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Commutative Objects, Central Morphisms and Subtractors in Subtractive Categories

Vaino Tuhafeni Shaumbwa[1,2]

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Abstract
We give some characterizations of commutative objects in a subtractive category and central morphisms in a regular subtractive category. In particular, we show that commutative objects, i.e., internal unitary magmas, are the same as internal abelian groups in a subtractive category and that analogously, centrality has an alternative description in terms of so-called "subtractors" in a regular subtractive category.

Keywords Abelian object · Commutative object · Central morphism · Internal subtraction structure · Subtractor · Subtractive category

Mathematics Subject Classification(2020) 18C40 · 08C05 · 18E08 · 18E13

1 Introduction

A morphism $\varphi : Y \times X \longrightarrow Z$ in a pointed category is a *subtractor* of $f : X \longrightarrow Y$ along $g : Y \longrightarrow Z$, if φ makes the diagram

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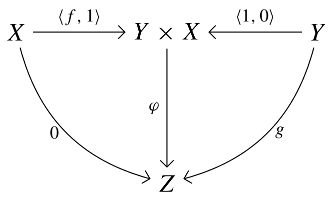
✉ Vaino Tuhafeni Shaumbwa
vshaumbwa@unam.na

1 Department of Computing, Mathematical and Statistical Sciences, University of Namibia, Private Bag 13301, Windhoek, Namibia

2 Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland 7602, South Africa



commute. When g is the identity morphism of Y, we will just call φ a subtractor of f. Here and throughout the rest of this paper, 1 and 0 denote the suitable identity and zero morphisms respectively. Furthermore, we follow the standard convention that for a span

$$X \xleftarrow{\;r\;} A \xrightarrow{\;s\;} Y$$

in a category $\mathbb{C}$ with finite products, $\langle r, s\rangle$ denotes the induced morphism $A \longrightarrow X \times Y$.

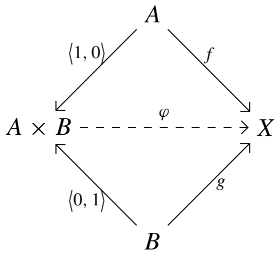
The notion of a subtractor generalises the notion of an *internal subtraction structure* [5] defined in a pointed category $\mathbb{C}$ to be a morphism $s : X \times X \longrightarrow X$ such that $s\langle 1, 1\rangle = 0$ and $s\langle 1, 0\rangle = 1$. Also following [5], an object X in a pointed category $\mathbb{C}$ is said to admit a *subtraction structure* when there is an internal subtraction s on X. We may sometimes refer to an internal subtraction structure as just a subtraction structure. Indeed, a subtractor of the identity morphism of X along itself is just an internal subtraction on X.

A pointed finitely complete category $\mathbb{C}$ is a subtractive category [8] if every left punctual reflexive relation is right punctual, i.e., for every relation $\langle r_1, r_2\rangle : R \rightarrowtail X \times X$ on an object X, if $\langle 1, 1\rangle : X \longrightarrow X \times X$ and $\langle 1, 0\rangle : X \longrightarrow X \times X$ factor through $\langle r_1, r_2\rangle$, then $\langle 0, 1\rangle : X \longrightarrow X \times X$ factors through $\langle r_1, r_2\rangle$ as well. Equivalently (see e.g., [9, Theorem 4.1]), a subtractive category can be defined to be a pointed finitely complete category $\mathbb{C}$ satisfying that for every relation $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ and a pair of morphisms $f : A \longrightarrow X$ and $g : A \longrightarrow Y$, if $\langle f, g\rangle : A \longrightarrow X \times Y$ and $\langle f, 0\rangle : A \longrightarrow X \times Y$ factor through $\langle r_1, r_2\rangle$, then $\langle 0, g\rangle : A \longrightarrow X \times Y$ factors through $\langle r_1, r_2\rangle$ as well. Recall that the notion of a subtractive category generalizes the notion of a pointed subtractive variety [11]; that is, a variety of universal algebras that has a unique constant 0 and a binary term s in its theory, such that $s(x, x) = 0$ and $s(x, 0) = x$.

In a pointed category $\mathbb{C}$, an approximate subtraction [4] on an object X is a morphism $s : X \times X \to Y$ such that $s\langle 1, 1\rangle = 0$. The composite $a = s\langle 1, 0\rangle : X \to Y$ is called the approximation morphism of s. The notion of a subtractor generalizes an approximate subtraction. Indeed, it can be easily observed that when there is an approximate subtraction s on X and a is the approximation morphism of s, it precisely means that the identity morphism of X admits a subtractor s along $a : X \to Y$.

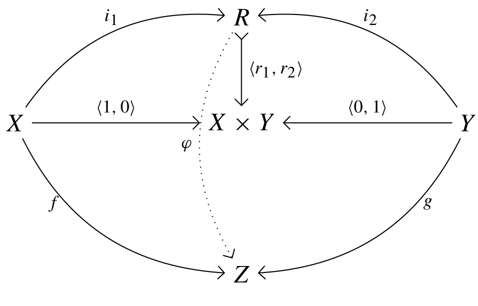
In this paper we use the notion of a subtractor to describe, in a regular subtractive category, *central morphisms* and *commutative objects*, which are based on the notion of *commuting morphisms* [6]. We recall the definitions of these notions below, but we state them in a more general context than where these notions are usually considered.

In a pointed category $\mathbb{C}$ with finite products, a pair of morphisms $f : A \longrightarrow X$ and $g : B \longrightarrow X$ are said to *commute* (or Huq-commute) if there is a morphism φ making the diagram



commute. A morphism f is *central* if it commutes with the identity morphism of its codomain, while an object X is *commutative* if the identity morphism on X is central. Commuting morphisms (and centrality) were first considered by S. A. Huq [6] in a context closely related to that of a semi-abelian category [7]. Commuting morphisms were further investigated by Bourn [2] and other authors in a lighter context, namely, that of a unital category [3]; that is, a pointed finitely complete category $\mathbb{C}$ with the property that, for every pair of objects X and Y, the pair of morphisms $\langle 1, 0\rangle : X \longrightarrow X \times Y$ and $\langle 0, 1\rangle : Y \longrightarrow X \times Y$ are jointly extremal-epimorphic. The notion of commuting morphisms can be thought of as an extension of classical commutativity to general categories. Indeed, it can be easily observed that when $\mathbb{C}$ is the category of groups, two group homomorphisms $f : A \longrightarrow X$ and $g : B \longrightarrow X$ commute in the above sense, if and only if $f(a)g(b) = g(b)f(a)$ for all $a \in A$ and $b \in B$. The same description applies in the category of monoids. Thus, commutative objects in the category of groups are abelian groups, while in the category monoids they are commutative monoids.

In the present paper, we say a pair of morphisms $f : X \to Z$ and $g : Y \to Z$ in a pointed category $\mathbb{C}$ with finite products *partially commute* if there is a subobject $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ and a morphism $\varphi : R \to Z$, such that the morphisms $\langle 1, 0\rangle : X \to X \times Y$ and $\langle 0, 1\rangle : Y \to X \times Y$ factor through $\langle r_1, r_2\rangle$ as $\langle r_1, r_2\rangle i_1 = \langle 1, 0\rangle$ and $\langle r_1, r_2\rangle i_2 = \langle 0, 1\rangle$ respectively, and the diagram



commutes. We will say f and g partially commute with respect to $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ with *cooperator* φ (following the terminology of D. Bourn in the absolute case [2], who calls φ a cooperator of f and g when $\langle r_1, r_2\rangle$ is an identity morphism). When a morphism $f : X \longrightarrow Z$ partially commutes with the identity morphism of Z, we say f is *partially central*. In a similar way, we say X is *partially commutative* when the identity morphism of X partially commutes with itself. Clearly, if $\mathbb{C}$ is a unital category then partial commute coincides with Huq-commute.

In our main results we use subtractors to establish that partially central morphisms are precisely central morphisms in a regular subtractive category, and partially commutative objects are precisely commutative objects in a subtractive category.

The paper is organized as follows: in Sect. 2 we describe what it means for a pair of morphisms to partially commute in two contrasting examples. In Sect. 3 we slightly generalize the notion of a *partial subtraction structure* which happens to be related to the notion of partially commuting morphisms in a subtractive category. It has been shown in [5, Corollary 2.7] that abelian objects in subtractive categories are precisely those objects which admit partial subtraction structures. We prove (in Theorem 3.1) that commutative objects in subtractive categories are exactly abelian objects. In Sect. 4 we characterize central morphisms in regular subtractive categories in terms of subtractors (Theorem 4.1).

2 Two Examples of Partially Commuting Morphisms

In this section we briefly describe partially commuting morphisms in the following examples:

Example 2.1 Let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be morphisms in a pointed category $\mathbb{C}$ with finite products. If $\mathbb{C}$ admits join $X{\star}Y$ of the canonical morphisms $\langle 1, 0\rangle : X \to X \times Y$ and $\langle 0, 1\rangle : Y \to X \times Y$, then f and g partially commute if and only if they commute with respect to the punctual relation $X{\star}Y$. For example, consider the category $\mathcal{I}$ whose objects are sets X equipped with a binary operation $-$ and a unique constant 0, satisfying (i) $x - 0 = x$, (ii) $x - x = 0$, and (iii) $0 - x = 0$. Morphisms in this category are maps which preserve the binary operation $-$ and the constant 0. The variety of (nonempty) implication algebras can be interpreted in $\mathcal{I}$ as a subvariety. A nonempty implication algebra [1] gives rise to a binary "subtraction" $-$, with $x - y := y \to x$ and a unique constant $0 := x \to x = y \to y$, satisfying (i)–(iii) above (see e.g., [10] for a detailed explanation). Now for two morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ in $\mathcal{I}$, the join $X{\star}Y$ is given by the union

$$X{\star}Y = \{(x, 0)|x \in X\} \cup \{(0, y)|y \in Y\}.$$

The map $\varphi : X{\star}Y \longrightarrow Z$ defined by

$$\varphi(x, 0) = f(x) \quad \text{and} \quad \varphi(0, y) = g(y),$$

is a morphism in $\mathcal{I}$ if and only if

$$f(x) = f(x) - g(y) \quad \text{and} \quad g(y) = g(y) - f(x)$$

for all $x \in X$ and $y \in Y$. This means that f and g partially commute if and only if $f(x) = f(x) - g(y)$ and $g(y) = g(y) - f(x)$ for all $x \in X$ and $y \in Y$.

Example 2.2 In the category $\mathbf{Set}_*$ of pointed sets, the induced morphism $X + Y \longrightarrow X \times Y$ is a monomorphism for any two pointed sets X and Y. Hence in $\mathbf{Set}_*$, every two morphisms having the same codomain partially commute.

3 Partial Subtractors and Commutative Objects in Subtractive Categories

Let $\mathcal{S}$ be the variety of subtraction algebras, i.e., algebras whose signature consists of one binary operation $-$ and one nullary operation 0, satisfying $x - x = 0$ and $x - 0 = 0$. In the next proposition we describe what it means for a morphism to admit a subtractor in $\mathcal{S}$.

Proposition 3.1 *A homomorphism $f : X \longrightarrow Y$ in $\mathcal{S}$ admits a subtractor along $g : Y \longrightarrow Z$, if and only if, for every $x, x' \in X$ and $y, y' \in Y$, one has*

$$g(y - f(x)) - g(y' - f(x')) = g(y - y') - g(f(x) - f(x')).$$

Proof If $f : X \longrightarrow Y$ admits a subtractor $\varphi : Y \times X \longrightarrow Z$ along $g : Y \longrightarrow Z$, then it means for every $x \in X$ and $y \in Y$, $\varphi(y, 0) = g(y)$ and $\varphi(f(x), x) = 0$. Now it follows that

$$\begin{aligned}\varphi(y, x) &= \varphi(y, x) - \varphi(f(x), x)\\ &= \varphi(y - f(x), 0)\\ &= g(y - f(x)).\end{aligned}$$

Moreover one can observe that

$$\begin{aligned}g(y - f(x)) - g(y' - f(x')) &= \varphi(y, x) - \varphi(y', x')\\ &= \varphi(y - y', x - x')\\ &= g(y - y') - g(f(x) - f(x')).\end{aligned}$$

Conversely, let us suppose for every $x, x' \in X$ and $y, y' \in Y$, one has

$$g(y - f(x)) - g(y' - f(x')) = g(y - y') - g(f(x) - f(x')).$$

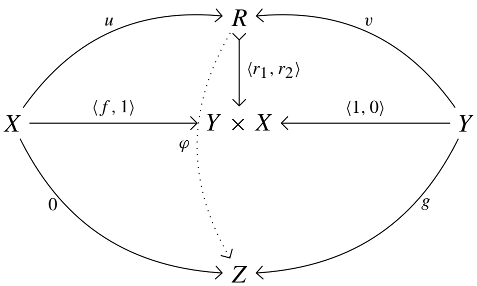
Define $\varphi : Y \times X \longrightarrow Z$ by $\varphi(y, x) = g(y - f(x))$. Clearly, $\varphi(f(x), x) = 0$ and $\varphi(y, 0) = g(y)$. Furthermore, for every $y, y' \in Y$ and $x, x' \in X$,

$$\begin{aligned}\varphi(y - y', x - x') &= g(y - y') - g(f(x) - f(x'))\\ &= g(y - f(x)) - g(y' - f(x'))\\ &= \varphi(y, x) - \varphi(y', x'),\end{aligned}$$

and this means φ is a homomorphism. Therefore φ is a subtractor of f along g. □

We give in the next definition a slight generalization of *partial subtraction structures*, initially defined only on objects in [5].

Definition 3.1 Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms in a subtractive category $\mathbb{C}$. A partial subtractor of f along g is a morphism $\varphi : R \longrightarrow Z$, where $\langle r_1, r_2 \rangle : R \rightarrowtail Y \times X$ is a monomorphism such that the morphisms $\langle f, 1 \rangle : X \longrightarrow Y \times X$ and $\langle 1, 0 \rangle : Y \longrightarrow Y \times X$ factor through $\langle r_1, r_2 \rangle$, and the diagram



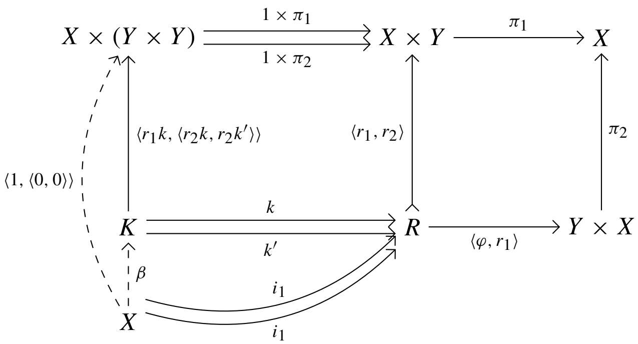
commutes. For that we will say f admits a partial subtractor φ along g (with respect to a subobject $\langle r_1, r_2 \rangle : R \rightarrowtail Y \times X$) or equivalently, f admits a partial subtraction structure along g. We will mostly require g to be the identity morphism of Y, and for that we will just say f admits a partial subtraction structure, and write a triple $(f, (R, r_1, r_2), \varphi)$ to denote a partial subtraction structure on f.

Remark 3.1 Note that when an object X in a subtractive category $\mathbb{C}$ admits a (partial) subtraction structure it means that the identity morphism of X admits a (partial) subtractor along itself.

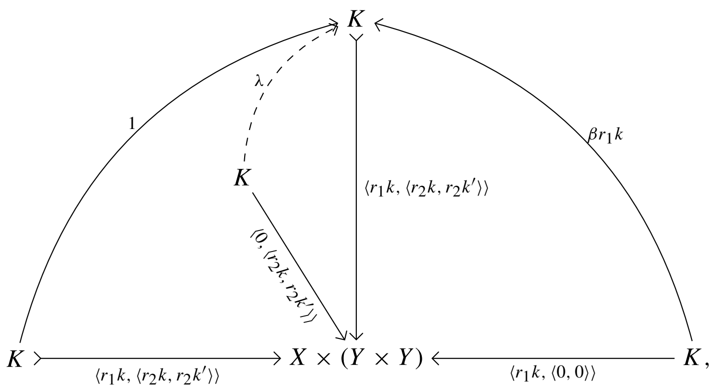
In the main result of this section we will describe *partially commutative objects* in subtractive categories in terms of partial subtraction structures.

Lemma 3.1 *In a subtractive category $\mathbb{C}$, if $f : X \longrightarrow Y$ is partially central with respect to a subobject $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ and with cooperator $\varphi : R \longrightarrow Y$, then the morphism $\langle \varphi, r_1\rangle : R \longrightarrow Y \times X$ is a monomorphism.*

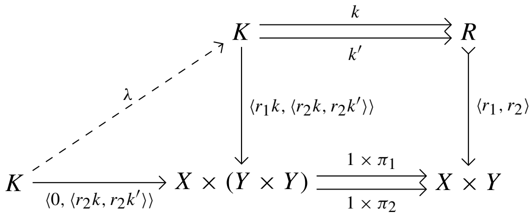
Proof Let $\langle k, k'\rangle : K \rightarrowtail R \times R$ denote the kernel pair relation of $\langle \varphi, r_1\rangle$ and i_1, i_2 be morphisms such that $\langle r_1, r_2\rangle i_1 = \langle 1, 0\rangle$, $\langle r_1, r_2\rangle i_2 = \langle 0, 1\rangle$, $\varphi i_1 = f$, and $\varphi i_2 = 1$. We will show that $k = k'$. Since $\langle \varphi, r_1\rangle k = \langle \varphi, r_1\rangle k'$, we see that $r_1 k = r_1 k'$. In the diagram



in which β is a morphism such that $k\beta = i_1 = k'\beta$, we see that $\langle 1, \langle 0, 0\rangle\rangle$ factors through $\langle r_1 k, \langle r_2 k, r_2 k'\rangle\rangle$ by β. Therefore, we obtain the following commutative diagram



from which by using the subtractivity of $\mathbb{C}$, we conclude that $\langle 0, \langle r_2 k, r_2 k'\rangle\rangle$ factors through $\langle r_1 k, \langle r_2 k, r_2 k'\rangle\rangle$. Using the diagram

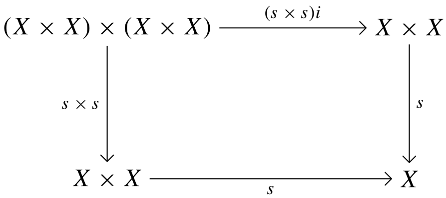


we see that $\langle r_1, r_2\rangle k\lambda = \langle 0, r_2k\rangle = \langle r_1, r_2\rangle i_2r_2k$, and similarly $\langle r_1, r_2\rangle k'\lambda = \langle r_1, r_2\rangle i_2r_2k'$, which imply that $k\lambda = i_2r_2k$ and $k'\lambda = i_2r_2k'$ respectively. But since $\varphi k = \varphi k'$, we obtain $r_2k = \varphi i_2r_2k = \varphi k\lambda = \varphi k'\lambda = \varphi i_2r_2k' = r_2k'$, and, together with $r_1k = r_1k'$, we have $k = k'$. □

Before we state the main result of this section, we recall the following fundamental fact:

Lemma 3.2 [5, Theorem 2.5] *In a subtractive category $\mathbb{C}$, if the identity morphism of X admits a partial subtraction structure $(1, (R, r_1, r_2), \varphi)$, then the morphism $\langle r_1, r_2\rangle$ is an isomorphism, in other words, X admits a subtraction structure as soon as it admits a partial subtraction structure.*

As explained in [5], an immediate consequence of the above lemma is that an internal subtraction $s : X \times X \longrightarrow X$ on an object X in a subtractive category $\mathbb{C}$ is necessarily a homomorphism of subtractions, i.e., the diagram



in which $i = \langle\langle \pi_1\pi_1, \pi_1\pi_2\rangle, \langle \pi_2\pi_1, \pi_2\pi_2\rangle\rangle$ is the middle-interchange isomorphism, commutes. This yields the following calculation:

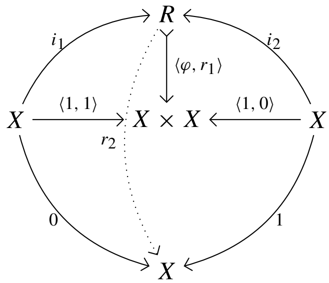
$$s\langle 0, s\langle 0, 1\rangle\rangle = s\langle s\langle 1, 1\rangle, s\langle 0, 1\rangle\rangle = s\langle s\langle 1, 0\rangle, s\langle 1, 1\rangle\rangle = 1.$$

Now the main result of this section.

Theorem 3.1 *In a subtractive category $\mathbb{C}$, the following statements are equivalent for an object X:*

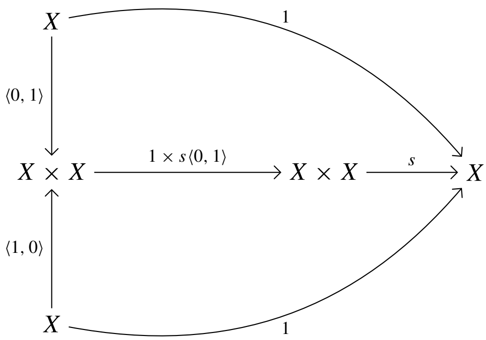
(a) *X is partially commutative.*
(b) *X admits an internal subtraction structure.*
(c) *X is commutative.*

Proof (a) $\Rightarrow$ (b) Let X be partially commutative with respect to a subobject $\langle r_1, r_2\rangle : R \rightarrowtail X \times X$, and with cooperator $\varphi : R \longrightarrow X$. Furthermore, let i_1, i_2 be morphisms such that $\langle r_1, r_2\rangle i_1 = \langle 1, 0\rangle$, $\langle r_1, r_2\rangle i_2 = \langle 0, 1\rangle$, $\varphi i_1 = 1$, and $\varphi i_2 = 1$. Applying Lemma 3.1, the morphism $\langle \varphi, r_1\rangle$ is a monomorphism. Since the diagram



commutes, we see that r_2 is a partial subtractor of the identity morphism of X with respect to $\langle\varphi, r_1\rangle$. Now the result follows from Lemma 3.2.

(b)⇒ (c) Let $s : X \times X \longrightarrow X$ be a subtraction on X. Since $s\langle 0, s\langle 0, 1\rangle\rangle = s\langle s\langle 1, 1\rangle, s\langle 0, 1\rangle\rangle = s\langle s\langle 1, 0\rangle, s\langle 1, 1\rangle\rangle = 1$, it is not difficult to see that the diagram



commutes. Hence, X is commutative.

(c) ⇒ (a) Clearly if X is commutative then it is partially commutative. □

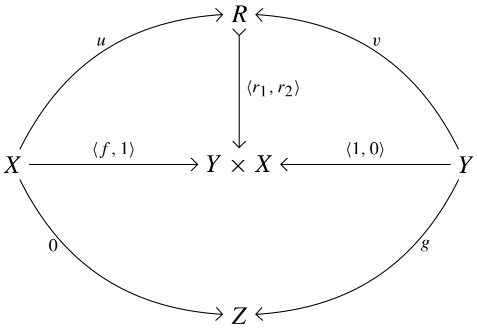
Remark 3.2 Since objects that admit internal abelian group structures in a subtractive category are exactly those that admits internal subtraction structures [5, Corollary 2.7], then commutative objects are precisely abelian objects in a subtractive category.

4 Characterization of Central Morphisms in Regular Subtractive Category

In this section we characterize central morphisms in terms of subtractors. We shall first prove a series of preliminary results about subtractors.

In the next lemma we establish the uniqueness of (partial) subtractors.

Lemma 4.1 *In a regular subtractive category $\mathbb{C}$, for each commutative diagram*



if there is a morphism $\varphi : R \longrightarrow Z$ such that $\varphi u = 0$ and $\varphi v = g$ (i.e., φ is a partial subtractor of f along g with respect to a monomorphism $\langle r_1, r_2\rangle$), then φ is necessarily unique.

Proof Suppose $\varphi' : R \longrightarrow Z$ and $\varphi : R \longrightarrow Z$ are two morphisms such that $\varphi' u = 0,\ \varphi' v = g$ and $\varphi u = 0,\ \varphi v = g$. Let $\langle s_1, s_2\rangle : S \rightarrowtail R \times R$ be the joint kernel pair relation of φ and φ', i.e., (S, s_1, s_2) is the kernel pair relation of $\langle \varphi, \varphi'\rangle : R \longrightarrow Z \times Z$. Let us consider a binary relation $\langle t_1, t_2\rangle : T \rightarrowtail R \times X$ given by the regular image of $\langle s_1, s_2\rangle$ along the morphism $1 \times r_2$. Using generalized elements,

$$((y_1, x_1), x_2) \in T \Leftrightarrow \exists y_2\ ((y_1, x_1), (y_2, x_2)) \in S.$$

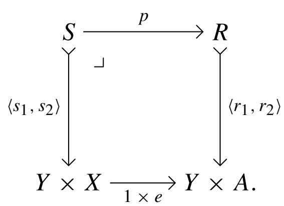
For each $(y_1, x_1) \in R$, since $((y_1, x_1), (y_1, x_1)) \in S$ and $((0, 0), (f(x_1), x_1)) \in S$, it follows that $((y_1, x_1), x_1), ((0, 0), x_1) \in T$, and so by subtractivity, one has $((y_1, x_1), 0) \in T$. But

$$\begin{aligned} ((y_1, x_1), 0) \in T &\Longrightarrow ((y_1, x_1), (y, 0)) \in S \\ &\Longrightarrow \varphi(y_1, x_1) = \varphi(y, 0) = g(y) = \varphi'(y, 0) = \varphi'(y_1, x_1). \end{aligned}$$

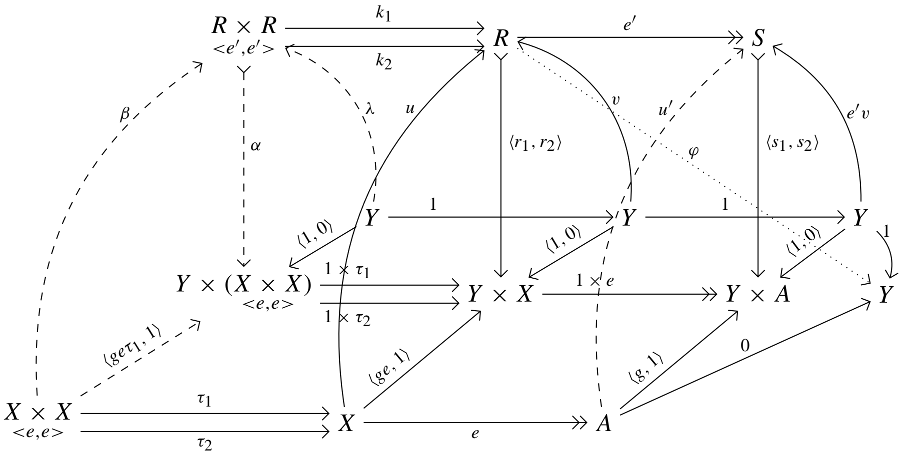
□

Proposition 4.1 *Let $\mathbb{C}$ be a regular subtractive category. If a morphism $g : A \longrightarrow Y$ admits a partial subtraction structure, then for any morphism $e : X \longrightarrow A$, the composite ge also admits a partial subtraction structure. The converse is true when e is a regular epimorphism.*

Proof If $g : A \longrightarrow Y$ admits a partial subtraction structure $(g, (R, r_1, r_2), \varphi)$, it can be easily shown that the composite ge admits a partial subtraction structure $(ge, (S, s_1, s_2), \varphi p)$, where $\langle s_1, s_2\rangle$ is the pullback of $\langle r_1, r_2\rangle$ along $1 \times e$ in the diagram



To prove the second part of the proposition, let us suppose ge, where e is a regular epimorphism, admits a partial subtraction structure $(ge, (R, r_1, r_2), \varphi)$ in the diagram



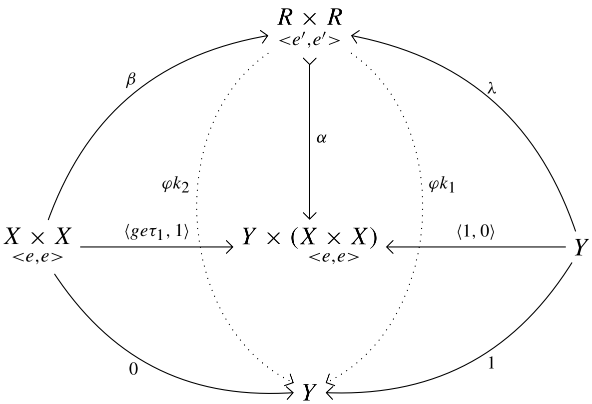
where u and v are respective factorizations of $\langle ge, 1\rangle$ and $\langle 1, 0\rangle$ through $\langle r_1, r_2\rangle$, $\langle s_1, s_2\rangle$ is the regular image of $\langle r_1, r_2\rangle$ along the morphism $1 \times e$, and

$$(\underset{<e',e'>}{R \times R}, k_1, k_2) \quad \text{and} \quad (\underset{<e,e>}{X \times X}, \tau_1, \tau_2)$$

are the kernel pair relations of e' and e respectively. Since e is a strong epimorphism, it is not difficult to see that $\langle g, 1\rangle$ factors through $\langle s_1, s_2\rangle$ by some morphism u'. Now the morphisms α, β, and λ are obtained as canonical morphisms between kernel pairs. It can be seen that

$$\varphi k_1 \beta = \varphi u \tau_1 = 0 = \varphi k_2 \beta \quad \text{and} \quad \varphi k_1 \lambda = \varphi v = 1 = \varphi k_2 \lambda.$$

Now we have obtained the following commutative diagram



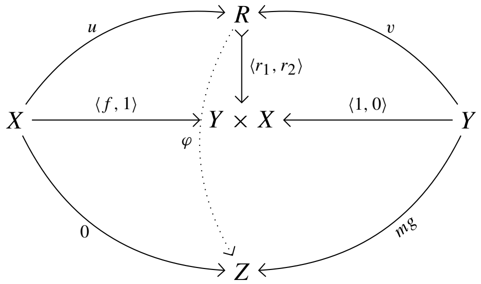
from which the uniqueness of partial subtractors implies $\varphi k_1 = \varphi k_2$. But since e' is the coequalizer of the pair k_1, k_2, there is a unique morphism $\phi : S \longrightarrow Y$ such that $\varphi = \phi e'$, and it can be easily seen that ϕ is a partial subtractor of g. □

In the next proposition we establish, amongst other things, further "cancellation" properties of partial subtractors.

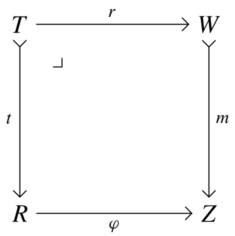
Proposition 4.2 *In a subtractive category* $\mathbb{C}$*, the following hold:*

(a) *If a morphism* $f : X \longrightarrow Y$ *admits a partial subtraction structure along a composite* $mg : Y \longrightarrow Z$*, where* $m : W \rightarrow Z$ *is a monomorphism and* $g : Y \rightarrow W$ *is any morphism, then* f *also admits a partial subtraction structure along* g*.*
(b) *If a composite* $mf : X \longrightarrow Z$*, where* $m : Y \longrightarrow Z$ *is a monomorphism and* $f : X \rightarrow Y$ *is any morphism, admits a partial subtraction structure then* f *also admits a partial subtraction structure.*
(c) *If a monomorphism* $f : X \rightarrowtail Y$ *admits a partial subtraction structure* $(f, (R, r_1, r_2), \varphi)$*, then* f *admits a subtractor.*

Proof (a) Suppose a morphism f admits a partial subtraction structure along a composite mg as shown in the diagram

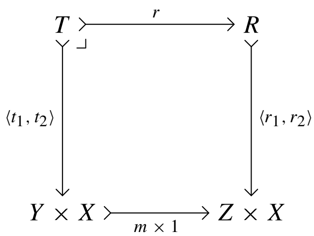


with m a monomorphism. Using the pullback diagram



it can be shown that r is the partial subtractor of f along g with respect to a subobject $\langle r_1, r_2\rangle t : T \rightarrowtail Y \times X$.

(b) If mf admits a partial subtraction structure $(mf, (R, r_1, r_2), \varphi)$, then f admits a partial subtraction structure along m; the composite φr, where r is the pullback of $m \times 1$ along $\langle r_1, r_2\rangle$ in the diagram



is a partial subtractor of f along m with respect to $\langle t_1, t_2 \rangle$. And applying (a) above, it follows that f admits a partial subtraction structure.

(c) Let $(f, (R, r_1, r_2), \varphi)$ be a partial subtraction structure on a monomorphism f. Applying (b) above and Lemma 3.2, the identity morphism of X admits a subtraction $s : X \times X \longrightarrow X$. Now consider the relation $\langle t_1, t_2 \rangle : T \rightarrowtail Y \times (X \times X)$, obtained from pulling back $\langle r_1, r_2 \rangle : R \rightarrowtail Y \times X$ along $1 \times s : Y \times (X \times X) \longrightarrow Y \times X$. Using generalized elements, T is defined as follows: for $y \in Y$ and $x, x' \in X$,

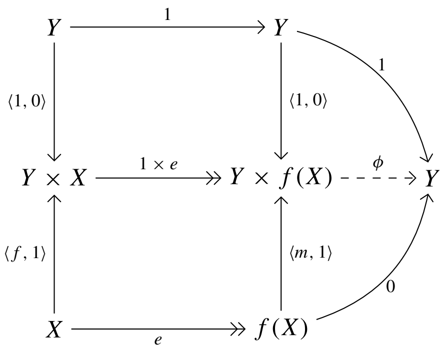
$$(y, (x, x')) \in T \Leftrightarrow (y, s(x, x')) \in R.$$

For every pair $(y, x) \in Y \times X$, since $(y, (x, x)) \in T$ and $(0, (0, x)) \in T$ (since for every $x \in X$, $(f(x), x), (f(x), 0) \in R$ imply $(0, x) \in R$), then by subtractivity, $(y, (x, 0)) \in T$, and this implies $(y, s(x, 0)) = (y, x) \in R$. Hence, $\langle r_1, r_2 \rangle$ is an isomorphism. It can now be easily shown that $\varphi \langle r_1, r_2 \rangle^{-1}$, where $\langle r_1, r_2 \rangle^{-1}$ denotes the inverse of $\langle r_1, r_2 \rangle$, is a subtractor of f. □

Note that by taking $f : X \longrightarrow Y$ to be the identity morphism of X in Proposition 4.2(c), one recovers Lemma 3.2. In the context of a regular subtractive category, the following proposition generalizes Lemma 3.2.

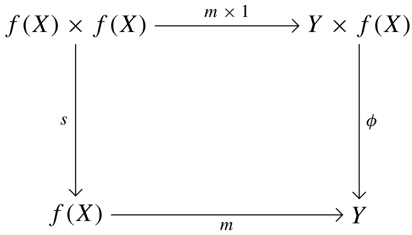
Proposition 4.3 *Let $\mathbb{C}$ be a regular subtractive category. A morphism $f : X \longrightarrow Y$ admits a subtractor as soon as it admits a partial subtraction structure.*

Proof Let $(f, (R, r_1, r_2), \varphi)$ be a partial subtraction structure on f, and $f = me$ be the (regular epi, mono)-factorization of f. Using Proposition 4.1, it can be concluded that the image $m : f(X) \rightarrowtail Y$ admits a partial subtraction structure. By further applying Proposition 4.2(c), it follows that m admits a subtractor ϕ. Hence, as seen in the diagram



the morphism $\phi(1 \times e)$ is a subtractor of f. □

Remark 4.1 As observed in the previous proposition, for a morphism $f : X \longrightarrow Y$ in a regular subtractive category $\mathbb{C}$, with (regular epi, mono)-factorization $f = me$, when it admits a subtractor φ, the image $m : f(X) \rightarrowtail Y$ also admits a subtractor ϕ, and $\varphi = \phi(1 \times e)$. In addition, $f(X)$ admits a subtraction $s : f(X) \times f(X) \longrightarrow f(X)$, and it is not difficult to see that the two composites $f(X) \times f(X) \longrightarrow Y$ on the rectangle



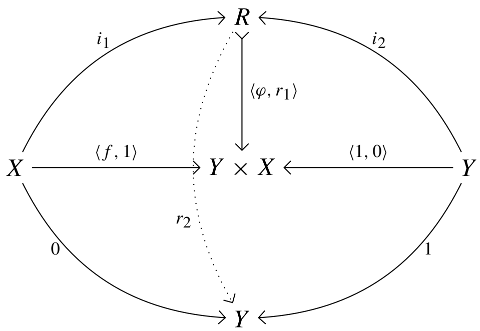
are both subtractors of the identity morphism of $f(X)$ along m. Hence by the uniqueness of subtractors (Lemma 4.1), $ms = \phi(m \times 1)$.

Now we are ready to state the main result of this section.

Theorem 4.1 *In a regular subtractive category* $\mathbb{C}$*, the following statements are equivalent for a morphism* $f : X \longrightarrow Y$:

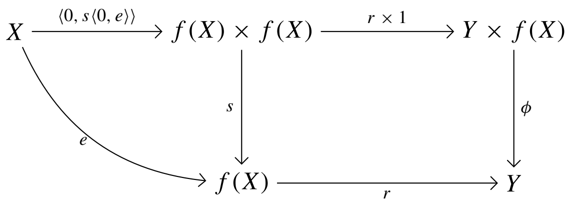
(a) f *is partially central.*
(b) f *admits a subtractor.*
(c) f *is central.*

Proof $(a) \Rightarrow (b)$ Let $f : X \longrightarrow Y$ be partially central with respect to a subobject $\langle r_1, r_2 \rangle : R \rightarrowtail X \times Y$, and with cooperator $\varphi : R \longrightarrow Y$. Furthermore, let i_1, i_2 be morphisms such that $\langle r_1, r_2 \rangle i_1 = \langle 1, 0 \rangle$, $\langle r_1, r_2 \rangle i_2 = \langle 0, 1 \rangle$, $\varphi i_1 = f$, and $\varphi i_2 = 1$. According to Lemma 3.1, the morphism $\langle \varphi, r_1 \rangle$ is a monomorphism. Therefore, as shown in the commutative diagram



f admits a partial subtraction structure, and hence, by applying Proposition 4.3, f admits a subtractor.

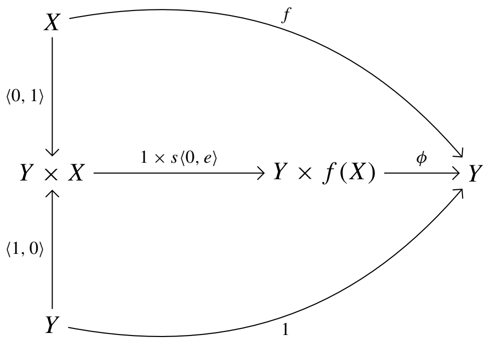
$(b) \Rightarrow$ (c) Let $f = re$ be the (regular epi, mono)-factorization of f. If f admits a subtractor, then according to Proposition 4.1, $r : f(X) \rightarrowtail Y$ admits a subtractor $\phi : Y \times f(X) \longrightarrow Y$, and so the image $f(X)$ also admits a subtraction $s : f(X) \times f(X) \longrightarrow f(X)$ by applying Proposition 4.2(b). As seen in Remark 4.1, $\phi(r \times 1) = rs$ and this means the rectangle in the diagram



commutes. Furthermore, using the fact that s is a homomorphism of subtraction, we have

$$\begin{aligned} s\langle 0, s\langle 0, e\rangle\rangle &= s\langle s\langle 1, 1\rangle, s\langle 0, 1\rangle\rangle e \\ &= s\langle s\langle 1, 0\rangle, s\langle 1, 1\rangle\rangle e \\ &= s\langle 1, 0\rangle e \\ &= e, \end{aligned}$$

which means the left triangle commutes. Using the commutativity of the previous diagram, since $f = re = \phi(r \times 1)\langle 0, s\langle 0, e\rangle\rangle = \phi\langle 0, s\langle 0, e\rangle\rangle$, we see that the diagram



commutes. Hence f is central.

$(c) \Rightarrow (a)$ Clearly, if f is central then it is partially central. □

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Data availability Not applicable.

Declarations

Conflict of interest The author declare that there are no competing interests.

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