

How students view the general nature of their errors

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Published online: 15 February 2007
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Abstract In this study we examined how two students viewed the general nature of their proportional reasoning errors as they attempted to generalize numeric situations. Using a teaching experiment methodology we studied the reasoning of two students over 18 instructional sessions. One student, Dallas, appeared to recognize that the proportional reasoning error applied to all cases of a particular problem situation and began to apply this reasoning across problems. The other student, Lloyd, exhibited difficulty seeing the generality of his mistaken use of proportional reasoning and regularly repeated this error during the study. From the data we developed a schematized description of how students view the generality of their errors and tracked the changes in these views over the course of the study. We analyze how Dallas and Lloyd's perception of errors shaped their understanding of proportional reasoning and provide suggestions for the role errors play in restructuring a student's conceptual schema.

Key words student errors · algebraic reasoning · generalization

Mistakes are almost always of a sacred nature. Never try to correct them. On the contrary: rationalize them, understand them thoroughly. After that, it will be possible for you to sublimate them.

Salvador Dali

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1 Introduction

The quote above by Salvador Dali describes errors as opportunities for deepening one's understanding and as important components of the learning process. The view of errors as a vehicle for learning, rather than an activity to eradicate, continues to gain momentum in mathematics education. In the past, errors have been portrayed as a medical condition deserving of a particular treatment. As illustrated by Meyerson (1976), "(A student's mistake) is a symptom of what may be a serious disease or even more than one disease" (p. 38). Research on student errors has an extensive history in mathematics education as early researchers (e.g., Buswell & Judd, 1925) were cognizant of various student mathematical errors. Researchers have diagnosed a variety of resilient errors in various mathematical domains including whole number operations (e.g., Brueckner & Bond, 1955; Phelps, 1913; Verschaffel, De Corte, & Vierstraete, 1999), algebra (e.g., Booth, 1984; Clement, 1982), geometry (e.g., Clements & Battista, 1992), and probability and statistics (e.g., Shaughnessy, 1992).

A variety of theoretical perspectives have been offered for how teachers should view the errors that students make. In the United States of America, many teachers continue to apply a behaviorist view of learning in their attempts to eliminate student errors (Arbaugh, Lannin, Jones, & Park Rogers, 2006). They provide positive reinforcement when students yield correct answers and initiate negative reinforcement or withhold positive reinforcement in an effort to reduce student errors (Miller, 1983). Buswell (1926) provided an illustration of a typical teacher's attempt to eradicate student errors as she discusses her student's difficulties with subtraction as follows,

In the case of the process of subtraction I found that some of the pupils in the class read the problems backward, so I explained the meaning of the term 'subtract' and showed them that the number subtracted was always smaller. Then I gave them groups of examples in which they merely read the example correctly. (p. 190)

As in this quotation, from a behaviorist perspective, errors are viewed as incorrect behaviors that need to be replaced with correct behaviors. Instructional strategies for correcting student mistakes involve clearly defining terms, performing repeated practice with correct procedures, and decomposing procedures into small, manageable bites.

Besides the behaviorist theory, other theories have emerged regarding how errors are viewed. Repair theory (Van Lehn, 1983; Woodward & Howard, 1994), using an information processing lens, provides a view of errors as student attempts to deal with situations with which they are unfamiliar. Unlike computers that "lock up" when faced with a new situation, students "repair" these new situations by substituting what they view as a reasonable strategy. For example, students facing a subtraction with regrouping situation (e.g., $42-27$) may simply ignore the difficulties that occur when taking away 7 from 2, resulting in the erroneous result of 25. Although the student's strategy appears to have its origins in a correct procedure, an incorrect, yet sensible to the student, adaptation has been made to account for the unfamiliar situation.

Another application of information processing theory related to student errors involves examination of the limitations of working memory on students during problem solving. Ayres (2001) considered the cognitive load placed on students when simplifying expressions involving positive and negative symbols [e.g., $-4(3-6)-2(6\times-1)$]. Repair theory and working memory explain some of the difficulty students experience when engaged in problem solving. An important finding of error research is that errors are not random, mindless attempts on the part of the student, but that errors "frequently result from systematic strategies that often have sensible origins" (Perso, 1992, p. 12).

Another perspective that builds on student sense making utilizes errors as instructional opportunities to promote student learning (Brousseau, 1997; Hiebert et al., 1997; Perso, 1992; Siegler, 2003), allowing teachers to use errors to delve into deeper conceptual issues (Borasi, 1987, 1996). Eggleton and Moldavan (2001) suggest that teachers “allow students to ‘own’ their errors and wrestle with the concepts and principles involved as they work their way to fuller understanding” (p. 42). Such recommendations represent a substantial shift in the typical roles of teachers and students (Brousseau).

To inform instructional decision making regarding student errors, research must be conducted that describes, in detail, the reasoning that students use when attempting to reconcile their errors. Further understanding of how students view the general nature of their errors can provide insight into how students use the errors that they make to deepen understanding. In our study, we examined how students dealt with errors related to proportional reasoning. Specifically, we posed the following research questions: (a) To what extent do students perceive generality of the errors they identify? and (b) How do students’ perceptions of their errors affect their views of a particular concept?

2 Theoretical perspectives

Within this section, we provide background literature regarding the knowledge of student thinking related to errors, proportional reasoning errors, and how researchers have characterized generality. We suggest that research needs to be conducted on the extent to which students perceive generality of the errors they commit.

2.1 Knowledge of student thinking related to errors

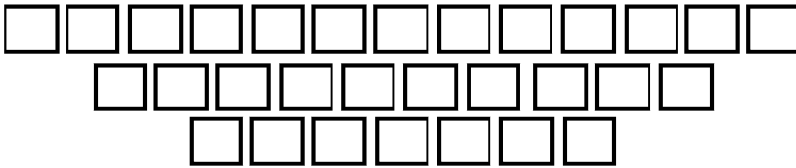
Research-based knowledge of student thinking is useful for teachers as they plan and implement instruction (Fennema et al., 1996; Fennema, Franke, Carpenter, & Carey, 1993). Just as knowledge of student thinking in specific content domains can inform instruction, knowledge of how students view their errors can also guide instructional decision-making. The view of errors as “sites for learning” is essential to the classroom that builds on student sense-making (Hiebert et al., 1997). “Making mistakes is a natural part of the (problem solving) process; it even may be essential sometimes” (p. 48). As Smith, diSessa, and Roschelle (1993) state, “it is time to move beyond simple models of knowledge and learning in which novice misconceptions are replaced by appropriate expert concepts” (p. 117). Instead researchers must seek to understand the nature of student errors and document how students move from novice conceptions toward deeper understanding.

2.2 Proportional reasoning errors

One common student error involves incorrectly applying proportional reasoning (de Bock, Verschaffel, & Janssens, 1998; Hart, 1984; Stacey, 1989; Swafford & Langrall, 2000). An example of this occurs as a student attempts to find the number of seats in the 10th row of the Theater Seats Problem (see Fig. 1). The student could use the fact that 19 seats are needed for the fifth row, doubling this amount to find the number of seats in the 10th row, which generates an incorrect number of seats.

Vergnaud (1983) characterized this type of proportional reasoning, when used correctly, within the larger conceptual field of proportional reasoning as an isomorphism of measures situation. The student knows three quantities that he assumes are related multiplicatively

In a theater there are 7 seats in the first row. The increase in the number of seats is the same from row to row. Below is a diagram of the first three rows in the theater.



How many seats are there in the 4th row of the theater? In the 5th row? In the 10th row? In the 23rd row? In the 38th row? In the 138th row of the theater? Write a rule that would allow you to calculate the number of seats in any row.

Fig. 1 The theater seats problem

and desires to find a missing fourth quantity. This strategy has also been described as a “within” strategy (Lamon, 1993; Lo & Watanabe, 1997; Noelting, 1980) as the student assumes a relationship between one measure, in this case the row numbers, and applies this multiplicative relationship to another measure (i.e., the number of seats).

An instructional focus on multiplicative reasoning can lead to an overextended view of the applicability of proportional reasoning (Freudenthal, 1983). Such overgeneralization has been referred to as the *illusion of linearity* (de Bock et al., 1998), which has often been observed in geometric situations (Freudenthal, 1983; Stacey, 1989). Students assume that “ m times A therefore, m times B ”, follows from the relationship that “more A implies more B ” (de Bock, Verschaffel, & Janssens, 2002). In the Theater Seats problem, students may recognize that as the row number increases the number of seats also increases and assume that applying proportional reasoning will provide the correct number of seats. De Bock et al. provided visual representations and a metacognitive scaffold in order to confront the error and encourage students to further examine the potential for misapplied proportional reasoning. However, they found that they “still did not succeed in really defeating the (proportionality) illusion by changing different elements of the experimental setting” (p. 86). Such results indicate the difficulty that students have recognizing when to apply and when not to apply proportional reasoning. We believe the degree to which their perceptions of their proportional reasoning errors are general may account for some of this difficulty. This study sought to further understand this generality in greater detail.

2.3 Generality

As described by Davydov (1990), “generalization is regarded, as a rule, as inseparably linked to the process of abstracting” (p. 13). Abstracting involves identifying characteristics of particular cases and examining these characteristics across cases (Dienes, 1961; Dörfler, 1991). Generalization involves the following:

deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or

lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures, and the relationship across and among them. (Kaput, 1999, p. 136)

Though Kaput describes an individual's focus on the pattern or procedure without reference to particular cases, others (e.g., Davydov, 1990; Mason, 1996) emphasize the repeated movement between the particular and the general during the process of generalizing. As such, generalizing involves "seeing a generality through a particular and seeing the particular in the general" (Mason, 1996, p. 65).

Despite the emphasis on generalization as an essential component of mathematical activity, students often remain unaware of the generalizability of the procedures they learn. Instruction often masks the general character of their mathematical activity because teachers focus on applying rules to particular cases (Mason, 1996) rather than bringing out the commonalities among students' prior, current, and future work. A considerable body of research documents the challenges that students face as they attempt to apply their reasoning in a general manner (e.g., Erickson, 1991; Kuchemann, 1981). We use the phrase *general nature of the error* to describe our perceptions of a student's domain for his or her errors. We elaborate further on our schematization for illustrating this domain in the next section.

3 Methodology, data sources, and analysis

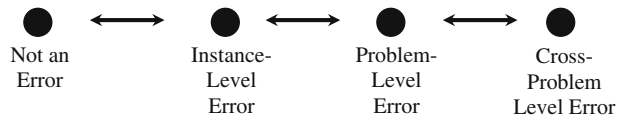
The research detailed in this paper is part of a larger body of work that investigated student cognition of algebraic generalization. In our study, eight students were purposefully selected from the fifth grade population of 80 students in a US elementary school using a pretest involving algebraic generalization tasks. The eight participants represented a range of ability levels and strategy use. They were grouped into two high/medium and two low/medium pairs. One high/medium pair, Lloyd and Dallas, serves as the focus for this paper.

A teaching experiment in the model of Steffe and Thompson (2000) was utilized throughout 18 instructional sessions occurring weekly over a 4-month period. The first author served the role of teacher and was assisted by the co-authors, who provided alternative perspectives on the students' thinking. The lead author attended all of the instructional sessions, whereas one of the coauthors was present at any one session. Our role during these sessions was to elicit and build on students' natural ways of reasoning. Tasks were purposefully chosen to facilitate the generation and testing of hypotheses about "students' unanticipated ways and means of operating as well as their unexpected mistakes" (p. 277). Each week the authors met to determine the task for the following week based on the interactions that occurred in the previous session. The tasks were selected to assist students in developing a more sophisticated understanding of the generalizations and justifications that they constructed.

Each episode was captured on video with a separate camera focused on each participant. Further evidence included researcher field notes, student written work, and video screen-capture of student computer spreadsheets.

Following completion of the 18-week teaching experiment, the data were retrospectively analyzed using a data-reduction approach (Miles & Huberman, 1994). Initially, a descriptive account of each session was constructed. We used these rich descriptive accounts to identify and code the occurrence of each error using tables to document and organize this information. For each error that occurred, we described how the students

Fig. 2 The error generality framework



viewed the general nature of their errors. One theme that emerged from this process was that students held different views of the range of applicability for their errors. These different views were further analyzed using elements of grounded theory (Strauss & Corbin, 1998), resulting in four levels for students' views of their errors: not an error, instance-level, problem-level, and cross-problem level.

Not An Error: The student does not see the strategy as an error.

Instance-Level Errors: The student views the error as applicable only to a particular instance or a few isolated instances of the situation (e.g., the student views proportional reasoning as an error for a jump from $n=4$ to $n=8$, but does not view proportional reasoning as an error for other instances.)

Problem-Level Errors: The student views the error as incorrect for any application in a particular problem context.

Cross-Problem Level Errors: The student views the error as incorrect for a particular class of problems (e.g., the student recognizes that proportional reasoning will produce incorrect results for a group of problem situations).

Following this, a constant comparative method (Glaser & Strauss, 1967) was applied to test and revise these categories (see Fig. 2).

After developing our error-generality framework, all occurrences of proportional reasoning were coded using the revised schematized description. Using this framework we characterized how Dallas and Lloyd viewed the general nature of their errors and documented the changes over time that occurred in these views.

4 Data interpretations

In the following paragraphs we describe the views of proportional reasoning errors exhibited by Lloyd and Dallas during four instructional sessions and detail the changes that occurred among these sessions.

4.1 Session 6, December 6

In the Theater Seats Problem (Fig. 1), both students correctly determined the number of seats in the fourth row (16) by adding 3 to the number of seats in the third row (13). When calculating the number of seats in the 10th row, Dallas incorrectly used proportional reasoning, doubling the result for row five (19 seats) to obtain 38. Lloyd continued adding 3 repeatedly, but left off one group of three seats to arrive at 31 seats. A dialogue began about their strategies for finding the number of seats in the 10th row.

Dallas: I wasn't going to keep adding all the way up to get to the tenth row. But I knew that 5 times 2 was 10, so I just multiplied 19 by 2, and got 38.

Teacher: Lloyd, how many seats do you think there are for the tenth row?

Lloyd: I think there are 31.

Dallas: I got 38. Nineteen times 2 is 38.

Lloyd: So how did you get the 19 times 2?

Dallas: Because in the fifth row there were 19. The fourth row had 16 (seats), so I added 3 again to get 19. Then I timesed that by 2 to get 38.

Teacher: So we disagree on the answer for the tenth row.

Dallas: I think we're both wrong. I timesed my number by 2, but I have to be adding 3 every time. I don't think I could just times it by 2.

Dallas explained that he could take 19 and add 15 (3 times 5) because each row added 3 and there were five more rows. He recognized that his doubling strategy did not provide the correct number of seats when moving from the fifth to the 10th row (Instance-Level Error). At this point, he did not explore why applying proportional reasoning was incorrect and could have believed that proportional reasoning provided correct values for other instances.

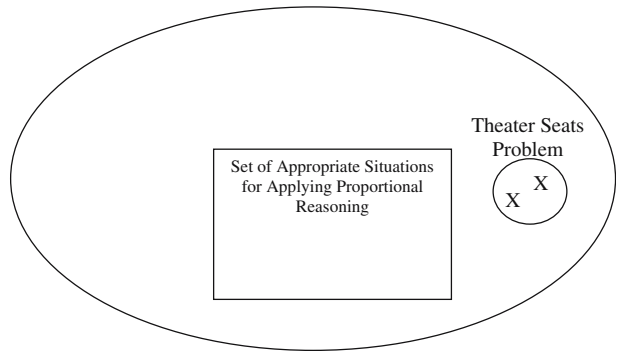
For the 23rd row, Dallas discontinued using proportional reasoning. He calculated the number of seats by starting with 38 seats (the incorrect value he previously obtained for the 10th row) and adding 39 (i.e., 13 times 3) for the extra 13 rows that were added to the 10th row, arriving at a value of 77. Lloyd started with 34 seats in the 10th row, doubling that value to arrive at 68 seats in the 20th row. For the remaining three rows, Lloyd added an additional nine seats. Neither student questioned the other's strategy as both obtained the same, though incorrect, value of 77 seats in the 23rd row. When questioned about Dallas' previous use of proportional reasoning (row five to row ten), Lloyd said, "maybe you (Dallas) did it wrong." When calculating the number of seats for the 23rd row, Lloyd appeared to believe that Dallas' error was not an error in proportional reasoning, but a mistake with his calculations.

To determine the number of seats in the 50th row, Dallas returned to using proportional reasoning, doubling the number of seats in the 23rd row (77) in an attempt to obtain the number of seats in row 46. He then added 12 more seats for the four additional rows, arriving at 166 seats for the 50th row. Lloyd multiplied the number of seats in the 10th row (34) by 5 to obtain the result of 170 seats for the 50th row. When discussing their results, Dallas stated that both strategies should be correct, but was unsure why his answer differed from Lloyd's by four seats. At this point, both students appeared to believe that proportional reasoning was either not an error or was an Instance-Level Error that occurred for a previous particular case.

When asked to determine the number of seats in row 20, Lloyd returned to using proportional reasoning, doubling the number of seats in row 10 (34) to arrive at 68 seats for row 20. Lloyd also said that he could add 19 more threes to the seven seats in the first row to obtain the number of seats in row 20. After computing $7 + 3 \times 19$, Lloyd and Dallas found that there were 64 seats in the 20th row. Lloyd stated that the answer must be 64, but was unsure why doubling did not provide the same result. At this point, both students abandoned the use of proportional reasoning, but appeared unsure why this strategy did not provide correct answers for these instances. During the remainder of the class period, Lloyd and Dallas used recursive or explicit rules.

At the end of the session, Lloyd and Dallas appeared uncertain about their use of proportional reasoning. They either believed that proportional reasoning provided incorrect values for every instance of the problem (Problem-Level Error) or that it introduced errors for enough instances that it could not be reliably used in this problem (Instance-Level Error). As shown in Fig. 3, Dallas and Lloyd appeared to view the Theater Seats problem as a situation for which they could not apply proportional reasoning based on a few particular instances. However, they did not appear to obtain a deep understanding of why this was the case.

Fig. 3 Session 6 schematic for Lloyd and Dallas



Each X represents a particular instance of a problem situation.
Each circular region represents all instances of a problem situation.

4.2 Session 8, December 20

For the Beam Design Problem (see Fig. 4), Dallas and Lloyd did not apply proportional reasoning until they were asked to find the number of rods for a length-20 beam; they previously determined the number of rods for a length-10 beam. Below is an excerpt of their discussion.

Teacher: What would you do for a beam of length 20?

Lloyd: I think you just do 39 plus 39, see if that might do it. Both (beams) are (length) ten, and I know 10 plus 10 is 20.

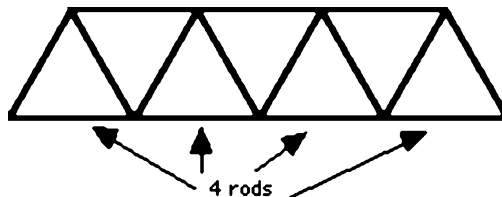
Teacher: Explain why you would take 39 plus 39 in terms of the beam.

Lloyd: Because it might be a quicker way.

Dallas: I think you would have to add one more (rod) on the top to go with the one (part of the beam) before it. I think you have to add one more (rod) after you times it.

Teacher: Why did you add one?

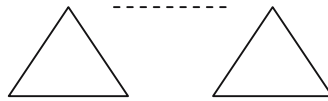
Beams are designed as a support for various bridges. The number of rods used to build the beam bottom determines the beam length. Below is a length-4 beam.



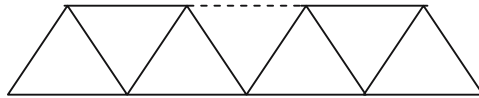
How many rods are needed to make a beam of length 5? Of length 8? Of length 10? Of length 20? Of length 34? Of length 76? Of length 223? Write a rule for how you could find the number of rods needed to make a beam of any length.

Fig. 4 The beam problem (adapted from Wijers et al., 1998)

Fig. 5 **a** Dallas’ diagram for the Beam Problem, **b** Dallas’ second diagram for the Beam Problem



a. Dallas’ Diagram for the Beam Problem



b. Dallas’ Second Diagram for the Beam Problem

Dallas: Say you (want) a rod length of two; you have to add this one (a connector rod). I think you have to add one after you times it. (See Fig. 5a)

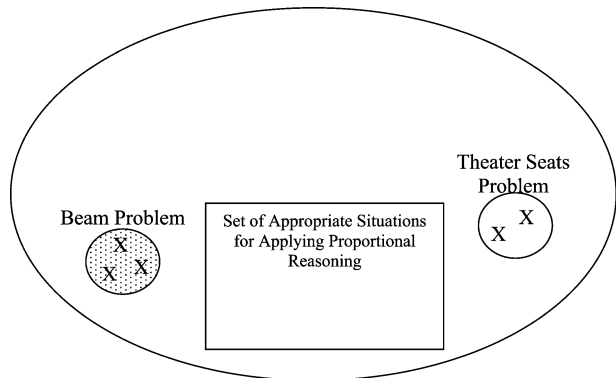
Lloyd: I don’t understand what you are saying.

(Dallas draws two beams of length-two and shows the connecting rod between the length-two sections.) (See Fig. 5b.)

Note that Dallas discussed doubling a length-10 beam by providing examples of doubling a length-one beam and a length-two beam. Dallas viewed the error as a Problem-Level Error, implying that you can examine any “doubling” case for this situation and the same error will occur.

When asked about a length-37 beam, Lloyd multiplied 39 by 3 and added 2 to connect the three groups, stating that he needed to count how many rods were required for a beam of length seven and add that on as well. Lloyd’s thought process illustrates that he moved to thinking about his use of proportional reasoning at the problem-level. Figure 6 represents the view of Dallas and Lloyd regarding the use of proportional reasoning following this session. At this point, both students appeared to have a deep understanding (represented by the shading in the diagram) of why they could not directly apply proportional reasoning to this situation. The shading represents the recognition by Dallas and Lloyd that they could

Fig. 6 Session 8 schematic for Lloyd and Dallas



Each X represents a particular instance of a problem situation.
Each circular region represents all instances of a problem situation.

not apply proportional reasoning for all possible instances of the Beam Problem. In contrast, for the Theater Seats problem, Dallas and Lloyd recognized that they could not apply proportional reasoning for only a few instances. However, for the Beam Problem they appeared to recognize that their error generalized to any instance (i.e., to the problem level).

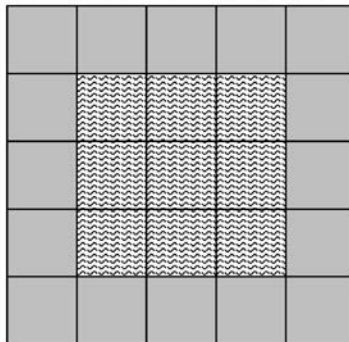
4.3 Session 15, March 6

Initially, Dallas and Lloyd were able to successfully apply two different explicit rules for the Floor Design Problem (see Fig. 7). At this point, Dallas and Lloyd were shown the strategy of a fictitious student who applied proportional reasoning to the situation, doubling the number of gray tiles (44) for a design of length-10 and arriving at 88 for the number of gray tiles for a design of length-20.

Lloyd stated that this strategy was incorrect as the number of gray tiles was always less than the amount that you would obtain by doubling. When asked why this was the case, Lloyd noted that this had occurred when they found the number of tiles for a design of length-20 previously. He mentioned that he computed this value previously to arrive at 84 gray tiles but was unable to explain why the doubling strategy did not apply to other instances. Lloyd appeared to view his use of the doubling strategy at the problem level, but did not have a deep understanding of why the strategy was incorrect.

Dallas noted that if you doubled the number of tiles for a floor design of length-10 that you would double the number of tiles for the sides ($4n$) and for the corners (4), resulting in a value that was four more than the correct number of tiles. He explained that doubling the number of gray tiles for a border of length n in the expression $4n+4$ doubled both of these quantities. Lloyd stated that he was unsure of Dallas' explanation, but believed that the doubling strategy could not be applied to this situation because it resulted in fewer gray tiles for the instances discussed.

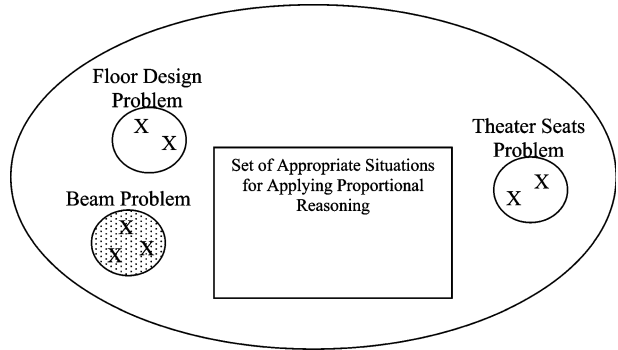
The AKME floor design company creates square floor patterns made of shaded square tiles surrounded by a gray border. The size of the floor is determined by the side length of the shaded square in the centre of the pattern. Below is the example of a floor of length 3 (shaded tiles).



1. How many gray tiles would you need to make a floor of length 6 (shaded tiles)? 10 (shaded tiles)? 20 (shaded tiles)? 47 (shaded tiles)? 139 (shaded tiles)?
2. Write a rule or a formula for how you could find the number of gray tiles needed to make a floor of any length (in shaded tiles). Explain your rule or formula.

Fig. 7 Floor design problem

Fig. 8 Session 15 schematic for Lyod



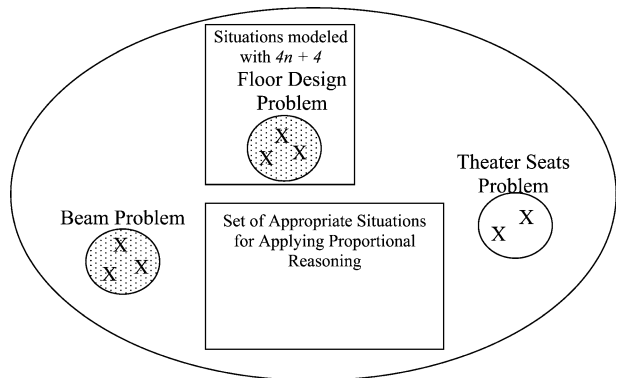
Each X represents a particular instance of a problem situation.
 Each circular region represents all instances of a problem situation.

Lloyd stated that applying proportional reasoning resulted in incorrect values for any instance. However, he based his reasoning on a few instances rather than a deep understanding of the strategy in relation to the situation. We indicate this in Fig. 8 by listing the few instances that Lloyd constructed his generalization about the use of proportional reasoning. Dallas clearly explained, in relation to the situation, why doubling the number of tiles for a design was incorrect. He noted that the doubling strategy doubled the number of tiles for each side, but also doubled the number of tiles in the corners, resulting in overcounting the number of gray tiles by 4. In addition, Dallas began to view the general nature of the doubling strategy in relation to the expression, $4n + 4$, that he created. He said that doubling the quantity $4n + 4$ doubled both the quantity $4n$ for the number of tiles in the sides of the design and the 4 tiles in the corners. We denote this in Fig. 9 by creating a rectangle to represent the situations that could involve the expression $4n + 4$, which includes the Floor Design problem.

4.4 Session 18, April 4

During Session 18, Lloyd added 4 repeatedly for the Straw problem (see Fig. 10) to correctly determine the number of straws need for five squares (16 straws), seven squares

Fig. 9 Session 15 schematic for Dallas



Each X represents a particular instance of a problem situation.
 Each circular region represents all instances of a problem situation.

The fifth grade class is making a design like the one below out of straws. The design looks like a string of squares. Below is an example of a string of five squares that uses 16 straws.



1. How many straws are needed to make 7 squares? 10 squares? 20 squares? 32 squares? 245 squares?
2. Write a rule or a formula to find the number of straws needed to make any number of squares. Explain your rule or formula.

Fig. 10 The straw problem

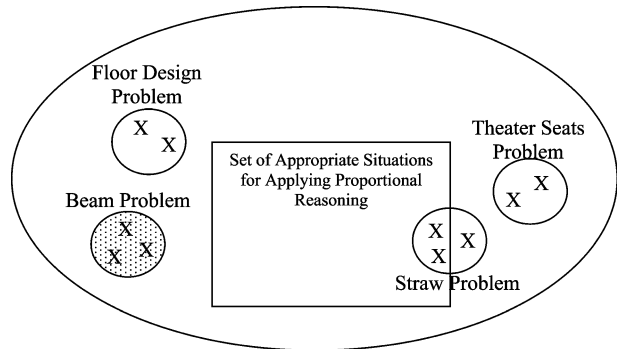
(22 straws), and 10 squares (31 straws). For 20 squares, Lloyd applied the doubling strategy, multiplying the number of straws for 10 squares (31) to arrive at 62 squares. He justified his doubling strategy by stating that “10 plus 10 equals 20” referring to the 10 squares combined with the next 10 squares. Lloyd also used proportional reasoning to determine the number of straws for 245 squares.

When asked to examine the instance of doubling the number of straws from five squares to 10 squares, Lloyd stated that the doubling strategy should provide the correct number of straws. After the teacher pointed out that he arrived at 31 straws when using his “add 3” rule, Lloyd was puzzled by the difference between the results he obtained when using his “add 3” rule (31 straws) and using the doubling strategy (32 straws). Lloyd drew a picture of 10 squares and counted them one at a time, again arriving at 31 straws. When asked why the doubling strategy did not produce the correct answer, Lloyd stated that he was “not sure.” He added 16 and 16 again on his calculator, continuing to arrive at 32.

As part of the final session, Lloyd examined the work of various fictitious students. Lloyd first examined Diane’s work for which she used the doubling strategy just as Lloyd used previously in the session. Lloyd stated that the doubling strategy will work for 10 squares (to double to 20), but it would not work when jumping from five squares to 10 squares. It was unclear why Lloyd believed that using the proportional strategy produced incorrect results when moving from 5 squares to 10 squares whereas a jump from 10 squares to 20 squares was appropriate. However, it appeared that Lloyd saw the proportional strategy as incorrect for only one instance, believing that this strategy would provide correct results for many instances of the Straw problem.

The teacher instructed Lloyd to sketch a picture of 20 squares and count the number of straws. Lloyd did so, arriving at the correct value of 61 straws. At this point, Lloyd suddenly stated that 61 was the correct number of straws for 20 squares and adjusted his proportional strategy so that he subtracted one after doubling. When questioned why he subtracted one, Lloyd said that he was unsure, but that this strategy provided the correct result for 20 squares. He later stated that, “(the) multiplying by 2 (rule) is not helpful because you might not get the right answer.” We denote Lloyd’s reasoning for this situation by placing some instances within the region of appropriate use for using proportional reasoning and some instances outside this box (see Fig. 11).

Fig. 11 Session 18 schematic for Lloyd

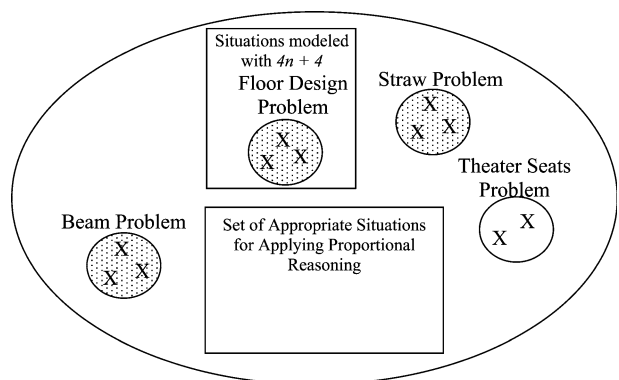


Each X represents a particular instance of a problem situation. Each circular region represents all instances of a problem situation.

When Dallas examined the work of a fictitious student using a doubling strategy to find the number of straws for a length-20 rod using a length-10 rod, he stated that this strategy would not provide the correct number of straws “because you have to minus one, because you have the end of (pointing to a particular straw) so you cannot count this straw twice.” Dallas noted that he could make the doubling strategy produce correct values if he subtracted one for the extra straw.

Throughout Session 18, Lloyd continued to struggle with the use of the proportional strategy. Initially, he appeared to view the proportional strategy as “Not an Error” for the Straw problem. When counting the number of straws resulted in discrepant results with his doubling strategy, he changed his direct application of proportional reasoning by subtracting one to match the results provided when he counted. Lloyd moved from viewing proportional reasoning as not an error to viewing it as an error at the problem level; however, he appeared to have little understanding of why he applied this adjustment. Dallas viewed the doubling strategy as an error for any instance for the Straw problem, noting the adjustment that must be made due to the overlap when joining separate segments. Dallas’ viewed the proportional strategy as an error for all cases in the Straw problem. We represent Dallas’ view of the Straw problem in Fig. 12.

Fig. 12 Session 18 schematic for Dallas



Each X represents a particular instance of a problem situation. Each circular region represents all instances of a problem situation.

5 Discussion

That data that we analyzed focused on Dallas and Lloyd’s errors. No assumptions can be made regarding the generalizability of this analysis to other student populations. However, we believe that our schematized descriptions provide a useful lens for examining student errors in other settings. In the following sections we further describe the theoretical and instructional implications related to understanding the extent to which students recognize the generality of their errors.

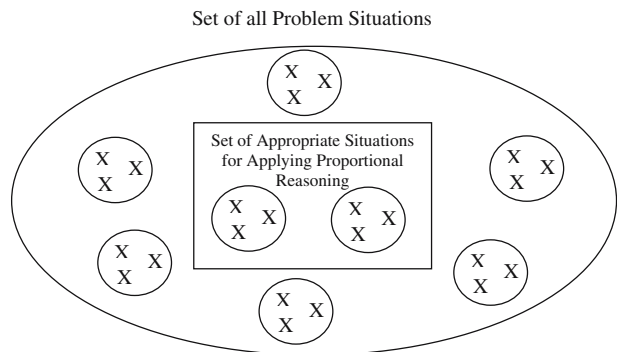
5.1 Theoretical implications

During this study Dallas and Lloyd applied proportional reasoning to a variety of particular instances. Initially it appeared that Dallas and Lloyd overgeneralized the use of proportional reasoning. As described by Smith et al. (1993), the students’ knowledge was “extended beyond its productive range of application” (p. 152) causing errors in their use of proportional reasoning. Eventually, when Dallas and Lloyd were faced with situations that did not allow for the direct application of proportional reasoning, they grappled with their errors at the instance level, and began to extend their reasoning to the problem and cross-problem levels. However, these students examined the range of applicability of their errors in differing ways.

When confronted with examples of his mistaken use of proportional reasoning, Lloyd often applied numeric adjustments to address his errors. He recognized that the proportional strategy did not apply for particular instances (e.g., from $n=5$ from $n=10$), but struggled to generalize his reasoning to other instances or to the problem level. Due to Lloyd’s struggles to understand his errors at the instance level, he continued to mistakenly apply proportional reasoning throughout the study.

In contrast, Dallas critically examined his proportional reasoning errors by connecting to pictorial representations of the problems. Dallas recognized the error of directly applying proportional reasoning to particular instances (instance level) and extended his reasoning to the problem and cross-problem levels. Although Dallas initially appeared to overgeneralize the applicability of proportional reasoning, by seeking to further understand his errors for particular cases he was able to apply his understanding to the problem and cross-problem levels, altering his conception of the use of proportional reasoning. As such, Dallas appeared to engage in reconstructive generalization (Harel & Tall, 1991) in which he

Fig. 13 A potential perspective for the use of the proportional strategy



Each X represents a particular instance of a problem situation.
 Each circular region represents all instances of a problem situation.

restructured his schema of proportional reasoning, extending the range of applicability of the mistaken use of proportional reasoning and restricting the applicability of proportional reasoning simultaneously.

Figure 13 represents how students can use their errors to construct boundaries for the application of particular concepts, in our case the application of proportional reasoning. By extending the range of applicability of the incorrect use of proportional reasoning, the student reduces the range of applicability of the correct use of proportional reasoning. As such, this process shrinks the inner rectangle and expands the area outside the rectangle. The student does this through recognition of particular instances (represented by an 'X' in Fig. 13) and problem situations (small circular regions in Fig. 13) for which proportional reasoning can and cannot be applied. When students seek to understand the general nature of their errors, they engage in a critical boundary-defining process that can lead toward a normative view of the application of a particular concept.

When students engage in considering the applicability of a particular concept, they define the boundary for when to and when not to apply a particular idea. Initially, students create a tentative boundary for applying a concept, such as linearity, to a range of problem situations (e.g., cost of a telephone call, cost of steak for a given weight). Through applying linearity inappropriately to other situations, such as projectile motion, they can further clarify when to apply and when not to apply linearity. Dallas began to recognize that proportional reasoning was inappropriate for the class of situations that could be modeled with the expression $4n+4$, moving toward consideration of the application of proportionality to situations that could be modeled with expressions of the form $an+b$.

5.2 Implications

Another important component of our generality frameworks involves the implications that students' views of the generality of their errors have for instruction. Though teachers have been encouraged to allow their students to use errors as opportunities for learning, other considerations regarding students' views of their errors can further clarify how errors can be used to deepen student understanding. When a teacher interacts with a student who sees errors only at the instance level, the student and the teacher may discuss an error, both coming to the conclusion that an error exists, but leave with differing interpretations of the general nature of the error; the teacher believing the student sees the error as applying to all instances of a problem, whereas the student views the error as applying only to a particular instance. The lack of a shared understanding of the general nature of the error can cause confusion for the teacher when the student makes the error again for other specific instances. This may partly explain the observation that some students seem to repeat the "same mistakes" over and over.

Encouraging students to consider the general nature of their errors may deepen their understanding of the underlying concept. Students can move from focusing on particular instances toward considering errors at the problem and cross-problem levels (as illustrated in Dallas' case). Teachers may encourage students to consider whether an error applies to other instances or to related problem situations, clarifying the boundaries for when to apply and when not to apply particular ideas.

5.3 Conclusion

If student errors are often the result of overgeneralizations of concepts, it is essential that they use their errors to define the boundaries of particular concepts. Thus, it is important

that teachers and students not seek simply to correct their errors, but in the spirit of the initial quote from Salvador Dali, to rationalize them and to understand them thoroughly.

Further research is needed to describe the process that students use when questioned about the general nature of their errors and how teachers can use student responses in classroom settings. Understanding how students view and reconcile their errors is an emerging research area deserving of further attention from various perspectives.

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