{M-1}),gg_{\boldsymbol{\psi}_{M-1}}(\boldsymbol{\rho}_{M-1},\boldsymbol{\psi}^0_{M-1}))}=\mathrm{e}^{\mathrm{i}(g_{\boldsymbol{\varphi}_{M-1}}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\varphi}^0_{M-1}),g_{\boldsymbol{\psi}_{M-1}}(\boldsymbol{\rho}_{M-1},\boldsymbol{\psi}^0_{M-1}))},\quad g\in K.$$

Denote by $\hat{K}_{H_{M-1}}$ the set of all equivalence classes of irreducible representations of K of class 1 with respect to H_{M-1}, and let the restriction of the representation $U^q\in\hat{K}_{H_{M-1}}$ to H_{M-1} contains d_q copies of the trivial representation of H_{M-1}. By the Fine Structure Theorem [17], there are some numbers $d'_q\le d_q$ such that the set

$$\left\{\dim U^q\cdot U^q_{ll'}(\boldsymbol{\varphi}_{M-1})U^q_{ll'}(\boldsymbol{\psi}_{M-1})\colon U^q\in\hat{K}_{H_{M-1}},1\le l\le\dim U^q,1\le l'\le d'_q\right\}$$

is the orthonormal basis in the Hilbert space $L^2((K/H_{M-1})^2,\mathrm{d}\boldsymbol{\varphi}_{M-1}\,\mathrm{d}\boldsymbol{\psi}_{M-1})$. Let

$$\begin{aligned}j^q_{ll'}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\rho}_{M-1})=\dim U^q\int_{(K/H_{M-1})^2}&\mathrm{e}^{\mathrm{i}(g_{\boldsymbol{\varphi}_{M-1}}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\varphi}^0_{M-1}),g_{\boldsymbol{\psi}_{M-1}}(\boldsymbol{\rho}_{M-1},\boldsymbol{\psi}^0_{M-1}))}\\&\times U^q_{ll'}(\boldsymbol{\varphi}_{M-1})U^q_{ll'}(\boldsymbol{\psi}_{M-1})\,\mathrm{d}\boldsymbol{\varphi}_{M-1}\,\mathrm{d}\boldsymbol{\psi}_{M-1}\end{aligned}$$

be the corresponding Fourier coefficients. The uniformly convergent Fourier expansion takes the form

$$\begin{aligned}\mathrm{e}^{\mathrm{i}(g_{\boldsymbol{\varphi}_{M-1}}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\varphi}^0_{M-1}),g_{\boldsymbol{\psi}_{M-1}}(\boldsymbol{\rho}_{M-1},\boldsymbol{\psi}^0_{M-1}))}=&\sum_{U^q\in\hat{K}_{H_{M-1}}}\sum_{l=1}^{\dim U^q}\sum_{l'=1}^{d'_q}\dim U^q\\&\times j^q_{ll'}(\boldsymbol{\lambda}_{M-1},\boldsymbol{\rho}_{M-1})U^q_{ll'}(\boldsymbol{\varphi}_{M-1})U^q_{ll'}(\boldsymbol{\psi}_{M-1}).\end{aligned}\tag{14}$$

This expansion is defined on the dense set

$$\left(\hat{\mathbb{R}}^3/K\right)_{M-1}\times(K/H_{M-1})\times\left(\mathbb{R}^3/K\right)_{M-1}\times(K/H_{M-1})$$

and may be extended to all of $\hat{\mathbb{R}}^3\times\mathbb{R}^3$ by continuity. Substituting the extended expansion to (13), we obtain the expansion

$$\begin{aligned}\langle\mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\rangle_{ij}=\sum_{m=0}^{M-1}\sum_{k=1}^{k_m}\sum_{l=1}^{\dim U^{m,k}}\sum_{l'=1}^{d'_{mk}}\sum_{n=1}^{c_{mk}}c^{mkln}_{ij}\int_{(\hat{\mathbb{R}}^3/K)_m}&j^q_{ll'}(\boldsymbol{\lambda}_m,\boldsymbol{\rho}_0)\\&\times U^{m,k}_{ll'}(\boldsymbol{\psi}_m)f_{l'n}\big(\boldsymbol{\lambda}_m,\boldsymbol{\varphi}^0_m\big)\,\mathrm{d}\Phi_m(\boldsymbol{\lambda}_m).\end{aligned}\tag{15}$$

Theorem 0 *Let $-I\in K$. The one-point correlation tensor of a homogeneous and (K,U)-isotropic random field lies in the space of the isotypic component of the representation U that corresponds to the trivial representation of K and is equal to $\mathsf{0}$ if no such isotypic component exists. Its two-point correlation tensor is given by* (15).

Remark 2 The results by [20, 22, 31, 36, 37] as well as Theorems 1–16 below are particular cases of Theorem 0. The expansion (15) is the first necessary step in studying random fields connected to Hooke's law.

Later we will see that it is easy to write the spectral expansion of the field directly if the group K is finite. Otherwise, we write the Fourier expansion (14) for plane waves $\mathrm{e}^{\mathrm{i}(\mathbf{p},\mathbf{y})}$ and $\mathrm{e}^{-\mathrm{i}(\mathbf{p},\mathbf{x})}$ separately and substitute both expansions to (13). As a result, we obtain the expansion of the two-point correlation tensor of the field in the form

$$\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y})\rangle_{ij} = \int_{\Lambda} \overline{u(\mathbf{x},\lambda)}u(\mathbf{y},\lambda)\,\mathrm{d}\Phi_{ij}(\lambda),$$

where Λ is a set, and where F is a measure on a σ-field $\mathfrak{L}$ of subsets of Λ taking values in the set of Hermitian nonnegative-definite operators on $\mathsf{V}^{\mathbb{C}}$. Moreover, the set $\{u(\mathbf{x},\lambda) \colon \mathbf{x} \in \mathbb{R}^3\}$ is *total* in the Hilbert space $L^2(\Lambda, \Phi)$ of the measurable complex-valued functions on Λ that are square-integrable with respect to the measure Φ, that is, the set of finite linear combinations $\sum c_n u(\mathbf{x}_n, \lambda)$ is dense in the above space. By Karhunen's theorem [18], the field $\mathsf{C}(\mathbf{x})$ has the following spectral expansion:

$$\mathsf{C}(\mathbf{x}) = \mathsf{E}\big[\mathsf{C}(\mathbf{0})\big] + \int_{\Lambda} u(\mathbf{x},\lambda)\,\mathrm{d}\mathsf{Z}(\lambda), \tag{16}$$

where Z is a measure on the measurable space $(\Lambda, \mathfrak{L})$ taking values in the Hilbert space of random tensors $\mathsf{Z} \colon \Omega \to \mathsf{V}^{\mathbb{C}}$ with $\mathsf{E}[\mathsf{Z}] = \mathsf{0}$ and $\mathsf{E}[\|\mathsf{Z}\|^2] < \infty$. The measure F is the *control measure* of the measure Z, i.e.,

$$\mathsf{E}\big[J\mathsf{Z}(A)\mathsf{Z}^{\top}(B)\big] = \Phi(A \cap B), \quad A, B \in \mathfrak{L}.$$

The components of the random tensor $\mathsf{Z}(A)$ are correlated, which creates difficulties when one tries to use (16) for computer simulation. It is possible to use Cholesky decomposition and to write the expansion of the field using uncorrelated random measures, see details in [22].

4 Preliminary Calculations

The possibilities for the group K are as follows. In the triclinic class, there exist infinitely many groups between Z_2^c and O(3), we put $K_1 = Z_2^c$ and $K_2 = \mathrm{O}(3)$. Similarly, for the monoclinic class put $K_3 = Z_2 \times Z_2^c$ and $K_4 = \mathrm{O}(2) \times Z_2^c$. The possibilities for the orthotropic class are $K_5 = D_2 \times Z_2^c$, $K_6 = D_4 \times Z_2^c$, $K_7 = D_6 \times Z_2^c$, $K_8 = \mathscr{T} \times Z_2^c$, and $K_9 = \mathscr{O} \times Z_2^c$. Here $\mathscr{T}$ is the *tetrahedral group* which fixes a tetrahedron. In the trigonal class, we have $K_{10} = D_3 \times Z_2^c$ and $K_{11} = D_6 \times Z_2^c$. In the tetragonal class, the possibilities are $K_{12} = D_4 \times Z_2^c$ and $K_{13} = D_8 \times Z_2^c$. In the three remaining classes, the possibilities are $K_{14} = \mathrm{O}(2) \times Z_2^c$, $K_{15} = \mathscr{O} \times Z_2^c$, and $K_{16} = \mathrm{O}(3)$. The intermediate groups were determined using [4, Vol. 1, Fig. 10.1.3.2]. For each group K_i, $1 \le i \le 16$, we formulate Theorem number i below.

4.1 The Structure of the Representation U

The notation for irreducible orthogonal representation is as follows. If K_i is a *finite* group, we use the *Mulliken notation* [24], see also [1, Chapter 14] to denote the irreducible *unitary* representation of K_i. For an irreducible orthogonal representation, consider its complexification. A standard result of representation theory, see, for example, [6, Proposition 4.8.4], states that there are three possibilities:

Table 2 The structure of the representation U

K_i	Table number	The structure of U
$K_1 = Z_2^c$	11	$21A_g$
$K_2 = \mathrm{O}(3)$	–	$2U^{0g} \oplus 2U^{2g} \oplus U^{4g}$
$K_3 = Z_2 \times Z_2^c$	60	$13A_g$
$K_4 = \mathrm{O}(2) \times Z_2^c$	–	$5U^{0gg} \oplus 3U^{2g} \oplus U^{4g}$
$K_5 = D_2 \times Z_2^c$	31	$9A_g$
$K_6 = D_4 \times Z_2^c$	33	$6A_{1g} \oplus 3B_{1g}$
$K_7 = D_6 \times Z_2^c$	35	$5A_{1g} \oplus 2E_{2g}$
$K_8 = \mathscr{T} \times Z_2^c$	72	$3A_g \oplus 3({}^1E_g \oplus {}^2E_g)$
$K_9 = \mathscr{O} \times Z_2^c$	71	$3A_{1g} \oplus 3E_g$
$K_{10} = D_3 \times Z_2^c$	42	$6A_{1g}$
$K_{11} = D_6 \times Z_2^c$	35	$5A_{1g} \oplus B_{1g}$
$K_{12} = D_4 \times Z_2^c$	33	$6A_{1g}$
$K_{13} = D_8 \times Z_2^c$	37	$5A_{1g} \oplus B_{2g}$
$K_{14} = \mathrm{O}(2) \times Z_2^c$	–	$5U^{0gg}$
$K_{15} = \mathscr{O} \times Z_2^c$	71	$3A_{1g}$
$K_{16} = \mathrm{O}(3)$	–	$2U^{0g}$

– The complexification is irreducible, say U. Then, it is a sum of two equivalent orthogonal representations, and we denote each of them by U.
– The complexification is a direct sum of two mutually conjugate representation U_1 and U_2, that is, $U_2(g) = \overline{U_1(g)}$. We denote the orthogonal representation by $U_1 \oplus U_2$.
– The complexification is a direct sum of two copies of an irreducible representation U. We denote the orthogonal representation by $U \oplus U$.

For infinite groups, the notation is as follows. For $K_2 = \mathrm{O}(3)$, we denote the representations by $U^{\ell g}$ (the tensor product of the representation U^ℓ of the group SO(3) and the trivial representation A_g of Z_2^c) and $U^{\ell u}$ (that of U^ℓ and the nontrivial representation A_u of Z_2^c). For $K_4 = K_{14} = \mathrm{O}(2) \times Z_2^c$ the notation is $U^{0gg} = U^{0g} \otimes A_g$, $U^{0gu} = U^{0g} \otimes A_u$, $U^{0ug} = U^{0u} \otimes A_g$, $U^{0uu} = U^{0u} \otimes A_u$, $U^{\ell g} = U^\ell \otimes A_g$, and $U^{\ell u} = U^\ell \otimes A_u$, where U^{0g} is the trivial representation of O(2), $U^{0u}(g) = \det g$, and

$$U^\ell\left(\begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix}\right) = \begin{pmatrix} \cos(\ell\varphi) & \sin(\ell\varphi) \\ -\sin(\ell\varphi) & \cos(\ell\varphi) \end{pmatrix},$$

$$U^\ell\left(\begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{pmatrix}\right) = \begin{pmatrix} \cos(\ell\varphi) & \sin(\ell\varphi) \\ \sin(\ell\varphi) & -\cos(\ell\varphi) \end{pmatrix}.$$

Fist, we determine the structure of the representation $g \mapsto g$ of the group K_i. For finite groups, the above structure is given in Table n.10 in [1], where n in the number given in the second column of Table 2. For K_2 and K_{16}, this representation is U^{1u}, for K_4 and K_{14} it is $U^{1u} \oplus U^{0uu}$. Then we determine the structure of the representations $\mathsf{S}^2(g)$ and $\mathsf{S}^2(\mathsf{S}^2(g))$. For finite groups, we use Table n.8. For infinite groups, we use the following multiplication rules. The product of two isomorphic irreducible representations of Z_2^c is A_g, that of two different representations is A_u. For SO(3), we have

$$U^{\ell_1} \otimes U^{\ell_2} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \oplus U^\ell.$$

Table 3 The isotropy subgroups of the groups K_i

K_i	H_1	H_2	H_3	H_4	H_5	H_6
K_1	I					
K_2, K_{16}	O(2)					
K_3	Z_2	Z_2^-	I			
K_4, K_{14}	O(2)	$Z_2^- \times Z_2^c$	Z_2^-			
K_5	Z_2	Z_2^-	I			
K_6, K_{12}	Z_4	Z_2	Z_2^-	I		
K_7, K_{11}	$Z_3 \times Z_2^c$	Z_2	Z_2^-	I		
K_8	D_3	D_2	Z_2^c	I		
K_9, K_{15}	D_3	D_4	D_2	$\tilde{Z}_2$	Z_2	I
K_{10}	Z_3	Z_2^-	Z_2	I		
K_{13}	Z_8	Z_2	Z_2^-	I		

For O(2), we have $U^\ell \otimes U^\ell = U^{2\ell} \oplus U^{0g} \oplus U^{0u}$ and $U^{\ell_1} \otimes U^{\ell_2} = U^{\ell_1+\ell_2} \oplus U^{|\ell_1-\ell_2|}$ for $\ell_2 \neq \ell_1$.

If $K_i = H$, then the space V is spanned by the spaces of the copies of all trivial representations of K_i that belong to $\mathsf{S}^2(\mathsf{S}^2(g))$. This gives us a method for calculation of the dimension $\dim \mathsf{V}$ alternative to that in [2]. Otherwise, it is spanned by the spaces of all irreducible representations of K_i that contain at least one copy of the trivial representation of H. To determine such representations, we use Table n.9 from [1].

4.2 The Basis of the Space V^H for Different Groups

We start from the basis for K_2. Gordienko [10] proposed a basis $\{\mathbf{h}_\ell^m : -\ell \leq m \leq \ell\}$ in the space of the irreducible representation U^ℓ of the group SO(3) in which all matrix entries of the representation's matrices become real-valued functions. Godunov and Cordienko [8] found the coefficients $g_{\ell[\ell_1,\ell_2]}^{m[m_1,m_2]}$ of the expansion

$$\mathbf{h}_{\ell_1}^{m_1} \otimes \mathbf{h}_{\ell_2}^{m_2} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{m=-\ell}^{\ell} g_{\ell[\ell_1,\ell_2]}^{m[m_1,m_2]} \mathbf{h}_\ell^m .$$

We call them the *Godunov–Gordienko coefficients*. Malyarenko and Ostoja–Starzewski [22] calculated the tensors of the basis of the 21-dimensional space $\mathsf{S}^2(\mathsf{S}^2(\mathbb{R}^3))$ for the group K_2 in terms of the above coefficients, see Table 3 in the electronic version of this paper [21]. Using MATLAB Symbolic Math Toolbox, we calculate the elements of the bases for the groups K_1, K_3–K_{16} as linear combinations of the tensors of the basis for the group K_2, see [21, Table 3].

4.3 The Isotropy Subgroups for the Groups K_i

Table 3 shows the isotropy subgroups of the groups K_i. In this table, Z_2^- is the order 2 group generated by the reflection through the yz-plane, and $\tilde{Z}_2$ is the group generated by a reflection leavind an edge of a cube invariant [9]. The group H_0 is always equal to K_i and therefore is omitted.

4.4 The Orbit Type Stratification

The following formulae describe the orbit type stratification of the orbit space $\hat{\mathbb{R}}^3/K_i$. The zeroth stratum is always equal to $\{\mathbf{0}\}$ and therefore is omitted.

$\hat{\mathbb{R}}^3/K_1$:

$$\left(\hat{\mathbb{R}}^3/K_1\right)_1=\{p_3>0\}\cup\{p_2>0,\,p_3=0\}\cup\left\{\,(p_1,0,0)\colon\,p_1>0\,\right\}.$$

$\hat{\mathbb{R}}^3/K_2$, $\hat{\mathbb{R}}^3/K_{16}$:

$$\left(\hat{\mathbb{R}}^3/K_2\right)_1=\left\{\,(0,0,p_3)\colon\,p_3>0\,\right\}.$$

$\hat{\mathbb{R}}^3/K_3$:

$$\begin{aligned}\left(\hat{\mathbb{R}}^3/K_3\right)_1&=\left\{\,(0,0,p_3)\colon\,p_3>0\,\right\},\\ \left(\hat{\mathbb{R}}^3/K_3\right)_2&=\left\{(p_1\neq 0,0,p_3>0)\right\},\\ \left(\hat{\mathbb{R}}^3/K_3\right)_3&=\left\{\,(p_1,p_2>0,p_3>0)\colon\,p_3>0\,\right\}.\end{aligned}$$

$\hat{\mathbb{R}}^3/K_4$, $\hat{\mathbb{R}}^3/K_{14}$:

$$\begin{aligned}\left(\hat{\mathbb{R}}^3/K_4\right)_1&=\left\{\,(p_1,0,0)\colon\,p_1>0\,\right\},\\ \left(\hat{\mathbb{R}}^3/K_4\right)_2&=\left\{(0,0,p_3)\colon\,p_3>0\right\},\\ \left(\hat{\mathbb{R}}^3/K_4\right)_3&=\left\{\,(p_1,0,p_3)\colon\,p_1>0,\,p_3>0\,\right\}.\end{aligned}$$

$\hat{\mathbb{R}}^3/K_5$:

$$\begin{aligned}\left(\hat{\mathbb{R}}^3/K_5\right)_1&=\left\{\,(\lambda,0,0)\colon\,\lambda>0\,\right\},\\ \left(\hat{\mathbb{R}}^3/K_5\right)_2&=\left\{\,(\lambda,\theta_p,0)\colon\,\lambda>0,0<\theta_p<\pi/2\,\right\},\\ \left(\hat{\mathbb{R}}^3/K_5\right)_3&=\left\{\,(\lambda,\pi/2,\varphi_p)\colon\,\lambda>0,0<\varphi_p<\pi/m\,\right\},\\ \left(\hat{\mathbb{R}}^3/K_5\right)_4&=\left\{\,(\lambda,\theta_p,\varphi_p)\colon\,\lambda>0,0<\theta_p<\pi/2,0<\varphi_p<\pi/m\,\right\}\end{aligned} \tag{17}$$

for $m=1$, where $(\lambda,\theta_p,\varphi_p)$ are the spherical coordinates in $\hat{\mathbb{R}}^3$.

$\hat{\mathbb{R}}^3/K_6$, $\hat{\mathbb{R}}^3/K_{12}$: (17) with $m=2$.

$\hat{\mathbb{R}}^3/K_7$, $\hat{\mathbb{R}}^3/K_{11}$: (17) with $m=3$.

$\hat{\mathbb{R}}^3/K_8$:

$$\begin{aligned}\left(\hat{\mathbb{R}}^3/K_8\right)_1&=\left\{\,(\lambda,\pi/4,0)\colon\,\lambda>0\,\right\},\\ \left(\hat{\mathbb{R}}^3/K_8\right)_2&=\left\{\,(\lambda,0,0)\colon\,\lambda>0\,\right\},\\ \left(\hat{\mathbb{R}}^3/K_8\right)_3&=\left\{\,(\lambda,\theta_p,0)\colon\,\lambda>0,0<\theta_p<\pi/4\,\right\},\\ \left(\hat{\mathbb{R}}^3/K_8\right)_3&=\left\{\,(\lambda,\theta_p,\varphi)\colon\,\lambda>0,0<\varphi_p<\pi/2,0<\theta_p<\cot^{-1}(\sqrt{2}\cos(\varphi_p-\pi/4))\,\right\}.\end{aligned}$$

$\hat{\mathbb{R}}^3/K_9$, $\hat{\mathbb{R}}^3/K_{15}$:

$$\begin{aligned}
\left(\hat{\mathbb{R}}^3/K_9\right)_1 &= \left\{\,(p_1,p_2,p_3)\colon 0<p_1=p_2=p_3\,\right\},\\
\left(\hat{\mathbb{R}}^3/K_9\right)_2 &= \left\{\,(0,0,p_3)\colon p_3>0\,\right\},\\
\left(\hat{\mathbb{R}}^3/K_9\right)_3 &= \left\{\,(0,p_2,p_3)\colon 0<p_2=p_3\,\right\},\\
\left(\hat{\mathbb{R}}^3/K_9\right)_4 &= \left\{\,(0,p_2,p_3)\colon 0<p_2<p_3\,\right\},\\
\left(\hat{\mathbb{R}}^3/K_9\right)_5 &= \left\{\,(p_1,p_2,p_3)\colon 0<p_1=p_2<p_3\,\right\},\\
\left(\hat{\mathbb{R}}^3/K_9\right)_6 &= \left\{\,(p_1,p_2,p_3)\colon 0<p_1<p_2<p_3\,\right\}.
\end{aligned}$$

$\hat{\mathbb{R}}^3/K_{10}$:

$$\begin{aligned}
\left(\hat{\mathbb{R}}^3/K_{10}\right)_1 &= \left\{\,(\lambda,0,0)\colon \lambda>0\,\right\},\\
\left(\hat{\mathbb{R}}^3/K_{10}\right)_2 &= \left\{\,(\lambda,\theta_p,0)\colon \lambda>0, 0<\theta_p<\pi/2\,\right\},\\
\left(\hat{\mathbb{R}}^3/K_{10}\right)_3 &= \left\{\,(\lambda,\pi/2,\varphi_p)\colon \lambda>0, 0<\varphi_p<\pi/3\,\right\},\\
\left(\hat{\mathbb{R}}^3/K_{10}\right)_4 &= \left\{\,(\lambda,\theta/2,\varphi_p)\colon \lambda>0, 0<\theta_p<\pi/3, 0<\varphi_p<\pi/3\,\right\}.
\end{aligned}$$

$\hat{\mathbb{R}}^3/K_{13}$: (17) with $m=4$.

5 The Results

In Theorem m below we denote by ${}_{K_m}\mathsf{T}_{ijkl}$ the tensors of the basis given in [21, Table 3] in the lines marked by K_m, $1\le m\le 16$. We say "a triclinic (orthotropic, etc.) random field" instead of more rigourous "a random field with triclinic (orthotropic, etc.) symmetry". A sketch of proofs is given in [21, Section 6].

5.1 The Triclinic Class

Theorem 1 (A triclinic random field in the triclinic class) *The one-point correlation tensor of a homogeneous and* $(Z_2^c, 21A_g)$*-isotropic random field* $\mathsf{C}(\mathbf{x})$ *is*

$$\langle \mathsf{C}(\mathbf{x})\rangle_{ijkl} = \sum_{m=1}^{21} C_m \, {}_{Z_2^c}\mathsf{T}_{ijkl}^{A_g,m,1},$$

where $C_m\in\mathbb{R}$. *Its two-point correlation tensor has the form*

$$\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y})\rangle = \int_{\hat{\mathbb{R}}^3/Z_2^c} \cos(\mathbf{p},\mathbf{y}-\mathbf{x}) f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p}),$$

where $f(\mathbf{p})$ *is a* Φ*-equivalence class of measurable functions acting from* $\hat{\mathbb{R}}^3/Z_2^c$ *to the set of nonnegative-definite symmetric linear operators on* $\mathsf{V}^{Z_2^c}$ *with unit trace, and* Φ *is a finite* *measure on* $\hat{\mathbb{R}}^3/Z_2^c$. *The field has the form*

$$\begin{aligned}\mathsf{C}_{ijkl}(\mathbf{x}) &= \sum_{m=1}^{21} C_m\, {}_{Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl} + \sum_{m=1}^{21}\int_{\hat{\mathbb{R}}^3/Z_2^c} \cos(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^1_m(\mathbf{p})\, {}_{Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl}\\ &\quad + \sum_{m=1}^{21}\int_{\hat{\mathbb{R}}^3/Z_2^c} \sin(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^2_m(\mathbf{p})\, {}_{Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl},\end{aligned}$$

where $(Z^m_1(\mathbf{p}),\dots,Z^m_{21}(\mathbf{p}))^\top$ *are two centred uncorrelated* $\mathsf{V}^{Z_2^c}$*-valued random measures on* $\hat{\mathbb{R}}^3/Z_2^c$ *with control measure* $f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$.

Theorem 2 (An isotropic random field in the triclinic class) *The one-point correlation tensor of a homogeneous and* $(\mathrm{O}(3), 2U^{0g}\oplus 2U^{2g}\oplus U^{4g})$*-isotropic random field* $\mathsf{C}(\mathbf{x})$ *is*

$$\langle\mathsf{C}(\mathbf{x})\rangle_{ijkl} = C_1\mathsf{T}^{U^{0g},1,1}_{ijkl} + C_2\mathsf{T}^{U^{0g},2,1}_{ijkl},$$

where $C_1, C_2\in\mathbb{R}$. *Its two-point correlation tensor has the spectral expansion*

$$\langle\mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\rangle_{ijkli'j'k'l'} = \sum_{n=1}^{3}\int_0^\infty\sum_{q=1}^{29} N_{nq}(\lambda,\rho)L^q_{iikli'j'k'l'}(\mathbf{y}-\mathbf{x})\,\mathrm{d}\Phi_n(\lambda).$$

The measures $\Phi_n(\lambda)$ *satisfy the condition*

$$\Phi_2(\{0\}) = 2\Phi_3(\{0\}).$$

The spectral expansion of the field has the form

$$\begin{aligned}\mathsf{C}_{ijkl}(\rho,\theta,\varphi) &= C_1\mathsf{T}^{U^{0g},1,1}_{ijkl} + C_2\mathsf{T}^{U^{0g},2,1}_{ijkl}\\ &\quad + 2\sqrt{\pi}\sum_{m=1}^{13}\sum_{t=0}^{\infty}\sum_{u=-t}^{t}\int_0^\infty j_t(\lambda\rho)\,\mathrm{d}Z_{mtuijkl}(\lambda)S^u_t(\theta,\varphi),\end{aligned}$$

where $S^u_t(\theta,\varphi)$ *are real-valued spherical harmonics.*

The functions $N_{nq}(\lambda,\rho)$ and $L^q_{iikli'j'k'l'}(\mathbf{y}-\mathbf{x})$ are given in [21, Tables 5, 6].

5.2 The Monoclinic Class

Theorem 3 (A monoclinic random field in the monoclinic class) *The one-point correlation tensor of a homogeneous and* $(Z_2\times Z_2^c, 13A_g)$*-isotropic random field* $\mathsf{C}(\mathbf{x})$ *is*

$$\langle\mathsf{C}(\mathbf{x})\rangle_{ijkl} = \sum_{m=1}^{13} C_m\, {}_{Z_2\times Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl},$$

where $C_m\in\mathbb{R}$. *Its two-point correlation tensor has the form*

$$\langle\mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\rangle = \frac{1}{2}\int_{\hat{\mathbb{R}}^3/Z_2\times Z_2^c}\cos\bigl(p_1(y_1-x_1)+p_2(y_2-x_2)\bigr)\cos\bigl(p_3(y_3-x_3)\bigr)f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p}),$$

where $f(\mathbf{p})$ *is a* Φ*-equivalence class of measurable functions acting from* $\hat{\mathbb{R}}^3/Z_2 \times Z_2^c$ *to the set of nonnegative-definite symmetric linear operators on* $\mathsf{V}^{Z_2 \times Z_2^c}$ *with unit trace, and* Φ *is a finite Radon measure on* $\hat{\mathbb{R}}^3/Z_2 \times Z_2^c$*. The field has the form*

$$
\begin{aligned}
\mathsf{C}(\mathbf{x})_{ijkl} &= \sum_{m=1}^{13} C_{m\ Z_2 \times Z_2^c} \mathsf{T}_{ijkl}^{A_g,m,1} \\
&+ \frac{1}{\sqrt{2}} \sum_{m=1}^{13} \int_{\hat{\mathbb{R}}^3/Z_2 \times Z_2^c} \cos(p_1 x + p_2 y) \cos(p_3 z)\, \mathrm{d}\mathbf{Z}_m^1(\mathbf{p})_{Z_2 \times Z_2^c} \mathsf{T}_{ijkl}^{A_g,m,1} \\
&+ \frac{1}{\sqrt{2}} \sum_{m=1}^{13} \int_{\hat{\mathbb{R}}^3/Z_2 \times Z_2^c} \sin(p_1 x + p_2 y) \sin(p_3 z)\, \mathrm{d}Z_m^2(\mathbf{p})_{Z_2 \times Z_2^c} \mathsf{T}_{ijkl}^{A_g,m,1} \\
&+ \frac{1}{\sqrt{2}} \sum_{m=1}^{13} \int_{\hat{\mathbb{R}}^3/Z_2 \times Z_2^c} \cos(p_1 x + p_2 y) \sin(p_3 z)\, \mathrm{d}Z_m^3(\mathbf{p})_{Z_2 \times Z_2^c} \mathsf{T}_{ijkl}^{A_g,m,1} \\
&+ \frac{1}{\sqrt{2}} \sum_{m=1}^{13} \int_{\hat{\mathbb{R}}^3/Z_2 \times Z_2^c} \sin(p_1 x + p_2 y) \cos(p_3 z)\, \mathrm{d}Z_m^4(\mathbf{p})_{Z_2 \times Z_2^c} \mathsf{T}_{ijkl}^{A_g,m,1},
\end{aligned}
$$

where $(Z_1^n(\mathbf{p}), \dots, Z_{13}^n(\mathbf{p}))^\top$ *are four centred uncorrelated* $\mathsf{V}^{Z_2 \times Z_2^c}$*-valued random measures on* $\hat{\mathbb{R}}^3/Z_2 \times Z_2^c$ *with control measure* $f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$.

Theorem 4 (A transverse isotropic random field in the monoclinic class) *The one-point correlation tensor of a homogeneous and* $(\mathrm{O}(2) \times Z_2^c, 5U^{0gg} \oplus 3U^{2g} \oplus U^{4g})$*-isotropic random field* $\mathsf{C}(\mathbf{x})$ *is*

$$
\langle \mathsf{C}(\mathbf{x}) \rangle_{ijkl} = \sum_{m=1}^{5} C_{m\ \mathrm{O}(2) \times Z_2^c} \mathsf{T}_{ijkl}^{U^{0gg},m,1},
$$

where $C_m \in \mathbb{R}$*. Its two-point correlation tensor has the form*

$$
\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y}) \rangle = \int_{\hat{\mathbb{R}}^3/\mathrm{O}(2) \times Z_2^c} J_0\left(\sqrt{(p_1^2 + p_2^2)(z_1^2 + z_2^2)}\right) \cos(p_3 z_3) f(\mathbf{p})\, \mathrm{d}\Phi(\mathbf{p}),
$$

where Φ *is a measure on* $\hat{\mathbb{R}}^3/\mathrm{O}(2) \times Z_2^c$*, and* $f(\mathbf{p})$ *is a* Φ*-equivalence class of measurable functions on* $\hat{\mathbb{R}}^3/\mathrm{O}(2) \times Z_2^c$ *with values in the compact set of all nonnegative-definite linear operators in the space* $\mathsf{V}^{\mathrm{O}(2) \times Z_2^c}$ *with unit trace of the form*

$$
\begin{pmatrix}
A & 0 & 0 & 0 & 0 \\
0 & B_1 & B_2 & B_3 & 0 \\
0 & B_2 & B_4 & B_5 & 0 \\
0 & B_3 & B_5 & B_6 & 0 \\
0 & 0 & 0 & 0 & B_7
\end{pmatrix},
$$

where A is a nonnegative-definite 5×5 matrix, and B_m, $1\le m\le 7$ are 2×2 matrices proportional to the identity matrix. The field has the form

$$
\begin{aligned}
\mathsf{C}(\mathbf{x}) &= \sum_{m=1}^{5} C_m \mathsf{T}^m_{ijkl} \\
&+ \sum_{m=1}^{13} \int_{\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c} J_0\Bigl(\sqrt{\bigl(p_1^2+p_2^2\bigr)\bigl(z_1^2+z_2^2\bigr)}\Bigr) \\
&\times \bigl(\cos(p_3 z)\mathrm{d}Z^{01}_m(\mathbf{p})\mathsf{T}^m_{ijkl} + \sin(p_3 z)\mathrm{d}Z^{02}_m(\mathbf{p})\mathsf{T}^m_{ijkl}\bigr) \\
&+ \sqrt{2}\sum_{\ell=1}^{\infty}\sum_{m=1}^{13} \int_{\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c} J_\ell\Bigl(\sqrt{\bigl(p_1^2+p_2^2\bigr)\bigl(z_1^2+z_2^2\bigr)}\Bigr) \\
&\times \bigl(\cos(p_3 z)\cos(\ell\varphi_p)\mathrm{d}Z^{\ell 1}_m(\mathbf{p})\mathsf{T}^m_{ijkl} + \cos(p_3 z)\sin(\ell\varphi_p)\mathrm{d}Z^{\ell 2m}(\mathbf{p})\mathsf{T}^m_{ijkl} \\
&+ \sin(p_3 z)\cos(\ell\varphi_p)\mathrm{d}Z^{\ell 3}_m(\mathbf{p})\mathsf{T}^m_{ijkl} + \sin(p_3 z)\sin(\ell\varphi_p)\mathrm{d}Z^{\ell 4}_m(\mathbf{p})\mathsf{T}^m_{ijkl}\bigr),
\end{aligned}
$$

where $(Z^{\ell i}_1(\mathbf{p}),\dots,Z^{\ell i}_{13}(\mathbf{p}))^\top$ are centred uncorrelated $\mathsf{V}^{\mathrm{O}(2)\times Z_2^c}$-valued random measures on $\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c$ with control measure $f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, and

$$
\mathsf{T}^m_{ijkl} = \begin{cases} {}_{\mathrm{O}(2)\times Z_2^c}\mathsf{T}^{U^{0gg},m,1}, & \text{if } 1\le m\le 5, \\ {}_{\mathrm{O}(2)\times Z_2^c}\mathsf{T}^{U^{2g},\lfloor m/2\rfloor-2,m \bmod 2+1}, & \text{if } 6\le m\le 11, \\ {}_{\mathrm{O}(2)\times Z_2^c}\mathsf{T}^{U^{4g},1,m-11}, & \text{if } 12\le m\le 13. \end{cases}
$$

5.3 The Orthotropic Class

Theorem 5 (An orthotropic random field in the orthotropic class) *The one-point correlation tensor of a homogeneous and $(D_2\times Z_2^c, 9A_g)$-isotropic random field $\mathsf{C}(\mathbf{x})$ is*

$$
\bigl\langle \mathsf{C}(\mathbf{x})\bigr\rangle_{ijkl} = \sum_{m=1}^{9} C_m\, {}_{D_2\times Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl},
$$

where $C_m\in\mathbb{R}$. Its two-point correlation tensor has the form

$$
\bigl\langle \mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\bigr\rangle = \int_{\hat{\mathbb{R}}^3/D_2\times Z_2^c} \cos\bigl(p_1(y_1-x_1)\bigr)\cos\bigl(p_2(y_2-x_2)\bigr)\cos\bigl(p_3(y_3-x_3)\bigr) f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p}),
$$

where $f(\mathbf{p})$ is a Φ-equivalence class of measurable functions acting from $\hat{\mathbb{R}}^3/D_2\times Z_2^c$ to the set of nonnegative-definite symmetric linear operators on $\mathsf{V}^{D_2\times Z_2^c}$ with unit trace, and Φ is a finite measure on $\hat{\mathbb{R}}^3/D_2\times Z_2^c$. The field has the form

$$
\mathsf{C}(\mathbf{x})_{ijkl} = \sum_{m=1}^{9} C_m\, {}_{D_2\times Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl} + \sum_{m=1}^{9}\sum_{n=1}^{8} \int_{\hat{\mathbb{R}}^3/D_2\times Z_2^c} u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^n_m(\mathbf{p})\, {}_{D_2\times Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl},
$$

where $(Z^n_1(\mathbf{p}),\dots,Z^n_9(\mathbf{p}))^\top$ are eight centred uncorrelated $\mathsf{V}^{D_2\times Z_2^c}$-valued random measures on $\hat{\mathbb{R}}^3/D_2\times Z_2^c$ with control measure $f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, and where $u_n(\mathbf{p},\mathbf{x})$ are eight different product of sines and cosines of $p_r x_r$.

Consider a 9×9 symmetric nonnegative-definite matrix with the unit trace of the following structure:

$$\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix},$$

where A is a 6×6 matrix. Introduce the following notation:

$$j_1(\mathbf{p},\mathbf{z})=\cos(p_1z_1)\cos(p_2z_2)\cos(p_3z_3),$$
$$j_2(\mathbf{p},\mathbf{z})=\cos(p_1z_2)\cos(p_2z_1)\cos(p_3z_3).$$

Let Φ be a finite measure on $\hat{\mathbb{R}}^3/D_4\times Z^c_2$. Let $f^0(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_4\times Z^c_2)_m$, $0\le m\le 1$ to the set of nonnegative-definite symmetric matrices with unit trace satisfying $B=0$. Let $f^+(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_4\times Z^c_2)_m$, $2\le m\le 4$ to the set of nonnegative-definite symmetric linear operators on $\mathsf{V}^{D_4\times Z^c_2}$ with unit trace, and let $f^-(\mathbf{p})$ is obtained from $f^+(\mathbf{p})$ by multiplying B and $B^\top$ by -1.

Theorem 6 (A tetragonal random field in the orthotropic class) *The one-point correlation tensor of a homogeneous and $(D_4\times Z^c_2, 6A_{1g}\oplus 3B_{1g})$-isotropic random field $\mathsf{C}(\mathbf{x})$ is*

$$\langle\mathsf{C}(\mathbf{x})\rangle_{ijkl}=\sum_{m=1}^{6}C_m\,{}_{D_4\times Z^c_2}\mathsf{T}^{A_{g1},m,1}_{ijkl},$$

where $C_m\in\mathbb{R}$. Its two-point correlation tensor has the form

$$\begin{aligned}\langle\mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\rangle&=\frac{1}{2}\sum_{m=0}^{1}\int_{(\hat{\mathbb{R}}^3/D_4\times Z^c_2)_m}\left[j_1(\mathbf{p},\mathbf{y}-\mathbf{x})+j_2(\mathbf{p},\mathbf{y}-\mathbf{x})\right]f^0(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})\\&\quad+\frac{1}{2}\sum_{m=2}^{4}\int_{(\hat{\mathbb{R}}^3/D_4\times Z^c_2)_m}\left[j_1(\mathbf{p},\mathbf{y}-\mathbf{x})f^+(\mathbf{p})+j_2(\mathbf{p},\mathbf{y}-\mathbf{x})f^-(\mathbf{p})\right]\mathrm{d}\Phi(\mathbf{p}).\end{aligned}$$

The field has the form

$$\begin{aligned}\mathsf{C}(\mathbf{x})_{ijkl}&=\sum_{m=1}^{6}C_m\,{}_{D_4\times Z^c_2}\mathsf{T}^{A_{1g},m,1}_{ijkl}\\&\quad+\frac{1}{\sqrt{2}}\sum_{q=1}^{9}\sum_{n=1}^{16}\sum_{m=0}^{1}\int_{(\hat{\mathbb{R}}^3/D_4\times Z^c_2)_m}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{n0}_q(\mathbf{p})\,{}_{D_4\times Z^c_2}\mathsf{T}^q_{ijkl}\\&\quad+\frac{1}{\sqrt{2}}\sum_{q=1}^{9}\sum_{n=1}^{8}\sum_{m=2}^{4}\int_{(\hat{\mathbb{R}}^3/D_4\times Z^c_2)_m}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{n+}_q(\mathbf{p})\,{}_{D_4\times Z^c_2}\mathsf{T}^q_{ijkl}\\&\quad+\frac{1}{\sqrt{2}}\sum_{q=1}^{9}\sum_{n=9}^{16}\sum_{m=2}^{4}\int_{(\hat{\mathbb{R}}^3/D_4\times Z^c_2)_m}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{n-}_q(\mathbf{p})\,{}_{D_4\times Z^c_2}\mathsf{T}^q_{ijkl},\end{aligned}$$

where $(Z^{n0}_1(\mathbf{p}),\dots,Z^{n0}_9(\mathbf{p}))^\top$ (resp. $(Z^{n+}_1(\mathbf{p}),\dots,Z^{n+}_9(\mathbf{p}))^\top$, resp. $(Z^{n-}_1(\mathbf{p}),\dots,Z^{n-}_9(\mathbf{p}))^\top$) are centred uncorrelated $\mathsf{V}^{D_4\times Z^c_2}$-valued random measures on the spaces $(\hat{\mathbb{R}}^3/D_4\times Z^c_2)_m$,

Table 4 The functions $j_n(\mathbf{p},\mathbf{z})$ for the tetrahedral group

n	$j_n(\mathbf{p},\mathbf{z})$
1	$[\cos(p_1z_1+p_2z_2)+\cos(p_1z_2+p_2z_1)]\cos(p_3z_3)$
2	$\cos[\frac{1}{2}((p_1+\sqrt{2}p_3)z_1+p_2z_2+\sqrt{2}p_1z_3)]\cos[\frac{1}{2}(p_2z_1+(-p_1+\sqrt{2}p_3)z_2-\sqrt{2}p_2z_3)]$ $+\cos[\frac{1}{2}(z_1(p_1-\sqrt{2}p_3)-p_2z_2-\sqrt{2}p_1z_3)]\cos[\frac{1}{2}(p_2z_1+(-p_1-\sqrt{2}p_3)z_2+\sqrt{2}p_2z_3)]$
3	$\cos[\frac{1}{2}(-p_1z_1+(p_2+\sqrt{2}p_3)z_2+\sqrt{2}p_2z_3)]\cos[\frac{1}{2}((p_2-\sqrt{2}p_3)z_1-p_1z_2+\sqrt{2}p_1z_3)]$ $+\cos[\frac{1}{2}(-p_1z_1+(p_2-\sqrt{2}p_3)z_2-\sqrt{2}p_2z_3)]\cos[\frac{1}{2}((p_2+\sqrt{2}p_3)z_1-p_1z_2-\sqrt{2}p_1z_3)]$

$0\le m\le 1$ (*resp.* $2\le m\le 4$) *with control measure* $f^0(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$ (*resp.* $f^+(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, *resp.* $f^-(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$), $u_n(\mathbf{p},\mathbf{x})$ *are different product of sines and cosines of* p_rx_r *for* $1\le n\le 8$ *and eight different product of sines and cosines of* p_1x_2, p_2x_1, *and* p_3x_3 *for* $9\le n\le 16$, *and*

$$\mathsf{T}^q_{ijkl}=\begin{cases}\mathsf{T}^{A_{1g},q,1}_{ijkl}, & \text{if } 1\le q\le 6,\\ \mathsf{T}^{B_{1g},q-6,1}_{ijkl}, & \text{otherwise.}\end{cases}$$

Consider a 9×9 symmetric nonnegative-definite matrix with unit trace of the following structure

$$\begin{pmatrix} * & * & * & \mathbf{c}_1^\top & \mathbf{c}_2^\top & \mathbf{c}_3^\top\\ * & * & * & \mathbf{c}_4^\top & \mathbf{c}_5^\top & \mathbf{c}_6^\top\\ * & * & * & \mathbf{c}_7^\top & \mathbf{c}_8^\top & \mathbf{c}_9^\top\\ \mathbf{c}_1 & \mathbf{c}_4 & \mathbf{c}_7 & A_1 & A_2 & A_3\\ \mathbf{c}_2 & \mathbf{c}_5 & \mathbf{c}_8 & A_2 & A_4 & A_5\\ \mathbf{c}_3 & \mathbf{c}_6 & \mathbf{c}_9 & A_3 & A_5 & A_6 \end{pmatrix},$$

where stars are arbitrary numbers, $\mathbf{c}_i$ are vectors with two components, and A_i are 2×2 matrices. Let Φ be a finite measure on $\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c$. Let $f^0(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c)_m$, $0\le m\le 1$ to the set of nonnegative-definite symmetric linear operators on $\mathsf{V}^{\mathscr{T}\times Z_2^c}$ with unit trace such that $\mathbf{c}_i=\mathbf{0}$ and A_i are proportional to the identity matrix. Let $f^1(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c)_m$, $2\le m\le 4$ to the set of nonnegative-definite symmetric linear operators on $\mathsf{V}^{\mathscr{T}\times Z_2^c}$ with unit trace. Denote

$$g=\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}.$$

Let $f^+(\mathbf{p})$ (resp. $f^-(\mathbf{p})$) is obtained from $f^1(\mathbf{p})$ by replacing all $\mathbf{c}_i$ with $g\mathbf{c}_1$ (resp. with $g^{-1}\mathbf{c}_i$) and all A_i with gA_ig^{-1} (resp. $g^{-1}A_ig$). Finally, let $j_m(\mathbf{p},\mathbf{z})$ be functions from Table 4.

Theorem 7 (A cubic random field in the orthotropic class) *The one-point correlation tensor of a homogeneous and* $(\mathscr{T}\times Z_2^c, 3A_g\oplus 3({}^1E_g\oplus {}^2E_g))$*-isotropic random field* $\mathsf{C}(\mathbf{x})$ *is*

$$\langle\mathsf{C}(\mathbf{x})\rangle_{ijkl}=\sum_{m=1}^{3}C_{m\ \mathscr{T}\times Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl},$$

where $C_m \in \mathbb{R}$. *Its two-point correlation tensor has the form*

$$
\begin{aligned}
\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y}) \rangle = {} & \frac{1}{6} \sum_{m=0}^{1} \int_{(\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c)_m} \sum_{n=1}^{3} j_n(\mathbf{p}, \mathbf{y}-\mathbf{x}) f^0(\mathbf{p}) \,\mathrm{d}\Phi(\mathbf{p}) \\
& + \frac{1}{6} \sum_{m=2}^{4} \int_{(\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c)_m} \big[j_1(\mathbf{p}, \mathbf{y}-\mathbf{x}) f^1(\mathbf{p}) + j_2(\mathbf{p}, \mathbf{y}-\mathbf{x}) f^+(\mathbf{p}) \\
& + j_3(\mathbf{p}, \mathbf{y}-\mathbf{x}) f^-(\mathbf{p}) \big] \,\mathrm{d}\Phi(\mathbf{p}).
\end{aligned}
$$

The field has the form

$$
\begin{aligned}
\mathsf{C}(\mathbf{x}) = {} & \sum_{m=1}^{3} C_m \, {}_{\mathscr{T}\times Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl} + \frac{1}{\sqrt{6}} \Bigg(\sum_{q=1}^{9} \sum_{n=1}^{24} \sum_{m=0}^{1} \int_{(\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c)_m} u_n(\mathbf{p}, \mathbf{x}) \,\mathrm{d}Z^{n0}_q(\mathbf{p}) \mathsf{T}^q_{ijkl} \\
& + \sum_{q=1}^{9} \sum_{n=1}^{8} \sum_{m=2}^{4} \int_{(\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c)_m} u_n(\mathbf{p}, \mathbf{x}) \,\mathrm{d}Z^{n1}_q(\mathbf{p}) \mathsf{T}^q_{ijkl} \\
& + \sum_{q=1}^{9} \sum_{n=9}^{16} \sum_{m=2}^{4} \int_{(\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c)_m} u_n(\mathbf{p}, \mathbf{x}) \,\mathrm{d}Z^{n+}_q(\mathbf{p}) \mathsf{T}^q_{ijkl} \\
& + \sum_{q=1}^{9} \sum_{n=17}^{24} \sum_{m=2}^{4} \int_{(\hat{\mathbb{R}}^3/\mathscr{T}\times Z_2^c)_m} u_n(\mathbf{p}, \mathbf{x}) \,\mathrm{d}Z^{n-}_q(\mathbf{p}) \mathsf{T}^q_{ijkl} \Bigg),
\end{aligned}
$$

where $u_n(\mathbf{p}, \mathbf{x})$ *are various products of sines and cosines of angles from Table 4,*

$$
\mathsf{T}^q_{ijkl} = \begin{cases} {}_{\mathscr{T}\times Z_2^c}\mathsf{T}^{A_g,q,1}_{ijkl}, & \text{if } 1 \le q \le 3 \\ {}_{\mathscr{T}\times Z_2^c}\mathsf{T}^{E^{2g},\lfloor q/2 \rfloor -1, q \bmod 2+1}_{ijkl}, & \text{otherwise,} \end{cases}
$$

and where $(Z^{n0}_1(\mathbf{p}), \dots, Z^{n0}_9(\mathbf{p}))^\top$ *(resp.* $(Z^{n1}_1(\mathbf{p}), \dots, Z^{n1}_9(\mathbf{p}))^\top$*, resp.* $(Z^{n+}_1(\mathbf{p}), \dots, Z^{n+}_9(\mathbf{p}))^\top$ *resp.* $(Z^{n-}_1(\mathbf{p}), \dots, Z^{n-}_9(\mathbf{p}))^\top$*) are centred uncorrelated* $\mathsf{V}^{\mathscr{T}\times Z_2^c}$*-valued random measures on* $(\hat{\mathbb{R}}^3/\mathscr{T} \times Z_2^c)_m$ *for* $0 \le m \le 1$ *(resp.* $2 \le m \le 4$*) with control measure* $f^0(\mathbf{p}) \,\mathrm{d}\Phi(\mathbf{p})$ *(resp.* $f^1(\mathbf{p}) \,\mathrm{d}\Phi(\mathbf{p})$*, resp.* $f^+(\mathbf{p}) \,\mathrm{d}\Phi(\mathbf{p})$*, resp.* $f^-(\mathbf{p}) \,\mathrm{d}\Phi(\mathbf{p})$*).*

Consider a 9×9 symmetric nonnegative-definite matrix with unit trace of the following structure

$$
\begin{pmatrix}
* & * & * & * & * & \mathbf{c}_1^\top & \mathbf{c}_2^\top \\
* & * & * & * & * & \mathbf{c}_3^\top & \mathbf{c}_4^\top \\
* & * & * & * & * & \mathbf{c}_5^\top & \mathbf{c}_6^\top \\
* & * & * & * & * & \mathbf{c}_7^\top & \mathbf{c}_8^\top \\
* & * & * & * & * & \mathbf{c}_9^\top & \mathbf{c}_{10}^\top \\
\mathbf{c}_1 & \mathbf{c}_3 & \mathbf{c}_5 & \mathbf{c}_7 & \mathbf{c}_9 & A_1 & A_2 \\
\mathbf{c}_2 & \mathbf{c}_4 & \mathbf{c}_6 & \mathbf{c}_8 & \mathbf{c}_{10} & A_2 & A_3
\end{pmatrix},
$$

Table 5 The matrices g_n and the functions $j_n(\mathbf{p}, \mathbf{z})$ for the group $D_6 \times Z_2^c$

n	g_n	$j_n(\mathbf{p}, \mathbf{z})$
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\cos(p_3z_3)\cos(p_1z_1 + p_2z_2)$
2	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\cos(p_3z_3)\cos(p_1z_1 - p_2z_2)$
3	$\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$	$\cos(p_3z_3)\cos[(p_1 + \sqrt{3}p_2)z_1 + (-\sqrt{3}p_1 + p_2)z_2]/2$
4	$\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$\cos(p_3z_3)\cos[(p_1 - \sqrt{3}p_2)z_1 + (\sqrt{3}p_1 + p_2)z_2]/2$
5	$\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$\cos(p_3z_3)\cos[(-p_1 + \sqrt{3}p_2)z_1 + (\sqrt{3}p_1 + p_2)z_2]/2$
6	$\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$	$\cos(p_3z_3)\cos[(p_1 + \sqrt{3}p_2)z_1 + (\sqrt{3}p_1 - p_2)z_2]/2$

where stars are arbitrary numbers, $\mathbf{c}_i$ are vectors with two components, and A_i are 2×2 matrices of the form

$$A_j = \begin{pmatrix} a-b & c+d \\ c-d & a+b \end{pmatrix}. \tag{18}$$

Let Φ be a finite measure on $\hat{\mathbb{R}}^3/D_6 \times Z_2^c$. Let $f^0(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m$, $0 \le m \le 1$ to the set of nonnegative-definite symmetric matrices with unit trace such that $\mathbf{c}_i = \mathbf{0}$ and A_i are proportional to the identity matrix. Let $f^-(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_2$ to the set of nonnegative-definite symmetric matrices with unit trace such that A_i are symmetric. Let $f^+(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m$, $3 \le m \le 4$ to the set of nonnegative-definite symmetric matrices with unit trace. Consider matrices and functions of Table 5.

Let $f^{-i}(\mathbf{p})$ is obtained from $f^-(\mathbf{p})$ by replacing all $\mathbf{c}_j$ with $g_i\mathbf{c}_j$ and the vectors $(b, c)^\top$ in all A_j with $g_i(b, c)^\top$. Let $f^{+i}(\mathbf{p})$ is obtained from $f^+(\mathbf{p})$ by replacing all $\mathbf{c}_j$ with $g_i\mathbf{c}_j$ and all A_j with $g_iA_jg_i^{-1}$.

Theorem 8 (A hexagonal random field in the orthotropic class) *The one-point correlation tensor of a homogeneous and $(D_6 \times Z_2^c, 5A_{1g} \oplus 2E_{2g})$-isotropic random field $\mathsf{C}(\mathbf{x})$ is*

$$\langle \mathsf{C}(\mathbf{x}) \rangle_{ijkl} = \sum_{m=1}^{5} C_m \, {}_{D_6 \times Z_2^c}\mathsf{T}^{A_{1g},m,1}_{ijkl},$$

where $C_m \in \mathbb{R}$. Its two-point correlation tensor has the form

$$\begin{aligned} \langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y}) \rangle = \frac{1}{6}\Bigg(& \sum_{m=0}^{1} \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m} \sum_{n=1}^{6} j_n(\mathbf{p}, \mathbf{y} - \mathbf{x}) f^0(\mathbf{p}) \, \mathrm{d}\Phi(\mathbf{p}) \\ & + \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_2} \sum_{n=1}^{6} j_n(\mathbf{p}, \mathbf{y} - \mathbf{x}) f^{-n}(\mathbf{p}) \, \mathrm{d}\Phi(\mathbf{p}) \\ & + \sum_{m=3}^{4} \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m} \sum_{n=1}^{6} j_n(\mathbf{p}, \mathbf{y} - \mathbf{x}) f^{+n}(\mathbf{p}) \, \mathrm{d}\Phi(\mathbf{p}) \Bigg). \end{aligned}$$

The field has the form

$$\begin{aligned}\mathsf{C}(\mathbf{x})_{ijkl}&=\sum_{m=1}^{5}C_{m\;D_6\times Z_2^c}\mathsf{T}^{A_{1g},m,1}_{ijkl}+\frac{1}{\sqrt{6}}\sum_{q=1}^{9}\sum_{n=1}^{24}\sum_{m=0}^{1}\int_{(\hat{\mathbb{R}}^3/D_6\times Z_2^c)_m}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{0n}_q(\mathbf{p})_{D_6\times Z_2^c}T^q_{ijkl}\\&+\frac{1}{\sqrt{6}}\sum_{q=1}^{9}\sum_{s=1}^{6}\sum_{n=4s-3}^{4s}\int_{(\hat{\mathbb{R}}^3/D_6\times Z_2^c)_2}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{-ns}_q(\mathbf{p})_{D_6\times Z_2^c}T^q_{ijkl}\\&+\frac{1}{\sqrt{6}}\sum_{q=1}^{9}\sum_{s=1}^{6}\sum_{n=4s-3}^{4s}\sum_{m=3}^{4}\int_{(\hat{\mathbb{R}}^3/D_6\times Z_2^c)_m}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{+ns}_q(\mathbf{p})_{D_6\times Z_2^c}T^q_{ijkl},\end{aligned}$$

where $(Z^{0n}_1(\mathbf{p}),\dots,Z^{0n}_9(\mathbf{p}))^\top$ (*resp.* $(Z^{-ns}_1(\mathbf{p}),\dots,Z^{-ns}_9(\mathbf{p}))^\top$, *resp.* $(Z^{+ns}_1(\mathbf{p}),\dots,Z^{+ns}_9(\mathbf{p}))^\top$) *are centred uncorrelated* $\mathsf{V}^{D_6\times Z_2^c}$*-valued random measures on* $(\hat{\mathbb{R}}^3/D_6\times Z_2^c)_m$, $0\le m\le 1$ (*resp. on* $(\hat{\mathbb{R}}^3/D_6\times Z_2^c)_2$, *resp. on* $(\hat{\mathbb{R}}^3/D_6\times Z_2^c)_2$, $3\le m\le 4$) *with control measure* $f^0(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$ (*resp.* $f^{-s}(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, *resp.* $f^{+s}(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$), $u_n(\mathbf{p},\mathbf{x})$, $1\le n\le 8$ *are different product of sines and cosines of angles in Table 5, and where*

$$\mathsf{T}^q_{ijkl}=\begin{cases}{}_{D_6\times Z_2^c}\mathsf{T}^{A_g,q,1}_{ijkl}, & \text{if } 1\le q\le 5\\ {}_{D_6\times Z_2^c}\mathsf{T}^{E^{2g},\lfloor q/2\rfloor-1,q \bmod 2+1}_{ijkl}, & \text{otherwise.}\end{cases}$$

Consider a 9×9 symmetric nonnegative-definite matrix with unit trace of the following structure

$$\begin{pmatrix} * & * & * & \mathbf{c}_1^\top & \mathbf{c}_2^\top & \mathbf{c}_3^\top\\ * & * & * & \mathbf{c}_4^\top & \mathbf{c}_5^\top & \mathbf{c}_6^\top\\ * & * & * & \mathbf{c}_7^\top & \mathbf{c}_8^\top & \mathbf{c}_9^\top\\ \mathbf{c}_1 & \mathbf{c}_4 & \mathbf{c}_7 & A_1 & A_2 & A_3\\ \mathbf{c}_2 & \mathbf{c}_5 & \mathbf{c}_8 & A_2 & A_4 & A_5\\ \mathbf{c}_3 & \mathbf{c}_6 & \mathbf{c}_9 & A_3 & A_5 & A_6\end{pmatrix},$$

where stars are arbitrary numbers, $\mathbf{c}_i$ are vectors with two components, and A_i are 2×2 matrices of the form (18). Let Φ be a finite measure on $\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c$. Let $f^0(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_m$, $0\le m\le 1$ to the set of nonnegative-definite symmetric matrices with unit trace such that $\mathbf{c}_i=\mathbf{0}$ and A_i are proportional to the identity matrix. Let $f^-(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_2$ to the set of nonnegative-definite symmetric matrices with unit trace such that A_i are symmetric. Let $f^+(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_m$, $3\le m\le 6$ to the set of nonnegative-definite symmetric matrices with unit trace. Consider matrices and functions of Table 6.

Let $f^{-i}(\mathbf{p})$ is obtained from $f^-(\mathbf{p})$ by replacing all $\mathbf{c}_j$ with $g_i\mathbf{c}_i$ and the vectors $(b,c)^\top$ in all A_j with $g_i(b,c)^\top$. Let $f^{+i}(\mathbf{p})$ is obtained from $f^+(\mathbf{p})$ by replacing all $\mathbf{c}_j$ with $g_i\mathbf{c}_i$ and all A_j with $g_iA_jg_i^{-1}$.

Table 6 The matrices g_n and the functions $j_n(\mathbf{p},\mathbf{z})$ for the group $\mathscr{O}\times Z_2^c$

n	g_n	$j_n(\mathbf{p},\mathbf{z})$
1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\cos(p_3z_3)[\cos(p_1z_1+p_2z_2)+\cos(p_1z_2+p_2z_1)]$
2	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\cos(p_3z_3)[\cos(-p_1z_2+p_2z_1)+\cos(p_2z_2-p_1z_1)]$
3	$\frac{1}{2}\begin{pmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{pmatrix}$	$2\cos[\sqrt{2}p_3(z_2-z_1)/2]\cos[(p_2-p_1)(z_1+z_2)/2]\cos[\sqrt{2}(p_1+p_2)z_3/2]$
4	$\frac{1}{2}\begin{pmatrix}-1&\sqrt{3}\\\sqrt{3}&-1\end{pmatrix}$	$\cos[\sqrt{2}(p_1-p_2)z_3/2]\{\cos[(-p_1-p_2+\sqrt{2}p_3)z_1/2+(p_1+p_2+\sqrt{2}p_3)z_2/2]$ $+\cos[(-p_1-p_2-\sqrt{2}p_3)z_1/2+(p_1+p_2-\sqrt{2}p_3)z_2/2]\}$
5	$\frac{1}{2}\begin{pmatrix}1&\sqrt{3}\\\sqrt{3}&-1\end{pmatrix}$	$2\cos[\sqrt{2}p_3(z_1+z_2)/2]\cos[(p_2-p_1)(z_2-z_1)/2]\cos[\sqrt{2}(p_1+p_2)z_3/2]$
6	$\frac{1}{2}\begin{pmatrix}1&-\sqrt{3}\\-\sqrt{3}&-1\end{pmatrix}$	$\cos[\sqrt{2}(p_1-p_2)z_3/2]\{\cos[(p_1+p_2-\sqrt{2}p_3)z_1/2+(p_1+p_2+\sqrt{2}p_3)z_2/2]$ $+\cos[(p_1+p_2+\sqrt{2}p_3)z_1/2+(p_1+p_2-\sqrt{2}p_3)z_2/2]\}$

Theorem 9 (A cubic random field in the orthotropic class) *The one-point correlation tensor of a homogeneous and $(\mathscr{O}\times Z_2^c, 3A_{1g}\oplus 3E_g)$-isotropic random field $\mathsf{C}(\mathbf{x})$ is*

$$\langle\mathsf{C}(\mathbf{x})\rangle_{ijkl}=\sum_{m=1}^{3}C_m\,{}_{\mathscr{O}\times Z_2^c}\mathsf{T}^{A_{1g},m,1}_{ijkl},$$

where $C_m\in\mathbb{R}$. Its two-point correlation tensor has the form

$$\begin{aligned}\langle\mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\rangle=\frac{1}{12}\Bigg(&\sum_{m=0}^{1}\int_{(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_m}\sum_{n=1}^{6}j_n(\mathbf{p},\mathbf{y}-\mathbf{x})f^0(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})\\&+\int_{(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_2}\sum_{n=1}^{6}j_n(\mathbf{p},\mathbf{y}-\mathbf{x})f^{-n}(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})\\&+\sum_{m=3}^{6}\int_{(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_m}\sum_{n=1}^{6}j_n(\mathbf{p},\mathbf{y}-\mathbf{x})f^{+n}(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})\Bigg).\end{aligned}$$

The field has the form

$$\begin{aligned}\mathsf{C}(\mathbf{x})_{ijkl}=&\sum_{m=1}^{3}C_m\,{}_{\mathscr{O}\times Z_2^c}\mathsf{T}^{A_{1g},m,1}_{ijkl}+\frac{1}{\sqrt{12}}\sum_{q=1}^{6}\sum_{n=1}^{48}\sum_{m=0}^{1}\int_{(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_m}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{0n}_q(\mathbf{p})\,{}_{\mathscr{O}\times Z_2^c}T^q_{ijkl}\\&+\frac{1}{\sqrt{12}}\sum_{q=1}^{6}\sum_{s=1}^{6}\sum_{n=8s-7}^{8s}\int_{(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_2}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{-ns}_q(\mathbf{p})\,{}_{\mathscr{O}\times Z_2^c}T^q_{ijkl}\\&+\frac{1}{\sqrt{12}}\sum_{q=1}^{6}\sum_{s=1}^{6}\sum_{n=8s-7}^{8s}\sum_{m=3}^{6}\int_{(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_m}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{+ns}_q(\mathbf{p})\,{}_{\mathscr{O}\times Z_2^c}T^q_{ijkl},\end{aligned}$$

where $u_n(\mathbf{p},\mathbf{x})$, $1\le n\le 8$ are different products of sines and cosines of angles in Table 6, $(Z^{0n}_1(\mathbf{p}),\dots,Z^{0n}_9(\mathbf{p}))^\top$ (resp. $(Z^{-ns}_1(\mathbf{p}),\dots,Z^{-ns}_9(\mathbf{p}))^\top$, resp. $(Z^{+ns}_1(\mathbf{p}),\dots,Z^{+ns}_9(\mathbf{p}))^\top$) are centred uncorrelated $\mathsf{V}^{\mathscr{O}\times Z_2^c}$-valued random measures on $(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_m$, $0\le m\le 1$ (resp.

on $(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_2$, *resp. on* $(\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c)_2$, $3\le m\le 6$) *with control measure* $f^0(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$ (*resp.* $f^{-s}(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, *resp.* $f^{+s}(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$), *and where*

$$\mathsf{T}^m_{ijkl}=\begin{cases}{}_{\mathscr{O}\times Z_2^c}\mathsf{T}^{A_g,m,1}_{ijkl}, & \text{if } 1\le m\le 3\\ {}_{\mathscr{O}\times Z_2^c}\mathsf{T}^{E^{2g},\lfloor m/2\rfloor-1,m \bmod 2+1}_{ijkl}, & \text{if } 4\le m\le 9.\end{cases}$$

5.4 The Trigonal Class

Introduce the following notation:

$$\begin{aligned}j_{10}(\mathbf{p},\mathbf{z})&=\cos(p_1z_1+p_3z_3)\cos(p_2z_2)\\&\quad+\cos\left[\frac{1}{2}(p_1+\sqrt{3}p_2)z_1\right]\cos\left[\frac{1}{2}(\sqrt{3}p_1-p_2)z_2+p_3z_3\right]\\&\quad+\cos\left[\frac{1}{2}(p_1-\sqrt{3}p_2)z_1\right]\cos\left[\frac{1}{2}(-\sqrt{3}p_1+p_2)z_2+p_3z_3\right].\end{aligned}$$

Theorem 10 (A trigonal random field in the trigonal class) *The one-point correlation tensor of a homogeneous and* $(D_3\times Z_2^c, 6A_{1g})$*-isotropic random field* $\mathsf{C}(\mathbf{x})$ *is*

$$\langle\mathsf{C}(\mathbf{x})\rangle_{ijkl}=\sum_{m=1}^{6}C_m\,{}_{D_3\times Z_2^c}T^{A_{1g},m,1}_{ijkl},$$

where $C_m\in\mathbb{R}$. *Its two-point correlation tensor has the form*

$$\langle\mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\rangle=\frac{1}{3}\int_{\hat{\mathbb{R}}^3/D_3\times Z_2^c}j_{10}(\mathbf{p},\mathbf{y}-\mathbf{x})f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p}),$$

where $f(\mathbf{p})$ *is the* Φ*-equivalence class of measurable functions acting from* $\hat{\mathbb{R}}^3/D_3\times Z_2^c$ *to the set of nonnegative-definite symmetric linear operators on* $\mathsf{V}^{D_3\times Z_2^c}$ *with unit trace, and* Φ *is a finite measure on* $\hat{\mathbb{R}}^3/D_3\times Z_2^c$. *The field has the form*

$$\mathsf{C}(\mathbf{x})_{ijkl}=\sum_{m=1}^{6}C_m\,{}_{D_3\times Z_2^c}T^{A_{1g},m,1}_{ijkl}+\frac{1}{\sqrt{3}}\sum_{m=1}^{6}\sum_{n=1}^{12}\int_{\hat{\mathbb{R}}^3/D_3\times Z_2^c}u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{mn}(\mathbf{p})\,{}_{D_3\times Z_2^c}T^{A_{1g},m,1}_{ijkl},$$

where $(Z^{1n}(\mathbf{p}),\dots,Z^{6n}(\mathbf{p}))^\top$ *are* 12 *centred uncorrelated* $\mathsf{V}^{D_3\times Z_2^c}$*-valued random measures on* $\hat{\mathbb{R}}^3/D_3\times Z_2^c$ *with control measure* $f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, *and where* $u_n(\mathbf{p},\mathbf{x})$, $1\le n\le 4$ *are four different products of sines and cosines of* $p_1x_1+p_3x_3$ *and* p_2x_2, $u_n(\mathbf{p},\mathbf{x})$, $5\le n\le 8$ *are four different product of sines and cosines of* $\frac{1}{2}(p_1+\sqrt{3}p_2)x_1$ *and* $\frac{1}{2}(\sqrt{3}p_1-p_2)x_2+p_3x_3$, $u_n(\mathbf{p},\mathbf{x})$, $9\le n\le 12$ *are four different product of sines and cosines of* $\frac{1}{2}(p_1-\sqrt{3}p_2)x_1$ *and* $\frac{1}{2}(-\sqrt{3}p_1+p_2)x_2+p_3x_3$.

Consider a 6×6 symmetric nonnegative-definite matrix with unit trace of the following structure

$$\begin{pmatrix} * & * & * & * & * & c_1 \\ * & * & * & * & * & c_2 \\ * & * & * & * & * & c_3 \\ * & * & * & * & * & c_4 \\ * & * & * & * & * & c_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 & * \end{pmatrix}, \tag{19}$$

where stars and c_i are arbitrary numbers. Let Φ be a finite measure on $\hat{\mathbb{R}}^3/D_6 \times Z_2^c$. Let $f^0(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m$, $0 \le m \le 2$ to the set of nonnegative-definite symmetric matrices with unit trace such that $c_i = 0$. Let $f^+(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m$, $3 \le m \le 4$ to the set of nonnegative-definite symmetric matrices with unit trace, and let $f^-(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m$, $3 \le m \le 4$ to the set of nonnegative-definite symmetric matrices with unit trace such that all c_is are multiplied by -1.

Theorem 11 (A hexagonal random field in the trigonal class) *The one-point correlation tensor of a homogeneous and $(D_6 \times Z_2^c, 5A_{1g} \oplus B_{1g})$-isotropic random field $\mathsf{C}(\mathbf{x})$ is*

$$\langle \mathsf{C}(\mathbf{x}) \rangle_{ijkl} = \sum_{m=1}^{5} C_m\, {}_{D_6 \times Z_2^c} T^{A_{1g},1,1}_{ijkl},$$

where $C_m \in \mathbb{R}$. Its two-point correlation tensor has the form

$$\begin{aligned} \langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y}) \rangle = \frac{1}{6} \Bigg(& \sum_{m=0}^{2} \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m} \sum_{n=1}^{6} j_n(\mathbf{p}, \mathbf{y}-\mathbf{x}) f^0(\mathbf{p})\, \mathrm{d}\Phi(\mathbf{p}) \\ & + \sum_{m=3}^{4} \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m} \sum_{n=1}^{3} j_n(\mathbf{p}, \mathbf{y}-\mathbf{x}) f^+(\mathbf{p})\, \mathrm{d}\Phi(\mathbf{p}) \\ & + \sum_{m=3}^{4} \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m} \sum_{n=4}^{6} j_n(\mathbf{p}, \mathbf{y}-\mathbf{x}) f^-(\mathbf{p})\, \mathrm{d}\Phi(\mathbf{p}) \Bigg). \end{aligned}$$

The field has the form

$$\begin{aligned} \mathsf{C}(\mathbf{x})_{ijkl} = & \sum_{m=1}^{5} C_m\, {}_{D_6 \times Z_2^c} \mathsf{T}^{A_{1g},m,1}_{ijkl} + \frac{1}{\sqrt{6}} \sum_{q=1}^{6} \sum_{n=1}^{24} \sum_{m=0}^{2} \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m} u_n(\mathbf{p}, \mathbf{x})\, \mathrm{d}Z^{0n}_q(\mathbf{p})\, {}_{D_6 \times Z_2^c} T^q_{ijkl} \\ & + \frac{1}{\sqrt{6}} \sum_{q=1}^{6} \sum_{s=1}^{6} \sum_{n=4s-3}^{4s} \sum_{m=3}^{4} \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m} u_n(\mathbf{p}, \mathbf{x})\, \mathrm{d}Z^{+ns}_q(\mathbf{p})\, {}_{D_6 \times Z_2^c} T^q_{ijkl} \\ & + \frac{1}{\sqrt{6}} \sum_{q=1}^{9} \sum_{s=1}^{6} \sum_{n=4s-3}^{4s} \sum_{m=3}^{4} \int_{(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m} u_n(\mathbf{p}, \mathbf{x})\, \mathrm{d}Z^{-ns}_q(\mathbf{p})\, {}_{D_6 \times Z_2^c} T^q_{ijkl}, \end{aligned}$$

where $(Z^{0n}_1(\mathbf{p}), \dots, Z^{0n}_6(\mathbf{p}))^\top$ (resp. $(Z^{+ns}_1(\mathbf{p}), \dots, Z^{+ns}_6(\mathbf{p}))^\top$, resp. $(Z^{-ns}_1(\mathbf{p}), \dots, Z^{-ns}_6(\mathbf{p}))^\top$) are centred uncorrelated $\mathsf{V}^{D_6 \times Z_2^c}$-valued random measures on $(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m$,

$0 \le m \le 2$ (*resp. on* $(\hat{\mathbb{R}}^3/D_6 \times Z_2^c)_m$, $3 \le m \le 4$) *with control measure* $f^0(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$ (*resp.* $f^+(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, *resp.* $f^-(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$), $u_n(\mathbf{p},\mathbf{x})$, $1 \le n \le 8$ *are different product of sines and cosines of angles in Table 5, and where*

$$T^q_{ijkl} = \begin{cases} {}_{D_6\times Z_2^c}\mathsf{T}^{A_{1g},q,1}_{ijkl}, & \text{if } 1 \le q \le 5, \\ {}_{D_6\times Z_2^c}\mathsf{T}^{B_{1g},m,1}_{ijkl}, & \text{otherwise.} \end{cases}$$

5.5 The Tetragonal Class

Theorem 12 (A tetragonal random field in the tetragonal class) *The one-point correlation tensor of a homogeneous and* $(D_4 \times Z_2^c, 6A_{1g})$*-isotropic random field* $\mathsf{C}(\mathbf{x})$ *is*

$$\langle \mathsf{C}(\mathbf{x})\rangle_{ijkl} = \sum_{m=1}^{6} C_m\, {}_{D_4\times Z_2^c}\mathsf{T}^{A_{1g},m,1}_{ijkl},$$

where $C_m \in \mathbb{R}$. *Its two-point correlation tensor has the form*

$$\begin{aligned}\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y})\rangle = \frac{1}{2}\int_{\hat{\mathbb{R}}^3/D_4\times Z_2^c} &\bigl[\cos\bigl(p_1(x_1-y_1)\bigr)\cos m\bigl(p_2(x_2-y_2)\bigr) \\ &+ \cos\bigl(p_1(x_2-y_2)\bigr)\cos\bigl(p_2(x_1-y_1)\bigr)\bigr]\cos\bigl(p_3(x_3-y_3)\bigr) f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p}),\end{aligned}$$

where $f(\mathbf{p})$ *is a* Φ*-equivalence class of measurable functions acting from* $\hat{\mathbb{R}}^3/D_4 \times Z_2^c$ *to the set of nonnegative-definite symmetric linear operators on* $\mathsf{V}^{D_4\times Z_2^c}$ *with unit trace, and* Φ *is a finite measure on* $\hat{\mathbb{R}}^3/D_4 \times Z_2^c$. *The field has the form*

$$\mathsf{C}(\mathbf{x})_{ijkl} = \sum_{m=1}^{6} C_m\, {}_{D_4\times Z_2^c}\mathsf{T}^{A_{1g},m,1}_{ijkl} + \frac{1}{\sqrt{2}}\sum_{m=1}^{6}\sum_{n=1}^{16}\int_{\hat{\mathbb{R}}^3/D_4\times Z_2^c} u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{mn}(\mathbf{p})\, {}_{D_4\times Z_2^c}\mathsf{T}^{A_{1g},m,1}_{ijkl},$$

where $(Z^{1n}(\mathbf{p}),\dots,Z^{6n}(\mathbf{p}))^\top$ *are* 16 *centred uncorrelated* $\mathsf{V}^{D_4\times Z_2^c}$*-valued random measures on* $\hat{\mathbb{R}}^3/D_4 \times Z_2^c$ *with control measure* $f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, *and where* $u_n(\mathbf{p},\mathbf{x})$ *are eight different product of sines and cosines of* $p_r x_r$ *for* $1 \le n \le 8$ *and eight different product of sines and cosines of* p_1x_2, p_2x_1, *and* p_3x_3 *for* $9 \le n \le 16$.

Consider a 6×6 symmetric nonnegative-definite matrix with unit trace of the structure (19). Let Φ be a finite measure on $\hat{\mathbb{R}}^3/D_8 \times Z_2^c$. Let $f^0(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_8 \times Z_2^c)_m$, $0 \le m \le 1$ to the set of nonnegative-definite symmetric matrices with unit trace such that $c_i = 0$. Let $f^+(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_8 \times Z_2^c)_m$, $2 \le m \le 4$ to the set of nonnegative-definite symmetric matrices with unit trace, and let $f^-(\mathbf{p})$ be a Φ-equivalence class of measurable functions acting from $(\hat{\mathbb{R}}^3/D_8 \times Z_2^c)_m$, $2 \le m \le 4$ to the set of nonnegative-definite symmetric matrices with unit trace such that all c_is are multiplied by -1.

Introduce the following notation.

$$
\begin{aligned}
j^+_{13}(\mathbf{p},\mathbf{z}) &= 2\cos(p_3z_3)\bigl[\cos(p_1z_1+p_2z_2)+\cos(p_2z_1-p_1z_2)\\
&\quad+\cos\bigl((p_1+p_2)(z_1+z_2)/\sqrt{2}\bigr)\\
&\quad+\cos\bigl((p_2z_2-p_1z_1)/\sqrt{2}-p_3z_3\bigr)\cos\bigl((p_1z_2+p_2z_1)/\sqrt{2}\bigr)\bigr],\\
j^-_{13}(\mathbf{p},\mathbf{z}) &= \cos(p_3z_3)\bigl[2\cos(p_1z_1-p_2z_2)+2\cos(p_2z_1+p_1z_2)\\
&\quad+\cos\bigl((p_1z_1+p_2z_2)/\sqrt{2}\bigr)\cos\bigl((p_2z_1-p_1z_2)/\sqrt{2}\bigr)\bigr].
\end{aligned}
$$

Theorem 13 (An octagonal random field in the tetragonal class) *The one-point correlation tensor of a homogeneous and $(D_8\times Z^c_2, 5A_{1g}\oplus B_{1g})$-isotropic random field $\mathsf{C}(\mathbf{x})$ is*

$$
\langle \mathsf{C}(\mathbf{x})\rangle_{ijkl}=\sum_{m=1}^{5} C_m\,{}_{D_8\times Z^c_2}T^{A_{1g},1,1}_{ijkl},
$$

where $C_m\in\mathbb{R}$. Its two-point correlation tensor has the form

$$
\begin{aligned}
\langle \mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\rangle &= \frac{1}{4}\Biggl(\sum_{m=0}^{1}\int_{(\hat{\mathbb{R}}^3/D_8\times Z^c_2)_m}\bigl(j^+_{13}(\mathbf{p},\mathbf{y}-\mathbf{x})+j^-_{13}(\mathbf{p},\mathbf{y}-\mathbf{x})\bigr)f^0(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})\\
&\quad+\sum_{m=2}^{4}\int_{(\hat{\mathbb{R}}^3/D_8\times Z^c_2)_m} j^+_{13}(\mathbf{p},\mathbf{y}-\mathbf{x})f^+(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})\\
&\quad+\sum_{m=2}^{4}\int_{(\hat{\mathbb{R}}^3/D_8\times Z^c_2)_m} j^-_{13}(\mathbf{p},\mathbf{y}-\mathbf{x})f^-(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})\Biggr).
\end{aligned}
$$

The field has the form

$$
\begin{aligned}
\mathsf{C}(\mathbf{x})_{ijkl} &= \sum_{m=1}^{5} C_m\,{}_{D_8\times Z^c_2}\mathsf{T}^{A_{1g},m,1}_{ijkl}+\frac{1}{2}\sum_{q=1}^{6}\sum_{n=1}^{32}\sum_{m=0}^{1}\int_{(\hat{\mathbb{R}}^3/D_8\times Z^c_2)_m} u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{0n}_q(\mathbf{p})\,{}_{D_8\times Z^c_2}T^q_{ijkl}\\
&\quad+\frac{1}{2}\sum_{q=1}^{6}\sum_{n=1}^{16}\sum_{m=2}^{4}\int_{(\hat{\mathbb{R}}^3/D_8\times Z^c_2)_m} u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{+n}_q(\mathbf{p})\,{}_{D_8\times Z^c_2}T^q_{ijkl}\\
&\quad+\frac{1}{2}\sum_{q=1}^{6}\sum_{n=17}^{32}\sum_{m=2}^{4}\int_{(\hat{\mathbb{R}}^3/D_8\times Z^c_2)_m} u_n(\mathbf{p},\mathbf{x})\,\mathrm{d}Z^{-n}_q(\mathbf{p})\,{}_{D_8\times Z^c_2}T^q_{ijkl},
\end{aligned}
$$

where $(Z^{0n}_1(\mathbf{p}),\dots,Z^{0n}_6(\mathbf{p}))^\top$ (resp. $(Z^{+n}_1(\mathbf{p}),\dots,Z^{+n}_6(\mathbf{p}))^\top$, resp. $(Z^{-n}_1(\mathbf{p}),\dots,Z^{-n}_6(\mathbf{p}))^\top$) are centred uncorrelated $\mathsf{V}^{D_8\times Z^c_2}$-valued random measures on $(\hat{\mathbb{R}}^3/D_8\times Z^c_2)_m$, $0\le m\le 1$ (resp. on $(\hat{\mathbb{R}}^3/D_8\times Z^c_2)_m$, $2\le m\le 4$) with control measure $f^0(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$ (resp. $f^+(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$, resp. $f^-(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$), $u_n(\mathbf{p},\mathbf{x})$, $1\le n\le 8$ are different product of sines and cosines of angles in Table 5, and where

$$
T^q_{ijkl}=\begin{cases}{}_{D_8\times Z^c_2}\mathsf{T}^{A_{1g},q,1}_{ijkl}, & \text{if } 1\le q\le 5,\\ {}_{D_8\times Z^c_2}\mathsf{T}^{B_{1g},m,1}_{ijkl}, & \text{otherwise.}\end{cases}
$$

5.6 The Transverse Isotropic Class

Theorem 14 (A transverse isotropic random field in the transverse isotropic class) *The one-point correlation tensor of the homogeneous and* $(\mathrm{O}(2)\times Z_2^c, 5U^{0gg})$*-isotropic mean-square continuous random field* $\mathsf{C}(\mathbf{x})$ *has the form*

$$\langle\mathsf{C}(\mathbf{x})\rangle=\sum_{m=1}^{5}C_{m\ \mathrm{O}(2)\times Z_2^c}T^{U^{0gg},m,1}_{ijkl},$$

where $C_m\in\mathbb{R}$. *Its two-point correlation tensor has the form*

$$\langle\mathsf{C}(\mathbf{x}),\mathsf{C}(\mathbf{y})\rangle=\int_{\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c}J_0\Big(\sqrt{(p_1^2+p_2^2)\big((y_1-x_1)^2+(y_2-x_2)^2\big)}\Big)\times\cos\big(p_3(y_3-x_3)\big)f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p}),$$

where Φ *is a measure on* $\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c$, *and* $f(\mathbf{p})$ *is a* Φ*-equivalence class of measurable functions on* $\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c$ *with values in the compact set of all nonnegative-definite linear operators in the space* $\mathsf{V}^{\mathrm{O}(2)\times Z_2^c}$ *with unit trace. The field has the form*

$$\begin{aligned}\mathsf{C}(\mathbf{x})&=\sum_{m=1}^{5}C_{m\ \mathrm{O}(2)\times Z_2^c}T^{0\otimes A,m,1}_{ijkl}\\&+\sum_{m=1}^{5}\int_{\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c}J_0\Big(\sqrt{(p_1^2+p_2^2)(x_1^2+x_2^2)}\Big)\\&\times\big(\cos(p_3x_3)\mathrm{d}Z^{01m}(\mathbf{p})_{\mathrm{O}(2)\times Z_2^c}T^{U^{0gg},m,1}_{ijkl}+\sin(p_3x_3)\mathrm{d}Z^{02m}(\mathbf{p})_{\mathrm{O}(2)\times Z_2^c}T^{U^{0gg},m,1}_{ijkl}\big)\\&+\sqrt{2}\sum_{\ell=1}^{\infty}\sum_{m=1}^{5}\int_{\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c}J_\ell\Big(\sqrt{(p_1^2+p_2^2)(x_1^2+x_2^2)}\Big)\\&\times\big(\cos(p_3x_3)\cos(\ell\varphi_p)\mathrm{d}Z^{\ell1m}(\mathbf{p})_{\mathrm{O}(2)\times Z_2^c}T^{U^{0gg},m,1}_{ijkl}\\&+\cos(p_3x_3)\sin(\ell\varphi_p)\mathrm{d}Z^{\ell2m}(\mathbf{p})_{\mathrm{O}(2)\times Z_2^c}T^{U^{0gg},m,1}_{ijkl}\\&+\sin(p_3x_3)\cos(\ell\varphi_p)\mathrm{d}Z^{\ell3m}(\mathbf{p})_{\mathrm{O}(2)\times Z_2^c}T^{U^{0gg},m,1}_{ijkl}\\&+\sin(p_3x_3)\sin(\ell\varphi_p)\mathrm{d}Z^{\ell4m}(\mathbf{p})_{\mathrm{O}(2)\times Z_2^c}T^{U^{0gg},m,1}_{ijkl}\big),\end{aligned}$$

where $(Z^{\ell i1}(\mathbf{p}),\dots,Z^{\ell i5}(\mathbf{p}))^\top$ *are centred uncorrelated* $\mathsf{V}^{\mathrm{O}(2)\times Z_2^c}$*-valued random measures on* $\hat{\mathbb{R}}^3/\mathrm{O}(2)\times Z_2^c$ *with control measure* $f(\mathbf{p})\,\mathrm{d}\Phi(\mathbf{p})$.

5.7 The Cubic Class

Theorem 15 (A cubic random field in the cubic class) *The one-point correlation tensor of the homogeneous and* $(\mathscr{O}\times Z_2^c, 3A_{1g})$*-isotropic mean-square continuous random field* $\mathsf{C}(\mathbf{x})$ *has the form*

$$\langle\mathsf{C}(\mathbf{x})\rangle=\sum_{m=1}^{3}C_{m\ \mathscr{O}\times Z_2^c}\mathsf{T}^{A_{1g},m,1}_{ijkl},$$

where $C_m \in \mathbb{R}$. Its two-point correlation tensor has the form

$$\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y}) \rangle = \int_{\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c} \sum_{m=0}^{8} j_m(\mathbf{x}-\mathbf{y}, \mathbf{p}) f(\mathbf{p})\, \mathrm{d}\Phi(\mathbf{p}),$$

where the functions $j_m(\mathbf{z}, \mathbf{p})$ are shown in Table 6, Φ is a measure on $\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c$, and $f(\mathbf{p})$ is a Φ-equivalence class of measurable functions on $\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c$ with values in the compact set of all nonnegative-definite linear operators in the space $\mathsf{V}^{\mathscr{O}\times Z_2^c}$ with unit trace. The field has the form

$$\mathsf{C}(\mathbf{x}) = \sum_{m=1}^{3} C_m\, {}_{\mathscr{O}\times Z_2^c}\mathsf{T}_{ijkl}^{A_{1g},m,1} + \sum_{m=1}^{3}\sum_{n=1}^{48} \int_{\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c} u_n(\mathbf{x}, \mathbf{p})\, \mathrm{d}Z^{mn}(\mathbf{p})\, {}_{\mathscr{O}\times Z_2^c}\mathsf{T}_{ijkl}^{A_{1g},m,1},$$

where $(Z^{1n}(\mathbf{p}), \dots, Z^{3n}(\mathbf{p}))^\top$ are 48 centred uncorrelated $\mathsf{V}^{\mathscr{O}\times Z_2^c}$-valued random measures on $\hat{\mathbb{R}}^3/\mathscr{O}\times Z_2^c$ with control measure $f(\mathbf{p})\,\mathrm{d}\mu(\mathbf{p})$, and where $u_n(\mathbf{x}, \mathbf{p})$ are different products of sines and cosines of angles from Table 6.

5.8 The Isotropic Class

Theorem 16 (An isotropic random field in the isotropic class) *The one-point correlation tensor of the homogeneous and $(\mathrm{O}(3), 2U^{0g})$-isotropic mean-square continuous random field $\mathsf{C}(\mathbf{x})$ has the form*

$$\langle \mathsf{C}(\mathbf{x}) \rangle = C_1\delta_{ij}\delta_{kl} + C_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \qquad C_m \in \mathbb{R}.$$

Its two-point correlation tensor has the form

$$\langle \mathsf{C}(\mathbf{x}), \mathsf{C}(\mathbf{y}) \rangle = \int_0^\infty \frac{\sin(\lambda\|\mathbf{y}-\mathbf{x}\|)}{\lambda\|\mathbf{y}-\mathbf{x}\|} f(\lambda)\, \mathrm{d}\Phi(\lambda),$$

where $\Phi(\lambda)$ is a finite measure on $[0, \infty)$,

$$f(\lambda) = \begin{pmatrix} v_1(\lambda) & v_2(\lambda) \\ v_2(\lambda) & 1 - v_1(\lambda) \end{pmatrix},$$

and where $\mathbf{v}(\lambda) = (v_1(\lambda), v_2(\lambda))^\top$ is a Φ-equivalence class of measurable functions on $[0, \infty)$ taking values in the closed disk $(v_1(\lambda) - 1/2)^2 + v_2^2(\lambda) \le 1/4$. The field itself has the form

$$\mathsf{C}_{ijkl}(\rho, \theta, \varphi) = C_1\delta_{ij}\delta_{kl} + C_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + 2\sqrt{\pi} \sum_{\ell=0}^{\infty}\sum_{m=\ell}^{\ell} S_\ell^m(\theta, \varphi) \int_0^\infty j_\ell(\lambda\rho) \times \left({}_{\mathrm{O}(3)}\mathsf{T}_{ijkl}^{0,1,0}\, \mathrm{d}Z_{\ell 1}^m(\lambda) + {}_{\mathrm{O}(3)}\mathsf{T}_{ijkl}^{0,2,0}\, \mathrm{d}Z_{\ell 2}^m(\lambda) \right),$$

where $(Z_{\ell 1}^m, Z_{\ell 2}^m)^\top$ is the set of mutually uncorrelated $\mathsf{V}^{\mathrm{O}(3)}$-valued random measures with $f(\lambda)\,\mathrm{d}\Phi(\lambda)$ as their common control measure.

6 Conclusions

Hooke's law describes the physical phenomenon of *elasticity* and belongs to the family of *linear constitutive laws*, see [25]. In general, a linear constitutive law is an element of a subspace of the tensor product $V^{\otimes(p+q)}$, where p (resp. q) is the rank of tensors in the first (resp. second) state tensor space. Denote by U the restriction of the representation $g \mapsto g^{\otimes(p+q)}$ to the above subspace. Consider U as a group action. The orbit types of this action are called the classes of the phenomenon under consideration (e.g., photoelasticity classes, piezoelectricity classes and so on). All symmetry classes of all possible linear constitutive laws were described in [25, 26].

For each class, one can consider its fixed point set $\mathsf{V}^H \subset V^{\otimes(p+q)}$, a group K with $H \subseteq K \subseteq N(H)$, and the restriction U of the representation $g \mapsto g^{\otimes(p+q)}$ of the group K to V^H. Calculating the general form of the one-point and two-point correlation tensors of the corresponding homogeneous and (K, U)-isotropic random field and the spectral expansion of the field in terms of stochastic integrals with respect to orthogonal scattered random measures is an interesting research question.

There are two principal uses of the results obtained here. The first one is to model and simulate any statistically wide-sense homogeneous and isotropic, linear hyperelastic, random medium. One example is a polycrystal made of grains belonging to a specific crystal class, while another example is a mesoscale continuum defined through upscaling of a random material on scales smaller than the RVE; if the upscaling is conducted on the RVE level, there is no spatial randomness and the continuum model is deterministic. Here one would proceed in the following steps:

– for a given microstructure, determine the one- and two-point statistics using some experimental and/or image-based computational methods;
– calibrate the entire correlation structure of the elasticity TRF;
– simulate the realisations of this TRF.

The second application of our results is their use as input of a random mesoscale continuum (Fig. 1(c)) into stochastic field equations such as SPDEs and SFEs.

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