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# Mathematical Modeling of Volumetric Material Growth in Thermoelasticity

**Jean-François Ganghoffer · Pavel I. Plotnikov · Jan Sokołowski**

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**Abstract** The model of volumetric material growth is introduced in the framework of finite elasticity. The state variables include the deformations, temperature and the transplant matrix function. The wellposedness of the proposed model is shown. The existence of local in time classical solutions for the quasistatic deformations boundary value problem coupled with the energy balance and the growth evolution of the transplant is obtained. The new mathematical results for a broad class of growth models in mechanics and biology are presented with complete proofs.

**Keywords** Volumetric growth · Evolutionary problem · Existence of local solutions · Finite elasticity

**Mathematics Subject Classification** 35Q74 · 35J60 · 35K61 · 34B15 · 35Q92

## 1 Introduction

*Motivation and main contribution* The present work deals with the mathematical modeling of volumetric growth in thermoelastic bodies. The mechanical models are based on

J.-F. Ganghoffer
LEMTA, Nancy Université, 2, Avenue de la Forêt de Haye, TSA 60604, 54518 Vandoeuvre-Lès-Nancy, France
e-mail: jean-francois.ganghoffer@univ-lorraine.fr

P.I. Plotnikov
Lavrentyev Institute of Hydrodynamics, Lavrentyev pr. 15, 630090 Novosibirsk, Russia
e-mail: plotnikov@hydro.nsc.ru

J. Sokołowski (✉)
Laboratoire de Mathématiques, Institut Élie Cartan Nancy, UMR7502 (Université de Lorraine, CNRS, INRIA), TSA 70239, 54506 Vandoeuvre-Lès-Nancy Cedex, France
e-mail: Jan.Sokolowski@univ-lorraine.fr

J. Sokołowski
Systems Research Institute of the Polish Academy of Sciences, ul. Newelska 6, 01-447 Warszawa, Poland

the general idea that growth can be taken into account by considering that deformations of a growing solid body are due to both changes of mass and elastic deformation. The most important statement of the theory from a kinematic viewpoint ([18]) is that the geometric deformation tensor is decomposed into the product of a growth tensor describing the local addition of material and an elastic tensor characterizing the reorganization of the body. The rigorous foundation of the volumetric growth theory was given in [6], with the so-called transplant tensor representing the growth transformation.

Our developments are based on the equations formulated in [6]. The following issue is addressed in this paper:

**Problem 1** Determine the state variables of growing elastic body in the specific framework of finite elasticity [24]. The state variables include the deformation vector field $\mathbf{u}$, the scalar temperature $\theta$ and the so-called *transplant mapping* $\mathbf{K}$. The associated model of growing elastic body contains the momentum balance equation, the energy balance equation, and the nonlinear evolutionary equation for the transplant field $\mathbf{K}$ supplemented with initial and boundary conditions.

*Note 1* It is shown that there are local, classical solutions to the model of growing, elastic body (3.4a)–(3.5c), for the initial transplant field given by the sum of a rotation and a bounded mapping. This new mathematical result with the full proof is given by Theorem 1.

*Short literature review* Growth (resp. atrophy) describes the physical processes by which a material or solid body increases (resp. decreases) its size by addition (resp. removal) of mass. A clear distinction is generally made between growth per se, remodeling (change of properties), and morphogenesis (shape changes), a classification suggested by [22] Taber (1995). The advantages and drawbacks of the existing growth models are exposed in the recent contribution [15] (Menzel and Kuhl, 2012). A first class of models is the kinematic models describing an evolution towards an homeostatic state rely on the kinematic decomposition of the transformation gradient into a generally incompatible mapping and an elastic mapping; they were historically introduced by [18] Rodriguez et al. (1994). The growth transformation evolves in time as a function of the difference between a stress measure and a corresponding measure associated to the surmised homeostatic state ([22] Taber 1998; [19] Rodriguez et al. 2007; [1] Alford et al. 2008; [25] Vignes and Papadopoulos 2010). This first class of models is criticized due to the absence of a rational mechanical framework. Approaches analogous to elastoplasticity have then been developed as a second class of models in a rational framework basing on the writing of the second principle of thermodynamics for open systems, in order to identify the evolution laws of growth ([13] Kuhl et al. 2007; [14] Menzel 2007; [16] Olsson and Klarbring 2008). It is important to note there the prominent role of Eshelby stress in relation to the material driving forces for growth ([8, 9] Ganghoffer 2010, 2011; [13] Kuhl et al. 2007), relying on Eshelby pioneering approach ([7] Eshelby 1957). Central here is the idea to separate the shape variation due to the physical motion from the microstructural evolutions due to growth and remodeling phenomena occurring in the evolutive reference configuration.

## 2 Mechanical Background

*Finite elasticity* In this section, we briefly discuss some basic facts from finite elasticity theory. Throughout the paper, we shall assume that $\Omega \subset \mathbb{R}^3$ is a bounded reference domain with the boundary $\partial\Omega$ of class $C^\infty$ in the space variable $x$. The state of an elastic material is characterized by a deformation field $\mathbf{u} = (u_1, u_2, u_3) : \Omega \times [0, T] \to \mathbb{R}^3$ and the Kelvin temperature $\theta : \Omega \times [0, T] \to \mathbb{R}^+$. The elastic distortion tensor $D\mathbf{u}$ is the Jacobi matrix of the mapping $\mathbf{u}$ with the entries

$$D\mathbf{u}_{ij}(x,t) = \partial_j u_i(x,t), \quad (x,t) \in \Omega \times [0, T].$$

Here the notation

$$\partial_i := \partial_{x_i} = \partial/\partial x_i,$$

stands for the spatial derivatives. We will assume that the material is hyperelastic and its properties are described by the specific free energy density $\Psi(\theta, D\mathbf{u})$. In particular, a stress tensor $\mathbf{T}(\theta, D\mathbf{u})$ and internal energy $e(\theta, D\mathbf{u})$ are defined by

$$\mathbf{T}(\theta, D\mathbf{u}) = \frac{\partial \Psi(\theta, D\mathbf{u})}{\partial(D\mathbf{u})}, \qquad e = \Psi(\theta, D\mathbf{u}) - \theta \frac{\partial \Psi(\theta, D\mathbf{u})}{\partial \theta}. \tag{2.1}$$

Here $\partial\Psi(\theta, \Phi)/\partial\Phi$ denotes the matrix with the entries

$$\left(\frac{\partial \Psi(\theta, \Phi)}{\partial \Phi}\right)_{ij} = \frac{\partial \Psi(\theta, \Phi)}{\partial \Phi_{ij}}.$$

In many applications, it is sufficient to take the specific free energy density in the form

$$\Psi(\theta, D\mathbf{u}) = -c_T \theta \log\theta + \theta W(D\mathbf{u}), \tag{2.2}$$

where $W$ is the stored elastic energy. The specific free energy density satisfies the two following conditions

$$\mathbf{T}(\theta, \Phi)\Phi^\top = \Phi \mathbf{T}(\theta, \Phi)^\top, \tag{2.3}$$

$$\mathbf{T}(\theta, \mathbf{R}\Phi) = \mathbf{R}\mathbf{T}(\theta, \Phi) \tag{2.4}$$

for all $\theta$, for all matrices $\Phi$, and for all orthogonal matrices $\mathbf{R}$. Relation (2.3) expresses the angular momentum conservation law, and relation (2.4) expresses the observer independence principle. We assume that the reference configuration is unstressed, i.e.,

$$\mathbf{T}(\theta, \mathbf{I}) = 0 \quad \text{for all } \theta. \tag{2.5}$$

It follows from (2.4) that

$$\mathbf{T}(\theta, \mathbf{R}) = 0 \tag{2.6}$$

for all $\theta$ and all orthogonal matrices $\mathbf{R}$. In order to characterize stability properties of the reference configuration, it is convenient to introduce the linear matrix-valued form $L(\theta, \Phi)$ defined on the linear space of $3 \times 3$ matrices $\boldsymbol{\xi}$ by

$$L(\theta, \Phi)\boldsymbol{\xi} = \frac{\partial \mathbf{T}(\theta, \Phi)}{\partial \Phi}\boldsymbol{\xi} = \frac{\partial \mathbf{T}(\theta, \Phi)}{\partial \Phi_{pq}}\xi_{pq}. \tag{2.7}$$

Notice that $L(\theta, \Phi)\boldsymbol{\xi}$ is a matrix with the entries

$$\big(L(\theta, \Phi)\boldsymbol{\xi}\big)_{ij} = l_{ijpq}(\theta, \Phi)\xi_{pq}, \quad \text{where} \quad l_{ijpq}(\theta, \Phi) = \frac{\partial^2 \Psi(\theta, \Phi)}{\partial \Phi_{ij}\partial \Phi_{pq}}. \tag{2.8}$$

The linear form $L$ is associated with the bilinear form

$$L(\theta,\Phi)\boldsymbol{\xi}\cdot\boldsymbol{\eta}=l_{ijpq}(\theta,\Phi)\xi_{pq}\eta_{ij}. \tag{2.9}$$

The following lemma constitutes the basic properties of the linear form $L$.

**Lemma 1** *For all $\theta$, for all matrices $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, for all orthogonal matrices $\mathbf{R}$, and for all skew-symmetric matrices $\boldsymbol{\zeta}$,*

$$L(\theta,\mathbf{R})(\boldsymbol{\xi}\mathbf{R})\cdot(\boldsymbol{\eta}\mathbf{R})=L(\theta,\mathbf{I})\big(\mathbf{R}^\top\boldsymbol{\xi}\mathbf{R}\big)\cdot\big(\mathbf{R}^\top\boldsymbol{\eta}\mathbf{R}\big), \tag{2.10}$$

$$L(\theta,\mathbf{R})(\boldsymbol{\zeta}\mathbf{R})=0, \tag{2.11}$$

$$l_{ijpq}(\theta,\mathbf{R})=l_{\alpha j\sigma q}(\theta,\mathbf{I})R_{p\sigma}R_{i\alpha}, \tag{2.12}$$

$$l(\theta,\mathbf{I})_{ijpq}=l(\theta,\mathbf{I})_{pqij}=l(\theta,\mathbf{I})_{jipq}=l(\theta,\mathbf{I})_{ijqp}. \tag{2.13}$$

*Proof* Identities (2.10)–(2.13) are a straightforward consequence of conditions (2.3) and (2.4). See [3] Ch. 4, [10], and [24] Ch. 3 Sect. 3 for details. □

We will assume throughout the paper that the specific energy satisfies the following stability condition

$$L(\vartheta,\mathbf{I})\boldsymbol{\xi}\cdot\boldsymbol{\xi}\ge c(\theta)\big|\boldsymbol{\xi}+\boldsymbol{\xi}^\top\big|^2, \tag{2.14}$$

for all matrices $\boldsymbol{\xi}$. Here the constant $c(\theta)$ is strongly positive and bounded for positive and bounded $\theta$.

*Remark 1* It follows from the stability condition (2.14) that for every orthogonal $\mathbf{R}$,

$$L(\theta,\mathbf{R})(\boldsymbol{\xi}\mathbf{R})\cdot(\boldsymbol{\xi}\mathbf{R})\ge c(\theta)\big|\boldsymbol{\xi}+\boldsymbol{\xi}^\top\big|^2. \tag{2.15}$$

Indeed, in view of (2.10) and (2.14) we have

$$\begin{aligned}L(\theta,\mathbf{R})(\boldsymbol{\xi}\mathbf{R})\cdot(\boldsymbol{\xi}\mathbf{R})&=L(\theta,\mathbf{I})\big(\mathbf{R}^\top\boldsymbol{\xi}\mathbf{R}\big)\cdot\big(\mathbf{R}^\top\boldsymbol{\xi}\mathbf{R}\big)\\&\ge c(\theta)\big|\mathbf{R}^\top\boldsymbol{\xi}\mathbf{R}+\big(\mathbf{R}^\top\boldsymbol{\xi}\mathbf{R}\big)^\top\big|^2=c(\theta)\big|\mathbf{R}^\top\big(\boldsymbol{\xi}+\boldsymbol{\xi}^\top\big)\mathbf{R}\big|^2=c(\theta)\big|\boldsymbol{\xi}+\boldsymbol{\xi}^\top\big|^2.\end{aligned}$$

*Growing material* The main hypothesis of the volumetric growth theory is that a material consists of infinitesimally small particles $\mathcal{O}(x,t)$ labeled by the reference coordinate $x$ and $t$. The growth of each particle is determined by the transplant matrix $\mathbf{K}(x,t):\mathcal{O}(x,t)\to\mathbf{K}(x,t)\mathcal{O}(x,t)$. On the other hand, the growing particles are subjected to elastic deformations characterized by the Jacobi matrix $D\mathbf{u}(x,t)$. This leads to the following diagram

$$\mathcal{O}\xrightarrow{K}\mathcal{O}_g=\mathbf{K}(\mathcal{O})\xrightarrow{D\mathbf{u}}\mathcal{O}_{real}=D\mathbf{u}\mathbf{K}(\mathcal{O}). \tag{2.16}$$

Such an interpretation of the volumetric growth theory is widely distributed in the literature. Notice that there is no growth at a reference point $(x,t)$ if $\mathbf{K}(x,t)$ coincides with some rotation matrix $\mathbf{R}(x,t)$. Hence the rotation transplant matrices corresponds to non-growing homeostatic states.

However, the diagram (2.16) is misleading since the deformation of the growing elastic body is completely determined by the deformation field $\mathbf{u}(x,t)$, and the domain $\mathcal{D}_t$, occupied by the growing body in the real Euclidean space, is given by $\mathcal{D}_t=\mathbf{u}(\Omega,t)$. In fact, the transplant $\mathbf{K}$ is a dynamical characteristic. It has no direct effect on the kinematic of the process, but participates in formation of the shape of elastic body via the governing equations.

Thus the distortion tensor has the form of the product $D\mathbf{u}\mathbf{K}$ of the elastic distortion tensor $D\mathbf{u}$ and the transplant $\mathbf{K}$. The transplant tensor is responsible for material growth.

For growing materials, the specific free energy $\Psi_g(\theta, \mathbf{K}, D\mathbf{u})$, the stress tensor $\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u})$, and the internal energy $e_g(\theta, \mathbf{K}, D\mathbf{u})$ are defined as follows.

$$\Psi_g(\theta, \mathbf{K}, D\mathbf{u}) = \frac{1}{J_K}\Psi(\theta, D\mathbf{u}\mathbf{K}), \tag{2.17a}$$

$$\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) = \frac{1}{J_K}\frac{\partial\Psi(\theta, D\mathbf{u}\mathbf{K})}{\partial(D\mathbf{u})}, \tag{2.17b}$$

$$e_g(\theta, \mathbf{K}, D\mathbf{u}) = \frac{1}{J_K}\left(\Psi(\theta, D\mathbf{u}\mathbf{K}) - \theta\frac{\partial\Phi(\theta, D\mathbf{u}\mathbf{K})}{\partial\theta}\right). \tag{2.17c}$$

Here, $J_K = \det\mathbf{K}$, $\Psi$ is the specific free energy density of the basic elastic material. It is easy to see that

$$\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) = \frac{1}{J_K}\frac{\partial\Psi(\theta, \Phi)}{\partial\Phi}\mathbf{K}^\top, \quad \text{where } \Phi = D\mathbf{u}\mathbf{K}. \tag{2.18}$$

If we take the specific free energy in the form

$$\Psi_g(\theta, \mathbf{K}, D\mathbf{u}) = \frac{1}{J_K}\big(-c_T\theta\log\theta + \theta W(D\mathbf{u}\mathbf{K})\big), \tag{2.19}$$

then we get the following expression for the stress tensor and the internal energy

$$\mathbf{T}_g = \frac{\theta}{J_K}\frac{\partial W(\Phi)}{\partial\Phi}\mathbf{K}^\top, \quad \text{where} \quad \Phi = D\mathbf{u}\mathbf{K}, \quad e_g = \frac{c_T}{J_K}\theta. \tag{2.20}$$

The reference configuration is unstressed if and only if $\mathbf{K} = \mathbf{R}(x, t)$, where $\mathbf{R}$ is an orthogonal matrix. We stress that the tensor $\mathbf{K}$ is not a potential, and $\mathbf{R}$ is an arbitrary orthogonal matrix depending on $(x, t)$. For given $\theta$, $\mathbf{K}$ and $D\mathbf{u}$, let define the linear matrix-valued form $L_g(\theta, \mathbf{K}, D\mathbf{u})$ by the equality

$$L_g(\theta, \mathbf{K}, D\mathbf{u})\boldsymbol{\xi} = \lim_{\tau\to 0}\frac{1}{\tau}\big\{\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u} + \tau\boldsymbol{\xi}) - \mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u})\big\}. \tag{2.21}$$

Calculations show that

$$\big(L_g(\theta, \mathbf{K}, D\mathbf{u})\boldsymbol{\xi}\big)_{ij} = l_{g,ijpq}(\theta, \mathbf{K}, D\mathbf{u})\xi_{pq}, \tag{2.22}$$

where

$$l_{g,ijpq}(\theta, \mathbf{K}, D\mathbf{u}) = \frac{1}{J_K}\frac{\partial^2\Psi(\theta, \Phi)}{\partial\Phi_{i\alpha}\partial\Phi_{p\beta}}K_{j\alpha}K_{q\beta}, \quad \Phi = D\mathbf{u}\mathbf{K}. \tag{2.23}$$

It follows from (2.23) that the forms $L_g$ and $L$ are connected by the relations

$$L_g(\theta, \mathbf{K}, D\mathbf{u})\boldsymbol{\xi}\cdot\boldsymbol{\eta} = L(\theta, \Phi)(\boldsymbol{\xi}\mathbf{K})\cdot(\boldsymbol{\eta}\mathbf{K}), \quad \Phi = D\mathbf{u}\mathbf{K}. \tag{2.24}$$

The following lemma is the extension of Lemma 1 to the case of growing materials.

**Lemma 2** *Let $\Psi$ satisfies symmetry conditions (2.3)–(2.4), the equilibrium condition (2.5), and the stability condition (2.14). Then, for all $\theta$, for all matrices $\boldsymbol{\xi}$, for all orthogonal matrices $\mathbf{R}$, and for all skew-symmetric matrices $\boldsymbol{\zeta}$, it holds that*

$$L_g(\theta, \mathbf{R}, \mathbf{I})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq c(\theta)\left|\boldsymbol{\xi} + \boldsymbol{\xi}^\top\right|^2, \tag{2.25}$$

$$L_g(\theta, \mathbf{R}, \mathbf{I})\boldsymbol{\zeta} = 0, \tag{2.26}$$

$$l_{g,ijpq}(\theta, \mathbf{R}, \mathbf{I}) = l_{\alpha m \sigma n}(\theta, \mathbf{I}) R_{i\alpha} R_{jm} R_{p\sigma} R_{qn}, \tag{2.27}$$

$$l_{g,ijpq}(\theta, \mathbf{R}, \mathbf{I}) = l_{g,pqij}(\theta, \mathbf{R}, \mathbf{I}) = l_{g,jipq}(\theta, \mathbf{R}, \mathbf{I}) = l_{g,ijqp}(\theta, \mathbf{R}, \mathbf{I}). \tag{2.28}$$

*Proof* Notice that $J_K = 1$ for $\mathbf{K} = \mathbf{R}$. It follows from this, (2.10) and (2.24) that

$$\begin{aligned} L_g(\theta, \mathbf{R}, \mathbf{I})\boldsymbol{\xi} \cdot \boldsymbol{\xi} &= L(\theta, I)\big(\mathbf{R}^\top \boldsymbol{\xi} \mathbf{R}\big) \cdot \big(\mathbf{R}^\top \boldsymbol{\xi} \mathbf{R}\big) \\ &\geq c(\theta)\left|\mathbf{R}^\top \boldsymbol{\xi} \mathbf{R} + \big(\mathbf{R}^\top \boldsymbol{\xi} \mathbf{R}\big)\top\right|^2 = c(\theta)\left|\mathbf{R}^\top\big(\boldsymbol{\xi} + \boldsymbol{\xi}^\top\big)\mathbf{R}\right|^2 = c(\theta)\left|\boldsymbol{\xi} + \boldsymbol{\xi}^\top\right|^2, \end{aligned}$$

which leads to (2.25). Next, representation (2.22) implies

$$\big(L_g(\theta, \mathbf{K}, D\mathbf{u})\boldsymbol{\xi}\big)_{ij} = (J_K)^{-1}\big(L(\theta, \Phi)(\boldsymbol{\xi}\mathbf{K})\big)_{i\alpha} K_{j\alpha},$$

where $\Phi = D\mathbf{u}\mathbf{K}$. Setting $\mathbf{K} = \mathbf{R}$, $D\mathbf{u} = \mathbf{I}$, and $\boldsymbol{\xi} = \boldsymbol{\zeta}$, we obtain

$$\big(L_g(\theta, \mathbf{R}, \mathbf{I})\boldsymbol{\zeta}\big)_{ij} = \big(L(\theta, \mathbf{R})(\boldsymbol{\zeta}\mathbf{R})\big)_{i\alpha} R_{j\alpha},$$

which along with (2.11) yields (2.26). Next, it follows from representation (2.23) that

$$l_{g,ijpq}(\theta, \mathbf{R}, \mathbf{I}) = l_{i\alpha p\beta}(\theta, \mathbf{R}) R_{j\alpha} R_{q\beta}.$$

Combining this result with (2.12), we obtain (2.27). It remains to note that (2.27) and the symmetry relations (2.13) imply (2.28), and the lemma follows. □

It is a remarkable fact of the theory that the form $L_g$, obtained by linearization of $\mathbf{T}_g$ on unstressed deformation field with an arbitrary transplant $\mathbf{K} = \mathbf{R}$, satisfies the symmetry relations (2.28), which are similar to the symmetry relations of the classical linear elasticity theory. This means that we can apply the main tools of linear elasticity theory, such as the Korn inequalities, to the theory of growing materials.

## 3 Problem Formulation. Assumptions. Results

The problem consists of finding a deformation $\mathbf{u}$, a temperature $\theta$ and a transplant $\mathbf{K}$ satisfying the quasi-stationary momentum balance equation

$$\operatorname{div} \mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) + \mathbf{f} = 0, \tag{3.1}$$

the energy balance equation

$$\frac{\partial}{\partial t} e_g(\theta, \mathbf{K}, D\mathbf{u}) + \operatorname{div} \mathbf{q} = \mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) \cdot \frac{\partial}{\partial t}(D\mathbf{u}), \tag{3.2}$$

and the evolutionary equation for $\mathbf{K}$,

$$\frac{\partial}{\partial t}\mathbf{K} = \mathbf{g}(\theta, \mathbf{K}, D\mathbf{u}). \tag{3.3}$$

Here $\mathbf{q} = \mathbf{q}(\nabla\theta, \theta, \mathbf{K}, D\mathbf{u})$ is a given heat flux, $\mathbf{g}$ a given matrix-valued function, $\mathbf{f}$ a given bulk exterior dead force; $\mathbf{h}$ is a given boundary load, and the stress tensor $\mathbf{T}_g$ and the internal energy $e_g$ are defined by (2.17a)–(2.17c). Further, we assume that the free energy density is in the form (2.19) and takes the heat flux in the simplest thermodynamically consistent form $\mathbf{q} = \nabla(\theta^{-1})$. Thus, we obtain the following system of differential equations in the cylinder $\Omega \times (0, T)$

$$\operatorname{div} \mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) + \mathbf{f} = 0 \quad \text{in } \Omega \times (0, T), \tag{3.4a}$$

$$\frac{\partial}{\partial t}\left(\frac{c_T\theta}{J_K}\right) + \Delta\left(\frac{1}{\theta}\right) = \mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) \cdot \frac{\partial}{\partial t}(D\mathbf{u}) \quad \text{in } \Omega \times (0, T), \tag{3.4b}$$

$$\frac{\partial}{\partial t}\mathbf{K} = \mathbf{g}(\theta, \mathbf{K}, D\mathbf{u}) \quad \text{in } \Omega \times (0, T). \tag{3.4c}$$

Here

$$\mathbf{T}_g = \frac{\theta}{J_K}\frac{\partial W(\Phi)}{\partial \Phi}\mathbf{K}^\top, \quad \Phi = D\mathbf{u}\mathbf{K}.$$

These equations should be supplemented with boundary and initial conditions. For growing materials, the problem of place with a fixed deformations of the boundary is not natural and we will instead consider the traction problem for the momentum equation

$$-\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u})\mathbf{n} + \mathbf{h} = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{3.5a}$$

For simplicity reasons, we assume that there is no heat flux through the boundary, which leads to the following boundary condition for the temperature

$$\nabla\theta \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0, T). \tag{3.5b}$$

At the initial moment, the temperature and the transplant should be prescribed

$$\theta(x, 0) = \Theta(x), \qquad \mathbf{K}(x, 0) = \mathbf{K}_0(x) \quad \text{in } \Omega. \tag{3.5c}$$

Here $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$, $\mathbf{h}$, $\Theta$ and $\mathbf{K}_0$ are given functions.

Relation (3.4a)–(3.5c) form the closed system of equations and boundary conditions for $\mathbf{u}$, $\theta$ and $\mathbf{K}$. Here the important function $\mathbf{g}$ should be defined by experimental data. The lack of information about this function is a weak point of the theory. It is known, see [6] and the discussion in the beginning of Sect. 7, that $\mathbf{g}$ should satisfy some structural conditions. Physically relevant examples can be found in [12].

The main distinction between problem (3.4a)–(3.5c) and the classic problems of thermoelasticity is the presence of the evolutionary equation (3.4c) with the strongly nonlinear right-hand side $\mathbf{g}$, and the dependence of the stored energy on the solutions of (3.4c). Calculations show that in general the solutions to problem (3.4a)–(3.5c) blow up in finite time. The simplest trivial example is the case of isotropic growth with the transplant $\mathbf{K} = K(x, t)\mathbf{I}$ and the scalar function $\mathbf{g} = |K|^\alpha K g_0(K, D\mathbf{u})\mathbf{I}$, $\alpha > 0$, $g_0 > c > 0$. In this particular case solution may exist only on a small interval, depending on initial data. The fracture and collapse of growing biological objects are not rare phenomena, and the problem of finite time blow-up deserves more detailed investigation, which is beyond the scope of this work. In the present paper we are focusing on the local existence theory for small time intervals. However, the problem remains nontrivial even in this refined case. Among the difficulties are the strong nonlinearity, the composed character of equations, and the traction boundary conditions (3.5a).

Before the formulation of results, we introduce necessary notation and formulate the assumptions on the boundary and initial data.

**Definition 1** A couple $(\mathbf{f}, \mathbf{h}) \in L^1(\Omega) \times L^1(\partial\Omega)$ is said to be equilibrated if

$$\int_\Omega \mathbf{f}\, dx + \int_{\partial\Omega} \mathbf{h}\, ds = 0, \qquad \int_\Omega (x_i f_j - x_j f_i)\, dx + \int_{\partial\Omega} (x_i h_j - x_j h_i)\, ds = 0 \tag{3.6}$$

for all $i, j$. Following [5], we introduce the astatic matrix $\mathbf{C}$ with the entries

$$C_{ij} = \int_\Omega x_i f_j\, dx + \int_{\partial\Omega} x_i h_j\, ds. \tag{3.7}$$

If the couple $(\mathbf{f}, \mathbf{h})$ is equilibrated, then the astatic matrix is symmetric.

**Definition 2** An equilibrated couple $(\mathbf{f}, \mathbf{h}) \in L^2(\Omega) \times L^2(\partial\Omega)$ is said to be non-degenerate if there is a positive $c^*$ such that

$$|\mu_i + \mu_j| \ge c^* \|\mathbf{f}, \mathbf{h}\|_{L^2} \quad \text{for all } i \ne j, \tag{3.8}$$

where $\mu_i$ are eigenvalues of the matrix $\mathbf{C}$, and

$$\|\mathbf{f}, \mathbf{h}\|_{L^2} = \left( \int_\Omega |\mathbf{f}|^2\, dx + \int_{\partial\Omega} |\mathbf{h}|^2\, ds \right)^{1/2}. \tag{3.9}$$

**Definition 3** For a given constant $c^* > 0$, we denote by $\mathcal{F}_c \subset L^2(\Omega \times \partial\Omega)$ the set all equilibrated non-degenerate couples $(\mathbf{f}, \mathbf{h}) \ne 0$ satisfying inequality (3.8). It is easily seen that zero is a limiting point of $\mathcal{F}_c$ and the set $\mathcal{F}$ is star-shaped, i.e., if $(\mathbf{f}, \mathbf{h}) \in \mathcal{F}_c$, then $\epsilon(\mathbf{f}, \mathbf{h}) \in \mathcal{F}_c$ for all $\epsilon \ne 0$.

Finally, we denote by $\mathcal{A}$ the annulus $\{1/2 \le ||\Phi| - 1| \le 2\}$. We assume that the specific free energy, the function $\mathbf{g}$ and the initial and boundary data satisfy the following conditions.

H.1 The specific free energy density $\Psi_g$ has the form

$$\Psi_g(\theta, \mathbf{K}, D\mathbf{u}) = J_K^{-1}\big(-c_T \theta \log\theta + \theta W(D\mathbf{u}\mathbf{K})\big).$$

The elastic stored energy $W \in C^\infty(\mathcal{A})$, and the matrix valued function $\mathbf{g} \in C^\infty(\mathbb{R} \times \mathcal{A} \times \mathcal{A})$.

H.2 Let $\mathbf{T}$ be an elastic energy tensor with the entries $T_{ij} = \theta \partial W(\Phi)/\partial \Phi_{ij}$. Then for all $\Phi \in \mathcal{A}$ and all orthogonal matrices $\mathbf{R}$,

$$\mathbf{T}(\theta, \Phi)\Phi^\top = \Phi \mathbf{T}(\theta, \Phi)^\top, \qquad \mathbf{T}(\theta, \mathbf{R}\boldsymbol{\Phi}) = \mathbf{R}\mathbf{T}(\theta, \Phi), \qquad \mathbf{T}(\theta, \mathbf{R}) = 0.$$

H.3 Let the linear form $L(\theta, \Phi)$ be defined by (2.8)–(2.9). Then

$$L(\theta, \mathbf{I})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \ge c\theta \left|\boldsymbol{\xi} + \boldsymbol{\xi}^\top\right|^2 \quad \text{for all matrices } \boldsymbol{\xi}.$$

H.4 There is $c_0 > 0$ such that $(\mathbf{f}(t), \mathbf{h}(t)) \in \mathcal{F}_c$ for every $t \in [0, T]$ and

$$\big\|\partial_t \mathbf{f}(t), \partial_t \mathbf{h}(t)\big\|_{L^2} \le c_0 \big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}. \tag{3.10}$$

H.5 The function $\Theta \in C^\infty(\Omega)$ satisfies the conditions

$$0 < c^{-1} < \Theta < c < \infty, \qquad \partial_n \Theta = 0 \quad \text{on } \partial\Omega.$$

It is worth noting that Condition (H.4) eliminates the case when the couple $(\mathbf{f}, \mathbf{h})$ vanishes identically at some moment $t$. It seems that a solution may develop a singularity at such a moment. The following theorem is the main result of this paper

**Theorem 1** *Let $p > 4$ and $T > 0$. Let conditions* (H.1)–(H.5) *be satisfied. Furthermore assume that $\mathbf{K}_0(x) = \mathbf{R}(x) + \mathbf{k}_0(x)$, where $\mathbf{R} \in C^\infty(\Omega)$ is an arbitrary orthogonal matrix and $\mathbf{k}_0 \in C^\infty(\Omega)$ is an arbitrary perturbation of $\mathbf{R}$. Then, there are positive $\varepsilon_0$ and $T_0 \in (0, T]$ such that for all $(\mathbf{f}, \mathbf{h}) \in \mathcal{F}_c$, and for all $\mathbf{k}_0$ satisfying*

$$\begin{aligned} \|\mathbf{f}\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\mathbf{h}\|_{L^\infty(0,T;B_p^{2-1/p}(\partial\Omega))} + \|\mathbf{k}_0\|_{W^{2,p}(\Omega)} \le \varepsilon_0, \\ \|\partial_t \mathbf{f}\|_{L^p(0,T;L^p(\Omega))} + \|\partial_t \mathbf{h}\|_{L^p(0,T;B_p^{1-1/p}(\partial\Omega))} \le \varepsilon_0 \end{aligned} \tag{3.11}$$

*problem* (3.4a)–(3.5c) *has a locally unique solution*

$$\begin{aligned} &\mathbf{u} \in L^\infty\big(0, T; W^{3,p}(\Omega)\big) \cap W^{1,p}\big(0, T_0; W^{2,p}(\Omega)\big), \\ &\theta \in L^p\big(0, T; W^{3,p}(\Omega)\big) \cap W^{1,p}\big(0, T_0; W^{1,p}(\Omega)\big), \\ &\mathbf{K} \in W^{1,p}\big(0, T_0; W^{2,p}(\Omega)\big). \end{aligned}$$

The rest of the paper is devoted to the proof of this theorem. The mathematical difficulties are the complexity of the nonlinear traction problem and the inconsistency between mathematical tools needed for solving the static and evolutionary parts of the governing equations. In order to cope with these difficulties, we use the modification of the method proposed in [5, 21, 24] for analysis of the static nonlinear traction problem, and replace the governing equations by extended system (4.4a)–(4.4i). In Sect. 4, we deduce the extended system and formulate the main result on existence and local uniqueness of solutions to the boundary value problem for the extended system in Sobolev spaces. In Sect. 5, we reduce this boundary value problem to an operator equation and analyze the smoothness properties of the corresponding operator. In Sect. 7, we deduce the equations of linear theory for growing materials, and prove the well posedness of boundary value problem for these equations. Finally, we employ the Newton-Kantorovich iteration scheme to obtain a solution of the operator equation and by doing so complete the proof of Theorem 1.

## 4 Modified Problem

In this section, we formulate the extension of the basic equations (3.4a)–(3.5c). To this end, we introduce some auxiliary constructions. Following [5], we define the special nonlinear projection of the stress tensor on the space of equilibrated vector fields. Let us choose an arbitrary vector field $\boldsymbol{\varphi} \in C^\infty(\Omega)$ such that

$$\int_\Omega \boldsymbol{\varphi}\,dx = 0, \qquad \int_\Omega x_i \varphi_j(x) = 0 \quad \text{for all } i \neq j,$$
$$\int_\Omega \big(x_i\varphi_i(x) + x_j\varphi_j(x)\big)\,dx = 1 \quad \text{for all } i, j. \tag{4.1}$$

Next we define the matrix-valued integral operator

$$\mathcal{E}(\theta, \mathbf{K}, D\mathbf{u}) = \mathbf{E}\boldsymbol{\varphi}, \qquad \mathbf{E} = \int_\Omega \big(\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) - \mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u})^\top\big)dx. \tag{4.2}$$

Finally, for every positive $T_0$, we set

$$\chi(t) = 1 \quad \text{for } t \le T_0, \qquad \chi(t) = 0 \quad \text{for } t > T_0. \tag{4.3}$$

We are now in a position to formulate the modified system of equations and boundary and initial conditions. Since $\theta(x,0) = \Theta(x)$ and $\mathbf{K}(x,0) = \mathbf{R}(x) + \mathbf{k}_0(x)$, it is convenient to formulate the problem in terms of the perturbations $(\mathbf{u}, \theta, \mathbf{K}) - (\mathbf{Id}, \Theta, \mathbf{R})$. We thus come to the following

**Problem M** For given $T \ge T_0 > 0$, $(\mathbf{f}, \mathbf{h})$, $\Theta(x)$, $\mathbf{R}(x)$, and $\mathbf{k}_0(x)$, find a deformation field $\mathbf{u}$, a temperature $\theta$, and a transplant $\mathbf{K}$ which admits the representation

$$\mathbf{u}(x,t) = x + \mathbf{S}(t)x + \mathbf{v}(x,t) \quad \text{in } \Omega \times (0,T),$$
$$\theta(x,t) = \Theta(x) + \vartheta(x,t), \qquad \mathbf{K}(x,t) = \mathbf{R}(x) + \mathbf{k}(x,t) \quad \text{in } \Omega \times (0,T), \tag{4.4a}$$

where $\mathbf{S}(t)$ is a skew-symmetric matrix, and $\mathbf{v}$ satisfies the conditions

$$\int_\Omega \mathbf{v}(x,t)\,dx = 0, \qquad \int_\Omega \big(\partial_i v_j(x,t) - \partial_j v_i(x,t)\big)\,dx = 0 \tag{4.4b}$$

for all $i$, $j$ and all $t \in [0, T]$. The unknowns $\vartheta$, $\mathbf{k}$, $\mathbf{v}$, and $\mathbf{S}$ should satisfy the static equations

$$\Xi_1(\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k}) \equiv \operatorname{div} T_g(\theta, \mathbf{K}, D\mathbf{u}) + \mathcal{E}(\theta, \mathbf{K}, D\mathbf{u}) + f = 0 \quad \text{in } \Omega \times (0,T), \tag{4.4c}$$

$$\Xi_2(\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k}) \equiv -T_g(\theta, \mathbf{K}, D\mathbf{u})\mathbf{n} + \mathbf{h} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{4.4d}$$

$$\Xi_3(\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k}) \equiv \frac{1}{\|\mathbf{f}(t), \mathbf{h}(t)\|_{L^2}} \big\{\mathbf{SC} - (\mathbf{SC})^\top + \mathbf{D}(\mathbf{v}) - \mathbf{D}^\top(\mathbf{v})\big\} = 0 \quad \text{in } (0,T), \tag{4.4e}$$

and the evolutionary equations

$$\Xi_4(\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k}) \equiv c_T \frac{\partial\vartheta}{\partial t} - \Delta\left(\frac{\vartheta}{\Theta^2}\right) - \chi(t)\left\{c_T \frac{\partial\vartheta}{\partial t} - \Delta\left(\frac{\vartheta}{\Theta^2}\right) - c_T \frac{\partial}{\partial t}\left(\frac{\theta}{J_K}\right)\right.$$
$$\left. - \Delta\left(\frac{1}{\theta}\right) + \mathbf{T}(\vartheta, \mathbf{K}, D\mathbf{u}) \cdot D\frac{\partial\mathbf{u}}{\partial t}\right\} = 0 \quad \text{in } \Omega \times (0,T), \tag{4.4f}$$

$$\nabla\vartheta \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \vartheta(x,0) = 0 \quad \text{in } \Omega, \tag{4.4g}$$

$$\Xi_5(\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k}) \equiv \frac{\partial\mathbf{k}}{\partial t} - \chi(t)\mathbf{g}(\theta, \mathbf{R} + \mathbf{k}, D\mathbf{u}) = 0 \quad \text{in } \Omega \times (0,T),$$
$$\Xi_6(\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k}) \equiv \mathbf{k}(x,0) - \mathbf{k}_0(x) = 0 \quad \text{in } \Omega. \tag{4.4h}$$

Here $\mathbf{C}(t)$ and $\mathbf{D}$ are matrices with the entries

$$C_{ij} = \int_{\Omega} x_i f_j \, dx + \int_{\partial\Omega} x_i h_j \, ds, \qquad D_{ij} = \int_{\Omega} v_i f_j \, dx + \int_{\partial\Omega} v_i h_j \, ds, \tag{4.4i}$$

and the norm $\|\mathbf{f}, \mathbf{h}\|_{L^2}$ is defined by (3.9). Notice that in view of Definition 3, the quantity $\|\mathbf{f}(t), \mathbf{h}(t)\|_{L^2}$ is strictly positive for all $(\mathbf{f}(t), \mathbf{h}(t)) \in \mathcal{F}_c$. Equations (4.4a)–(4.4i) form a closed system of integro-differential equations for the perturbations $\mathbf{S}$, $\mathbf{v}$, $\vartheta$, and $\mathbf{k}$.

Our goal is to prove the local solvability of problem (4.4a)–(4.4i) and to show that its solution satisfies equations (3.4a)–(3.5c) on the interval $(0, T_0)$. Notice that we are looking for strong bounded solution to problem (4.4a)–(4.4i). In order to formulate the existence result we introduce appropriate Banach spaces.

For an integer $l \geq 0$ and for an exponent $p \in [1, \infty]$, we denote by $W^{l,p}(\Omega)$ the Sobolev space of functions having all weak derivatives up to order $l$ in $L^p(\Omega)$. Endowed with the norm $\|u\|_{W^{l,p}(\Omega)} = \sup_{|\alpha| \leq l} \|\partial^\alpha u\|_{L^p(\Omega)}$, it becomes a Banach space. We also denote by $B^s_p(\partial\Omega)$, $s > 0$, $1 < p < \infty$, the Besov space of functions defined on $\partial\Omega$. For non-integer $l - 1/p > 0$, the space $B^{l-1/p}_p(\partial\Omega)$ is the trace space of $W^{l,p}(\Omega)$ and its norm is equivalent to the following

$$\|u\|_{B^{l-1/p}_p(\partial\Omega)} = \inf_{v=u \text{ at } \partial\Omega} \|v\|_{W^{l,p}(\Omega)}.$$

Next, for arbitrary Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, we will consider the intersection $\mathbb{X} \cap \mathbb{Y}$ as a Banach space endowed with the norm

$$\|z\|_{\mathbb{X}\cap\mathbb{Y}} = \|z\|_{\mathbb{X}} + \|z\|_{\mathbb{Y}}.$$

**Definition 4** For every $p \in (1, \infty)$, we denote by $\mathbb{V}_p$ a closed subspace of $W^{1,p}(0, T; W^{2,p}(\Omega)) \cap L^\infty(0, T; W^{3,p}(\Omega))$ which consists of all functions $\mathbf{v} : \Omega \times (0, T) \to \mathbb{R}^3$ satisfying condition (4.4b).

**Definition 5** For every $p \in (1, \infty)$, we denote by $\mathbb{S}_p$ the Banach space of all skew-symmetric matrix-valued functions $\mathbf{S}(t)$ with the finite norm

$$\|\mathbf{S}\|_{\mathbb{S}_p} = \|\partial_t \mathbf{S}\|_{L^p(0,T)} + \|\mathbf{S}\|_{L^\infty(0,T)}. \tag{4.5}$$

**Definition 6** For every $p \in (1, \infty)$, we denote by $\mathbb{T}_p$ a closed subspace of $W^{1,p}(0, T; W^{1,p}(\Omega)) \cap L^p(0, T; W^{3,p}(\Omega))$ which consists of all function $\vartheta$ satisfying condition (4.4g).

**Definition 7** For every $p \in (1, \infty)$, we denote by $\mathbb{K}_p$ the Banach space $L^\infty(0, T; W^{2,p}(\Omega)) \cap W^{1,p}(0, T; W^{2,p}(\Omega))$.

The spaces $\mathbb{V}_p$, $\mathbb{S}_p$, $\mathbb{T}_p$, and $\mathbb{K}_p$ determine the class of solution to problem (4.4a)–(4.4i). Next, we introduce Banach spaces which characterize the class of given data.

**Definition 8** For every $p \in (1, \infty)$, we denote by $\mathbb{F}_p$ the Banach space which consists of all couples $(\mathbf{f}, \mathbf{h}) : (\Omega \times \partial\Omega) \times (0, T) \to \mathbb{R}^6$ which are equilibrated and have the finite norm

$$\begin{aligned}\|\mathbf{f}, \mathbf{h}\|_{\mathbb{F}_p} = {} & \|\mathbf{f}\|_{L^\infty(0,T; W^{1,p}(\Omega))} + \|\partial_t \mathbf{f}\|_{L^p(0,T; L^p(\Omega))} \\ & + \|\mathbf{h}\|_{L^\infty(0,T; B^{2-1/p}_p(\partial\Omega))} + \|\partial_t \mathbf{h}\|_{L^p(0,T; B^{1-1/p}_p(\partial\Omega))}.\end{aligned}$$

**Definition 9** For every $p \in (1, \infty)$, we denote by $\mathbb{H}_p$, $\mathbb{G}_p$, and $\mathbb{E}_p$ the Banach spaces

$$\mathbb{H}_p = L^p\big(0, T; W^{1,p}(\Omega)\big), \qquad \mathbb{G}_p = L^p\big(0, T; W^{2,p}(\Omega)\big), \qquad \mathbb{E}_p = W^{2,p}(\Omega).$$

We are now in a position to formulate the main existence and local uniqueness result for the modified problem (4.4a)–(4.4i).

**Theorem 2** *Let* $p > 4$. *Let conditions* (H.1)–(H.5) *be satisfied. Then, there are* $\varepsilon_0 > 0$, *and* $T^* \in (0, T]$ *such that for all* $T_0 \in (0, T^*]$, *for all* $(\mathbf{f}, \mathbf{h}) \in \mathcal{F}_c$, *and for all matrix-valued functions* $\mathbf{k}_0$ *satisfying*

$$\|\mathbf{f}, \mathbf{h}\|_{\mathbb{F}_p} + \|\mathbf{k}_0\|_{\mathbb{E}_p} \le \varepsilon, \tag{4.6}$$

*problem* (4.4a)–(4.4i) *has a locally unique solution* $(\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k}) \in \mathbb{V}_p \times \mathbb{S}_p \times \mathbb{T}_p \times \mathbb{K}_p$ *with*

$$\|\mathbf{v}\|_{\mathbb{V}_p} + \|\mathbf{S}\|_{\mathbb{S}_p} + \|\vartheta\|_{\mathbb{T}_p} + \|\mathbf{k}\|_{\mathbb{K}_p} \le \rho\big(\varepsilon_0, T^*\big) \equiv c\big(\varepsilon_0 + {T^*}^{1/p}\big). \tag{4.7}$$

The following three sections are devoted to the proof of this theorem.

## 5 Operator Equation. Iteration Scheme

First, we reduce the modified problem (4.4a)–(4.4i) to a nonlinear operator equation. In order to do this, we introduce the vector function $\Upsilon = (\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k})$ and the Banach spaces

$$\mathbb{U}_p = \mathbb{V}_p \times \mathbb{S}_p \times \mathbb{T}_p \times \mathbb{K}_p, \qquad \mathbb{W}_p = \mathbb{F}_p \times \mathbb{S}_p \times \mathbb{H}_p \times \mathbb{G}_p \times \mathbb{E}_p.$$

For every $\rho > 0$, denote by $\mathcal{B}(\rho)$ the ball $\{\Upsilon : \|\Upsilon\|_{\mathbb{U}_p} \le \rho\}$. Equations, boundary and initial conditions (4.4a)–(4.4i) imply that the modified Problem (**M**) can be written in the form of an operator equation for the vector function $\Upsilon$,

$$\varXi(\Upsilon) \equiv \big(\varXi_i(\mathbf{v}, \mathbf{S}, \vartheta, \mathbf{k})\big) = 0, \quad 1 \le i \le 6. \tag{5.1}$$

The following lemma constitutes the smoothness properties of the operator $\varXi$.

**Lemma 3** *Under the assumptions of Theorem* 2, *there is* $r > 0$ *such that the operator* $\varXi : \mathcal{B}(r) \to \mathbb{W}_p$ *is infinitely differentiable.*

*Proof* We begin with the observation that in view of formula (4.2) for the projection $\mathcal{E}$, the couple $(\varXi_1, \varXi_2)$ is automatically equilibrated. Next, since $p > 4$, the embedding $\mathbb{V}_p \hookrightarrow C^1(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow C(\Omega)$ is continuous. Hence, we can choose $r$ so small that

$$\big|(\mathbf{R} + \mathbf{k})^{\pm 1}\big| \le 2, \qquad 1/2 \le \big|(\mathbf{I} + \mathbf{S} + D\mathbf{v})(\mathbf{R} + \mathbf{k})\big| \le 2$$

for all $\Upsilon \in \mathcal{B}(r)$. Next notice that

$$\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) = \frac{\Theta + \vartheta}{\det(\mathbf{R} + \mathbf{k})} \frac{\partial W}{\partial \Phi}\big((\mathbf{I} + \mathbf{S} + D\mathbf{v})(\mathbf{R} + \mathbf{k})\big)(\mathbf{R} + \mathbf{k})^{\top}.$$

The right hand side can be regarded as a function of the entries of matrices $\mathbf{R}$, $\mathbf{k}$, $\mathbf{S}$, $D\mathbf{v}$. It follows from Condition (H.1) that this function is infinitely differentiable on the range of vectors $\Upsilon \in \mathcal{B}(r)$. Classical results from finite elasticity theory, see [3, 24], imply that the operator

$$\mathcal{B}(r) \ni \Upsilon \mapsto \mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u}) \in L^\infty\big(0, T; W^{2,p}(\Omega)\big) \cap W^{1,p}\big(0, T; W^{1,p}(\Omega)\big)$$

is infinitely differentiable. Since the embedding $W^{l,p}(\Omega) \hookrightarrow B_p^{l-1/p}(\partial\Omega)$ is continuous and the couple $(\Xi_1, \Xi_2)$ is equilibrated, the operator $(\Xi_1, \Xi_2) : \mathcal{B}(r) \to \mathbb{F}_p$ is infinitely differentiable. In view of Condition (H.4) the operator $\Xi_3 : \mathcal{B}(r) \to \mathbb{S}_p$ is linear and continuous. Hence it is obviously differentiable. Applying the same arguments, we conclude that the operators $\Xi_4 : \mathcal{B}(r) \to \mathbb{H}_p$ and $\Xi_5 : \mathcal{B}(r) \to \mathbb{G}_p$ are infinitely differentiable. The operator $\Xi_6 : \mathbb{K}_p \to \mathbb{E}_p$ is linear and bounded. Hence it is infinitely differentiable. □

The following corollary is a straightforward consequence of the lemma. Denote by $\Xi'(\Upsilon)$ the Frechet derivative of the operator $\Xi$ at a point $\Upsilon$.

**Corollary 1** *Let all assumptions of Theorem 2 be satisfied and $r$ be given by Lemma 5.1. Assume that*

$$\|\mathbf{f}, \mathbf{h}\|_{\mathbb{F}_p} \le 1, \qquad \|\mathbf{k}_0\|_{E_p} \le 1.$$

*Then there is $L > 0$ such that*

$$\big\|\Xi'(\Upsilon_1) - \Xi'(\Upsilon_2)\big\| \le L\|\Upsilon_1 - \Upsilon_2\|_{\mathbb{U}_p} \quad \textit{for all } \Upsilon_1, \Upsilon_2 \in \mathcal{B}(r). \tag{5.2}$$

We thus reduce the question of existence of solutions to problem (4.4a)–(4.4i) to the question of existence of solutions to the operator equation (5.1). The proof of solvability of this equation is based on the Newton-Kantorovich implicit function theorem which can be formulated as follows, see [11] Theorem 12.2.

**Lemma 4** *Assume that an operator $\Xi : \mathcal{B}(r) \to \mathbb{W}_p$ has the properties: The operator $\Xi$ is differentiable and its derivative satisfies the Lipschitz condition (5.2); there is a bounded operator $\Xi'(0)^{-1} : \mathbb{W}_p \to \mathbb{U}_p$; there are positive $b_0$ and $\eta_0$ such that*

$$\big\|\Xi'(0)^{-1}\big\| \le b_0, \qquad \big\|\Xi'(0)^{-1}\Xi(0)\big\|_{\mathbb{U}_p} \le \eta_0, \tag{5.3}$$

$$h_0 \equiv b_0 L \eta_0 < 1/2, \qquad \rho = \frac{1 - \sqrt{1 - 2h_0}}{h_0}\eta_0 < r. \tag{5.4}$$

*Then there exists a unique $\Upsilon^* \in \mathcal{B}(\rho)$ such that $\Xi(\Upsilon^*) = 0$.*

In view of Lemma 3 and Corollary 1, the operator $\Xi$ is differentiable and its derivative satisfies condition (5.2). Hence in order to prove Theorem 2, it suffices to show that the derivative $\Xi'(0)$ has a bounded inverse satisfying (5.3) and (5.4).

## 6 Auxiliary Propositions. Linear Elasticity. Parabolic Equation

The proof of invertibility of the linear operator $\Xi'(0)$ is based on two auxiliary lemmas. The first constitutes the existence and uniqueness of solutions to the linear traction problem. This result has a mechanical meaning, since it states that the linear theory of growing material is similar to classical linear elasticity. Recall the definition (2.22)–(2.23) of the linear form $L_g$, which determines the linearization of the nonlinear differential operator $\mathbf{T}_g(\theta, \mathbf{K}, \cdot)$.

**Lemma 5** *Let $\Theta \in C^\infty(\Omega)$, $0 < c^{-1} \le \Theta \le c$, and $\mathbf{R} \in C^\infty(\Omega)$ be a field of orthogonal matrices. Then there is $c > 0$ such that for every $(\mathbf{F}, \mathbf{H}) \in \mathbb{F}_p$ the boundary value problem*

$$\begin{aligned} &\operatorname{div}\big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w}\big) + \mathbf{F} = 0 \quad \text{in } \Omega \times (0, T), \\ &- L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w}\mathbf{n} + \mathbf{H} = 0 \quad \text{on } \partial\Omega \times (0, T) \end{aligned} \tag{6.1}$$

*has a unique solution $\mathbf{w} \in \mathbb{V}_p$. This solution admits the estimate*

$$\|\mathbf{w}\|_{\mathbb{V}_p} \le c \big\| (\mathbf{F}, \mathbf{H}) \big\|_{\mathbb{F}_p}. \tag{6.2}$$

*Proof* Fix $t \in (0, T)$. By virtue of (2.22), Eq. (6.1) forms a second order system of partial differential equations with the Neumann type boundary condition

$$\begin{aligned} &\partial_j\big(l_{g,ijpq}(\Theta, \mathbf{R}, \mathbf{I})\partial_q w_p(t)\big) + F_i(t) = 0 \quad \text{in } \Omega, \\ &-\big(l_{g,ijpq}(\Theta, \mathbf{R}, \mathbf{I})\partial_q w_p(t)\big)n_j + H_i(t) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6.3}$$

In view of the conditions of the lemma, the coefficients of these equations are smooth in $\Omega$ and are independent of $t$. Problems of this type were thoroughly investigated in [23] and we simply recall the corresponding result. Let show that problem (6.1) defines the nonnegative quadratic form. To this end, notice that in view of (2.25) the inequality

$$\begin{aligned} &\int_\Omega \operatorname{div}\big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\boldsymbol{\psi}\big) \cdot \boldsymbol{\psi}\, dx - \int_{\partial\Omega} \big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\boldsymbol{\psi}\big)\mathbf{n} \cdot \boldsymbol{\psi}\, ds \\ &\quad = \int_\Omega L_g(\Theta, \mathbf{R}, \mathbf{I})D\boldsymbol{\psi} \cdot D\boldsymbol{\psi}\, dx \ge \Theta \int_\Omega \big|D\boldsymbol{\psi} + (D\boldsymbol{\psi})^\top\big|^2 dx \end{aligned} \tag{6.4}$$

holds for all smooth functions $\boldsymbol{\psi}$. From this and Korn inequality, we obtain that for every smooth $\boldsymbol{\psi}$,

$$\begin{aligned} &\int_\Omega \operatorname{div}\big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\boldsymbol{\psi}\big) \cdot \boldsymbol{\psi}\, dx - \int_{\partial\Omega} \big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\boldsymbol{\psi}\big)\mathbf{n} \cdot \boldsymbol{\psi}\, ds \\ &\quad + \int_\Omega |\boldsymbol{\psi}|^2\, dx \ge c\|\boldsymbol{\psi}\|^2_{W^{1,2}(\Omega)}. \end{aligned}$$

From this and Theorem 12, [23], we conclude that boundary value problem (6.3) is elliptic and the boundary conditions satisfie the completing conditions. The general theory of elliptic boundary value problems implies that in this case any weak solution $\mathbf{w}(t) \in W^{1,2}(\Omega)$ to problem (6.3) satisfies the estimate

$$\big\|\mathbf{w}(t)\big\|_{W^{3,p}(\Omega)} \le c\Big(\big\|\mathbf{F}(t)\big\|_{W^{1,p}(\Omega)} + \big\|\mathbf{H}(t)\big\|_{B_p^{2-1/p}(\partial\Omega)} + \big\|\mathbf{w}(t)\big\|_{W^{1,2}(\Omega)}\Big), \tag{6.5}$$

where $c$ is independent of $t$, $\mathbf{w}$, $\mathbf{F}$, and $\mathbf{H}$. Moreover, see [23], problem (6.3) has a weak solution $\mathbf{w} \in W^{1,2}(\Omega)$ for every couple $(\mathbf{F}, \mathbf{H}) \in W^{1,2}(\Omega)' \times W^{-1/2,2}(\Omega)$ satisfying the solvability condition

$$\langle \mathbf{F}, \mathbf{w}^* \rangle + \langle \mathbf{H}, \mathbf{w}^* \rangle = 0, \tag{6.6}$$

for all solutions $\mathbf{w}^* \in W^{1,2}(\Omega)$ of the transposed homogeneous problem. Since the transposed problem to (6.3) is elliptic and satisfies the completing condition, we have $\mathbf{w}^* \in C^\infty(\Omega)$. On the other hand, problem (6.3) is symmetric and hence $\mathbf{w}^*$ satisfies the homogeneous equations and boundary conditions (6.3). From this and estimate (6.4), we conclude that $D\mathbf{w}^* + (D\mathbf{w}^*)^\top = 0$. Hence $\mathbf{w}^* = \text{const.} + \mathbf{S}^* x$, where $\mathbf{S}^*$ is an arbitrary skew symmetric matrix. On the other hand, in view of Lemma 2, we have $L_g(\Theta, \mathbf{R}, \mathbf{I})\mathbf{S}^* = 0$. Hence all solutions to the homogeneous transposed problem have the form $\mathbf{w}^* = \text{const.} + \mathbf{S}^* x$. In this case, the solvability condition (6.6) simply means that the couple $(\mathbf{F}(t), \mathbf{H}(t))$ is equilibrated. From this we conclude that for every equilibrated couple $(\mathbf{F}(t), \mathbf{H}(t)) \in W^{1,p}(\Omega) \times B_p^{2-1/p}(\partial\Omega)$, problem (6.3) has a solution $\mathbf{w}(t) \in W^{3,p}(\Omega)$ satisfying (6.5). This solution is not unique. However, we can choose $\mathbf{w}^*(t) = \text{const.} + \mathbf{S}^*(t)x$ in such a way that after change of variable $\mathbf{w}(t) \to \mathbf{w}(t) - \mathbf{w}^*(t)$, the function $\mathbf{w}$ will satisfy the conditions

$$\int_\Omega \mathbf{w}\, dx = 0, \qquad \int_\Omega D\mathbf{w}\, dx \quad \text{is symmetric,} \tag{6.7}$$

i.e., $\mathbf{w}$ satisfies the orthogonality condition (4.4b) in Definition 4 of the space $\mathbb{V}_p$. Because of the Korn inequality, such a solution is unique and hence

$$\|\mathbf{w}(t)\|_{W^{3,p}(\Omega)} \le c\|\mathbf{F}(t)\|_{W^{1,p}(\Omega)} + \|\mathbf{H}(t)\|_{B_p^{2-1/p}(\partial\Omega)}$$

which yields

$$\|\mathbf{w}\|_{L^\infty(0,T;W^{3,p}(\Omega))} \le c\|\mathbf{F}\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\mathbf{H}\|_{L^\infty(0,T;B_p^{2-1/p}(\partial\Omega))}. \tag{6.8}$$

Notice that the coefficients of Eqs. (6.1) are independent of $t$. Hence, we can differentiate both sides of (6.1) with respect to $t$ to obtain

$$\begin{aligned} &\partial_j\big(l_{g,ijpq}(\Theta, \mathbf{R}, \mathbf{I})\partial_q \partial_t w_p(t)\big) + \partial_t F_i(t) = 0 \quad \text{in } \Omega, \\ &-\big(l_{g,ijpq}(\Theta, \mathbf{R}, \mathbf{I})\partial_q \partial_t w_p(t)\big)n_j + \partial_t H_i(t) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6.9}$$

Arguing as before, we conclude that for every equilibrated couple $(\partial_t\mathbf{F}(t), \partial_t\mathbf{H}(t)) \in L^p(\Omega) \times B_p^{1-1/p}(\partial\Omega)$, problem (6.3) has a unique solution $\partial_t\mathbf{w}(t) \in W^{2,p}(\Omega)$ satisfying

$$\int_\Omega \partial_t\mathbf{w}(t)\, dx = 0, \qquad \int_\Omega D\partial_t\mathbf{w}(t)\, dx \quad \text{is symmetric.}$$

This solution admits the estimate

$$\|\partial_t\mathbf{w}(t)\|_{W^{2,p}(\Omega)} \le c\|\partial_t\mathbf{F}(t)\|_{L^p(\Omega)} + \|\partial_t\mathbf{H}(t)\|_{B_p^{1-1/p}(\partial\Omega)}$$

which yields

$$\|\partial_t\mathbf{w}\|_{L^p(0,T;W^{2,p}(\Omega))} \le c\|\partial_t\mathbf{F}\|_{L^p(0,T;L^p(\Omega))} + \|\partial_t\mathbf{H}\|_{L^p(0,T;B_p^{1-1/p}(\partial\Omega))}. \tag{6.10}$$

Combining (6.8) and (6.10), we obtain (6.2). This completes the proof. □

The next lemma presents maximal regularity results for the heat equation in Sobolev spaces.

**Lemma 6** *Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^\infty$ boundary $\partial\Omega$ and $\Theta \in C^\infty(\Omega)$ be a strictly positive function. Then, for every $T > 0$ and $f \in L^p(Q)$, $Q = \Omega \times [0, T]$, $p \in (3, \infty)$, the problem*

$$\begin{aligned} &c_T \partial_t \vartheta - \Delta\left(\frac{\vartheta}{\Theta^2}\right) = f \quad \text{in } Q, \\ &\partial_n \vartheta = 0 \quad \text{on } \partial\Omega \times (0, T), \qquad \vartheta(x, 0) = 0 \quad \text{in } \Omega, \end{aligned} \tag{6.11}$$

*has a unique solution satisfying the inequality*

$$\|\vartheta\|_{\mathbb{T}_p} \le c\|f\|_{L^p(0,T;W^{1,p}(\Omega))}, \tag{6.12}$$

*where c depends only on $\Omega$, $T$, $p$ and $\Theta$.*

*Proof* The existence of a solution follows from the general theory of parabolic equations, see [20], Theorem 5.4. Hence it suffices to prove estimate (6.12). Since $\partial\Omega$ belongs to the class $C^\infty$, we can introduce the normal coordinates in a neighborhood $\partial\Omega$, see [17], Ch. 13. It follows that there is a collection of linearly independent differential operators $\mathbf{a}_i(x)\nabla$, $i = 1, 2, 3$, such that $\mathbf{a}_i \in C^2(\Omega)$, and

$$\mathbf{a}_1 = \mathbf{n}, \qquad \mathbf{a}_i \cdot \mathbf{n} = 0, \qquad \partial_n \mathbf{a}_i = 0, \quad i = 2, 3 \quad \text{on } \partial\Omega.$$

For every integer $l \ge 1$ and for all $\vartheta \in W^{l,p}(\Omega)$, we have

$$\|\vartheta\|_{W^{l,p}(\Omega)} \le c(l, \mathbf{a}_i) \sum_i \|\mathbf{a}_i \cdot \nabla\vartheta\|_{W^{l-1,p}(\Omega)} + c(l)\|\vartheta\|_{W^{l-1,p}(\Omega)}. \tag{6.13}$$

It follows from the maximal regularity results for parabolic boundary value problems, see [4], that for every $f \in L^p(Q)$, problem (6.11) has a unique solution satisfying the inequality

$$\|\vartheta\|_{L^p(0,T;W^{2,p}(\Omega))} + \|\partial_t\vartheta\|_{L^p(Q)} \le c\|f\|_{L^p(Q)}. \tag{6.14}$$

The same conclusion can be drawn if we replace the Neumann boundary condition in (6.11) by the Dirichlet boundary condition $\vartheta = 0$ on $\partial\Omega \times (0, T)$. Now introduce the functions $\vartheta_i = \mathbf{a}_i\nabla\vartheta$; It follows from (6.13) that

$$\begin{aligned} \|\vartheta\|_{\mathbb{T}_p} \le c &\sum_i \big(\|\vartheta_i\|_{L^p(0,T;W^{2,p}(\Omega))} + \|\partial_t\vartheta_i\|_{L^p(Q)}\big) \\ &+ c\big(\|\vartheta\|_{L^p(0,T;W^{2,p}(\Omega))} + \|\partial_t\vartheta\|_{L^p(Q)}\big). \end{aligned} \tag{6.15}$$

Next, applying the operators $\mathbf{a}_i \cdot \nabla$ to both sides of (6.11), we obtain the equations

$$\begin{aligned} &c_T \partial_t \vartheta_i - \Delta\left(\frac{\vartheta_i}{\Theta^2}\right) = f_i \quad \text{in } Q, \\ &\vartheta_1 = 0, \quad \partial_n\vartheta_i = 0, \quad i = 2, 3 \text{ on } \partial\Omega \times (0, T), \\ &\vartheta_i(x, 0) = 0, \quad i = 1, 2, 3 \text{ in } \Omega, \end{aligned} \tag{6.16}$$

where

$$f_i = \mathbf{a}_i\nabla f + \mathbf{a}_i\Delta\left(\vartheta\nabla\left(\frac{1}{\Theta^2}\right)\right) - \partial_k\left(\frac{1}{\Theta^2}\partial_k\mathbf{a}_i \cdot \nabla\vartheta\right) - \partial_k\mathbf{a}_i \cdot \partial_k\left(\frac{1}{\Theta^2}\nabla\vartheta\right).$$

Since $\mathbf{a}_i$ and $\Theta^{-2}$ belong to the class $C^\infty(\Omega)$, we have

$$\|f_i\|_{L^p(\Omega\times(0,T))} \le c\|f\|_{L^p(0,T;W^{1,p}(\Omega))} + c\|\vartheta\|_{L^p(0,T;W^{2,p}(\Omega))}.$$

From this and maximal regularity estimate (6.14) with $\vartheta$ and $f$ replaced by $\vartheta_i$ and $f_i$, we obtain

$$\|\vartheta_i\|_{L^p(0,T;W^{2,p}(\Omega))} + \|\partial_t\vartheta_i\|_{L^p(Q)} \le c\|f\|_{L^p(0,T;W^{1,p}(\Omega))} + c\|\vartheta\|_{L^p(0,T;W^{2,p}(\Omega))}$$

which along with (6.13) implies

$$\|\vartheta_i\|_{L^p(0,T;W^{2,p}(\Omega))} + \|\partial_t\vartheta_i\|_{L^p(Q)} \le c\|f\|_{L^p(0,T;W^{1,p}(\Omega))}.$$

Substituting this inequality and inequality (6.14) into (6.15), we obtain the desired estimate (6.12). □

## 7 Linearized Problem. Proof of Theorems 1 and 2

In this section, we prove that the operator $\Xi'(0)$ has a bounded inverse and by doing so complete the proof of Theorem 2. The results and the methods have a mechanical meaning: Lemmas 7 and 8 lead to the formulation of the linear theory for growing thermoelastic materials, see Remark 2. Moreover, the proof of Proposition 1 implies the well posedness of this problem. The main result of this section is the following

**Proposition 1** *Assume that $\Theta$, $\mathbf{R}$ and $\mathbf{h}$, $\mathbf{f}$ satisfy Conditions* (H.1)–(H.5) *and $p > 4$. Then there are $b_0 > 0$ and $T^* > 0$, depending only on $\Theta$ and $\mathbf{R}$, such that for $T_0 \in (0, T^*]$, the operator $\Xi'(0)$ has a bounded inverse $\Xi'(0)^{-1} : \mathbb{W}_p \to \mathbb{U}_p$ satisfying the inequality*

$$\left\|\Xi'(0)^{-1}\right\| \le b_0.$$

*Proof* It suffices to show that for every vector

$$\boldsymbol{\Psi} = (\mathbf{F}, \mathbf{H}, \boldsymbol{\Gamma}, Q, \mathbf{G}, \boldsymbol{\sigma}) \in \mathbb{W}_p,$$

the equation

$$\Xi'(0)\boldsymbol{\pi} = \boldsymbol{\Psi} \tag{7.1}$$

has a unique solution $\boldsymbol{\pi} = (\mathbf{w}, \boldsymbol{\zeta}, \psi, \boldsymbol{\lambda}) \in \mathbb{U}_p$ such that

$$\|\boldsymbol{\pi}\|_{\mathbb{U}_p} \le b_0\|\boldsymbol{\Psi}\|_{\mathbb{W}_p}. \tag{7.2}$$

We split the proof of solvability of Eq. (7.1) into a sequence of lemmas. First we reduce this equation to a system of linear PDE. □

**Lemma 7** *Under the assumptions of Proposition 1,*

$$\begin{aligned}&\big(\Xi_1'(0)\boldsymbol{\pi}, \Xi_2'(0)\boldsymbol{\pi}, \Xi_3'(0)\boldsymbol{\pi}\big)\\ &\quad = \big(\operatorname{div}\big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w} + \mathcal{M}\boldsymbol{\lambda}\big), -\big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w} + \mathcal{M}\boldsymbol{\lambda}\big)\mathbf{n}\big|_{\partial\Omega}, \Xi_3(\boldsymbol{\pi})\big).\end{aligned} \tag{7.3}$$

*where*

$$\mathcal{M}\boldsymbol{\lambda} = \big(L(\Theta, \mathbf{R})\boldsymbol{\lambda}\big)\mathbf{R}^\top \equiv \mathbf{R}\big(L(\Theta, \mathbf{I})\big(\mathbf{R}^\top\boldsymbol{\lambda}\big)\big)\mathbf{R}^\top, \tag{7.4}$$

*the linear forms $L$ and $L_g$ are given by* (2.7)–(2.8) *and* (2.22)–(2.23), *the linear operator $\Xi_3$ is given by* (4.4e).

*Proof* We have

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{1}{\tau}\big\{\mathbf{T}_g(\Theta + \tau\psi, \mathbf{R} + \tau\boldsymbol{\lambda}, \mathbf{I} + \tau\boldsymbol{\zeta} + \tau D\mathbf{w}) - \mathbf{T}_g(\Theta, \mathbf{R}, \mathbf{I})\big\}\\
&\quad\equiv L_g(\Theta, \mathbf{R}, \mathbf{I})(\boldsymbol{\zeta} + D\mathbf{w}) + \mathcal{M}\boldsymbol{\lambda} = L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w} + \mathcal{M}\boldsymbol{\lambda},
\end{aligned} \tag{7.5}$$

since $T_g(\Theta, \mathbf{R}, \mathbf{I}) = 0$ and $L_g(\Theta, \mathbf{R}, \mathbf{I})\boldsymbol{\zeta} = 0$ for any skew-symmetric matrix $\boldsymbol{\zeta}$. Let prove that

$$\lim_{\tau\to 0}\frac{1}{\tau}\big\{\mathcal{E}(\Theta + \tau\psi, \mathbf{R} + \tau\boldsymbol{\lambda}, \mathbf{I} + \tau\boldsymbol{\zeta} + \tau D\mathbf{w}) - \mathcal{E}(\Theta, \mathbf{R}, \mathbf{I})\big\} = 0. \tag{7.6}$$

It follows from (7.5) and (4.2) that

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{1}{\tau}\big\{\mathbf{E}(\Theta + \tau\psi, \mathbf{R} + \tau\boldsymbol{\lambda}, \mathbf{I} + \tau\boldsymbol{\zeta} + \tau D\mathbf{w}) - \mathbf{E}(\Theta, \mathbf{R}, \mathbf{I})\big\}\\
&\quad= \int_\Omega \Big(\big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w} + \mathcal{M}\boldsymbol{\lambda}\big) - \big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w} + \mathcal{M}\boldsymbol{\lambda}\big)^\top\Big)\,dx.
\end{aligned} \tag{7.7}$$

□

On the other hand, in view of symmetry relations (2.28) the matrix $L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w}$ is symmetric. The matrix $\mathcal{M}\boldsymbol{\lambda}$ also is symmetric since $L(\Theta, \mathbf{R})\boldsymbol{\xi}$ is symmetric for any $\boldsymbol{\xi}$. From this, (7.7), and expression (4.2) for $\mathcal{E}$ we obtain (7.6). Notice that

$$\big(\Xi_1'(0)\boldsymbol{\pi}, \Xi_2'(0)\boldsymbol{\pi}\big) = \lim_{\tau\to 0}\frac{1}{\tau}\big(\Xi_1(\tau\boldsymbol{\pi}) - \Xi_1(0), \Xi_2(\tau\boldsymbol{\pi}) - \Xi_2(0)\big).$$

Inserting in this relation the expressions (4.4c)–(4.4d) for $\Xi_1$, $\Xi_2$ and using (7.5)–(7.6), we arrive at representation (7.3) for $\Xi_1'(0)$ and $\Xi_2'(0)$. It remains to note that the operator $\Xi_3$ is linear and coincides with its derivative. This completes the proof.

**Lemma 8** *Under the assumptions of Proposition* 1,

$$\Xi_4'(0)\boldsymbol{\pi} = c_T\frac{\partial\psi}{\partial t} - \Delta\left(\frac{\psi}{\Theta^2}\right) + \chi(t)\Theta\,\mathrm{Tr}\left(\mathbf{R}^\top\frac{\partial\boldsymbol{\lambda}}{\partial t}\right), \qquad \Xi_6'\boldsymbol{\pi} = \boldsymbol{\lambda}\big|_{t=0}, \tag{7.8}$$

$$\Xi_5'(0)\boldsymbol{\pi} = \frac{\partial\boldsymbol{\lambda}}{\partial t} - \chi(t)\mathbf{a}(\boldsymbol{\lambda}) - \chi(t)\psi\mathbf{b} - \chi(t)\mathbf{c}(\boldsymbol{\zeta} + D\mathbf{w}), \tag{7.9}$$

*where the matrix valued linear forms* **a**, **b**, *and* **c** *are given by*

$$\begin{aligned}
&\mathbf{a}(\boldsymbol{\lambda}) = \mathbf{a}_{ij}\lambda_{ij}, \qquad \mathbf{a}_{ij} = \frac{\partial\mathbf{g}}{\partial K_{ij}}(\Theta, \mathbf{R}, \mathbf{I}), \qquad \psi\mathbf{b} = \psi\frac{\partial\mathbf{g}}{\partial\Theta}(\Theta, \mathbf{R}, \mathbf{I}),\\
&\mathbf{c}(\boldsymbol{\zeta} + D\mathbf{w}) = \mathbf{c}_{ij}(\zeta_{ij} + \partial_j w_i), \qquad \mathbf{c}_{ij} = \frac{\partial\mathbf{g}}{\partial_j u_i}(\Theta, \mathbf{R}, \mathbf{I})
\end{aligned} \tag{7.10}$$

*Proof* Notice that $\Theta$ and $\mathbf{R}$ are independent of $t$, and $T_g(\Theta, \mathbf{R}, I) = 0$. It follows from this and expression (4.4f) that

$$\Xi_4'(0)\boldsymbol{\pi} = c_T \frac{\partial \psi}{\partial t} - \Delta\left(\frac{\psi}{\Theta^2}\right) - \chi(t)\Theta \frac{\partial}{\partial t}\mathcal{N}, \tag{7.11}$$

where

$$\mathcal{N} = \lim_{\tau \to 0} \frac{1}{\tau}\left\{\det(\mathbf{R} + \tau\boldsymbol{\lambda})^{-1} - \det \mathbf{R}^{-1}\right\} = -R_{ij}\lambda_{ij}.$$

The latter relation follows from the equality $\det \mathbf{R} = 1$ and the identities $\Delta_{ij} = R_{ji}$, where $\Delta_{ij}$ are the cofactors of the orthogonal matrix $\mathbf{R}$. Substituting the expressions for $\mathcal{N}$ into (7.11), we arrive at (7.8). It remains to note that relations (7.9) and (7.10) obviously follow from (4.4h). $\square$

In view of Lemmas 7 and 8 linear operator equation (7.1) is equivalent to the following system of partial differential equations.

$$\begin{aligned} &\operatorname{div}\big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w} + \mathcal{M}\boldsymbol{\lambda}\big) = \mathbf{F} \quad \text{in } \Omega \times (0, T), \\ &-\big(L_g(\Theta, \mathbf{R}, \mathbf{I})D\mathbf{w} + \mathcal{M}\boldsymbol{\lambda}\big)\mathbf{n} = \mathbf{H} \quad \text{on } \partial\Omega \times (0, T). \end{aligned} \tag{7.12a}$$

$$\boldsymbol{\zeta}\mathbf{C} - (\boldsymbol{\zeta}\mathbf{C})^\top = \mathbf{D}(\mathbf{w})^\top - \mathbf{D}(\mathbf{w}) + \big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}\boldsymbol{\Gamma}, \tag{7.12b}$$

$$\begin{aligned} &c_T \frac{\partial \psi}{\partial t} - \Delta\left(\frac{\psi}{\Theta^2}\right) + \chi(t)\Theta\, \mathrm{Tr}\left(\mathbf{R}^\top \frac{\partial \boldsymbol{\lambda}}{\partial t}\right) = Q \quad \text{in } \Omega \times (0, T), \\ &\nabla\psi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad \psi(x, 0) = 0 \quad \text{in } \Omega, \end{aligned} \tag{7.12c}$$

$$\begin{aligned} &\frac{\partial \boldsymbol{\lambda}}{\partial t} - \chi(t)\mathbf{a}(\boldsymbol{\lambda}) - \chi(t)\psi\mathbf{b} - \chi(t)\mathbf{c}(\boldsymbol{\zeta} + D\mathbf{w}) = \mathbf{G} \quad \text{in } \Omega \times (0, T), \\ &\boldsymbol{\lambda}(x, 0) = \boldsymbol{\sigma}(x) \quad \text{in } \Omega. \end{aligned} \tag{7.12d}$$

*Remark 2* Equations (7.12a)–(7.12d) lead to the formulation of the linear theory for growing materials, but they are obtained by the linearization of the extended system (4.4a)–(4.4i) and inherit its structure. In order to derive the true formulation of the linear theory, it is necessary to make the following alterations in (7.12a)–(7.12d). First, we must take $\chi(t) = 1$, $(\mathbf{f}, \mathbf{h}) = 0$, and replace $\Theta$ by the equilibrium temperature $\Theta_c$. In this case, both sides of equation (7.12b) identically equal zero. The resulting system consists of static equations (7.12a) and evolutionary equations (7.12c)–(7.12d). In this framework, $\boldsymbol{\zeta}$ becomes an arbitrary skew symmetric matrix depending on $t$. Notice that it is present only in equation (7.12d) for the transplant $\boldsymbol{\lambda}$. It is unnatural that the evolution of the transplant depends on an arbitrary quantity. Hence, we have to impose the structural condition $\mathbf{c}(\boldsymbol{\zeta}) = 0$ for all skew symmetric $\boldsymbol{\zeta}$, which is equivalent to the symmetry of the matrix $\partial\mathbf{g}/\partial D\mathbf{u}(\Theta_c, \mathbf{R}, \mathbf{I})$.

Now our task is to prove that problem (7.12a)–(7.12d) is well posed. The proof is based on the following

**Lemma 9** *Let all assumptions of Proposition 1 be satisfied. Then for every* $(\mathbf{F}, \mathbf{H}, \boldsymbol{\Gamma}) \in \mathbb{F}_p \times \mathbb{S}_p$, *problem*

$$\begin{aligned} &\operatorname{div}\big(L_g(\Theta, \mathbf{R}, \mathbf{I}) D\mathbf{w}\big) = \mathbf{F} \quad \text{in } \Omega \times (0, T), \\ &-\big(L_g(\Theta, \mathbf{R}, \mathbf{I}) D\mathbf{w}\big)\mathbf{n} = \mathbf{H} \quad \text{on } \partial\Omega \times (0, T), \end{aligned} \tag{7.13}$$

$$\boldsymbol{\zeta}\mathbf{C} - (\boldsymbol{\zeta}\mathbf{C})^\top = \mathbf{D}(\mathbf{w})^\top - \mathbf{D}(\mathbf{w}) + \big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}\boldsymbol{\Gamma} \tag{7.14}$$

*has a unique solution* $(\mathbf{w}, \boldsymbol{\zeta}) \in \mathbb{V}_p \times \mathbb{S}_p$ *satisfying the inequality*

$$\|\mathbf{w}\|_{\mathbb{V}_p} + \|\boldsymbol{\zeta}\|_{\mathbb{S}_p} \le c\big(\|\mathbf{F}, \mathbf{H}\|_{\mathbb{F}_p} + \|\boldsymbol{\Gamma}\|_{\mathbb{S}_p}\big), \tag{7.15}$$

*where c depends on* $T$, $\Omega$, $\Theta$, $\mathbf{R}$ *and the constant* $c^*$ *in Definition 2.*

*Proof* In view of Lemma 5, problem (7.13) has a unique solution $\mathbf{w} \in \mathbb{V}_p$ satisfying the inequality

$$\|\mathbf{w}\|_{\mathbb{V}_p} \le c\|\mathbf{F}, \mathbf{H}\|_{\mathbb{F}_p}. \tag{7.16}$$

It remains to prove the solvability of equation (7.14). Introduce a skew-symmetric matrix $\mathbf{Y}$

$$\mathbf{Y} = \mathbf{D}(\mathbf{w})^\top - \mathbf{D}(\mathbf{w}) + \big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}\boldsymbol{\Gamma}.$$

Notice that we are looking for a skew-symmetric solution $\boldsymbol{\zeta} = -\boldsymbol{\zeta}^\top$. Since $\mathbf{C}$ is symmetric, there is an orthogonal matrix $\mathbf{U}$ such that $\mathbf{U}\mathbf{C}\mathbf{U}^\top = \mathbf{J}$, where $\mathbf{J} = \operatorname{diag}\{\mu_i(t)\}$. Thus we get $\boldsymbol{\zeta}\mathbf{U}^\top\mathbf{J}\mathbf{U} + \mathbf{U}^\top\mathbf{J}\mathbf{U}\boldsymbol{\zeta} = \mathbf{Y}$, which leads to

$$\mathbf{Z}\mathbf{J} + \mathbf{J}\mathbf{Z} = \mathbf{P},$$

with the skew symmetric matrices $\mathbf{Z} = \mathbf{U}\boldsymbol{\zeta}\mathbf{U}^\top$ and $\mathbf{P} = \mathbf{U}\mathbf{Y}\mathbf{U}^\top$. It remains to note that the latter equation can be written in the form

$$\big(\mu_1(t) + \mu_2(t)\big)Z_{12} = P_{12}, \qquad \big(\mu_1(t) + \mu_3(t)\big)Z_{13} = P_{13}, \qquad \big(\mu_2(t) + \mu_3(t)\big)Z_{23} = P_{23}.$$

Recall that $(\mathbf{f}, \mathbf{h})$ satisfies condition (H.4), and hence belongs to the set $\mathcal{F}_c$ given by Definition 3. It follows from this definition that

$$\big|\mu_i(t) + \mu_j(t)\big|^{-1} \le c\big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}^{-1} \quad \text{for } i \ne j.$$

Noting that $|\mathbf{Z}| = |\boldsymbol{\zeta}(t)|$ and $|\mathbf{P}| = |\mathbf{Y}(t)|$, we obtain

$$\big|\zeta(t)\big| \le c\big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}^{-1}\big|\mathbf{Y}(t)\big| \le c\big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}^{-1}\big|\mathbf{D}(\mathbf{w})(t)\big| + c\big|\boldsymbol{\Gamma}(t)\big|. \tag{7.17}$$

Expression (4.4i) for the matrix $\mathbf{D}$ implies

$$\begin{aligned} \big|\mathbf{D}(\mathbf{w})(t)\big| &\le \big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}\bigg(\int_\Omega \big|\mathbf{w}(t)\big|^2 dx + \int_{\partial\Omega} \big|\mathbf{w}(t)\big|^2 ds\bigg)^{1/2} \\ &\le c\big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}\big\|\mathbf{w}(t)\big\|_{W^{3,p}(\Omega)}. \end{aligned}$$

From this, estimate (7.17), and Definition 4 of the space $\mathbb{V}_p$, we conclude that

$$\|\boldsymbol{\zeta}\|_{L^\infty(0,T)} \le c\big(\|\mathbf{w}\|_{\mathbb{V}_p} + \|\boldsymbol{\Gamma}\|_{L^\infty(0,T)}\big). \tag{7.18}$$

Let us estimate the time derivative of $\boldsymbol{\zeta}$. Equation (7.14) yields

$$\partial_t\boldsymbol{\zeta}(t)\mathbf{C}(t) - \big(\partial_t\boldsymbol{\zeta}(t)\mathbf{C}(t)\big)^\top = \mathbf{Y}^{(1)}(t), \quad t \in (0,T) \tag{7.19}$$

where the skew-symmetric matrix $\mathbf{Y}^{(1)}$ is given by

$$\begin{aligned}\mathbf{Y}^{(1)} &= \mathbf{D}(\partial_t\mathbf{w})^\top - \mathbf{D}(\partial_t\mathbf{w}) + \|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}\partial_t\boldsymbol{\Gamma}\\ &\quad + \big(\boldsymbol{\zeta}(t)C_t\big)^\top - \boldsymbol{\zeta}(t)\mathbf{C}_t\boldsymbol{\zeta}(t) + \big(\mathbf{D}_t\big(\mathbf{w}(t)\big)\big)^\top - \mathbf{D}_t\big(\mathbf{w}(t)\big)\\ &\quad + \partial_t\big(\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}\big)\boldsymbol{\Gamma}(t).\end{aligned} \tag{7.20}$$

Here $\mathbf{C}_t(t)$ and $\mathbf{D}_t(\mathbf{w})$ are matrices with the entries

$$C_{t,ij} = \int_\Omega x_i\partial_t f_j\,dx + \int_{\partial\Omega} x_i\partial_t h_j\,ds, \qquad D_{t,ij} = \int_\Omega w_i\partial_t f_j\,dx + \int_{\partial\Omega} w_i\partial_t h_j\,ds. \tag{7.21}$$

Arguing as in the proof of (7.17) we obtain

$$\begin{aligned}\big|\partial_t\boldsymbol{\zeta}(t)\big| &\le c\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}^{-1}\big|\mathbf{Y}^{(1)}(t)\big|\\ &\le c|\partial_t\boldsymbol{\Gamma}| + c\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}^{-1}\\ &\quad\times\big\{\big|\mathbf{D}(\partial_t\mathbf{w})\big| + \big|\boldsymbol{\zeta}(t)C_t\big| + \big|\mathbf{D}_t\big(\mathbf{w}(t)\big)\big| + \big|\partial_t\big(\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}\big)\big|\big|\boldsymbol{\Gamma}(t)\big|\big\}.\end{aligned} \tag{7.22}$$

It follows from expression (4.4i) for the matrices $\mathbf{C}$ and $\mathbf{D}$ that

$$\begin{aligned}\big|\mathbf{D}(\partial_t\mathbf{w})(t)\big| &\le \|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}\bigg(\int_\Omega\big|\partial_t\mathbf{w}(t)\big|^2\,dx + \int_{\partial\Omega}\big|\partial_t\mathbf{w}(t)\big|^2\,ds\bigg)^{1/2}\\ &\le c\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}\|\partial_t\mathbf{w}(t)\|_{W^{2,p}(\Omega)}.\end{aligned} \tag{7.23}$$

Next, equalities (7.21) and (3.10) imply

$$\begin{aligned}\big|\mathbf{D}_t(\mathbf{w})(t)\big| &\le \|\partial_t\mathbf{f}(t),\partial_t\mathbf{h}(t)\|_{L^2}\bigg(\int_\Omega\big|\mathbf{w}(t)\big|^2\,dx + \int_{\partial\Omega}\big|\mathbf{w}(t)\big|^2\,ds\bigg)^{1/2}\\ &\le c\|\partial_t\mathbf{f}(t),\partial_t\mathbf{h}(t)\|_{L^2}\|\mathbf{w}(t)\|_{W^{2,p}(\Omega)} \le c\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}\|\mathbf{w}(t)\|_{W^{2,p}(\Omega)}.\end{aligned} \tag{7.24}$$

$$\begin{aligned}\big|\boldsymbol{\zeta}(t)\mathbf{C}_t(t)\big| &\le \big|\boldsymbol{\zeta}(t)\big|\big|\mathbf{C}_t(t)\big| \le c\big|\boldsymbol{\zeta}(t)\big|\|\partial_t\mathbf{f}(t),\partial_t\mathbf{h}(t)\|_{L^2}\\ &\le c\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}\big|\boldsymbol{\zeta}(t)\big|.\end{aligned} \tag{7.25}$$

Condition (H.4) implies

$$\begin{aligned}\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}^{-1}\big|\partial_t\big(\|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}\big)\big| &= \|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}^{-2}\bigg|\int_\Omega\mathbf{f}\cdot\partial_t\mathbf{f}\,dx + \int_{\partial\Omega}\mathbf{h}\cdot\partial_t\mathbf{h}\,ds\bigg|\\ &\le \|\mathbf{f}(t),\mathbf{h}(t)\|_{L^2}^{-1}\|\partial_t\mathbf{f}(t),\partial_t\mathbf{h}(t)\|_{L^2} \le c.\end{aligned} \tag{7.26}$$

Inserting (7.23)–(7.26) into (7.22), we arrive at

$$\big|\partial_t\boldsymbol{\zeta}(t)\big| \le c\big(\big|\boldsymbol{\zeta}(t)\big| + \|\mathbf{w}(t)\|_{W^{2,p}(\Omega)} + \|\partial_t\mathbf{w}(t)\|_{W^{2,p}(\Omega)} + \big|\boldsymbol{\Gamma}(t)\big| + \big|\partial_t\boldsymbol{\Gamma}(t)\big|\big). \tag{7.27}$$

Recalling Definitions 4 and 5 of spaces $\mathbb{V}_p$ and $\mathbb{S}_p$ we obtain

$$|\partial_t \boldsymbol{\zeta}|_{L^p(0,T)} \le c\big(\|\mathbf{w}\|_{\mathbb{V}_p} + \|\boldsymbol{\Gamma}\|_{\mathbb{S}_p}\big).$$

Combining this result with (7.18), we arrive at the estimate

$$\|\boldsymbol{\zeta}\|_{\mathbb{S}_p} \le c\big(\|\mathbf{w}\|_{\mathbb{V}_p} + \|\boldsymbol{\Gamma}\|_{\mathbb{S}_p}\big),$$

which along with (7.16) implies the desired estimate (7.15). □

Let us turn to the proof of Proposition 1. In order to solve problem (7.12a)–(7.12d) we apply the successive approximation method. Consider the sequence of boundary value problems which consist of the evolutionary part

$$\begin{aligned} &c_T \frac{\partial \psi_n}{\partial t} - \Delta\left(\frac{\psi_n}{\Theta^2}\right) = -\chi(t)\Theta \operatorname{Tr}\left(\mathbf{R}^\top \frac{\partial \boldsymbol{\lambda}_{n-1}}{\partial t}\right) + Q \quad \text{in } \Omega \times (0,T), \\ &\nabla \psi_n \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \psi_n(x,0) = 0 \quad \text{in } \Omega, \end{aligned} \tag{7.28a}$$

$$\begin{aligned} \boldsymbol{\lambda}_n = \boldsymbol{\sigma} &+ \int_0^t \chi(s)\big(\mathbf{a}(\boldsymbol{\lambda}_{n-1}) + \psi_{n-1}\mathbf{b}\big)\,ds \\ &+ \int_0^t \chi(s)\mathbf{c}(\boldsymbol{\zeta}_{n-1} + D\mathbf{w}_{n-1})\,ds + \int_0^t \mathbf{G}\,ds \quad \text{in } \Omega \times (0,T), \end{aligned}$$

and the static part

$$\begin{aligned} &\operatorname{div}\big(L_g(\Theta,\mathbf{R},\mathbf{I})D\mathbf{w}_n\big) = \mathbf{F} - \operatorname{div}(\mathcal{M}\boldsymbol{\lambda}_n) \quad \text{in } \Omega \times (0,T), \\ &-\big(L_g(\Theta,\mathbf{R},\mathbf{I})D\mathbf{w}_n\big)\mathbf{n} = \mathbf{H} + (\mathcal{M}\boldsymbol{\lambda}_n)\mathbf{n} \quad \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{7.28b}$$

$$\begin{aligned} &\boldsymbol{\zeta}_n\mathbf{C} - (\boldsymbol{\zeta}_n\mathbf{C})^\top = \mathbf{D}(\mathbf{w}_n)^\top - \mathbf{D}(\mathbf{w}_n) + \big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}\boldsymbol{\Gamma}, \\ &\mathbf{w}_0 = 0, \quad \boldsymbol{\zeta}_0 = 0, \quad \psi_0 = 0, \quad \boldsymbol{\lambda}_0 = 0, \quad n \ge 1. \end{aligned} \tag{7.28c}$$

Let us estimate solutions to problem (7.28a)–(7.28c). First, consider the case of $n = 1$. We have

$$c_T \frac{\partial \psi_1}{\partial t} - \Delta\left(\frac{\psi_1}{\Theta^2}\right) = Q \quad \text{in } \Omega \times (0,T), \tag{7.29a}$$

$$\nabla \psi_1 \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \psi_1(x,0) = 0 \quad \text{in } \Omega, \tag{7.29b}$$

$$\boldsymbol{\lambda}_1 = \boldsymbol{\sigma} + \int_0^t \mathbf{G}\,ds \quad \text{in } \Omega \times (0,T). \tag{7.29c}$$

$$\begin{aligned} &\operatorname{div}\big(L_g(\Theta,\mathbf{R},\mathbf{I})D\mathbf{w}_1\big) = \mathbf{F} - \operatorname{div}(\mathcal{M}\boldsymbol{\lambda}_1) \quad \text{in } \Omega \times (0,T), \\ &-\big(L_g(\Theta,\mathbf{R},\mathbf{I})D\mathbf{w}_1\big)\mathbf{n} = \mathbf{H} + (\mathcal{M}\boldsymbol{\lambda}_1)\mathbf{n} \quad \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{7.29d}$$

$$\boldsymbol{\zeta}_1\mathbf{C} - (\boldsymbol{\zeta}_1\mathbf{C})^\top = \mathbf{D}(\mathbf{w}_1)^\top - \mathbf{D}(\mathbf{w}_1) + \big\|\mathbf{f}(t), \mathbf{h}(t)\big\|_{L^2}\boldsymbol{\Gamma}. \tag{7.29e}$$

Applying Lemma 6 to the boundary value problem (7.29a)–(7.29b), we obtain

$$\alpha_1 \equiv \|\psi_1\|_{\mathbb{T}_p} \le c\|Q\|_{\mathbb{H}_p}. \tag{7.30}$$

In view of (7.10), the coefficients of the linear form **a**, **b**, **c** are infinitely differentiable and are independent on $t$. From this and (7.29c), we obtain

$$\|\boldsymbol{\lambda}_1\|_{L^\infty(0,T;W^{2,p}(\Omega))} \le \|\boldsymbol{\sigma}\|_{W^{2,p}(\Omega)} + T^{p/(p-1)}\|\mathbf{G}\|_{L^p(0,T;W^{2,p}(\Omega))}.$$
$$\|\partial_t\boldsymbol{\lambda}_1\|_{L^p(0,T;W^{2,p}(\Omega)} \le c\|\mathbf{G}\|_{L^p(0,T;W^{2,p}(\Omega))}.$$

Combining this inequalities and recalling Definitions 4, 5, and 7 of the spaces $\mathbb{V}_p$, $\mathbb{S}_p$, and $\mathbb{K}_p$ we get

$$\beta_1 \equiv \|\boldsymbol{\lambda}_1\|_{\mathbb{K}_p} \le c\big(\|\boldsymbol{\sigma}\|_{\mathbb{E}_p} + \|\mathbf{G}\|_{\mathbb{G}_p}\big). \tag{7.31}$$

It follows that

$$\|\mathcal{M}\boldsymbol{\lambda}_1\|_{L^\infty(0,T;W^{2,p}(\Omega))} + \|\partial_t\mathcal{M}\boldsymbol{\lambda}_1\|_{L^p(0,T;W^{1,p}(\Omega))} \le c\big(\|\boldsymbol{\sigma}\|_{\mathbb{E}_p} + \|\mathbf{G}\|_{\mathbb{G}_p}\big),$$

which yields

$$\big\|\operatorname{div}(\mathcal{M}\boldsymbol{\lambda}_1), \mathcal{M}\boldsymbol{\lambda}_1\big\|_{\mathbb{F}_p} \le c\big(\|\boldsymbol{\sigma}\|_{\mathbb{E}_p} + \|\mathbf{G}\|_{\mathbb{G}_p}\big).$$

Notice that the couple $(-\operatorname{div}\mathcal{M}\boldsymbol{\lambda}_1, \mathcal{M}\boldsymbol{\lambda}_1\mathbf{n})$ is equilibrated because of symmetry $\mathcal{M}\boldsymbol{\lambda}_1$. Applying Lemma 9 to problem (7.29d) we obtain

$$\gamma_1 \equiv \|\mathbf{w}_1\|_{\mathbb{V}_p} + \|\boldsymbol{\zeta}_1\|_{\mathbb{S}_p} \le c\big(\|\mathbf{F},\mathbf{H}\|_{\mathbb{F}_p} + \|\boldsymbol{\Gamma}\|_{\mathbb{S}_p} + \|\boldsymbol{\sigma}\|_{\mathbb{E}_p} + \|\mathbf{G}\|_{\mathbb{G}_p}\big). \tag{7.32}$$

Let us turn to the case of $n > 1$. Set

$$\begin{aligned} &\phi_n = \psi_n - \psi_{n-1}, \qquad \boldsymbol{\mu}_n = \boldsymbol{\lambda}_n - \boldsymbol{\lambda}_{n-1}, \qquad \mathbf{q}_n = \mathbf{w}_n - \mathbf{w}_{n-1}, \qquad \boldsymbol{\upsilon}_n = \boldsymbol{\zeta}_n - \boldsymbol{\zeta}_{n-1}, \\ &\alpha_n = \|\phi_n\|_{\mathbb{T}_p}, \qquad \beta_n = \|\boldsymbol{\mu}_n\|_{\mathbb{K}_p}, \qquad \gamma_n = \|\mathbf{q}_n\|_{\mathbb{V}_p} + \|\boldsymbol{\upsilon}_n\|_{\mathbb{S}_p}. \end{aligned} \tag{7.33}$$

These functions satisfy the equations

$$c_T\frac{\partial\phi_n}{\partial t} - \Delta\Big(\frac{\phi_n}{\Theta^2}\Big) = -\chi(t)\Theta\,\mathrm{Tr}\Big(\mathbf{R}^\top\frac{\partial\boldsymbol{\mu}_{n-1}}{\partial t}\Big) \quad \text{in } \Omega\times(0,T), \tag{7.34a}$$

$$\nabla\phi_n\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\times(0,T), \qquad \phi_n(x,0) = 0 \quad \text{in } \Omega, \tag{7.34b}$$

$$\boldsymbol{\mu}_n = \int_0^t \chi(s)\big(\mathbf{a}(\boldsymbol{\mu}_{n-1}) + \phi_{n-1}\mathbf{b} + \mathbf{c}(\boldsymbol{\upsilon}_{n-1} + \mathbf{q}_{n-1})\big)\,ds \quad \text{in } \Omega\times(0,T). \tag{7.34c}$$

$$\begin{aligned} &\operatorname{div}\big(L_g(\Theta,\mathbf{R},\mathbf{I})D\mathbf{q}_n\big) = -\operatorname{div}(\mathcal{M}\boldsymbol{\mu}_n) \quad \text{in } \Omega\times(0,T), \\ &-\big(L_g(\Theta,\mathbf{R},\mathbf{I})D\mathbf{w}_n\big)\mathbf{n} = (\mathcal{M}\boldsymbol{\mu}_n)\mathbf{n} \quad \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{7.34d}$$

$$\boldsymbol{\upsilon}_n\mathbf{C} - (\boldsymbol{\upsilon}_n\mathbf{C})^\top = \mathbf{D}(\mathbf{q}_n)^\top - \mathbf{D}(\mathbf{q}_n). \tag{7.34e}$$

Applying Lemma 6 to the boundary value problem (7.34a)–(7.34b), we arrive at the estimate

$$\alpha_n = \|\phi_n\|_{\mathbb{T}_p} \le c\|\partial_t\mu_{n-1}\|_{L^p(0,T;W^{1,p}(\Omega))} \le c\|\mu_{n-1}\|_{\mathbb{K}_p} = c\beta_{n-1}. \tag{7.35}$$

In view of the anisotropic embedding theorem, see [2], Theorem 10.2, the inequality

$$\|f\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \le c(\Omega,T,p)\big(\|\partial_t f\|_{L^p(\Omega\times(0,T))} + \|f\|_{L^p(0,T;W^{2,p}(\Omega))}\big)$$

holds for all $f \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ and for all $p > 4$. It follows from this and Definition 6 of the space $\mathbb{T}_p$ that for $p > 4$, we have

$$\|\phi_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))} \le c(T,\Omega,p)\|\phi_{n-1}\|_{\mathbb{T}_p} \quad \text{for all } \phi \in \mathbb{T}_p.$$

Recall that $\chi(s) = 1$ for $s \le T_0$ and $\chi(s) = 0$ for $s > T_0$. From this and (7.34c), we obtain

$$\begin{aligned}
&\|\boldsymbol{\mu}_n\|_{L^\infty(0,T;W^{2,p}(\Omega))} \\
&\quad \le c\big(\|\boldsymbol{\mu}_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))} + \|\phi_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))} \\
&\qquad + \|\boldsymbol{v}_{n-1} + \mathbf{q}_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))}\big)\int_0^T \chi(s)\,ds = cT_0\big(\|\boldsymbol{\mu}_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))} \\
&\qquad + \|\phi_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))} + \|\boldsymbol{v}_{n-1} + \mathbf{q}_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))}\big) \\
&\quad \le cT_0\big(\|\boldsymbol{\mu}_{n-1}\|_{\mathbb{K}_p} + \|\phi_{n-1}\|_{\mathbb{T}_p} + \|\boldsymbol{v}_{n-1}\|_{\mathbb{S}_p} + \|\mathbf{q}_{n-1}\|_{\mathbb{V}_p}\big) \\
&\quad = cT_0(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}).
\end{aligned} \tag{7.36}$$

Recalling relation (7.34c) we conclude that

$$\begin{aligned}
&\|\partial_t\boldsymbol{\mu}_n\|_{L^p(0,T;W^{2,p}(\Omega))} \\
&\quad \le c\big(\|\boldsymbol{\mu}_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))} + \|\phi_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))} \\
&\qquad + \|\boldsymbol{v}_{n-1} + \mathbf{q}_{n-1}\|_{L^\infty(0,T;W^{2,p}(\Omega))}\big)\left(\int_0^T \chi(s)^p\,ds\right)^{1/p} \\
&\quad \le cT_0^{1/p}(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}).
\end{aligned} \tag{7.37}$$

Combining (7.36) and (7.37) and recalling Definition 7 of the space $\mathbb{K}_p$, we finally arrive at

$$\begin{aligned}
\beta_n &= \|\boldsymbol{\mu}_n\|_{L^\infty(0,T;W^{2,p}(\Omega))} + \|\partial_t\boldsymbol{\mu}_n\|_{L^p(0,T;W^{2,p}(\Omega))} \\
&\le cT_0^{1/p}(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}).
\end{aligned} \tag{7.38}$$

It follows from this that

$$\|\mathcal{M}\boldsymbol{\mu}_n\|_{L^\infty(0,T;W^{2,p}(\Omega))} + \|\partial_t\mathcal{M}\boldsymbol{\mu}_n\|_{L^p(0,T;W^{1,p}(\Omega))} \le cT_0^{1/p}(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}),$$

which yields

$$\left\|\operatorname{div}(\mathcal{M}\boldsymbol{\mu}_n), \mathcal{M}\boldsymbol{\mu}_n\right\|_{\mathbb{F}_p} \le cT_0^{1/p}(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}).$$

The couple $(-\operatorname{div}\mathcal{M}\boldsymbol{\mu}_n, \mathcal{M}\boldsymbol{\mu}_n\mathbf{n})$ is equilibrated because of symmetry $\mathcal{M}\boldsymbol{\lambda}$. Now, we can apply Lemma 9 to problem (7.34d)–(7.34e) to obtain the estimate

$$\gamma_n = \|\mathbf{q}_n\|_{\mathbb{V}_p} + \|\boldsymbol{v}_n\|_{\mathbb{S}_p} \le cT_0^{1/p}(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}),$$

which along with estimates (7.35) and (7.38) leads to the recurrent system of inequalities

$$\alpha_n \le c\beta_{n-1}, \qquad (\beta_n + \gamma_n) \le cT_0^{1/p}(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}).$$

Without loss of generality, we can assume that $c > 1$ and $T_0 < 1$. We now proceed by induction to obtain

$$\alpha_n \le (2c)^n T_0^{(n-1)/2p}(\alpha_1+\beta_1+\gamma_1), \qquad \beta_n+\gamma_n \le (2c)^n T_0^{n/2p}(\alpha_1+\beta_1+\gamma_1). \tag{7.39}$$

Now choose $T^* > 0$ such that $2c(T^*)^{1/2p} < 1/2$. For all $T_0 \in (0, T^*)$, inequalities (7.39) and relations (7.33) imply

$$\|\phi_n\|_{\mathbb{T}_p} \le c2^{-n+2}(\alpha_1+\beta_1+\gamma_1),$$
$$\|\boldsymbol{\mu}_n\|_{\mathbb{K}_p} + \|\mathbf{q}_n\|_{\mathbb{V}_p} + \|\boldsymbol{\upsilon}_n\|_{\mathbb{S}_p} \le 2^{-n}(\alpha_1+\beta_1+\gamma_1).$$

Therefore the sequences

$$\mathbf{w}_n = \sum_1^n \mathbf{q}_k, \quad \boldsymbol{\zeta}_n = \sum_1^n \boldsymbol{\upsilon}_k, \quad \psi_n = \sum_1^n \phi_k, \quad \boldsymbol{\lambda}_n = \sum_1^n \boldsymbol{\mu}_k,$$

converge in $\mathbb{T}_p \times \mathbb{K}_p$ to a solution $(\mathbf{w}, \boldsymbol{\zeta}, \psi, \boldsymbol{\lambda})$ to problem (7.12c). This solution is unique and admits the estimate

$$\|\mathbf{w}\|_{\mathbb{V}_p} + \|\boldsymbol{\zeta}\|_{\mathbb{S}_p} + \|\psi\|_{\mathbb{T}_p} + \|\boldsymbol{\lambda}\|_{\mathbb{K}_p} \le c(\alpha_1+\beta_1+\gamma_1).$$

It remains to note that in view of (7.30), (7.31), and (7.32),

$$\alpha_1+\beta_1+\gamma_1 \le c\big(\|\mathbf{F},\mathbf{H}\|_{\mathbb{F}_p} + \|\boldsymbol{\Gamma}\|_{\mathbb{S}_p} + \|Q\|_{\mathbb{H}_p} + \|\mathbf{G}\|_{\mathbb{G}_p} + \|\boldsymbol{\sigma}\|_{\mathbb{E}_p}\big),$$

and the proposition follows. □

*Proof of Theorem 2* Assume that the given data satisfy the inequalities

$$\|\mathbf{f},\mathbf{h}\|_{\mathbb{F}_p} + \|\mathbf{k}_0\|_{\mathbb{E}_p} \le \varepsilon_0, \qquad 0 < T_0 \le T^*. \tag{7.40}$$

It suffices to show that the operator $\Xi$ given by (5.1) meets all requirements of Lemma 4 for all sufficiently small $\varepsilon_0$ and $T^*$. In view of Lemma 3 and Corollary 1, the operator $\Xi : \mathbb{U}_p \to \mathbb{W}_p$ is differentiable in the ball $\mathcal{B}(r) \subset \mathbb{U}_p$, and its derivative satisfies the Lipschitz condition (5.2). By virtue of Proposition 1, there exists $T^* > 0$ such that the norm of the operator $\Xi'(0)^{-1} : \mathbb{W}_p \to \mathbb{V}_p$ is bounded by the constant $b_0$ for all $T_0 \in (0, T^*)$. It remains to prove that the quantity $\|\Xi'(0)^{-1}\Xi(0)\|_{\mathbb{U}_p}$ is bounded by a constant $\eta_0$ satisfying inequalities (5.4). To this end notice that

$$\Xi(0) = \big(\mathbf{f}, \mathbf{h}, 0, \chi\Delta\Theta^{-1}, -\chi g(\Theta, \mathbf{R}, \mathbb{I}), -\mathbf{k}_0\big).$$

Since $\Theta$ and $\mathbf{R}$ are fixed infinitely differentiable functions, we have

$$\big\|\chi\Delta\Theta^{-1}\big\|_{\mathbb{H}_p} + \big\|\chi g(\Theta,\mathbf{R},\mathbb{I})\big\|_{\mathbb{G}_p} \le c\|\chi\|_{L^p(0,T)} \le cT_0^{1/p}.$$

We thus get

$$\|\Xi_0\|_{\mathbb{W}_p} \le c\big(\|\mathbf{f},\mathbf{h}\|_{\mathbb{F}_p} + \|\mathbf{k}_0\|_{\mathbb{E}_p} + T_0^{1/p}\big) \le c\big(\varepsilon_0 + T^{*1/p}\big).$$

Applying Proposition 1, we arrive at

$$\big\|\Xi'(0)^{-1}\Xi(0)\big\|_{\mathbb{U}_p} \le \eta_0 = cb_0\big(\varepsilon_0 + T^{*1/p}\big). \tag{7.41}$$

It is easy to check that $b_0$ and $\eta_0$ satisfy inequalities (5.4) for all sufficiently small $\varepsilon_0$ and $T^*$. Moreover, the quantity $\rho$ in these inequalities does not exceed $c(\varepsilon_0 + T^{*1/p})$. Applying Lemma 4, we conclude that operator equation (5.1) has a unique solution in the ball $\{\|\Upsilon\|_{U_p} \le \rho\}$. It remains to note that Eq. (5.1) is equivalent to boundary value problem (4.4a)–(4.4i), and the theorem follows.

*Proof of Theorem 1* Assume that the given data satisfy inequalities (7.40). Let $(\mathbf{u}, \theta, \mathbf{K})$ be a solution to problem (4.4a)–(4.4i) given by Theorem 2. Let us prove that for small $\varepsilon_0$ and $T^*$, this solution also satisfies equations and boundary conditions (3.4a)–(3.5c) on the interval $(0, T_0)$. Since $\chi(t) = 1$ for $t \in [0, T_0]$, equations (4.4f)–(4.4g) imply that $(\mathbf{u}, \theta, \mathbf{K})$ satisfies equations (3.4b)–(3.4c), boundary conditions (3.5b), and initial conditions (3.5c). It remains to prove that $(\mathbf{u}, \theta, \mathbf{K})$ satisfies Eq. (3.4a) and boundary condition (3.5a). It suffices to show that the term $\mathcal{E}$ in (4.4c) equals zero. To this end, we fix an arbitrary $t \in (0, T)$ and consider the static problem (4.4c)–(4.4e). System (4.4c) reads

$$\partial_p T_{g,ip}(\theta, \mathbf{K}, D\mathbf{u}) + \mathcal{E}_i(\theta, \mathbf{K}, D\mathbf{u}) + f_i = 0 \quad \text{in } \Omega \text{ for every } t \in (0, T).$$

Multiplying both the sides of this equality by $u_j$ and integrating the result by parts, we obtain

$$\begin{aligned} &\int_\Omega (\mathcal{E}_i u_j - \mathcal{E}_j u_i)\, dx + \int_\Omega (f_i u_j - f_j u_i)\, dx + \int_{\partial\Omega} (h_i u_j - h_j u_i)\, ds \\ &\quad = \int_\Omega (T_{g,ip}\partial_p u_j - T_{g,jp}\partial_p u_i)\, dx \equiv J_{ij}. \end{aligned} \tag{7.42}$$

Notice that the matrix $\mathbf{J} = (J_{ij})$ has the representation

$$\mathbf{J} = \int_\Omega \Big(\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u})(D\mathbf{u})^\top - \big(\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u})(D\mathbf{u})^\top\big)^\top\Big)\, dx.$$

Recalling expression (2.20) for $\mathbf{T}_g$, we obtain

$$\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u})(D\mathbf{u})^\top = (J_K)^{-1}\mathbf{T}(\theta, \Phi)\mathbf{K}^\top(D\mathbf{u})^\top = (J_K)^{-1}\mathbf{T}(\theta, \Phi)\Phi^\top.$$

From this and Condition (H.2), we conclude that the matrix $\mathbf{T}_g(\theta, \mathbf{K}, D\mathbf{u})(D\mathbf{u})^\top$ is symmetric. Hence $\mathbf{J} = 0$. Combining this result with (7.42), we arrive at

$$\int_\Omega (\mathcal{E}_i u_j - \mathcal{E}_j u_i)\, dx + \int_\Omega (f_i u_j - f_j u_i)\, dx + \int_{\partial\Omega} (h_i u_j - h_j u_i)\, ds = 0. \tag{7.43}$$

Recalling that $u_i = x_i + S_{ip}x_p + v_i$ and noting that the couple $(\mathbf{f}, \mathbf{h})$ is equilibrated, we obtain

$$\begin{aligned} &\int_\Omega (f_j u_i - f_i u_j)\, dx + \int_{\partial\Omega} (h_j u_j - h_i u_j)\, ds \\ &\quad = S_{ip}\Big(\int_\Omega x_p f_j\, dx + \int_{\partial\Omega} x_p h_j\, ds\Big) - S_{jp}\Big(\int_\Omega x_p f_i\, dx + \int_{\partial\Omega} x_p h_i\, ds\Big) \\ &\qquad + \int_\Omega (f_j v_i - f_i v_j)\, dx + \int_{\partial\Omega} (h_j v_j - h_i v_j)\, ds = S_{ip}C_{pj} - S_{jp}C_{pi} + D_{ij}(\mathbf{v}) - D_{ji}(\mathbf{v}) \\ &\quad = (\mathbf{SC})_{ij} - (\mathbf{SC})^\top_{ij} + D_{ij}(\mathbf{v}) - D^\top_{ij}(\mathbf{v}) = 0 \end{aligned}$$

because of (4.4e) and (4.4i). From this and (7.43), we obtain

$$\int_{\Omega} (\mathcal{E}_i u_j - \mathcal{E}_j u_i)\, dx = 0.$$

Recalling formula (4.2) for $\mathcal{E}$ and noting that the matrix $\mathbf{E}$ is skew-symmetric, we can rewrite this equality in the form $\mathbf{E} = \mathbf{O}^\top - \mathbf{O}$, where $\mathbf{O}$ is the matrix with the entries

$$O_{ij} = E_{ip} \int_{\Omega} \varphi_p (\mathbf{S}x + \mathbf{v})_j\, dx.$$

Obviously, $\mathbf{O}$ admits the estimate

$$\begin{aligned} |\mathbf{O}| &\le c|\mathbf{E}|\big(\|\mathbf{S}\|_{L^\infty(0,T)} + \|\mathbf{v}\|_{L^\infty(\Omega\times(0,T))}\big) \\ &\le c|\mathbf{E}|\big(\|\mathbf{S}\|_{\mathbb{S}_p} + \|\mathbf{v}\|_{\mathbb{V}_p}\big) \le c|\mathbf{E}|\rho \le c|\mathbf{E}|\big(\varepsilon_0 + T^{*1/p}\big). \end{aligned}$$

Choosing $\varepsilon_0$ and $T^*$ sufficiently small, we finally obtain $\mathbf{E} = 0$ which along with (4.2) yields $\mathcal{E} = 0$, and the theorem follows.

## 8 Concluding Remarks

The obtained results show that the nonlinear growth models proposed in mechanics and biology are well posed from the mathematical point of view. The models admit the local in time, classical solutions. To our best knowledge there are no such results in the mathematical literature of the subject due to the complexity of the nonlinear, coupled models. Subsequent papers will be devoted to the further analysis of the models and some applications as well as to the development of numerical solutions. This field of research is important for real life problems in mechanics, biology and medicine.

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