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RESEARCH

Closed and Open Maps for Partial Frames

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Abstract
This paper concerns the notions of closed and open maps in the setting of partial frames, which, in contrast to full frames, do not necessarily have all joins. Examples of these include bounded distributive lattices, σ- and κ-frames and full frames. We define closed and open maps using geometrically intuitively appealing conditions involving preservation of closed, respectively open, congruences under certain maps. We then characterize them in terms of algebraic identities involving adjoints. We note that partial frame maps need have neither right nor left adjoints whereas frame maps of course always have right adjoints. The embedding of a partial frame in either its free frame or its congruence frame has proved illuminating and useful. We consider the conditions under which these embeddings are closed, open or skeletal. We then look at preservation and reflection of closed or open maps under the functors providing the free frame or the congruence frame. Points arise naturally in the construction of the spectrum functor for partial frames to partial spaces. They may be viewed as maps from the given partial frame to the 2-chain or as certain kinds of filters; using the former description we consider closed and open points. Any point of a partial frame extends naturally to a point on its free frame and a point on its congruence frame; we consider the closedness or openness of these.

Keywords Frame · Partial frame · $\mathcal{S}$-frame · κ-Frame · σ-Frame · Free frame over partial frame · Congruence frame · Right adjoint · Left adjoint · Closed map · Open map · Points

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1 Introduction

Topological spaces and frames or locales have been contexts in which closed and open maps have proved to be important tools. For topological spaces, for example, the second projection from $X \times Y$ to Y is closed for any space Y iff X is compact. Another example states that if there exists an open map $f : X \to Y$ of a locally compact space X onto a Hausdorff space Y, then Y is locally compact. The uses to which closed and open mappings, respectively, have been put are somewhat different, but in this paper we begin by emphasizing their formally similar properties.

Our context will be that of partial frames. Partial frames are meet-semilattices where, in contrast with frames, not all subsets need have joins. A selection function, $\mathcal{S}$, specifies, for all meet-semilattices, certain subsets under consideration, which we call the "designated" ones; an $\mathcal{S}$-frame then must have joins of (at least) all such subsets and binary meet must distribute over these. A small collection of axioms suffices to specify our selection functions; these axioms are sufficiently general to include as examples of partial frames, bounded distributive lattices, σ-frames, κ-frames and frames.

In this paper we provide definitions of closed and open maps between partial frames. We use a geometrically intuitively appealing condition involving preservation of closed, respectively open, congruences under certain maps. We then characterize them in terms of algebraic identities involving adjoints. We follow the ideas of Chen (see [6]) whose work we gratefully acknowledge. We note however that partial frame maps need have neither right nor left adjoints whereas frame maps of course always have right adjoints.

A weakening of the notion of an open map leads to the idea of a skeletal map; these played an important rôle in out study of the Madden quotient in [12] and arise here also.

The embedding of a partial frame in either its free frame or its congruence frame has proved illuminating and useful. (See [9, 12].) We consider the conditions under which these embeddings are closed, open or skeletal. We then look at preservation and reflection of closed or open maps under the functors providing the free frame or the congruence frame.

Points arise naturally in the construction of the spectrum functor for partial frames to partial spaces. (See [13].) They may be viewed as maps from the given partial frame to the 2-chain or as certain kinds of filters; using the former description we consider closed and open points. Any point of a partial frame extends naturally to a point on its free frame and a point on its congruence frame; we consider the closedness or openness of these.

2 Background

See [16, 23] as references for frame theory; see [2, 3] for σ-frames; see [20, 21] for κ-frames; see [1, 19] for general category theory.

The basics of our approach to partial frames can be found in [7–9]. An example of a paper of ours with a more topological flavour is [11]. Our papers with a more algebraic flavour, especially relevant to the current topic, are [10, 12, 14, 15]. For earlier work by other authors in this field see [22, 24–26]. For those interested in a comparison of the various approaches, see [8].

A *meet-semilattice* is a partially ordered set in which all finite subsets have a meet. In particular, we regard the empty set as finite, so a meet-semilattice comes equipped with a top element, which we denote by 1. We do not insist that a meet-semilattice should have a bottom element, which, if it exists, we denote by 0. A function between meet-semilattices

$f : L \to M$ is a *meet-semilattice map* if it preserves finite meets, including the top element. A *sub meet-semilattice* is a subset for which the inclusion map is a meet-semilattice map.

The essential idea for a *partial frame* is that it should be "frame-like" but that not all joins need exist; only certain joins have guaranteed existence and binary meets should distribute over these joins. The guaranteed joins are specified in a global way on the category of meet-semilattices by specifying what is called a selection function; the details are given below.

Definition 2.1 A *selection function* is a rule, which we usually denote by $\mathcal{S}$, which assigns to each meet-semilattice A a collection $\mathcal{S}A$ of subsets of A, such that the following conditions hold (for all meet-semilattices A and B):

(S1) For all $x \in A$, $\{x\} \in \mathcal{S}A$.
(S2) If $G, H \in \mathcal{S}A$ then $\{x \wedge y : x \in G, y \in H\} \in \mathcal{S}A$.
(S2)′ If $G, H \in \mathcal{S}A$ then $\{x \vee y : x \in G, y \in H\} \in \mathcal{S}A$.
(S3) If $G \in \mathcal{S}A$ and, for all $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}A$, then

$$\bigcup_{x \in G} H_x \in \mathcal{S}A.$$

(S4) For any meet-semilattice map $f : A \to B$,

$$\mathcal{S}(f[A]) = \{f[G] : G \in \mathcal{S}A\} \subseteq \mathcal{S}B.$$

(SSub) For any sub meet-semilattice B of meet-semilattice A, if $G \subseteq B$ and $G \in \mathcal{S}A$, then $G \in \mathcal{S}B$.
(SFin) If F is a finite subset of A, then $F \in \mathcal{S}A$.
(SCov) If $G \subseteq H$ and $H \in \mathcal{S}A$ with $\bigvee H = 1$ then $G \in \mathcal{S}A$. (Such H are called $\mathcal{S}$-*covers*.)
(SRef) Let $X, Y \subseteq A$. If $X \leq Y$ with $X \in \mathcal{S}A$ there is a $C \in \mathcal{S}A$ such that $X \leq C \subseteq Y$. (By $X \leq Y$ we mean, as usual, that for each $x \in X$ there exists $y \in Y$ such that $x \leq y$.)

Of course (SFin) implies (S1) but there are situations where we do not impose (SFin) but insist on (S1). Note that we always have $\emptyset \in \mathcal{S}A$.
Once a selection function, $\mathcal{S}$, has been fixed, we speak informally of the members of $\mathcal{S}A$ as the *designated* subsets of A.

Definition 2.2 An $\mathcal{S}$-frame L is a meet-semilattice in which every designated subset has a join and for any such designated subset B of L and any $a \in L$

$$a \wedge \bigvee B = \bigvee_{b \in B} a \wedge b.$$

Of course such an $\mathcal{S}$-frame has both a top and a bottom element which we denote by 1 and 0 respectively.
A meet-semilattice map $f : L \to M$, where L and M are $\mathcal{S}$-frames, is an $\mathcal{S}$-frame *map* if $f(\bigvee B) = \bigvee_{b \in B} f(b)$ for any designated subset B of L. In particular such an f preserves the top and bottom element. If, in addition, $f(x) = 0$ implies that $x = 0$, we say that f is *dense*.
A *sub* $\mathcal{S}$-frame T of an $\mathcal{S}$-frame L is a subset of L such that the inclusion map $i : T \to L$ is an $\mathcal{S}$-frame map.
The category $\mathcal{S}$**Frm** has $\mathcal{S}$-frames as objects and $\mathcal{S}$-frame maps as arrows.

We use the terms "partial frame" and "$\mathcal{S}$-frame" interchangeably, especially if no confusion about the selection function is likely. We also use the term *full frame* in situations where we wish to emphasize that all joins exist.

Note 2.3 Here are some examples of different selection functions and their corresponding $\mathcal{S}$-frames.

1. In the case that all joins are specified, we are of course considering the notion of a frame.
2. In the case that (at most) countable joins are specified, we have the notion of a σ-frame.
3. In the case that joins of subsets with cardinality less than some (regular) cardinal κ are specified, we have the notion of a κ-frame.
4. In the case that only finite joins are specified, we have the notion of a bounded distributive lattice.

The remainder of this section gives a lot of information about $\mathcal{H}_{\mathcal{S}}L$, the free frame over the $\mathcal{S}$-frame L, as well as $\mathcal{C}_{\mathcal{S}}L$, the frame of $\mathcal{S}$-congruences of L, and the relationship between the two. These results come from [9] on $\mathcal{H}_{\mathcal{S}}L$, [10, 12, 14] on $\mathcal{C}_{\mathcal{S}}L$.

In the definition below, L is an $\mathcal{S}$-frame.

Definition 2.4 (a) A subset J of L is an $\mathcal{S}$-*ideal of* L if J is a non-empty downset closed under designated joins (the latter meaning that if $X \subseteq J$, for X a designated subset of L, then $\bigvee X \in J$).

(b) The collection of all $\mathcal{S}$-ideals of L will be denoted $\mathcal{H}_{\mathcal{S}}L$, and called the $\mathcal{S}$-ideal *frame of* L. It is in fact the free frame over L.

(c) We call $\theta \subseteq L \times L$ an $\mathcal{S}$-congruence *on* L if it satisfies the following:

- (C1) θ is an equivalence relation on L.
- (C2) $(a, b), (c, d) \in \theta$ implies that $(a \wedge c, b \wedge d) \in \theta$.
- (C3) If $\{(a_\alpha, b_\alpha) : \alpha \in \mathcal{A}\} \subseteq \theta$ and $\{a_\alpha : \alpha \in \mathcal{A}\}$ and $\{b_\alpha : \alpha \in \mathcal{A}\}$ are designated subsets of L, then $(\bigvee_{\alpha\in\mathcal{A}} a_\alpha, \bigvee_{\alpha\in\mathcal{A}} b_\alpha) \in \theta$.

(d) The collection of all $\mathcal{S}$-congruences on L is denoted by $\mathcal{C}_{\mathcal{S}}L$; we refer to it as the congruence frame of L. It is in fact a full frame with meet given by intersection.

(e) (i) Let $A \subseteq L \times L$. We use the notation $\langle A \rangle$ to denote the smallest $\mathcal{S}$-congruence containing A. This exists by completeness of $\mathcal{C}_{\mathcal{S}}L$.
- (ii) For $a \in L$ we define $\nabla_a = \{(x, y) : x \vee a = y \vee a\}$ and $\Delta_a = \{(x, y) : x \wedge a = y \wedge a\}$; these are $\mathcal{S}$-congruences on L.
- (iii) It is easily seen that $\nabla_a = \langle(0, a)\rangle$ and that $\Delta_a = \langle(a, 1)\rangle$.
- (iv) For $a \leq b$, it follows that $\Delta_a \cap \nabla_b = \langle(a, b)\rangle$.
- (v) The congruence $\nabla_1 = L \times L$ is the top element, also denoted ∇. Also $\nabla_0 = \{(x, x) : x \in L\}$ (called the *diagonal*, denoted Δ) is the bottom element of $\mathcal{C}_{\mathcal{S}}L$.

(f) The following hold in $\mathcal{C}_{\mathcal{S}}L$.

- (i) For any $\theta \in \mathcal{C}_{\mathcal{S}}L$, $\theta = \bigvee\{\nabla_b \wedge \Delta_a : (a, b) \in \theta, a \leq b\}$
- (ii) $\nabla_a \vee \theta = \{(x, y) : (x \vee a, y \vee a) \in \theta\}$
- (iii) $\Delta_a \vee \theta = \{(x, y) : (x \wedge a, y \wedge a) \in \theta\}$
- (iv) For any $I \in \mathcal{H}_{\mathcal{S}}L$, $\bigvee_{x\in I} \nabla_x = \bigcup\limits_{x\in I} \nabla_x$.

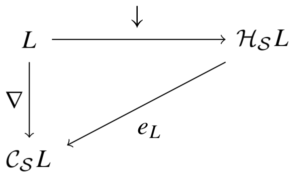
(g) The function $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ given by $\nabla(a) = \nabla_a$ is an $\mathcal{S}$-frame embedding. It has the universal property that if $f : L \to M$ is an $\mathcal{S}$-frame map into a frame M with complemented image, then there exists a frame map $\bar{f} : \mathcal{C}_{\mathcal{S}}L \to M$ such that $f = \bar{f} \circ \nabla$.

(h) We also note that for frame maps f and g with domain $\mathcal{C}_{\mathcal{S}}L$, if $f \circ \nabla = g \circ \nabla$ then $f = g$.

(i) A useful congruence for our purposes is the *Madden* congruence, π, described below.

   (i) For $x \in L$, set $P_x = \{t \in L : t \wedge x = 0\}$.
   (ii) For $x \in L$, P_x is an $\mathcal{S}$-ideal, and in $\mathcal{H}_{\mathcal{S}}L$, $P_x = (\downarrow x)^*$, the pseudocomplement of $\downarrow x$.
   (iii) We define $\pi = \{(x, y) : P_x = P_y\}$; π is an $\mathcal{S}$-congruence of L.
   (iv) The quotient map induced by the Madden congruence, $p : L \to L/\pi$ is dense, onto and the universal such. We refer to this as the *Madden quotient*. (See [12].)

Definition 2.5 For any $\mathcal{S}$-frame L, define $e_L : \mathcal{H}_{\mathcal{S}}L \to \mathcal{C}_{\mathcal{S}}L$ to be the unique frame map such that $e_L(\downarrow a) = \nabla_a$ for all $a \in L$; that is, making the following diagram commute:



That this map e_L exists follows from the freeness of $\mathcal{H}_{\mathcal{S}}L$ as a frame over L. See [9]. Where no confusion can arise, we omit the subscript L.

Remark 2.6 The range of the map $e_L : \mathcal{H}_{\mathcal{S}}L \to \mathcal{C}_{\mathcal{S}}L$ mentioned above consists of all the $\mathcal{S}$-congruences of L that can be written as arbitrary joins of ∇_a's, for $a \in L$. For a proof, see [14]. In the same place we show the following:

Note 2.7 For any $\mathcal{S}$-frame L, $\mathcal{H}_{\mathcal{S}}L$ is isomorphic to a subframe of $\mathcal{C}_{\mathcal{S}}L$; that is, the free frame over L is isomorphic to a subframe of the frame of $\mathcal{S}$-congruences of L.

3 Right and Left Adjoints

Much of what is mentioned in this section is well known but it is as well to remind the reader that, since $\mathcal{S}$-frames are in general not complete, some of what is used in frame theory concerning adjoints needs to be reconsidered.

Definition 3.1 Let $h : L \to M$ be an $\mathcal{S}$-frame map.
A function $r : M \to L$ is a *right adjoint* of h if

$$h(x) \leq m \iff x \leq r(m) \text{ for all } x \in L, m \in M.$$

A function $\ell : M \to L$ is a *left adjoint* of h if

$$\ell(m) \leq x \iff m \leq h(x) \text{ for all } x \in L, m \in M.$$

We make no claim that all $\mathcal{S}$-frame maps have right or left adjoints; this is false (see Example 3.4). However, clearly if an $\mathcal{S}$-frame map *has* a right or left adjoint, such is unique. The following are well-known.

Lemma 3.2 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(a) *Suppose that h has a right adjoint r. Then:*
(i) $hr \leq id_M$ and $rh \geq id_L$, (ii) h is one-one $\iff rh = id_L$ and (iii) h is onto $\iff hr = id_M$.
(b) *Suppose that h has a left adjoint ℓ. Then:*
(i) $id_M \leq h\ell$ and $id_L \geq \ell h$. (ii) h is one-one $\iff lh = id_L$ and (iii) h is onto $\iff h\ell = id_M$.

Since arbitrary meets or joins in $\mathcal{S}$-frames need not exist, the following prove useful.

Lemma 3.3 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(a) *If h has a right adjoint r then, for all $m \in M$*

$$r(m) = \bigvee \{x \in L : h(x) \leq m\}.$$

(b) *If h has a left adjoint ℓ then, for all $m \in M$*

$$\ell(m) = \bigwedge \{x \in L : m \leq h(x)\}.$$

Proof (a) By definition, $h(x) \leq m \iff x \leq r(m)$. Now $r(m)$ is an upper bound of $\{x \in L : h(x) \leq m\}$.
Let k be any upper bound of $\{x \in L : h(x) \leq m\}$. Since $h(r(m)) \leq m$, it follows that $r(m) \leq k$, so $r(m)$ is indeed the least upper bound required.
(b) Similar.

□

Example 3.4 This is an example of an (onto) $\mathcal{S}$-frame map which has neither a right nor a left adjoint.

Let L be the σ-frame consisting of all countable and co-countable subsets of $\mathbb{R}$, and $\mathbb{2}$ denote the 2-element chain. Define $h : L \to \mathbb{2}$ by $h(C) = 0$ if C is countable and $h(D) = 1$ if D is co-countable. Then h is a σ-frame map. However it has no right adjoint since there is no largest $A \in L$ with $h(A) = 0$. Similarly it has no left adjoint. (See Lemma 3.3) □

Proposition 3.5 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(a) *Suppose that h has a right adjoint, r. Then h preserves all existing joins and r preserves all existing meets.*
(b) *Suppose that h has a left adjoint, ℓ. Then h preserves all existing meets and ℓ preserves all existing joins.*

Proof This is a categorical fact about adjoints but can of course be checked directly. □

We mention some cases in which existence of adjoints ensures completeness. This result will prove useful when embedding an $\mathcal{S}$-frame in its free frame or congruence frame.

Corollary 3.6 *Suppose that $h : L \to N$ is a one-one $\mathcal{S}$-frame map, where L is an $\mathcal{S}$-frame and N is a (full) frame. If h has a right or left adjoint, then L is a complete lattice.*

Proof Assume h has a right adjoint r and $\{x_j : j \in J\}$ an arbitrary subset of L. Since N is a frame, $\bigwedge\limits_{j \in J} h(x_j)$ exists. Then $r(\bigwedge\limits_{j \in J} h(x_j)) = \bigwedge\limits_{j \in J} rh(x_j) = \bigwedge\limits_{j \in J} x_j$, using Proposition 3.5 and Lemma 3.2. The proof for left adjoints works similarly. □

A possible converse to Corollary 3.6 fails as the following example shows.

Example 3.7 Let L consist of all countable subsets of $\mathbb{R}$, with $\mathbb{R}$ itself added as top element. Let N be the power set of $\mathbb{R}$ and $h : L \to N$ the identical embedding. Then L is a σ-frame and a complete lattice while N is a frame, and h is a one-one σ-frame map. However h does not preserve all existing joins. This is demonstrated by considering the set $\{\{i\} : i \text{ is irrational}\}$:

$$h(\bigvee\{\{i\} : i \text{ is irrational}\}) = h(\mathbb{R}) = \mathbb{R} \text{ but } \bigvee\{h(\{i\}) : i \text{ is irrational}\}$$
$$= \bigcup\{\{i\} : i \text{ is irrational}\} \neq \mathbb{R}$$

So, by Proposition 3.5(a), h does not have a right adjoint. □

Considering the σ-frame consisting of all co-countable subsets of $\mathbb{R}$ with the empty set added as bottom element, provides a similar example, this time of a function with no left adjoint.

4 Closed and Open Maps

There are various equivalent characterizations of closed and open maps for frames, see for example [23] or [6]. For partial frames we choose a definition using closed and open congruences and then provide an equivalent approach using right and left adjoints.

Definition 4.1 Let $h : L \to M$ be an $\mathcal{S}$-frame map. We call h *closed* if, for all $m \in M$, there exists $x \in L$ with $(h \times h)^{-1}(\nabla_m) = \nabla_x$.
We call h *open* if, for all $m \in M$, there exists $x \in L$ with $(h \times h)^{-1}(\Delta_m) = \Delta_x$.

The above corresponds with one's intuition for topological spaces and closed and open continuous functions: for example, such a function $f : X \to Y$ is open if and only if $(h \times h)^{-1}(\Delta_U) = \Delta_{f[U]}$ for $h = f^{-1}$ and U open in X.

We note that (see [12]) $\mathcal{C}_\mathcal{S}$ is a functor from $\mathcal{S}$-frames to frames such that, for any $\mathcal{S}$-frame map $h : L \to M$ we have a frame map $\mathcal{C}_\mathcal{S}h : \mathcal{C}_\mathcal{S}L \to \mathcal{C}_\mathcal{S}M$ making the following diagram commute:

$$\begin{array}{ccc} L & \xrightarrow{\nabla_L} & \mathcal{C}_\mathcal{S}L \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathcal{C}_\mathcal{S}h} \\ M & \xrightarrow[\nabla_M]{} & \mathcal{C}_\mathcal{S}M \end{array}$$

Now $(h \times h)^{-1}$ is the right adjoint of $\mathcal{C}_\mathcal{S}h$, because, for $\theta \in \mathcal{C}_\mathcal{S}L$, $\mathcal{C}_\mathcal{S}h(\theta)$ is the $\mathcal{S}$-congruence of M generated by $(h \times h)[\theta]$, so for all $\theta \in \mathcal{C}_\mathcal{S}L$, $\phi \in \mathcal{C}_\mathcal{S}M$,

$$\mathcal{C}_\mathcal{S}h(\theta) \subseteq \phi \iff \theta \subseteq (h \times h)^{-1}(\phi).$$

In particular, notice that $\mathcal{C}_\mathcal{S}h(h \times h)^{-1}(\phi) \subseteq \phi$ for all $\phi \in \mathcal{C}_\mathcal{S}M$.

We now provide characterizations of closed and open maps in which right and left adjoints arise naturally.

Theorem 4.2 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(a) *The map h is closed iff h has a right adjoint, r, and for all $x \in L, m \in M$,*

$$r(h(x) \vee m) = x \vee r(m).$$

(b) *The map h is open iff h has a left adjoint, ℓ, and for all $x \in L, m \in M$,*

$$\ell(h(x) \wedge m) = x \wedge \ell(m).$$

Proof (a) ($\Rightarrow$) Suppose h is closed. Then, for all $m \in M$, there exists $x_m \in L$ such that $(h \times h)^{-1}(\nabla_m) = \nabla_{x_m}$. Define $r(m) = x_m$. This is clearly well defined, since $\nabla_a = \nabla_b$ implies that $a = b$. We now check that r is indeed a right adjoint of h. For $x \in L, m \in M$:

$$\begin{aligned} h(x) \leq m &\iff \nabla_{h(x)} \leq \nabla_m \\ &\iff \mathcal{C}_{\mathcal{S}} h(\nabla_x) \leq \nabla_m \\ &\iff \nabla_x \subseteq (h \times h)^{-1}(\nabla_m) \\ &\iff \nabla_x \subseteq \nabla_{x_m} \\ &\iff x \leq x_m = r(m) \end{aligned}$$

Next we check that, for all $x \in L, m \in M$ we get $r(h(x) \vee m) = x \vee r(m)$.
Let $x \in L, m \in M$. Since $hr \leq id_M$, we have $hr(h(x) \vee m) \leq h(x) \vee m$, so $hr(h(x) \vee m) \vee m \leq h(x) \vee m$. On the other hand, note that $h(x) \leq h(x) \vee m \Rightarrow x \leq r(h(x) \vee m) \Rightarrow h(x) \leq hr(h(x) \vee m)$, and so $h(x) \vee m \leq hr(h(x) \vee m) \vee m$. This establishes that $(r(h(x) \vee m), x) \in (h \times h)^{-1}(\nabla_m) = \nabla_{r(m)}$ by assumption, so we obtain $r(h(x) \vee m) \vee r(m) = x \vee r(m)$, so $r(h(x) \vee m) = x \vee r(m)$.
($\Leftarrow$) Under the given assumptions, we show that for all $m \in M$, $(h \times h)^{-1}(\nabla_m) = \nabla_{r(m)}$.
For $x, y \in L$

$$\begin{aligned} (x, y) \in (h \times h)^{-1}(\nabla_m) &\Rightarrow h(x) \vee m = h(y) \vee m \\ &\Rightarrow r(h(x) \vee m) = r(h(y) \vee m) \\ &\Rightarrow x \vee r(m) = y \vee r(m) \\ &\Rightarrow (x, y) \in \nabla_{r(m)} \end{aligned}$$

and

$$\begin{aligned} (x, y) \in \nabla_{r(m)} &\Rightarrow x \vee r(m) = y \vee r(m) \\ &\Rightarrow h(x) \vee hr(m) = h(y) \vee hr(m) \\ &\Rightarrow h(x) \vee m = h(y) \vee m \text{ since } hr(m) \leq m \\ &\Rightarrow (x, y) \in (h \times h)^{-1}(\nabla_m) \end{aligned}$$

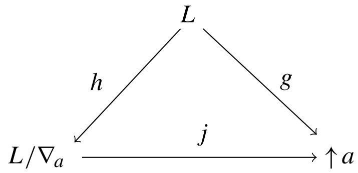
(b) Similar.

□

Note 4.3 Note that if r is the right adjoint of an $\mathcal{S}$-frame map $h : L \to M$, then for all $x \in L, m \in M$, we necessarily have $r(h(x) \vee m) \geq x \vee r(m)$, so only the reverse inequality is significant. Similarly if ℓ is the left adjoint of h then again we necessarily have $\ell(h(x) \wedge m) \leq x \wedge \ell(m)$.

Note 4.4 Let L be an $\mathcal{S}$-frame and $a \in L$. The following is well-known in the case of frames; for future reference we describe quotients using closed congruences explicitly.

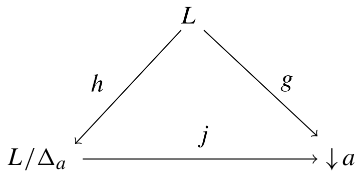
(a) Consider the diagram



Here $h(x) = [x]_{\nabla_a}$, $g(x) = x \vee a$, $j([x]) = x \vee a$. We note that j is an isomorphism making this diagram commute. The right adjoints of h and g are given explicitly by

$$r_h([x]) = x \vee a \text{ for } x \in L \text{ and } r_g(t) = t \text{ for } t \in \uparrow a.$$

(b) A similar situation applies for open quotients with $L/\Delta_a \simeq \downarrow a$. Here are the details. Consider the diagram



Here $h(x) = [x]_{\Delta_a}$, $g(x) = x \wedge a$, $j([x]) = x \wedge a$. Again j is an isomorphism making this diagram commute. The left adjoints of h and g are given explicitly by

$$\ell_h([x]) = x \wedge a \text{ for } x \in L \text{ and } \ell_g(t) = t \text{ for } t \in \downarrow a.$$

The following natural and useful results show that closed maps and closed congruences are related appropriately as are the open ones.

Theorem 4.5 *Let L be an $\mathcal{S}$-frame and θ an $\mathcal{S}$-congruence on L.*

(a) *The quotient map $q : L \to L/\theta$ is closed if and only if θ is a closed $\mathcal{S}$-congruence; that is, $\theta = \nabla_a$ for some $a \in L$.*
(b) *The quotient map $q : L \to L/\theta$ is open if and only if θ is an open $\mathcal{S}$-congruence; that is, $\theta = \Delta_a$ for some $a \in L$.*

Proof (a) ($\Rightarrow$) If q is closed, then $(q \times q)^{-1}(\nabla_0) = \nabla_{r(0)}$, where r is the right adjoint of q. But $(q \times q)^{-1}(\nabla_0)$ is just $\ker(q)$ and $\ker q = \theta$, so we obtain $\theta = \nabla_{r(0)}$.
($\Leftarrow$) If $\theta = \nabla_a$ then L/θ is isomorphic to $\uparrow a$; we use $h(x) = x \vee a$. By Note 4.4 the right adjoint of h is the identical embedding of $\uparrow a$ into L and, for $m \geq a$, $(h \times h)^{-1}(\nabla_m) = \nabla_m$.

The proof of (b) is similar to that of (a). For ($\Rightarrow$) one obtains $\theta = \Delta_{\ell(1)}$. □

The next two results provide some basic facts concerning closed and open maps.

Lemma 4.6 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(a) *If h is dense and closed, then h is one-one.*
If h is dense, closed and onto, then h is an isomorphism.
(b) *If h is codense and open, then h is one-one.*
If h is codense, open and onto, then h is an isomorphism.

Proof (a) Suppose that $h(x) = h(y)$ for some $x, y \in L$. Then $(x, y) \in (h \times h)^{-1}(\nabla_0) = \nabla_{r(0)}$, since h is closed. (Here r is the right adjoint of h.) So $x \vee r(0) = y \vee r(0)$. But if h is dense, $r(0) = 0$, so $x = y$.

(b) Similar.

□

The following result for frames appears in [6].

Lemma 4.7 *Suppose that $f : L \to M$ and $g : M \to N$ are $\mathcal{S}$-frame maps.*

(a) (i) *If f and g are both closed, then $g \circ f$ is closed.*
(ii) *If $g \circ f$ is closed and g is one-one, then f is closed.*
(iii) *If $g \circ f$ is closed and f is onto, then g is closed.*

(b) *As above but replace "closed" by "open".*

Proof (a) i) Clear from the definition of closed maps.

ii) By assumption, for all $n \in N$, $(gf \times gf)^{-1}(\nabla_n) = \nabla_{r(n)}$ where r is the right adjoint of gf. Now for $m \in M$ and $x, y \in L$,

$$\begin{aligned}(x, y) \in (f \times f)^{-1}(\nabla_m) &\iff f(x) \vee m = f(y) \vee m \\ &\iff g(f(x) \vee m) = g(f(y) \vee m) \text{ since } g \text{ is one-one} \\ &\iff (x, y) \in (gf \times gf)^{-1}\nabla_{g(m)} \\ &\iff (x, y) \in \nabla_{rg(m)}\end{aligned}$$

Thus f is closed. We note that (as can of course be checked directly) the right adjoint of f is rg.

iii) By assumption, for all $n \in N$, $(gf \times gf)^{-1}(\nabla_n) = \nabla_{r(n)}$, where r is the right adjoint of gf. Using f onto we need only consider pairs of the form $(f(x), f(y))$ for $x, y \in L$:

$$\begin{aligned}(f(x), f(y)) \in (g \times g)^{-1}(\nabla_n) &\iff gf(x) \vee n = gf(y) \vee n \\ &\iff x \vee r(n) = y \vee r(n) \\ &\Rightarrow \quad (f(x), f(y)) \in \nabla_{fr(n)}\end{aligned}$$

Also

$$\begin{aligned}(f(x), f(y)) \in \nabla_{fr(n)} &\Rightarrow f(x) \vee fr(n) = f(y) \vee fr(n) \\ &\Rightarrow gf(x) \vee gfr(n) = gf(y) \vee gfr(n) \\ &\Rightarrow gf(x) \vee n = gf(y) \vee n \text{ since } gfr(n) \leq n \\ &\Rightarrow (f(x), f(y)) \in (g \times g)^{-1}(\nabla_n)\end{aligned}$$

Thus g is closed. We note that the right adjoint of g is fr.

(b) Similar proof.

□

Note 4.8 Theorem 3.2.1 of [6] states that any frame map with a Boolean domain is closed. The corresponding statement for $\mathcal{S}$-frames is false, as Example 3.4 shows.

In Theorem 3.5 of [15] we noted that if $\mathcal{H}_{\mathcal{S}}L$ is Boolean then L is a Boolean frame but not conversely. The counterexample used there also shows that an $\mathcal{S}$-frame map with Boolean frame as domain need not be closed. However, an application of Lemma 4.7 shows that, if $\mathcal{H}_{\mathcal{S}}L$ is Boolean, any $\mathcal{S}$-frame map with domain L is closed.

The following result, due to [6] in the frame case, characterizes closed and open maps with domain $\mathfrak{Z}$.

Lemma 4.9 *Let $\mathfrak{Z}$ denote the 3-element frame with middle element μ, and let M denote any $\mathcal{S}$-frame.*

(a) *An $\mathcal{S}$-frame map $h : \mathfrak{Z} \to M$ is closed if and only if, for all $t \in M$, $h(\mu) \vee t = 1 \Rightarrow t = 1$.*
(b) *An $\mathcal{S}$-frame map $h : \mathfrak{Z} \to M$ is open if and only if, for all $t \in M$, $h(\mu) \wedge t = 0 \Rightarrow t = 0$.*

Proof (a) ($\Rightarrow$) Suppose that $h(\mu) \vee t = 1$ for some $t \in M$. By Theorem 4.2, $r(h(\mu) \vee t) = \mu \vee r(t)$, where r is the right adjoint of h. So $\mu \vee r(t) = 1$ which forces $r(t) = 1$, because $\mathfrak{Z}$ is a chain. So $t = 1$.
($\Leftarrow$) For $t \in M$, assume that $(h \times h)^{-1}(\nabla_t) \neq \nabla_0$ or ∇_1. We actually show that $(h \times h)^{-1}(\nabla_t) = \nabla_\mu$.
From our assumption, since $(h \times h)^{-1}(\nabla_t) \neq \nabla_1$, $(0, 1) \notin (h \times h)^{-1}(\nabla_t)$. Also $t \neq 1$. Since $(h \times h)^{-1}(\nabla_t) \neq \nabla_0$, either $(\mu, 1)$ or $(0, \mu)$ is in $(h \times h)^{-1}(\nabla_t)$.
Now, since $h(\mu) \vee t = h(1) \vee t$ implies $t = 1$ we must have $(\mu, 1) \notin (h \times h)^{-1}(\nabla_t)$. This forces $(0, \mu) \in (h \times h)^{-1}(\nabla_t)$; now it is clear that $\nabla_\mu \subseteq (h \times h)^{-1}(\nabla_t)$. For the reverse inclusion, use the fact that $(\mu, 1) \notin (h \times h)^{-1}(\nabla_t)$ again.
(b) Similar.

□

The final example of this section gives a simple example showing that maps that are both closed and open need not be isomorphisms.

Example 4.10 Let $\{L_i : i \in I\}$ be a collection of $\mathcal{S}$-frames, and $L = \prod L_i$ their product. The projection maps $p_j : L \to L_j$ are closed and open:
For $j \in I$, the right adjoint, r_j, of p_j is given by $r_j(x) = (a_\alpha)_{\alpha \in I}$ where $a_j = x$ and $a_k = 1$ for $k \neq j$. Then

$$r_j(p_j(a_\alpha) \vee b) = r_j(a_j \vee b) = (a_\alpha) \vee r_j(b).$$

Thus p_j is closed.
A similar idea shows that each p_j is open, using as left adjoint $\ell_j(x) = (a_\alpha)_{\alpha \in I}$ where $a_j = x$ and $a_k = 0$ for $k \neq j$. □

5 A Skeletal Interlude

Weakly open maps in the context of Boolean frames were considered by Banaschewski and Pultr in [4, 5]. Such maps are precisely the skeletal maps used in [17, 18]; the latter terminology is perhaps more widely known and we use it here. In [12] we considered the category of partial frames with skeletal maps and showed that the d-reduced partial frames form a reflective subcategory of this. In this paper we are interested in the fact that skeletal maps are a generalization of open ones.

Definition 5.1 An $\mathcal{S}$-frame map $h : L \to M$ is called *skeletal* if $(h \times h)[\pi_L] \subseteq \pi_M$ or equivalently $\pi_L \subseteq (h \times h)^{-1}(\pi_M)$. For the definition of the Madden congruences π_L, π_M see Definition 2.4.

Lemma 5.2 *An $\mathcal{S}$-frame map $h : L \to M$ is dense and skeletal if and only if $\pi_L = (h \times h)^{-1}(\pi_M)$.*

Proof The proof is straightforward and omitted. □

Lemma 5.3 *Every open $\mathcal{S}$-frame map is skeletal, but not conversely.*

Proof Suppose that $h : L \to M$ is open. By Theorem 4.2, h has a left adjoint ℓ and $l(h(x) \wedge m) = x \wedge \ell(m)$ for all $x \in L, m \in M$.
To show that h is skeletal, we assume $(x, y) \in \pi_L$ and show $(h(x), h(y)) \in \pi_M$. So suppose that $P_x = P_y$. Take $m \in M$ with $m \wedge h(x) = 0$. We show that $m \wedge h(y) = 0$. Applying ℓ gives $x \wedge l(m) = 0$, so $\ell(m) \in P_x = P_y$, so $y \wedge \ell(m) = 0$. Applying h gives $0 = h(y) \wedge h\ell(m) \geq h(y) \wedge m$, so $m \wedge h(y) = 0$.
Example 5.4 below shows that a skeletal map need not be open. □

Example 5.4 Let L be the σ-frame consisting of all countable subsets of $\mathbb{R}$ with $\mathbb{R}$ itself added as top element.
Let $h : L \to \mathbb{2}$ be given by $h(A) = 0$ for all countable A, and $h(\mathbb{R}) = 1$. Then h is a σ-frame map, with left adjoint ℓ given by $\ell(0) = \emptyset$ and $\ell(1) = \mathbb{R}$. However, for any countable A, $\ell(h(A)) = \ell(0) = \emptyset$ whereas $A \wedge \ell(1) = A \cap \mathbb{R} = A$, so h is not open. On the other hand, by [12], L is d-reduced, so every $\mathcal{S}$-frame map with domain L is skeletal. □

Lemma 5.5 *Let $\mathbb{3}$ denote the 3-element frame with middle element μ, and M any $\mathcal{S}$-frame. An $\mathcal{S}$-frame map $h : \mathbb{3} \to M$ is skeletal if and only if it is open.*

Proof For $L = \mathbb{3}$, we note that $\pi_L = \Delta \cup \{(\mu, 1), (1, \mu)\}$. So $h : L \to M$ is skeletal iff $(h(\mu), h(1)) \in \pi_M$; that is, $P_{h(\mu)} = P_1 = \{0\}$. However this is equivalent to the condition that, for all $t \in M$, $h(\mu) \wedge t = 0 \Rightarrow t = 0$. By Lemma 4.9, this is equivalent to h being open. □

Lemma 5.6 *Let $f : L \to M$ and $g : M \to N$ be $\mathcal{S}$-frame maps.*

(a) *If f, g are skeletal, so is gf.*
(b) *If gf is skeletal and g is one-one then f is skeletal.*
(c) *If gf is skeletal and g is dense then f is skeletal.*

Proof (a) See [12].
(b) See (c) below.
(c) Suppose $(x, y) \in \pi_L$. We show that $(f(x), f(y)) \in \pi_M$, that is $P_{f(x)} = P_{f(y)}$.
Take $s \in P_{f(x)}$. Then $s \wedge f(x) = 0$, so $g(s) \wedge g(f(x)) = 0$. So $g(s) \in P_{gf(x)}$. Since gf is skeletal, $g(s) \in P_{gf(y)}$. So $g(s) \wedge gf(y) = 0$; by density of g, $s \wedge f(y) = 0$, so $s \in P_{f(y)}$.

□

6 The Embedding of a Partial Frame into Its Free Frame and Its Congruence Frame

We investigate two important embeddings of a partial frame: first into its free frame and then into its congruence frame. In each case we characterize when the embeddings are closed, open and skeletal.

Proposition 6.1 *Let L be an $\mathcal{S}$-frame and $\downarrow: L \to \mathcal{H}_{\mathcal{S}}L$ the embedding into its free frame.*

(a) *The map $\downarrow$ has a right adjoint iff $\downarrow$ is an isomorphism.*

(b) *The map* $\downarrow$ *is closed iff* $\downarrow$ *is an isomorphism.*
(c) *The map* $\downarrow$ *has a left adjoint iff* L *is a complete lattice.*
(d) *The map* $\downarrow$ *is open iff* L *is a frame.*
(e) *The map* $\downarrow$ *is skeletal for any* L.

Proof (a) Suppose that $\downarrow$ has a right adjoint, r.
By Lemma 3.3, for all $I \in \mathcal{H}_{\mathcal{S}}L$, $r(I) = \bigvee\{x \in L : \downarrow x \subseteq I\} = \bigvee\{x \in L : x \in I\} = \bigvee I$. By Proposition 3.5, $\downarrow$ preserves all existing joins, so for all $I \in \mathcal{H}_{\mathcal{S}}L$, $\bigvee_{x \in I} \downarrow x = \downarrow (\bigvee I)$. This shows that all $\mathcal{S}$-ideals of L are principal, and so $\downarrow$ is an isomorphism.
(b) Follows directly from (a).
(c) ($\Rightarrow$) Corollary 3.6(b) applies, since $\downarrow$ is a one-one $\mathcal{S}$-frame map into the frame $\mathcal{H}_{\mathcal{S}}L$.
($\Leftarrow$) For $I \in \mathcal{H}_{\mathcal{S}}L$, define $\ell(I) = \bigvee I$. Then ℓ is a left adjoint of $\downarrow$, because $\bigvee I \leq x \iff I \subseteq \downarrow x$, for all $I \in \mathcal{H}_{\mathcal{S}}L$, $x \in L$.
(d) By Theorem 4.2, $\downarrow$ is open precisely when it has a left adjoint ℓ satisfying $\ell(\downarrow x \wedge J) = x \wedge \ell(J)$ for all $x \in L$, $J \in \mathcal{H}_{\mathcal{S}}L$. By (c) above, this amounts to $\bigvee\{j \in J : j \leq x\} = x \wedge \bigvee J$. The proof of the required frame distribution law is then straightforward using the idea of replacing the join of an arbitrary subset by the join of the $\mathcal{S}$-ideal generated by it.
(e) Suppose that $(x, y) \in \pi_L$, so that $P_x = P_y$. By Lemma 4.19 of [12], $P_x = (\downarrow x)^*$, the pseudocomplement taken in $\mathcal{H}_{\mathcal{S}}L$. So $(\downarrow x)^{**} = (\downarrow y)^{**}$, giving $(\downarrow x, \downarrow y) \in \pi_{\mathcal{H}_{\mathcal{S}}L}$. □

To put the above result into some context, we note that in [15] we established several conditions equivalent to the embedding $\downarrow : L \to \mathcal{H}_{\mathcal{S}}L$ being an isomorphism. These are

(a) Every $\mathcal{S}$-ideal of L is principal.
(b) L is a frame and every element of L is $\mathcal{S}$-Lindelöf.
(c) The frame of $\mathcal{S}$-congruences of L is isomorphic to the frame of congruences on $\mathcal{H}_{\mathcal{S}}L$.

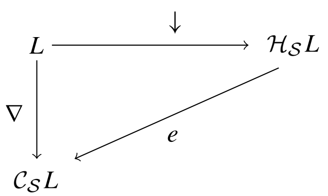
For details, see [15], Theorem 5.2.

We turn now to the analogous embedding of a partial frame into its frame of $\mathcal{S}$-congruences. See Definition 2.5 for some details.

Proposition 6.2 *Let* L *be an* $\mathcal{S}$*-frame and* $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ *the embedding into its congruence frame.*

(a) *The map* ∇ *is closed iff* ∇ *is an isomorphism.*
(b) *The map* ∇ *is open iff* L *is a Boolean frame.*
(c) *The map* ∇ *is skeletal iff* L *is a d-reduced* $\mathcal{S}$*-frame.*

Proof (a) Suppose that ∇ has right adjoint R and consider the commuting diagram



By Lemma 4.7, since e is one-one, ∇ closed implies $\downarrow$ closed. By Proposition 6.1, $\downarrow : L \to \mathcal{H}_{\mathcal{S}}L$ is an isomorphism. For $a \in L$, $a \wedge R(\Delta_a) = R(\nabla_a \wedge \Delta_a) = 0$. This uses the fact that

R preserves meets and ∇ is one-one. Since ∇ is also closed, $a \vee R(\Delta_a) = R(\nabla_a \vee \Delta_a) = 1$. As always, $R(1) = 1$ and $R(0) = 0$ since ∇ is dense. So $R(\Delta_a)$ is the complement of a in L. This shows that L is Boolean, so by Proposition 3.3 of [15] $e : \mathcal{H}_\mathcal{S}L \to \mathcal{C}_\mathcal{S}L$ is an isomorphism. So $\nabla : L \to \mathcal{C}_\mathcal{S}L$ is an isomorphism.

(b) ($\Rightarrow$) Suppose that ∇ is open and has left adjoint Λ. Using again the commuting diagram in (a) and e being one-one, Lemma 4.7 gives $\downarrow: L \to \mathcal{H}_\mathcal{S}L$ open. So L is in fact a frame, by Proposition 6.1.
For $a \in L$, $\Lambda(\nabla_a \wedge \Delta_a) = a \wedge \Lambda(\Delta_a) = 0$, since ∇ is open. Also $\Lambda(\nabla_a \vee \Delta_a) = \Lambda(\nabla_a) \vee \Lambda(\Delta_a) = 1$ using the fact that $\Lambda(\nabla_1) = 1$ since ∇ is one-one. Further, $\Lambda(\nabla_a) = a$, because ∇ is one-one; so $\Lambda(\Delta_a)$ is the complement of a in L. So L is Boolean.
($\Leftarrow$) By Proposition 6.1, since L is a frame, $\downarrow: L \to \mathcal{H}_\mathcal{S}L$ is open. By Proposition 3.3 of [15], L Boolean implies $e : \mathcal{H}_\mathcal{S}L \to \mathcal{C}_\mathcal{S}L$ is an isomorphism. So $\nabla : L \to \mathcal{C}_\mathcal{S}L$ is open.

(c) $\nabla : L \to \mathcal{C}_\mathcal{S}L$ is skeletal iff

$$\begin{aligned}
&\text{for all } x, y \in L,\ P_x = P_y \Rightarrow \nabla^*_x = \nabla^*_y \quad \text{iff}\\
&\text{for all } x, y \in L,\ P_x = P_y \Rightarrow \Delta_x = \Delta_y \quad \text{iff}\\
&\text{for all } x, y \in L,\ P_x = P_y \Rightarrow x = y \quad \text{iff}\\
&L \text{ is } d\text{-reduced}
\end{aligned}$$

□

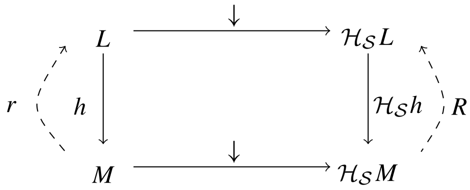
7 The Free Functor on Closed and Open Maps

We note that the map $\downarrow$ is a natural transformation from the identity functor on $\mathcal{S}$-frames to the functor $\mathcal{H}_\mathcal{S}$. (See [9].) So for any $\mathcal{S}$-frame map $h : L \to M$ there exists a unique frame map $\mathcal{H}_\mathcal{S}h : \mathcal{H}_\mathcal{S}L \to \mathcal{H}_\mathcal{S}M$ making the following diagram commute.

$$\begin{array}{ccc}
L & \xrightarrow{\ \downarrow\ } & \mathcal{H}_\mathcal{S}L \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathcal{H}_\mathcal{S}h} \\
M & \xrightarrow{\ \downarrow\ } & \mathcal{H}_\mathcal{S}M
\end{array}$$

We examine the relative strengths of h being closed versus $\mathcal{H}_\mathcal{S}h$ being closed. In fact the former condition is stronger; to ensure that h is closed, the right adjoint of $\mathcal{H}_\mathcal{S}h$ must send principal $\mathcal{S}$-ideals to principal ones as we show below.

Proposition 7.1 *Let $h : L \to M$ be an $\mathcal{S}$-frame map. Then h is closed iff $\mathcal{H}_\mathcal{S}h$ is a closed frame map and, for all $m \in M$, $R(\downarrow m) = \downarrow a$ for some $a \in L$, where R is the right adjoint of $\mathcal{H}_\mathcal{S}h$. When this condition holds, $a = r(m)$, where r is the right adjoint of h. The following diagram applies.*



Proof (⇒) Suppose that h is closed. Since $\mathcal{H}_{\mathcal{S}}h$ is a frame map, it automatically has a right adjoint R; we now give an explicit description of R. For $J \in \mathcal{H}_{\mathcal{S}}M$, define $R(J)$ to be the $\mathcal{S}$-ideal of L generated by $\{r(j) : j \in J\}$ where r is the right adjoint that must exist for h. We check that for all $K \in \mathcal{H}_{\mathcal{S}}L$, $J \in \mathcal{H}_{\mathcal{S}}M$ we have $\mathcal{H}_{\mathcal{S}}h(K) \subseteq J \iff K \subseteq R(J)$:
(⇒) Here $k \in K \Rightarrow h(k) \in J \Rightarrow k \leq rh(k) \in R(J)$.
(⇐) For $k \in K, k \leq \bigvee\{r(j_\alpha) : \alpha \in A\}$; the latter is a join of a designated subcollection of $\{r(j) : j \in J\}$. Then $h(k) \leq \bigvee\{hr(j_\alpha) : \alpha \in A\} \leq \bigvee\{j_\alpha : \alpha \in A\} \in J$ since J is an $\mathcal{S}$-ideal.

To now show that $\mathcal{H}_{\mathcal{S}}h$ is closed, we need, for all $K \in \mathcal{H}_{\mathcal{S}}L$, $J \in \mathcal{H}_{\mathcal{S}}M$, $R(\mathcal{H}_{\mathcal{S}}h(K) \vee J) \subseteq K \vee R(J)$. It suffices to show that $\{r(x) : x \in \mathcal{H}_{\mathcal{S}}h(K) \vee J\} \subseteq K \vee R(J)$ since this is a generating set for $R(\mathcal{H}_{\mathcal{S}}h(K) \vee J)$. So take $x \in \mathcal{H}_{\mathcal{S}}h(K) \vee J$. Then $x \leq h(k) \vee j$ for some $k \in K$, $j \in J$. Then $r(x) \leq r(h(k) \vee j) = k \vee r(j) \in K \vee R(J)$; this uses the fact that h is closed.

Finally $R(\downarrow m)$ is the $\mathcal{S}$-ideal generated by $\{r(x) : x \leq m\}$, so $R(\downarrow m) = \downarrow r(m)$.

(⇐) By assumption, if $m \in M$, then there is a (necessarily unique) $a \in L$ such that $R(\downarrow m) = \downarrow a$. We define $r(m) = a$. We check then that r is the right adjoint of h; for $x \in L, m \in M$:

$$\begin{aligned} h(x) \leq m &\iff \downarrow h(x) \subseteq \downarrow m \\ &\iff \mathcal{H}_{\mathcal{S}}h(\downarrow x) \subseteq \downarrow m \\ &\iff \downarrow x \subseteq R(\downarrow m) = \downarrow r(m) \\ &\iff x \leq r(m). \end{aligned}$$

Next we check that h is closed. Since $\mathcal{H}_{\mathcal{S}}h$ is closed, $R(\mathcal{H}_{\mathcal{S}}h(K) \vee J) = K \vee R(J)$ for all $K \in \mathcal{H}_{\mathcal{S}}L$, $J \in \mathcal{H}_{\mathcal{S}}M$. Applying this with $K = \downarrow x$ and $J = \downarrow m$ gives $R(\downarrow h(x) \vee \downarrow m) = \downarrow x \vee R(\downarrow m) = \downarrow (x \vee r(m))$. Since $r(h(x) \vee m)$ is the largest element of $R(\downarrow (h(x) \vee m))$, we obtain $r(h(x) \vee m) = x \vee r(m)$. □

We now consider the analogous question of an $\mathcal{S}$-frame map h being open versus $\mathcal{H}_{\mathcal{S}}h$ being open.

Note 7.2 (a) For $x, y \in L$ where L is an $\mathcal{S}$-frame we define $H_{(x,y)} = \{z \in L : z \wedge x \leq y\}$. It should be noted that $H_{(x,y)}$ is in fact an $\mathcal{S}$-ideal of L and $H_{(x,y)} = \downarrow x \to \downarrow y$ in $\mathcal{H}_{\mathcal{S}}L$. (See [12] for details.)

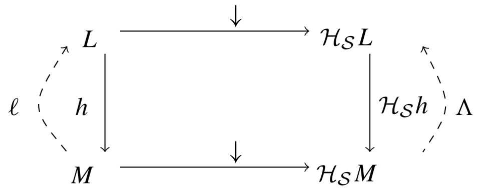
(b) It is a well-known fact that a full frame map f

- has a left adjoint iff f preserves arbitrary meets,
- is open iff f preserves arbitrary meets and Heyting arrows.

Of course a partial frame need not have all Heyting arrows. However openness of $\mathcal{S}$-frame maps is related to the preservation of Heyting arrows in the free frame as considered in the next proposition.

Proposition 7.3 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(a) *The map h is open if and only if $\mathcal{H}_{\mathcal{S}}h$ is an open frame map and, for all $m \in M$, $\Lambda(\downarrow m) = \downarrow a$ for some $a \in L$, where Λ is the left adjoint of $\mathcal{H}_{\mathcal{S}}h$. When this condition holds, $a = \ell(m)$, where ℓ is the left adjoint of h.*



(b) *The map h is open if and only if all the following conditions hold:*

(i) *$\mathcal{H}_{\mathcal{S}}h(\downarrow x \to \downarrow y) = \downarrow h(x) \to \downarrow h(y)$ for all $x, y \in L$.*
(ii) *$\mathcal{H}_{\mathcal{S}}h$ has a left adjoint Λ.*
(iii) *For all $m \in M$, $\Lambda(\downarrow m) = \downarrow a$ for some $a \in L$.*

Proof (a) This is analogous to the proof of Proposition 7.1. Given a left adjoint ℓ of h, define $\Lambda(J)$ to be the $\mathcal{S}$-ideal of L generated by $\{\ell(j) : j \in J\}$. Given a left adjoint Λ of $\mathcal{H}_{\mathcal{S}}h$ satisfying the given conditions, define $\ell(m) = a$ where $\Lambda(\downarrow m) = \downarrow a$. The details are omitted.

(b) ($\Rightarrow$) Apply (a). If h is open, $\mathcal{H}_{\mathcal{S}}h$ is an open frame map and so preserves all Heyting arrows. In particular $\mathcal{H}_{\mathcal{S}}h(\downarrow x \to \downarrow y) = \mathcal{H}_{\mathcal{S}}h(\downarrow x) \to \mathcal{H}_{\mathcal{S}}h(\downarrow y) = \downarrow h(x) \to \downarrow h(y)$.
($\Leftarrow$) As in (a), for $m \in M$ define the function ℓ by $\ell(m) = a$ where $\Lambda(\downarrow m) = \downarrow a$. That ℓ is the left adjoint of h is easily checked as in Proposition 7.1.
Further,

$$\begin{aligned}
&\downarrow x \wedge \Lambda(\downarrow m) \subseteq \Lambda(\downarrow (h(x) \wedge m))\\
\iff &\Lambda(\downarrow m) \subseteq \downarrow x \to \Lambda(\downarrow (h(x) \wedge m))\\
\iff &\downarrow m \subseteq \mathcal{H}_{\mathcal{S}}h(\downarrow x \to \Lambda(\downarrow (h(x) \wedge m)))\\
\iff &\downarrow m \subseteq \downarrow h(x) \to \mathcal{H}_{\mathcal{S}}h\Lambda(\downarrow (h(x) \wedge m))\\
\iff &\downarrow (h(x) \wedge m) \subseteq \mathcal{H}_{\mathcal{S}}h\Lambda(\downarrow (h(x) \wedge m))
\end{aligned}$$

which always holds. Now since $\ell(m) \in \Lambda(\downarrow m)$, this shows that $x \wedge \ell(m) \in \Lambda \downarrow (h(x) \wedge m)$, and so $x \wedge \ell(m) \leq \ell(h(x) \wedge m)$ as required.

□

Note 7.4 Let $h : L \to M$ be an $\mathcal{S}$-frame map. If $\mathcal{H}_{\mathcal{S}}h$ is skeletal then h is skeletal: By Proposition 6.1 (5), $\downarrow : L \to \mathcal{H}_{\mathcal{S}}L$ is always skeletal. Since a composite of skeletal maps is skeletal, $\downarrow h : L \to \mathcal{H}_{\mathcal{S}}M$ is skeletal. By Lemma 5.6, h is skeletal. We do not know if the converse holds.

8 Points of Partial Frames

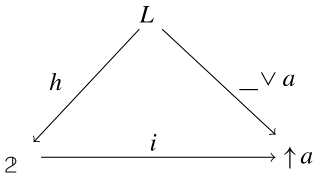
The classical adjunction between frames and topological spaces is given by an open set functor and a spectrum functor. The latter can be described using completely prime filters or, equivalently, frame maps to the 2-chain. These are usually referred to as the "points" of the frame; we call them frame points below.

The analogous situation for $\mathcal{S}$-frames was presented in [13], where the definition below appeared.

Definition 8.1 An $\mathcal{S}$-point of an $\mathcal{S}$-frame L is an $\mathcal{S}$-frame map $h : L \to \mathbb{2}$ where $\mathbb{2}$ is the two-element $\mathcal{S}$-frame.

We turn our attention to closed and open $\mathcal{S}$-points. Since points are obviously onto maps, Theorem 4.5 applies.

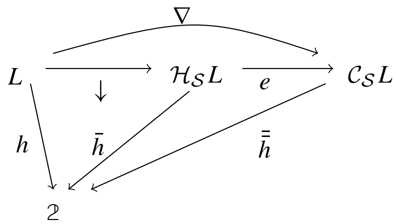
Note 8.2
- If $h : L \to \mathbb{2}$ is a closed $\mathcal{S}$-point, there exists $a \in L$ and an isomorphism i making this diagram commute:



So $\uparrow a = \{a, 1\}$, making a a co-atom of L.

- If a is a co-atom of L, then the $\mathcal{S}$-frame map $f : L \to \uparrow a$ given by $f(x) = x \vee a$, yields an $\mathcal{S}$-point of L which is clearly closed.

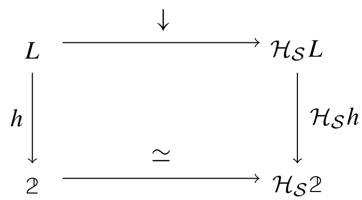
We note that, given an $\mathcal{S}$-point $h : L \to \mathbb{2}$, this extends quite naturally to a frame point $\bar{h} : \mathcal{H}_{\mathcal{S}}L \to \mathbb{2}$ such that $\bar{h}\downarrow = h$. This in turn extends naturally to a frame point $\bar{\bar{h}} : \mathcal{C}_{\mathcal{S}}L \to \mathbb{2}$ with $\bar{\bar{h}}e = \bar{h}$. The existence of $\bar{h}$ and $\bar{\bar{h}}$ follows from the universal properties of $\mathcal{H}_{\mathcal{S}}L$ and $\mathcal{C}_{\mathcal{S}}L$. This is made clearer in the following diagram in which all triangles commute:



Theorem 8.3 *Let $h : L \to \mathbb{2}$ be an $\mathcal{S}$-point of an $\mathcal{S}$-frame L. Using the notation above, we have:*

(a) *If h is closed, then $\bar{h}$ is closed, but not conversely.*
(b) *If $\bar{h}$ is closed then $\bar{\bar{h}}$ is closed, but not conversely.*

Proof (a) We note that, in the following diagram, $\bar{h}$ and $\mathcal{H}_{\mathcal{S}}h$ amount to the same map. So h closed implies $\bar{h}$ closed by Proposition 7.1.



A direct proof is also possible: If $h : L \to \uparrow a = \{a, 1\}$, one can show that $\downarrow a$ is a co-atom of $\mathcal{H}_{\mathcal{S}}L$ and $g : \mathcal{H}_{\mathcal{S}}L \to \uparrow(\downarrow a)$ is a closed frame point equal to $\bar{h}$.

For the lack of the converse, see Example 8.5.

(b) Suppose $\bar{h}$ is closed and is explicitly given by $\bar{h} : \mathcal{H}_{\mathcal{S}}L \to \uparrow J = \{J, \downarrow 1\}$ where $\bar{h}(I) = I \vee J$ for J the co-atom of $\mathcal{H}_{\mathcal{S}}L$ mentioned in Note 8.2.

We check that $e(J)$ is a co-atom of $\mathcal{C}_{\mathcal{S}}L$:

Writing $J = \bigvee_{j \in J} \downarrow j$, we have $e(J) = \bigvee e \downarrow j = \bigvee_{j \in J} \nabla_j$ which we denote by ∇_J. By Lemma 3.1 of [14] this join is actually a union, i.e. $e(J) = \bigcup_{j \in J} \nabla_j$.

- $\nabla_J \neq \nabla$ because $J \neq \downarrow 1$.
- If $\nabla_J \subseteq \theta$ and $\nabla_J \neq \theta$, for some $\theta \in \mathcal{C}_{\mathcal{S}}L$, then there exists $(s, t) \in \theta$ with $(s, t) \notin \nabla_J$; without loss of generality, $s \leq t$.
  If $s \notin J$, then $J \vee \downarrow s = \downarrow 1$, so there exists $j_0 \in J$ with $j_0 \vee s = 1$. Then $j_0 \vee t = 1$, contradicting $(s, t) \notin \nabla_{j_0}$. So $s \in J$. Now $(0, s) \in \nabla_s \subseteq \nabla_J$.
  However, $t \notin J$, since $s \vee j \neq t \vee j$ for all $j \in J$. So $J \vee \downarrow t = \downarrow 1$, so there exists $j_1 \in J$ with $j_1 \vee t = 1$; then $(t, 1) \in \nabla_{j_1}$.
  Putting these ingredients together gives $(0, s), (s, t), (t, 1)$ all in θ, so $\theta = \nabla$.

This makes the frame map $f : \mathcal{C}_{\mathcal{S}}L \to \uparrow \nabla_J$ given by $f(\theta) = \theta \vee \nabla_J$ a frame point; i.e. $\uparrow \nabla_J = \{\nabla_J, \nabla\} \simeq 2$ and f is clearly closed.

We now show that $\bar{\bar{h}}$ and f have the same kernel. First notice that

$$\bar{\bar{h}}(\nabla_J) = \bar{\bar{h}}(\bigvee_{j \in J} \nabla_j) = \bigvee_{j \in J} \bar{\bar{h}}(\nabla_j) = \bigvee_{j \in J} \bar{\bar{h}} e \downarrow (j) = \bigvee_{j \in J} \bar{h}(\downarrow j) = \bar{h}(J),$$

which is J, the bottom of $\uparrow J$.

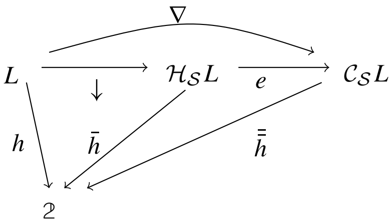
Now, for $\alpha, \beta \in \mathcal{C}_{\mathcal{S}}L$:

$$\begin{aligned}
(\alpha, \beta) \in \ker f &\iff \alpha \vee \nabla_J = \beta \vee \nabla_J \\
&\iff \alpha \vee \nabla_J = \beta \vee \nabla_J = \nabla_J \text{ or } \alpha \vee \nabla_J = \beta \vee \nabla_J = \nabla \\
&\iff \bar{\bar{h}}(\alpha \vee \nabla_J) = \bar{\bar{h}}(\beta \vee \nabla_J) \\
&\iff \bar{\bar{h}}(\alpha) \vee \bar{\bar{h}}(\nabla_J) = \bar{\bar{h}}(\beta) \vee \bar{\bar{h}}(\nabla_J) \\
&\iff \bar{\bar{h}}(\alpha) = \bar{\bar{h}}(\beta) \\
&\iff (\alpha, \beta) \in \ker \bar{\bar{h}}
\end{aligned}$$

For the lack of a converse, see Example 8.6 or 8.7.

□

The same diagram as before is used in the next result:



Theorem 8.4 *Let $h : L \to \mathbb{2}$ be an $\mathcal{S}$-point of an $\mathcal{S}$-frame L. Using the same notation as in the previous result, we have*

(a) *If h is open, then $\bar{h}$ is open and conversely.*
(b) *If $\bar{h}$ is open, then $\bar{\bar{h}}$ is open but not conversely.*

Proof (a) ($\Rightarrow$) This is similar to the proof of Theorem 8.3 (a). The open $\mathcal{S}$-point h can be explicitly given by the map $h : L \to \downarrow a = \{0, a\}$ where $h(x) = x \wedge a$, for an atom a of L. The fact that $\downarrow a$ is then an atom of $\mathcal{H}_{\mathcal{S}}L$ can be used to show that $\bar{h}$ is open.
($\Leftarrow$) Suppose that $\bar{h}$ is open and is explicitly given by $\bar{h} : \mathcal{H}_{\mathcal{S}}L \to \downarrow J$ for some atom J of $\mathcal{H}_{\mathcal{S}}L$. Since J is an atom, it is principal; in fact $J = \downarrow a$ for some atom a of L.
Then $h = \bar{h} \downarrow$ is given by $h(x) = \bar{h}(\downarrow x) = \downarrow x \cap \downarrow a = \downarrow (x \wedge a)$, for all $x \in L$. By Theorem 4.5(b) and Note 4.4(b), h is open.

(b) The logical strategy here is similar to that of Theorem 8.3(b); however, the proof that if J is an atom of $\mathcal{H}_{\mathcal{S}}L$, then $e(J)$ is an atom of $\mathcal{C}_{\mathcal{S}}L$ is different, so we provide that here: If J is an atom of $\mathcal{H}_{\mathcal{S}}L$, then $J = \downarrow c$, for some atom c of L. Then $e(J) = \nabla_c$. Suppose $\Delta \neq \theta \subseteq \nabla_c$ for some $\theta \in \mathcal{C}_{\mathcal{S}}L$. There exists $(s, t) \in \theta$ with $s \neq t$; without loss of generality, $s < t$. Now $s \vee c = t \vee c$. Further $(s \wedge c, t \wedge c) \in \theta$. Since c is an atom of L, $s \wedge c = 0$ or $s \wedge c = c$ and $t \wedge c = 0$ or $t \wedge c = c$. If $s \wedge c = t \wedge c$, then $s = t$ a contradiction. So $s \wedge c = 0$ and $t \wedge c = c$ which gives $(0, c) \in \theta$ and so $\nabla_c \subseteq \theta$.
We omit the isomorphism of the maps, since this is similar to Theorem 8.3(b). For the lack of a converse, see Example 8.6.

□

We conclude this paper with some illuminating examples.

Example 8.5 Let L consist of all countable subsets of $\mathbb{R}$, with $\mathbb{R}$ itself added as top element. Consider $\mathcal{S}$ that selects all countable subsets. Let J consist of all countable subsets of $\mathbb{R}$. Then J is an $\mathcal{S}$-ideal of L, and is in fact a co-atom of $\mathcal{H}_{\mathcal{S}}L$. The map $\bar{h} : \mathcal{H}_{\mathcal{S}}L \to \uparrow J$ is therefore a closed frame point. However, $h = \bar{h} \downarrow$ is not closed, because

$$h(A) = \begin{cases} \downarrow 1 \text{ if } A = \mathbb{R} \\ J \text{ if } A \text{ is countable} \end{cases}$$

so there is clearly no largest element A such that $h(A) = J$. □

Example 8.6 This example uses (full) frames and $\mathcal{S}$ selects all subsets. Let $L = \{0, a, 1\}$ with $0 < a < 1$. Define $h : L \to \mathbb{2}$ with $h(a) = 0$. Then h is a frame point of L and is not open. Here $\mathcal{C}_{\mathcal{S}}L = \{\Delta, \nabla_a, \Delta_a, \nabla\}$, the four-element Boolean algebra. Further $\bar{\bar{h}}$ is given by $\bar{\bar{h}}(\nabla) = \bar{\bar{h}}(\Delta_a) = 1$ and $\bar{\bar{h}}(\Delta) = \bar{\bar{h}}(\nabla_a) = 0$. So $\bar{\bar{h}}$ is open.

By defining $g : L \to \mathbb{2}$ by $g(a) = 1$ instead, one obtains a frame point that is not closed for which $\bar{g}$ is not closed but $\bar{\bar{g}}$ is closed. □

Example 8.7 Again, let L be the σ-frame as in the previous example.

In this example we provide an $\mathcal{S}$-point h which is not closed for which the corresponding frame point $\bar{h}$ is also not closed but the corresponding frame point $\bar{\bar{h}}$ is closed.

Define h by

$$h(A) = \begin{cases} 1 \text{ if } 4 \in A \\ 0 \text{ if } 4 \notin A \end{cases}$$

Then $h : L \to \mathbb{2}$ is an $\mathcal{S}$-point. However, since there is no largest countable subset of $\mathbb{R}$ missing 4, h has no right adjoint, so is not closed. The corresponding frame point $\bar{h}$ is given by $\bar{h}(I) = \bar{h}(\bigvee_{A \in I} \downarrow A) = \bigvee_{A \in I} h(A)$ for all $I \in \mathcal{H}_{\mathcal{S}}L$, so

$$\bar{h}(I) = \begin{cases} 1 \text{ if } 4 \in A \text{ for some } A \in I \\ 0 \text{ if } 4 \notin A \text{ for all } A \in I \end{cases}$$

This $\bar{h}$ of course has a right adjoint given by $\bar{r}(1) = \downarrow \mathbb{R}$ and $\bar{r}(0) = J$ where J is the $\mathcal{S}$-ideal of L consisting of all countable subsets of $\mathbb{R}$ missing 4. However, $\bar{h}$ is not closed, because J is not a co-atom of $\mathcal{H}_{\mathcal{S}}L$: $J \vee \downarrow \{4\}$ consists all countable subsets of $\mathbb{R}$, and so lies strictly between J and the top.

Finally we consider $\bar{\bar{h}}$ given by

$$\begin{aligned} \bar{\bar{h}}(\theta) &= \bar{\bar{h}}(\bigvee\{\nabla_A \cap \Delta_B : B \subseteq A, (A, B) \in \theta\}) \\ &= \bigvee\{\bar{\bar{h}}(\nabla_A) \wedge \bar{\bar{h}}(\Delta_B) : B \subseteq A, (A, B) \in \theta\} \\ &= \bigvee\{h(A) \wedge h(B)^c : B \subseteq A, (A, B) \in \theta\} \end{aligned}$$

where $h(B)^c$ denotes the complement of $h(B)$. This $\bar{\bar{h}}$ of course has a right adjoint, given by $\bar{\bar{r}}(1) = \nabla$ and $\bar{\bar{r}}(0) = \bigvee\{\nabla_A \cap \Delta_B : B \subseteq A, 4 \notin A \text{ or } 4 \in B\} = \alpha$, say. We show that $\bar{\bar{h}}$ is closed by showing that α is a co-atom of $\mathcal{C}_{\mathcal{S}}L$:

Suppose that $(C, D) \notin \alpha, C \subseteq D$. We show that the $\mathcal{S}$-congruence β generated by $\alpha \cup \{(C, D)\}$ is the top of $\mathcal{C}_{\mathcal{S}}L$. If $4 \in C$, then $(C, D) \in \nabla_D \cap \Delta_C \subseteq \alpha$, a contradiction. So $4 \notin C$. A similar argument shows $4 \in D$.

Now we have $\begin{cases} (\emptyset, C) \in \alpha \text{ because } 4 \notin C, \text{ so} \\ (\emptyset, C) \in \beta \\ (C, D) \in \beta \\ (D, \mathbb{R}) \in \alpha \text{ because } 4 \in D, \text{ so} \\ (D, \mathbb{R}) \in \beta \end{cases}$

Putting this together we get $\beta = \nabla$, the top of $\mathcal{C}_{\mathcal{S}}L$ as required. □

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Conflict of interest The authors declare that they have no financial or non-financial conflict of interest.

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