



2-Limits and 2-Terminal Objects are too Different

tslil clingman¹ · Lyne Moser²

Received: 17 March 2022 / Accepted: 3 August 2022 / Published online: 8 September 2022
© The Author(s) 2022

Abstract

In ordinary category theory, limits are known to be equivalent to terminal objects in the slice category of cones. In this paper, we prove that the 2-categorical analogues of this theorem relating 2-limits and 2-terminal objects in the various choices of slice 2-categories of 2-cones are false. Furthermore we show that, even when weakening the 2-cones to pseudo- or lax-natural transformations, or considering bi-type limits and bi-terminal objects, there is still no such correspondence.

Keywords 2-Dimensional limits · 2-Dimensional terminal objects · Slice 2-categories

Mathematics Subject Classification 18N10 · 18A30 · 18A25 · 18A05

1 Introduction

In this paper we address the question of whether the natural 2-categorical analogue to the 1-categorical result giving a correspondence between limits and terminal objects in a slice category of cones, holds.

1.1 Motivation, the 1-Dimensional Case and the Case of Cat

A *limit* of a functor $F : I \rightarrow \mathcal{C}$ comprises the data of an object $L \in \mathcal{C}$ together with a natural transformation $\lambda : \Delta L \Rightarrow F$, called the *limit cone*, which satisfies the following universal property: for each $X \in \mathcal{C}$, the map $\lambda_* \circ \Delta : \mathcal{C}(X, L) \rightarrow \text{Cat}(I, \mathcal{C})(\Delta X, F)$ given by post-composition with λ is an isomorphism of sets. We may also form the slice category $\Delta \downarrow F$

Communicated by Nicola Gambino.

✉ Lyne Moser
moser@mpim-bonn.mpg.de
tslil clingman
tslil@jhu.edu

¹ Johns Hopkins University, 3400 N. Charles St., Baltimore, MD, USA

² Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

of cones over F , and it is a folklore result that a limit of F is equivalently a terminal object in $\Delta \downarrow F$.

As an example, and in progressing up in dimension, let us now consider products in \mathbf{Cat} – the category of small categories and functors. The universal property of the product $(\mathcal{C} \times \mathcal{D}, \pi_{\mathcal{C}}, \pi_{\mathcal{D}})$ of two categories \mathcal{C} and \mathcal{D} gives, for each pair of functors $F: \mathcal{X} \rightarrow \mathcal{C}$ and $G: \mathcal{X} \rightarrow \mathcal{D}$, a functor $\langle F, G \rangle: \mathcal{X} \rightarrow \mathcal{C} \times \mathcal{D}$ unique among those satisfying $\pi_{\mathcal{C}} \langle F, G \rangle = F$ and $\pi_{\mathcal{D}} \langle F, G \rangle = G$.

$$\begin{array}{ccc}
 & \mathcal{X} & \\
 F \swarrow & \vdots & \searrow G \\
 \mathcal{C} & \xrightarrow{\exists! \langle F, G \rangle} & \mathcal{C} \times \mathcal{D} & \xrightarrow{\pi_{\mathcal{D}}} & \mathcal{D} \\
 \pi_{\mathcal{C}} \swarrow & & & & \searrow \pi_{\mathcal{D}}
 \end{array} \tag{1.1}$$

However, the category \mathbf{Cat} has further structure. Indeed, it is a 2-category with 2-morphisms the natural transformations between the functors. This 2-dimensional structure is compatible with the product of categories. More precisely, there is a bijection of natural transformations as depicted below, which is implemented by whiskering with the projection functors.

$$\left(F \begin{array}{c} \mathcal{X} \\ \xrightarrow{\alpha} \\ \mathcal{C} \end{array} \right) \cdot G \begin{array}{c} \mathcal{X} \\ \xrightarrow{\beta} \\ \mathcal{D} \end{array} \right) \rightsquigarrow \langle F, G \rangle \begin{array}{c} \mathcal{X} \\ \xrightarrow{\langle \alpha, \beta \rangle} \\ \mathcal{C} \times \mathcal{D} \end{array} \tag{1.2}$$

Observe that the natural transformations α and β correspond to functors $\alpha: \mathcal{X} \times \mathbb{2} \rightarrow \mathcal{C}$ and $\beta: \mathcal{X} \times \mathbb{2} \rightarrow \mathcal{D}$, where $\mathbb{2}$ is the category $\{0 \rightarrow 1\}$. In this light, the bijection (1.2) of natural transformations can be retrieved by applying the universal property of (1.1) to the functors $\alpha: \mathcal{X} \times \mathbb{2} \rightarrow \mathcal{C}$ and $\beta: \mathcal{X} \times \mathbb{2} \rightarrow \mathcal{D}$.

Taken together, the bijections of (1.1) and (1.2) assemble into an isomorphism of categories

$$(\pi_{\mathcal{C}})_*, (\pi_{\mathcal{D}})_*: \mathbf{Cat}(\mathcal{X}, \mathcal{C} \times \mathcal{D}) \xrightarrow{\cong} \mathbf{Cat}(\mathcal{X}, \mathcal{C}) \times \mathbf{Cat}(\mathcal{X}, \mathcal{D}).$$

This then is the defining feature of a 2-dimensional limit: there are two aspects of the universal property, one for morphisms and one for 2-morphisms.

In the case of the product above, the indexing category is just a 1-category. Since \mathbf{Cat} is a 2-category, one could instead consider indexing diagrams by a 2-category I . In order to define a general 2-dimensional limit in \mathbf{Cat} , we need a category of higher morphisms between two 2-functors. This is the category of 2-natural transformations and 3-morphisms, called *modifications*, between them. With these notions, a 2-limit of a 2-functor $F: I \rightarrow \mathbf{Cat}$ can be defined as a pair (\mathcal{L}, λ) of a category \mathcal{L} and a 2-natural transformation $\lambda: \Delta \mathcal{L} \Rightarrow F$ which are such that post-composition with λ gives an isomorphism of categories

$$\lambda_* \circ \Delta: \mathbf{Cat}(\mathcal{X}, \mathcal{L}) \xrightarrow{\cong} [I, \mathbf{Cat}](\Delta \mathcal{X}, F). \tag{1.3}$$

1.2 2-Dimensional Conjectures

A 2-limit of a general 2-functor $F: I \rightarrow \mathcal{A}$ is defined in the same fashion as indicated in (1.3) above; see Definition 2.4. This notion was first introduced, independently, by Auderset [1]

and Borceux-Kelly [2], and was further developed by Street [10], Kelly [7, 8] and Lack in [9]. Motivated by the 1-categorical case, it is natural to ask whether 2-limits can be characterised as 2-dimensional terminal objects in the slice 2-category of 2-cones. The appropriate notion of terminality here is that of a *2-terminal object* – an object such that every hom-category to this object is isomorphic to the terminal category $\mathbb{1}$.

Having seen all other concepts involved, let us introduce now the *slice 2-category* $\Delta \downarrow_s F$ of 2-cones over a 2-functor $F: I \rightarrow \mathcal{A}$. This 2-category has as objects pairs (X, μ) of an object $X \in \mathcal{A}$ together with a 2-cone $\mu: \Delta X \Rightarrow F$, and as morphisms those morphisms $f: X \rightarrow Y$ of \mathcal{A} making the 2-cones commute.

$$\begin{array}{ccc}
 \Delta X & \xrightarrow{\Delta f} & \Delta Y \\
 & \searrow \mu & \downarrow v \\
 & & F
 \end{array}
 \tag{1.4}$$

The 2-morphisms are given by 2-morphisms in \mathcal{A} which satisfy a certain whiskering identity.

This slice 2-category seems appropriate to our conjecture since the 1-dimensional aspect of the universal property of a 2-terminal object in there is exactly the same as the 1-dimensional aspect of the universal property of a 2-limit. In the special case of Cat , by generalising the argument we have seen for products, the 1-dimensional aspect of the universal property of a 2-limit in Cat suffices to reconstruct its 2-dimensional aspect. This holds more broadly in every 2-category admitting tensors by $\mathbb{2}$, as demonstrated in Proposition 2.11. Given this, we conjecture:

Conjecture 1 Let I and \mathcal{A} be 2-categories, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. Let $L \in \mathcal{A}$ be an object and $\lambda: \Delta L \Rightarrow F$ be a 2-natural transformation. The following two statements are equivalent:

- (i) The pair (L, λ) is a 2-limit of the functor F .
- (ii) The pair (L, λ) is a 2-terminal object in the slice 2-category $\Delta \downarrow_s F$ of 2-cones over F .

Although it gives the authors no great pleasure to mislead the reader so, such a conjecture is *false*. While a 2-limit is always a 2-terminal object in the slice 2-category of 2-cones (see Proposition 2.9), the converse is not necessarily true (see Counter-example 2.10). The reason for this failure is that the 2-dimensional aspect of the universal property of a 2-terminal object in the slice 2-category is weaker than the 2-dimensional aspect of the universal property of a 2-limit. This manifests in, among other things, the inability of the slice 2-category to detect enough modifications between two 2-cones with the same summit.

This is not, however, the last word for this conjecture. The theory of 2-categories affords us more room than that of categories to coherently weaken various notions. Instead of considering a 2-category of 2-cones where the morphisms render triangles like that of (1.4) commutative, we are free to ask that the data of a morphism comprises also the data of a general, or perhaps invertible, modification filling that triangle. This leads to the notions of *lax-slice* and *pseudo-slice* 2-categories of 2-cones, respectively. Unlike the original, *strict*, slice 2-category given above, the lax- and pseudo-slice 2-categories detect all, or more, modifications between 2-cones. With this in mind, we might conjecture:

Conjecture 2 Let I and \mathcal{A} be 2-categories, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. Let $L \in \mathcal{A}$ be an object and $\lambda: \Delta L \Rightarrow F$ be a 2-natural transformation. The following two statements are equivalent:

- (i) The pair (L, λ) is a 2-limit of the 2-functor F .
- (ii) The pair (L, λ) is a 2-terminal object in the lax-slice (or pseudo-slice) 2-category of 2-cones over F .

Unfortunately, this too is incorrect. The failure here is twofold: the pseudo-slice may still fail to detect enough modifications, as before, while simultaneously allowing too many new morphisms to appear. Similar issues plague the lax-slice, and as we will see in Sect. 3, 2-terminal objects in either are generally unrelated to 2-limits.

At this point, it is natural to ask whether the failure of Conjectures 1, 2 has something to do with the rigidity of the notion of 2-limits. We might, for instance, ask that our 2-cones have naturality triangles filled by a general 2-morphism, or perhaps any invertible 2-morphism, not just the identity. This leads us to consider *lax-limits* and *pseudo-limits*. One might imagine that these limits have special relationships with the lax- and pseudo-slices of matching cones, respectively. Specifically, one might hope that the peculiarities of these weaker notions of 2-dimensional limit conspire somehow to support the following conjectures.

Conjecture 3 Let I and \mathcal{A} be 2-categories, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. Let $L \in \mathcal{A}$ be an object and $\lambda: \Delta L \Rightarrow F$ be a pseudo-natural (resp. lax-natural) transformation. The following two statements are equivalent:

- (i) The pair (L, λ) is a pseudo-limit (resp. lax-limit) of the functor F .
- (ii) The pair (L, λ) is a 2-terminal object in the strict-/pseudo-/lax-slice 2-category of pseudo-cones (resp. lax-cones) over F .

As in the case of 2-limits, pseudo- and lax-limits are in particular 2-terminal objects in the strict-slice of appropriate cones (see Proposition 4.3). However, all other implications are generally false, as established in Sect. 4.

We might then ask whether the failure of Conjectures 1, 2, 3 has had, all along, something to do with the rigidity of universal properties expressed by isomorphisms of categories. One might, on this view, hope to generate analogous and valid conjectures by weakening these isomorphisms to equivalences of categories – conjectures concerning *bi*-type limits and *bi*-terminal objects. However, as discussed in Sect. 5, even these do not hold.

Finally, one might wonder about the case of weighted 2-limits, which is a well-established notion for limits in enriched category theory in the literature. The theory of weighted limits was developed by Auderset [1], Street [10], and Kelly [8] in the case of 2-categories, and by Borceux-Kelly [2] as well as Kelly [7, Chapter 3] in the case of general enriched categories. However, as conical 2-limits, pseudo-limits, and lax-limits are special cases of such weighted 2-limits as noted in [8, §3 and §5], we can see that the analogues of Conjectures 1, 2, 3 for weighted 2-limits must in general fail too.

1.3 Outline

The structure of the paper is as follows. In Sect. 2, we introduce the notions of 2-limits and strict-slices of 2-cones. We prove that a 2-limit is always 2-terminal in the strict-slice of 2-cones, but we provide a counter-example demonstrating that the converse fails in general. However, for the converse to hold, it is sufficient for the ambient 2-category to admit tensors by $\mathbb{2}$ —as is the case of the 2-category Cat . In Sect. 3, we turn our attention to the larger 2-categories of pseudo- and lax-slices of 2-cones. We provide counter-examples demonstrating that 2-limits are in fact unrelated to 2-terminal objects in these—neither notion generally implies the other. In Sect. 4, we introduce pseudo- and lax-limits, and investigate their relationships with 2-terminal objects in the different slices. Finally, in Sect. 5, we address the case

Table 1 2-type limits which are not 2-terminal

\rightarrow implies 2-terminal in \downarrow	2-limit	Pseudo-limit	Lax-limit
Strict-slice	✓ Prop. 2.9	✓ Prop. 4.3	✓ Prop. 4.3
Lax-slice	Counter-ex. 3.7	Reduction 4.9	Counter-ex. 4.10
Pseudo-slice	Reduction 3.8	Reduction 4.12	Reduction 4.12

Table 2 2-terminal objects which are not 2-type limits

2-terminal in \downarrow implies \rightarrow	2-limit	Pseudo-limit	Lax-limit
Strict-slice	Counter-ex. 2.10	Reduction 4.4	Counter-ex. 4.5
Lax-slice	Counter-ex. 3.9	Reduction 4.13	Reduction 4.14
Pseudo-slice	Reduction 3.10	Reduction 4.15	Reduction 4.15

of bi-type limits. We show that these are in particular always bi-terminal in the pseudo-slice of appropriate cones, and then adapt the results we have for the 2-type cases to the bi-type cases.

All of the counter-examples presented in this paper, with the exception of Example 4.5, are indexed by finite (1-)categories: $\mathbb{2}$ and the pullback shape specifically. These counter-examples were generally constructed to exhibit certain data, for example a modification that is not detected by the strict-slice or a 2-morphism which gives an errant morphism in the lax-slice. The resulting diagram shapes were inessential to this process, and there are certainly many more counter-examples yet.

In the Tables 1 and 2, we summarise our counter-examples and reductions for Conjectures 1, 2, 3—see Sect. 5 for the matching tables for the bi-type conjectures. Only results marked with a ✓ are true, everything else establishes a counter-example. Note that the objects in the slices considered vary by the column: the type of objects should match the type of the limit cone.

In consulting these tables, some readers might be confused that the adjectives “pseudo” and “lax” do not appear in the same order in the rows as they appear in the columns. We should be careful to consider that these adjectives play very different roles when attached to the various slice 2-categories as they do when attached to a notion of limit. Adding these adjectives to the labels of the rows changes the *morphisms* of the slices, while adding these adjectives to the labels of the columns changes the *type of the cones*, i.e. it changes the *objects* of the slices. The unexpected ordering of the columns of the tables has been chosen to be this way, since all counter-examples for pseudo-slices are reductions of counter-examples for lax-slices.

1.4 Positive Results for Characterisations of 2-Dimensional Limits

We may always view a 2-category as a horizontal double category with only trivial vertical morphisms, and in the double categorical setting we are now afforded a stronger notion of terminality. In this broader context, Grandis-Paré show in [5, 6] that (weighted) 2-limits of a 2-functor F are equivalently *double terminal* objects in the double category of (weighted) cones over the horizontal double functor induced by F ; see also [4, §5.6]. Similar work in this direction is done by Verity in his thesis [11]. With this proliferation of positive results,

it is surprising that the failures of Conjectures 1, 2, 3 are not documented in the literature. Grandis-Paré are certainly aware of such a failure as they write the following in their recent paper [6]:

On the other hand, there seems to be no natural way of expressing the 2-dimensional universal property of weighted (strict or pseudo) limits by terminality in a 2-category.

Unfortunately however, Grandis-Paré do not record their formulation of the “natural way” nor whatever obstacles they encountered. We feel that Conjectures 1, 2, 3 express a natural expectation of the relationship between 2-limits and 2-terminal objects in a 2-category, and we hope that our counter-examples illustrate clearly the failure of all such conjectures.

Closer examination of these counter-examples reveals the need to capture additional information not present in the slice 2-category of cones. The double categorical approach of Grandis, Paré, and Verity certainly suffices for this task, but in our paper [3] we give a purely 2-categorical characterisation of 2-limits by constructing two different slice 2-categories of cones which have the joint property that objects which are simultaneously 2-terminal in both correspond precisely to 2-limits. One of these slice 2-categories is predictably the slice 2-category of cones, but in fact the other slice 2-category alone succeeds in precisely characterising 2-limits through bi-initial objects of a specific form. This second slice 2-category, however, is a shifted version of the usual slice 2-category of cones: its *objects* are modifications between cones. An advantage of this approach will be highlighted in forthcoming work by the second author, where a notion of $(\infty, 2)$ -limits can then be defined in a fully $(\infty, 2)$ -categorical language without requiring the development of the accompany theory of double $(\infty, 1)$ -categories.

The counter-examples in this paper are indicative of a larger failure in the extension of 1-categorical theorems to the setting of 2-category theory. More generally, the existence and characterisations of bi-limits may be viewed as an instance of the corresponding problems for *bi-representations* of general pseudo-presheaves, and it is here that the analogy breaks down: while a representation for a presheaf corresponds to an initial object in the category of elements, the data of a bi-representation for a pseudo-presheaf is not wholly captured by a bi-initial object in the 2-category of elements.

At the level of 2-dimensional representations, in [3] we weaken the strict setting to that of pseudo-functors and pseudo-natural transformations and generalise the results of Grandis, Paré, and Verity to the case of bi-representations. In particular we give a double categorical characterisation of bi-representations of pseudo-presheaves in terms of double bi-initial objects in the double category of elements. Furthermore, we succeed in providing a purely 2-categorical characterisation of bi-representations in terms of objects which are simultaneously bi-initial in the familiar 2-category of elements and in a new 2-category of *morphisms*. In fact, we are able to demonstrate that bi-representations can actually be characterised as bi-initial objects of a specific form in the 2-category of morphisms alone. These results are the content of [3, Theorem 6.8]. The counter-examples of this paper establish the necessity of the presence of both 2-categories in the theorems there, as bi-limits are bi-representations. As a corollary of these theorems we obtain a purely 2-categorical characterisation of weighted bi-limits in [3, Theorem 7.19].

Finally, the positive results of Propositions 2.11, 5.5 are special cases of more general results for bi-representations: [3, Theorem 6.14] shows that in the presence of tensors by $\mathbb{2}$, if the pseudo-presheaf preserves such tensors, then bi-representations are precisely bi-initial objects in the 2-category of elements.

2 2-Limits do not Correspond to 2-Terminal Objects in the Strict-Slice

In this section, we start by comparing 2-limits with 2-terminal objects in the strict-slice 2-category of 2-cones. After introducing all the terms involved, we show that a 2-limit is in particular a 2-terminal object in the strict-slice, but we provide a counter-example for the other implication. However, when the ambient 2-category admits tensors by $\mathbb{2}$, such as is the case of \mathbf{Cat} , these two notions do coincide.

A 2-category has not only the structure of a category, with objects and morphisms, but additionally has 2-morphisms between parallel morphisms. These 2-morphisms may be composed both vertically, along a common morphism boundary, and horizontally, along a common object boundary. To differentiate on 2-morphisms, we write $*$ for horizontal composition and use juxtaposition to denote vertical composition. A 2-functor between 2-categories comprises maps of objects, morphisms, and 2-morphisms strictly compatible with the 2-categorical structures. There are also notions of 2- and 3-morphisms between 2-categories, which we introduce now.

Definition 2.1 Let $F, G: I \rightarrow \mathcal{A}$ be 2-functors. A **2-natural transformation** $\mu: F \Rightarrow G$ comprises the data of a morphism $\mu_i: Fi \rightarrow Gi$ of \mathcal{A} for each $i \in I$, which must satisfy

- (1) for all morphisms $f: i \rightarrow j$ of I , we have $(Gf)\mu_i = \mu_j(Ff)$, and
- (2) for all 2-morphisms $\alpha: f \Rightarrow g$ of I , we have $G\alpha * \mu_i = \mu_j * F\alpha$.

$$Fi \xrightarrow{\mu_i} Gi \begin{array}{c} \xrightarrow{Gf} \\ \Downarrow G\alpha \\ \xrightarrow{Gg} \end{array} Gj = Fi \begin{array}{c} \xrightarrow{Ff} \\ \Downarrow F\alpha \\ \xrightarrow{Fg} \end{array} Fj \xrightarrow{\mu_j} Gj$$

Definition 2.2 Let $F, G: I \rightarrow \mathcal{A}$ be 2-functors and let $\mu, \nu: F \Rightarrow G$ be 2-natural transformations. A **modification** $\varphi: \mu \Rightarrow \nu$ comprises the data of a 2-morphism $\varphi_i: \mu_i \Rightarrow \nu_i$ for each $i \in I$, which satisfy $Gf * \varphi_i = \varphi_j * Ff$, for all morphisms $f: i \rightarrow j$ of I .

$$Fi \begin{array}{c} \xrightarrow{\mu_i} \\ \Downarrow \varphi_i \\ \xrightarrow{\nu_i} \end{array} Gi \xrightarrow{Gf} Gj = Fi \xrightarrow{Ff} Fj \begin{array}{c} \xrightarrow{\mu_j} \\ \Downarrow \varphi_j \\ \xrightarrow{\nu_j} \end{array} Gj$$

With these definitions, 2-functors, 2-natural transformations, and modifications assemble into a 2-category.

Notation 2.3 Let I and \mathcal{A} be 2-categories. We denote by $[I, \mathcal{A}]$ the 2-category of 2-functors $I \rightarrow \mathcal{A}$, 2-natural transformations between them, and modifications.

We are now ready to define 2-dimensional limits.

Definition 2.4 Let I and \mathcal{A} be 2-categories, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. A **2-limit** of F comprises the data of an object $L \in \mathcal{A}$ together with a 2-natural transformation $\lambda: \Delta L \Rightarrow F$, which are such that, for each object $X \in \mathcal{A}$, the functor

$$\lambda_* \circ \Delta: \mathcal{A}(X, L) \rightarrow [I, \mathcal{A}](\Delta X, F)$$

given by post-composition with λ is an isomorphism of categories.

In what follows, we call a 2-natural transformation $\Delta X \Rightarrow F$ from a constant functor a 2-cone over F .

Remark 2.5 There are two aspects of the universal property of a 2-limit, which arise from the isomorphism of categories $\lambda_* \circ \Delta: \mathcal{A}(X, L) \rightarrow [I, \mathcal{A}](\Delta X, F)$ at the level of objects and at the level of morphisms. We reformulate this more explicitly as follows. For every $X \in \mathcal{A}$,

- (1) for every 2-cone $\mu: \Delta X \Rightarrow F$, there is a unique morphism $f_\mu: X \rightarrow L$ in \mathcal{A} such that $\lambda \Delta f_\mu = \mu$,
- (2) for every modification $\varphi: \mu \Rrightarrow \nu$ between 2-cones $\mu, \nu: \Delta X \Rightarrow F$, there is a unique 2-morphism $\alpha: f_\mu \Rightarrow f_\nu$ in \mathcal{A} such that $\lambda * \Delta \alpha = \varphi$.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \Delta X & \xrightarrow{\Delta f_\mu} & \Delta L \\
 \Downarrow \Delta \alpha & & \downarrow \lambda \\
 \Delta X & \xrightarrow{\Delta f_\nu} & \Delta L \\
 \Downarrow \Delta \alpha & & \downarrow \lambda \\
 \Delta X & \xrightarrow{\Delta f_\nu} & \Delta L
 \end{array} & \xrightarrow{\lambda} & F \\
 & = & \\
 \begin{array}{ccc}
 \Delta X & \xrightarrow{\mu} & F \\
 \Downarrow \varphi & & \downarrow \\
 \Delta X & \xrightarrow{\nu} & F
 \end{array}
 \end{array} \tag{2.5}$$

We now define strict-slice 2-categories of 2-cones and 2-terminal objects.

Definition 2.6 Let $F: I \rightarrow \mathcal{A}$ be a 2-functor. The **strict-slice** $\Delta \downarrow_s F$ of 2-cones over F is defined to be the following pullback in the (1-)category of 2-categories and 2-functors.

$$\begin{array}{ccc}
 \Delta \downarrow_s F & \longrightarrow & [\mathbb{2}, [I, \mathcal{A}]] \\
 \downarrow \lrcorner & & \downarrow (s, t) \\
 \mathcal{A} \times \mathbb{1} & \xrightarrow{(\Delta, F)} & [I, \mathcal{A}] \times [I, \mathcal{A}]
 \end{array}$$

This 2-category $\Delta \downarrow_s F$ is given by the following data:

- (i) an object in $\Delta \downarrow_s F$ is a pair (X, μ) of an object $X \in \mathcal{A}$ together with a 2-natural transformation $\mu: \Delta X \Rightarrow F$,
- (ii) a morphism $f: (X, \mu) \rightarrow (Y, \nu)$ consists of a morphism $f: X \rightarrow Y$ in \mathcal{A} such that $\nu \Delta f = \mu$,
- (iii) a 2-morphism $\alpha: f \Rrightarrow g$ between morphisms $f, g: (X, \mu) \rightarrow (Y, \nu)$ is a 2-morphism $\alpha: f \Rightarrow g$ in \mathcal{A} such that $\lambda * \Delta \alpha = \text{id}_\mu$.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \Delta X & \xrightarrow{\Delta f} & \Delta L \\
 \Downarrow \Delta \alpha & & \downarrow \lambda \\
 \Delta X & \xrightarrow{\Delta g} & \Delta L \\
 \Downarrow \Delta \alpha & & \downarrow \lambda \\
 \Delta X & \xrightarrow{\Delta g} & \Delta L
 \end{array} & \xrightarrow{\lambda} & F \\
 & = & \\
 \Delta X & \xrightarrow{\mu} & F
 \end{array} \tag{2.6}$$

Definition 2.7 Let \mathcal{A} be a 2-category. An object $L \in \mathcal{A}$ is 2-terminal if for all $X \in \mathcal{A}$ there is an isomorphism of categories $\mathcal{A}(X, L) \cong \mathbb{1}$.

As for 2-limits, there are also two aspects of the universal property of a 2-terminal object. Since we are interested here by 2-terminal objects in a strict-slice 2-category of 2-cones, we give a more explicit description of their universal property. We will then compare this description with the universal property of 2-limits (c.f. Remark 2.5).

Remark 2.8 Given a 2-functor $F: I \rightarrow \mathcal{A}$, we describe the two aspects of the universal property of a 2-terminal object in the strict-slice $\Delta \downarrow_s F$. Such a 2-terminal object comprises the data of an object $L \in \mathcal{A}$ together with a 2-cone $\lambda: \Delta L \Rightarrow F$ which satisfy the following:

- (1) for every $X \in \mathcal{A}$ and every 2-cone $\mu: \Delta X \Rightarrow F$, there is a unique morphism $f_\mu: X \rightarrow L$ in \mathcal{A} such that $\lambda \Delta f_\mu = \mu$,
- (2) for every $X \in \mathcal{A}$ and every 2-cone $\mu: \Delta X \Rightarrow F$, the unique 2-morphism $f_\mu \Rightarrow f_\mu$ in $\Delta \downarrow_s F$ is the identity id_{f_μ} .

In particular, we can see that the 2-dimensional aspect above seems somehow degenerate in comparison to (2) of Remark 2.5. However, the 1-dimensional aspect is the same as the one expressed in Remark 2.5 (1). This gives the following result.

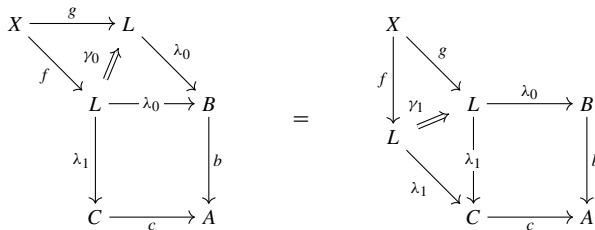
Proposition 2.9 *Let I and \mathcal{A} be 2-categories, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. If $(L, \lambda: \Delta L \Rightarrow F)$ is a 2-limit of F , then (L, λ) is 2-terminal in the strict-slice $\Delta \downarrow_s F$ of 2-cones over F .*

Proof By Remark 2.5 (1) and Remark 2.8 (1), we observe that the 1-dimensional aspects of the universal property of a 2-limit and of a 2-terminal object in the strict-slice coincide. Both say that, for every $X \in \mathcal{A}$ and every 2-cone $\mu: \Delta X \Rightarrow F$, there exists a unique morphism $f_\mu: X \rightarrow L$ in \mathcal{A} such that $\lambda \Delta f_\mu = \mu$.

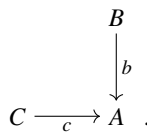
It remains to show (2) of Remark 2.8, that is, that the unique 2-morphism $f_\mu \Rightarrow f_\mu$ in $\Delta \downarrow_s F$ is the identity id_{f_μ} . Any 2-morphism $\alpha: f_\mu \Rightarrow f_\mu$ in $\Delta \downarrow_s F$ must satisfy $\lambda * \Delta \alpha = \text{id}_{\lambda \Delta f}$ by Definition 2.6 (iii). In particular, we also have $\lambda * \Delta \text{id}_f = \text{id}_{\lambda \Delta f} = \lambda * \Delta \alpha$. By the uniqueness in Remark 2.5 (2), it follows that $\alpha = \text{id}_f$. □

However, it is not true that every 2-terminal object in the strict-slice of 2-cones is a 2-limit. One reason for this is that the strict-slice only sees the identity modifications between 2-cones (compare (2.6) with (2.5)). With this in mind, to illustrate this failure we give an example of a modification between two 2-cones which does not arise from a 2-morphism.

Counter-example 2.10 Let I be the pullback shape diagram $\{\bullet \rightarrow \bullet \leftarrow \bullet\}$. Let \mathcal{A} be the 2-category generated by the data



subject to the relations $b \lambda_0 = c \lambda_1$ and $b * \gamma_0 = c * \gamma_1$. Take $F: I \rightarrow \mathcal{A}$ to be the diagram



Claim *The object $(L, \lambda: \Delta L \Rightarrow F)$ is 2-terminal in the strict-slice $\Delta \downarrow_s F$ of 2-cones over F , but the functor*

$$\lambda_* \circ \Delta: \mathcal{A}(X, L) \rightarrow [I, \mathcal{A}](\Delta X, F)$$

given by post-composition with λ is not surjective on morphisms, thus (L, λ) is not a 2-limit of F .

Proof The objects of the strict-slice $\Delta \downarrow_s F$ are given by the 2-cones over F :

$$(L, \lambda), \quad (X, \lambda * f), \quad \text{and} \quad (X, \lambda * g).$$

Each of these objects admits precisely one morphism to (L, λ) in $\Delta \downarrow_s F$ given by

$$\begin{aligned} \text{id}_L &: (L, \lambda) \rightarrow (L, \lambda) \\ f &: (X, \lambda * f) \rightarrow (L, \lambda) \\ g &: (X, \lambda * g) \rightarrow (L, \lambda). \end{aligned}$$

There are no non-trivial 2-morphisms to (L, λ) in $\Delta \downarrow_s F$, since there are no non-trivial 2-morphisms between X and L in \mathcal{A} . This proves that (L, λ) is 2-terminal in $\Delta \downarrow_s F$.

However, the 2-morphisms γ_0 and γ_1 give the data of a modification $\gamma: \lambda * f \rightrightarrows \lambda * g$, i.e. a morphism in $[I, \mathcal{A}](\Delta X, F)$. But there is no 2-morphism between f and g in \mathcal{A} that maps to γ via $\lambda_* \circ \Delta: \mathcal{A}(X, L) \rightarrow [I, \mathcal{A}](\Delta X, F)$. Hence (L, λ) is not the 2-limit of F . \square

A 2-terminal object in the strict-slice is, however, a 2-limit when the 2-category \mathcal{A} admits tensors by the category $\mathfrak{2} = \{0 \rightarrow 1\}$. Indeed, it follows from this condition that the 1-dimensional aspect of the universal property of a 2-limit implies the 2-dimensional one (see [8, §3]). A 2-category \mathcal{A} is said to *admit tensors by a category \mathcal{C}* when, for each object $X \in \mathcal{A}$, there exists an object $X \otimes \mathcal{C} \in \mathcal{A}$ together with isomorphisms of categories

$$\mathcal{A}(X \otimes \mathcal{C}, Y) \cong \text{Cat}(\mathcal{C}, \mathcal{A}(X, Y)),$$

2-natural in $X, Y \in \mathcal{A}$. In particular, this implies that there is a bijection between morphisms $X \otimes \mathcal{C} \rightarrow Y$ in \mathcal{A} and functors $\mathcal{C} \rightarrow \mathcal{A}(X, Y)$.

Proposition 2.11 *Suppose \mathcal{A} is a 2-category that admits tensors by $\mathfrak{2}$, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. Then an object $(L, \lambda: \Delta L \Rightarrow F)$ is a 2-terminal object in the strict-slice $\Delta \downarrow_s F$ if and only if it is a 2-limit of F .*

Proof We already saw one of the implications in Proposition 2.9. Let us prove the other.

Suppose that (L, λ) is a 2-terminal object in the strict-slice $\Delta \downarrow_s F$. We show that (L, λ) satisfies the two conditions (1) and (2) of Remark 2.5, expressing the two aspects of the universal property of a 2-limit. It is clear that (1) holds since it is the same condition as the one expressing the 1-dimensional aspect of the universal property of a 2-terminal object, as in Remark 2.8 (1). It remains to show (2).

Before proceeding, let us examine the effect of admitting tensors by $\mathfrak{2}$. Let $X \in \mathcal{A}$. The universal property of tensoring by $\mathfrak{2}$ in \mathcal{A} gives a canonical bijection between morphisms $X \otimes \mathfrak{2} \rightarrow L$ in \mathcal{A} and functors $\mathfrak{2} \rightarrow \mathcal{A}(X, L)$, but these functors coincide with 2-morphisms from X to L . The 2-category $[I, \mathcal{A}]$ is also tensored by $\mathfrak{2}$, since \mathcal{A} is, and the tensor is given object-wise [7, §3.3]. In particular, $\Delta(X \otimes \mathfrak{2}) = \Delta X \otimes \mathfrak{2}$ as constant functors, by the object-wise definition of tensoring by $\mathfrak{2}$. As $[I, \mathcal{A}]$ admits tensors by $\mathfrak{2}$, we have a canonical bijection between 2-cones $\Delta X \otimes \mathfrak{2} \Rightarrow F$ and functors $\mathfrak{2} \rightarrow [I, \mathcal{A}](\Delta X, F)$, which in turn coincide with modifications between ΔX and F .

By Remark 2.8 (1), for every 2-cone $\varphi: \Delta X \otimes \mathfrak{2} \Rightarrow F$, there exists a unique morphism $\alpha: X \otimes \mathfrak{2} \rightarrow L$ in \mathcal{A} such that $\lambda \Delta \alpha = \varphi$. Using the above, we can reformulate this statement as follows: for every modification φ between 2-cones $\Delta X \Rightarrow F$, there is a unique 2-morphism α between morphisms $X \rightarrow L$ such that $\lambda \Delta \alpha = \varphi$. But this is exactly (2) of Remark 2.5. \square

The category Cat of categories and functors is cartesian closed. Therefore, it is enriched over itself and so is, in particular, tensored over Cat . In other words, the 2-category Cat of categories, functors, and natural transformations admits tensors by all categories, and these tensors are given by cartesian products. In particular, Proposition 2.11 yields the following result.

Corollary 2.12 *Let $F: I \rightarrow \text{Cat}$ be a 2-functor into Cat . A pair $(\mathcal{L}, \lambda: \Delta\mathcal{L} \Rightarrow F)$ is a 2-terminal in the strict-slice $\Delta \downarrow_s F$ of 2-cones over F if and only if it is a 2-limit of F .*

3 2-Terminal Objects in Lax- and Pseudo-Slices are not Related to 2-Limits

We have seen in Sect. 2 that 2-terminal objects in the strict-slice of 2-cones over a 2-functor do not, in general, succeed in capturing both aspects of the universal properties of 2-limits. In particular, the problem is that the strict-slice of 2-cones does not see the modifications between two 2-cones with same summit. In attempt to rectify this, we might consider richer slice 2-categories containing more data in their morphisms: the *lax-slice* and the *pseudo-slice* of 2-cones. However, 2-terminal objects in these new slice 2-categories seem to be unrelated to 2-limits. As we present below, there are 2-limits that are not 2-terminal objects in the lax-slice (resp. pseudo-slice), and conversely so.

We start by introducing lax- and pseudo-natural transformations between 2-functors, and modifications between them.

Definition 3.1 Let I and \mathcal{A} be 2-categories, and let $F, G: I \rightarrow \mathcal{A}$ be 2-functors between them. A **lax-natural transformation** $\mu: F \Rightarrow G$ comprises the data of

- (i) a morphism $\mu_i: Fi \rightarrow Gi$, for each $i \in I$,
- (ii) a 2-morphism $\mu_f: (Gf)\mu_i \Rightarrow \mu_j(Ff)$, for each morphism $f: i \rightarrow j$ in I ,

$$\begin{array}{ccc}
 Fi & \xrightarrow{\mu_i} & Gi \\
 Ff \downarrow & \swarrow \mu_f & \downarrow Gf \\
 Fj & \xrightarrow{\mu_j} & Gj
 \end{array}$$

which satisfy the following conditions:

- (1) for all $i \in I$, $\mu_{id_i} = id_{\mu_i}$,
- (2) for all composable morphisms f, g in I , $\mu_{gf} = (\mu_g * Ff)(Gg * \mu_f)$,
- (3) for all 2-morphisms $\alpha: f \Rightarrow g$ in I , we have that $\mu_g(G\alpha * \mu_i) = (\mu_j * F\alpha)\mu_f$.

$$\begin{array}{ccc}
 Fi & \xrightarrow{\mu_i} & Gi \\
 Fg \downarrow & \swarrow \mu_g & \downarrow Gg \\
 Fj & \xrightarrow{\mu_j} & Gj
 \end{array}
 \begin{array}{c}
 \left. \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right\} G\alpha \\
 \left. \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right\} Gf
 \end{array}
 =
 \begin{array}{ccc}
 Fi & \xrightarrow{\mu_i} & Gi \\
 F\alpha \leftarrow \downarrow Ff & \swarrow \mu_f & \downarrow Gf \\
 Fj & \xrightarrow{\mu_j} & Gj
 \end{array}$$

A **pseudo-natural transformation** is a lax-natural transformation $\mu: F \Rightarrow G$ whose every 2-morphism component μ_f is invertible.

Definition 3.2 Let $F, G: I \rightarrow \mathcal{A}$ be 2-functors, and let $\mu, \nu: F \Rightarrow G$ be lax-natural transformations between them. A **modification** $\varphi: \mu \Rrightarrow \nu$ comprises the data of a 2-morphism $\varphi_i: \mu_i \Rrightarrow \nu_i$ for each $i \in I$, which satisfy $\nu_f(Gf * \varphi_i) = (\varphi_j * Ff)\mu_f$, for all morphisms $f: i \rightarrow j$ in I .

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \mu_i & \\
 & \Downarrow \varphi_i & \\
 Fi & \xrightarrow{v_i} & Gi \\
 Ff \downarrow & \swarrow \nu_f & \downarrow Gf \\
 Fj & \xrightarrow{v_j} & Gj
 \end{array} & = & \begin{array}{ccc}
 Fi & \xrightarrow{\mu_i} & Gi \\
 Ff \downarrow & \swarrow \mu_f & \downarrow Gf \\
 Fj & \xrightarrow{\mu_j} & Gj \\
 & \Downarrow \varphi_j & \\
 & \nu_j &
 \end{array}
 \end{array}$$

Similarly, we have a notion of modification between pseudo-natural transformations.

Remark 3.3 Note that a 2-natural transformation $\mu: F \Rightarrow G$ as defined in Definition 2.1 is precisely a lax-natural transformation whose every 2-morphism component μ_f is an identity. Moreover, modifications in the sense just defined between two lax-natural transformations which happen to be 2-natural coincide with the modifications of Definition 2.2.

As in the case of 2-natural transformations, lax- and pseudo-natural transformations and modifications assemble into 2-categories whose objects are 2-functors.

Notation 3.4 Let I and \mathcal{A} be 2-categories. We can define two 2-categories whose objects are the 2-functors $I \rightarrow \mathcal{A}$:

- (i) the 2-category $\text{Lax}[I, \mathcal{A}]$, whose 1- and 2-morphisms are lax-natural transformations and modifications,
- (ii) the 2-category $\text{Ps}[I, \mathcal{A}]$, whose 1- and 2-morphisms are pseudo-natural transformations and modifications.

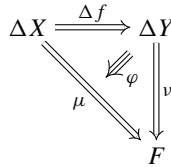
The lax-slice and pseudo-slice of 2-cones over a 2-functor $F: I \rightarrow \mathcal{A}$ can be defined as pullbacks, as in Definition 2.6, where we replace the upper-left corner with the 2-categories $\text{Lax}[\mathbb{2}, [I, \mathcal{A}]]$ and $\text{Ps}[\mathbb{2}, [I, \mathcal{A}]]$, respectively. These constructions do not change the objects of the slice, but add more morphisms between them.

Definition 3.5 Let $F: I \rightarrow \mathcal{A}$ be a 2-functor. The **lax-slice** $\Delta \downarrow_I F$ of 2-cones over F is defined to be the following pullback in the (1-)category of 2-categories and 2-functors.

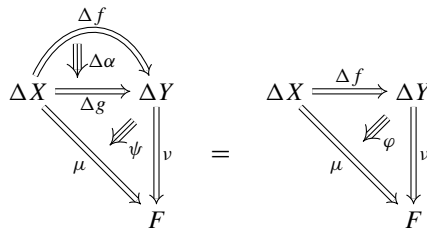
$$\begin{array}{ccc}
 \Delta \downarrow_I F & \longrightarrow & \text{Lax}[\mathbb{2}, [I, \mathcal{A}]] \\
 \downarrow \lrcorner & & \downarrow (s, t) \\
 \mathcal{A} \times \mathbb{1} & \xrightarrow{(\Delta, F)} & [I, \mathcal{A}] \times [I, \mathcal{A}]
 \end{array}$$

This 2-category $\Delta \downarrow_I F$ is given by the following data:

- (i) an object in $\Delta \downarrow_I F$ is a pair (X, μ) of an object $X \in \mathcal{A}$ together with a 2-natural transformation $\mu: \Delta X \Rightarrow F$,
- (ii) a morphism $(f, \varphi): (X, \mu) \rightarrow (Y, \nu)$ consists of a morphism $f: X \rightarrow Y$ in \mathcal{A} together with a modification $\varphi: \nu \Delta f \Rrightarrow \mu$,



(iii) a 2-morphism $\alpha: (f, \varphi) \Rightarrow (g, \psi)$ between morphisms $(f, \varphi), (g, \psi): (X, \mu) \rightarrow (Y, \nu)$ is a 2-morphism $\alpha: f \Rightarrow g$ in \mathcal{A} such that $\psi(\nu * \Delta\alpha) = \varphi$.



Similarly, we can define the **pseudo-slice** $\Delta \downarrow_p F$ of 2-cones over F by replacing the upper-left corner $\text{Lax}[2, [I, \mathcal{A}]]$ in the pullback above with $\text{Ps}[2, [I, \mathcal{A}]]$. The pseudo-slice corresponds to the sub-2-category of the lax-slice $\Delta \downarrow_l F$ containing all objects and only the morphisms (f, φ) for which the modification φ is invertible, and which is locally-full on 2-morphisms.

Remark 3.6 Note that the strict-slice $\Delta \downarrow_s F$ as defined in Definition 2.6 corresponds to the locally-full sub-2-category of the lax- or pseudo-slice containing all objects and only the morphisms (f, φ) for which the modification φ is an identity.

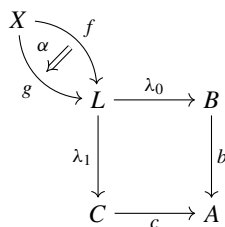
We now give two counter-examples which show that

- not every 2-limit is 2-terminal in the lax-slice of 2-cones (Counter-ex. 3.7),
- not every 2-terminal object in the lax-slice of 2-cones is a 2-limit (Counter-ex. 3.9).

These statements imply that, unlike in the case of strict-slices, 2-terminal objects in the lax-slice are not at all related to 2-limits. We derive counter-examples to show that the same is true for pseudo-slices, namely that 2-terminal objects in the pseudo-slice of 2-cones are not related to 2-limits.

We first give an example of a 2-limit that is not 2-terminal in the lax-slice of 2-cones. To illustrate this failure we seek a case where the lax-slice sees too many morphisms between the 2-cones. In the counter-example below, we show that a 2-morphism that is part of the 2-dimensional aspect of the universal property of a 2-limit might create undesirable morphisms in the lax-slice of 2-cones.

Counter-example 3.7 Let I be the pullback shape diagram $\{\bullet \longrightarrow \bullet \longleftarrow \bullet\}$. Let \mathcal{A} be the 2-category generated by the data



subject to the relation $b\lambda_0 = c\lambda_1$. Take $F: I \rightarrow \mathcal{A}$ to be the diagram

$$\begin{array}{ccc} & B & \\ & \downarrow b & \\ C & \xrightarrow{c} & A \end{array} .$$

Claim *The object $(L, \lambda: \Delta L \Rightarrow F)$ is the 2-limit of F , but it is not 2-terminal in the lax-slice $\Delta \downarrow_I F$ of 2-cones over F .*

Proof Let us begin by enumerating all the 2-cones over F :

$$(L, \lambda), \quad (X, \lambda * f), \quad \text{and} \quad (X, \lambda * g).$$

We can see that (L, λ) is a 2-limit of F , since we have

$$\mathcal{A}(X, L) = \{f \xrightarrow{\alpha} g\} \quad \text{and} \quad [I, \mathcal{A}](\Delta X, F) = \{\lambda * f \xrightarrow{\lambda * \alpha} \lambda * g\}$$

and $\mathcal{A}(L, L) = \{\text{id}_L\}$ and $[I, \mathcal{A}](\Delta L, F) = \{\lambda\}$.

However, there are two distinct morphisms from $(X, \lambda * g)$ to (L, λ) in the lax-slice $\Delta \downarrow_I F$, which are given by

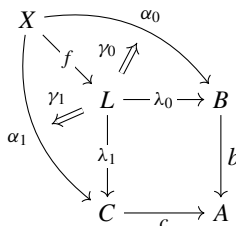
$$\begin{aligned} (g, (\text{id}_{\lambda_0 g}, \text{id}_{\lambda_1 g})) &: (X, \lambda * g) \rightarrow (L, \lambda) \\ (f, (\lambda_0 * \alpha, \lambda_1 * \alpha)) &: (X, \lambda * g) \rightarrow (L, \lambda). \end{aligned}$$

Therefore (L, λ) is not 2-terminal in $\Delta \downarrow_I F$. □

Reduction 3.8 By requiring α to be invertible in Counter-example 3.7, we can similarly show that (L, λ) is the 2-limit of F , but is not 2-terminal in the pseudo-slice $\Delta \downarrow_p F$ of 2-cones over F .

Next we give an example of a 2-terminal object in the lax-slice of 2-cones that is not a 2-limit. This counter-example is designed to capture a particular arrangement of two 2-cones over a 2-functor together with a single non-trivial modification between them. This modification gives rise to a morphism in the lax-slice between these two 2-cones, exhibiting the target 2-cone as 2-terminal in the lax-slice. However, the source 2-cone is not in the image of the post-composition functor by the target 2-cone, which shows that the latter is not a 2-limit.

Counter-example 3.9 Let I be the pullback shape diagram $\{\bullet \longrightarrow \bullet \longleftarrow \bullet\}$. Let \mathcal{A} be the 2-category generated by the data



subject to the relations $b\lambda_0 = c\lambda_1$ and $b\alpha_0 = c\alpha_1$. Take $F: I \rightarrow \mathcal{A}$ to be the diagram

$$\begin{array}{ccc}
 & & B \\
 & & \downarrow b \\
 C & \xrightarrow{c} & A
 \end{array}$$

Claim *The object $(L, \lambda: \Delta L \Rightarrow F)$ is 2-terminal in the lax-slice $\Delta \downarrow_l F$ of 2-cones over F , but the functor*

$$\lambda_* \circ \Delta: \mathcal{A}(X, L) \rightarrow [I, \mathcal{A}](\Delta X, F)$$

given by post-composition with λ is not surjective on objects, thus (L, λ) is not a 2-limit of F .

Proof The objects of the lax-slice $\Delta \downarrow_l F$ are given by the 2-cones over F :

$$(L, \lambda), \quad (X, \alpha), \quad \text{and} \quad (X, \lambda * f).$$

Each of these objects admits precisely one morphism to (L, λ) in $\Delta \downarrow_l F$ given by

$$\begin{aligned}
 (\text{id}_L, \text{id}_{\lambda_0}, \text{id}_{\lambda_1}) &: (L, \lambda) \rightarrow (L, \lambda) \\
 (f, \gamma_0, \gamma_1) &: (X, \alpha) \rightarrow (L, \lambda) \\
 (f, \text{id}_{\lambda_0 f}, \text{id}_{\lambda_1 f}) &: (X, \lambda * f) \rightarrow (L, \lambda).
 \end{aligned}$$

There are no non-trivial 2-morphisms to (L, λ) in $\Delta \downarrow_s F$, since there are no non-trivial 2-morphisms between X and L in \mathcal{A} . This proves that (L, λ) is 2-terminal in $\Delta \downarrow_l F$.

However, the 2-cone $\alpha: \Delta X \Rightarrow F$ is an object of $[I, \mathcal{A}](\Delta X, F)$, but it is not in the image of $\lambda_* \circ \Delta$. □

Reduction 3.10 By requiring γ_0 and γ_1 to be invertible in Counter-example 3.9, we can similarly show that (L, λ) is 2-terminal in the pseudo-slice $\Delta \downarrow_p F$ of 2-cones over F , but it is not a 2-limit of F .

4 The Cases of Pseudo- and Lax-Limits

Recall that part of the definition of a 2-limit involved 2-cones which were 2-natural transformations. In Sect. 3, we presented weaker notions of 2-dimensional natural transformations, namely pseudo- and lax-natural transformations. These give other ways of taking a 2-dimensional limit of a 2-functor, by changing the shape of the 2-dimensional cones. The corresponding notions are called *pseudo-limits* and *lax-limits*.

In this section, we show that results similar to those we have seen in Sects. 2, 3 hold for pseudo- and lax-limits. In the lax-limit case, we show that:

- every lax-limit is 2-terminal in the strict-slice of lax-cones (Remark 4.3),
- not every 2-terminal object in the strict-slice of lax-cones is a lax-limit (Counter-ex. 4.5),
- not every lax-limit is 2-terminal in the lax-slice of lax-cones (Counter-ex. 4.10),
- not every 2-terminal object in the lax-slice of lax-cones is a lax-limit (Counter-ex. 4.14).

From the last two, we also derive the result that lax-limits are not related to 2-terminal objects in the pseudo-slice of lax-cones. With all of the results and counter-examples we have established thus far, we are able to derive proofs and counter-examples covering the conjectures related to pseudo-limits.

We first introduce the notions of pseudo- and lax-limits.

Definition 4.1 Let I and \mathcal{A} be 2-categories, and let $F : I \rightarrow \mathcal{A}$ be a 2-functor.

- (i) A **pseudo-limit** of F comprises the data of an object $L \in \mathcal{A}$ together with a pseudo-natural transformation $\lambda : \Delta L \Rightarrow F$ such that, for each object $X \in \mathcal{A}$, the functor

$$\lambda_* \circ \Delta : \mathcal{A}(X, L) \rightarrow \text{Ps}[I, \mathcal{A}](\Delta X, F)$$

given by post-composition with λ is an isomorphism of categories.

- (ii) A **lax-limit** of F comprises the data of an object $L \in \mathcal{A}$ together with a lax-natural transformation $\lambda : \Delta L \Rightarrow F$ such that, for each object $X \in \mathcal{A}$, the functor

$$\lambda_* \circ \Delta : \mathcal{A}(X, L) \rightarrow \text{Lax}[I, \mathcal{A}](\Delta X, F)$$

given by post-composition with λ is an isomorphism of categories.

In order to consider slices in which these pseudo- and lax-limit cones live, we need to change the shape of the cone objects of the slices considered in Definitions 2.6, 3.5.

Remark 4.2 We can also define the strict-, pseudo-, and lax-slices of pseudo-cones (resp. lax-cones) over F , by considering objects of the form (X, μ) where $\mu : \Delta X \Rightarrow F$ is a pseudo-natural (resp. lax-natural) transformation. These constructions can be achieved by replacing $[I, \mathcal{A}]$ in the pullbacks of Definitions 2.6, 3.5 with $\text{Ps}[I, \mathcal{A}]$ (resp. $\text{Lax}[I, \mathcal{A}]$). For example, the pseudo-slice of lax-cones is the following pullback.

$$\begin{array}{ccc} \Delta \downarrow_p^{\text{lx}} F & \longrightarrow & \text{Ps}[\mathcal{A}, \text{Lax}[I, \mathcal{A}]] \\ \downarrow \lrcorner & & \downarrow (s, t) \\ \mathcal{A} \times \mathbb{1} & \xrightarrow{(\Delta, F)} & \text{Lax}[I, \mathcal{A}] \times \text{Lax}[I, \mathcal{A}] \end{array}$$

There is an analogue of Proposition 2.9 in the case of pseudo-limits (resp. lax-limits), whose proof may be derived by replacing 2-natural transformations with pseudo-natural ones (resp. lax-natural ones).

Proposition 4.3 A pseudo-limit (resp. lax-limit) of a 2-functor is 2-terminal in the strict-slice of pseudo-cones (resp. lax-cones).

However, not every 2-terminal object in the strict-slice of pseudo- or lax-cones is a pseudo- or lax-limit. In particular, Counter-example 2.10 exhibits such an object in the pseudo-limit case:

Reduction 4.4 Since there are no invertible 2-morphisms in Counter-example 2.10, this is also an example of a 2-terminal object in the strict-slice of pseudo-cones that is not a pseudo-limit.

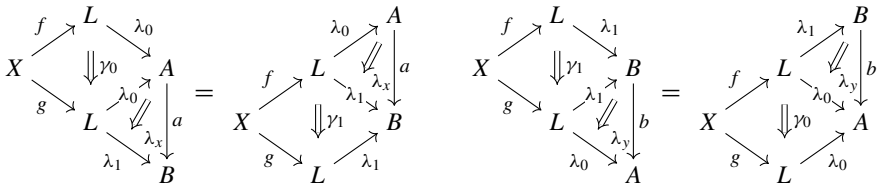
Let us recall that Counter-example 2.10 has 2-morphisms γ_0 and γ_1 that introduce two additional lax-cones with summit X over F . These new lax-cones do not admit a morphism to (L, λ) in the strict-slice of lax-cones. This arrangement demonstrates that (L, λ) is not 2-terminal in the strict-slice of lax-cones. Therefore, we can not use this counter-example for the lax-limit case.

The issue at heart here is that modifications between lax-cones over a 2-functor may be turned into lax-cones over this same 2-functor. Thus, to find an example of a 2-terminal object in the strict-slice of lax-cones that is not a lax-limit, we must find a case where such a transformation is not possible. In order to create such an example, we need the diagram shape to have objects that are both the source and the target of a non-trivial morphism.

Counter-example 4.5 Let I be the 2-category freely generated by the data

$$0 \begin{matrix} \xrightarrow{x} \\ \xleftarrow{y} \end{matrix} 1 ,$$

i.e. the non-trivial morphisms in I are given by all possible composites of x and y , e.g. $xyxy$. Let \mathcal{A} be the 2-category freely generated by the data



subject to the relations $(\lambda_x * g)(a * \gamma_0) = \gamma_1(\lambda_x * f)$, and $(\lambda_y * g)(b * \gamma_1) = \gamma_0(\lambda_y * f)$. Again, we have all possible composites of the morphisms a and b in \mathcal{A} , and all possible pastings of the 2-morphisms λ_x and λ_y . Take $F: I \rightarrow \mathcal{A}$ to be the diagram defined on the generators x and y of I by

$$A \begin{matrix} \xrightarrow{a} \\ \xleftarrow{b} \end{matrix} B .$$

Remark 4.6 Note that the morphisms $\lambda_0: L \rightarrow A$ and $\lambda_1: L \rightarrow B$ together with the 2-morphisms $\lambda_x: a\lambda_0 \Rightarrow \lambda_1$ and $\lambda_y: b\lambda_1 \Rightarrow \lambda_0$ suffice to give the data of a lax-natural transformation $\lambda: \Delta L \Rightarrow F$. Indeed, by Definition 3.1 (2), the 2-morphism component of λ at some composite of x and y is determined by the corresponding pasting of the 2-morphism components λ_x and λ_y .

Claim The object $(L, \lambda: \Delta L \Rightarrow F)$ is 2-terminal in the strict-slice $\Delta \downarrow_s^{lx} F$ of lax-cones over F , but the functor

$$\lambda_* \circ \Delta: \mathcal{A}(X, L) \rightarrow \text{Lax}[I, \mathcal{A}](\Delta X, F)$$

given by post-composition with λ is not surjective on morphisms, thus (L, λ) is not a lax-limit of F .

Proof The objects of the strict-slice $\Delta \downarrow_s^{lx} F$ are given by the lax-cones over F :

$$(L, \lambda), \quad (X, \lambda * f), \quad \text{and} \quad (X, \lambda * g).$$

Note that the 2-morphisms γ_0 and γ_1 do not induce lax-cones over F with summit X , since there are no 2-morphisms from $\lambda_1 g$ to $\lambda_0 f$, and from $\lambda_0 g$ to $\lambda_1 f$ in \mathcal{A} , respectively. There are also no lax-cones over F with summit A or B since there are no non-trivial 2-morphisms between any two composites of a and b in \mathcal{A} . Each of the objects above admits precisely one morphism to (L, λ) in $\Delta \downarrow_s^{lx} F$ given by

$$\begin{aligned} \text{id}_L: (L, \lambda) &\rightarrow (L, \lambda) \\ f: (X, \lambda * f) &\rightarrow (L, \lambda) \\ g: (X, \lambda * g) &\rightarrow (L, \lambda). \end{aligned}$$

There are no non-trivial 2-morphisms to (L, λ) in $\Delta \downarrow_s^{lx} F$, since there are no non-trivial 2-morphisms between X and L in \mathcal{A} . This proves that (L, λ) is 2-terminal in $\Delta \downarrow_s^{lx} F$.

However, the 2-morphisms γ_0 and γ_1 give the data of a modification $\gamma : \lambda * f \Rightarrow \lambda * g$, i.e. a morphism in $\text{Lax}[I, \mathcal{A}](\Delta X, F)$. But there is no 2-morphism between f and g in \mathcal{A} that maps to γ via $\lambda_* \circ \Delta : \mathcal{A}(X, L) \rightarrow \text{Lax}[I, \mathcal{A}](\Delta X, F)$. Hence (L, λ) is not the lax-limit of F . □

Remark 4.7 Note that, in the above counter-example, it is essential to have all free composites of x and y in I , and also of a and b in \mathcal{A} . Indeed, if we impose any conditions on the composites of x and y , e.g. x and y are mutual inverses, then the 2-functor F must preserve these, and these new conditions on a and b in \mathcal{A} add undesirable lax-cones with summits A and B .

However, when the 2-category \mathcal{A} admits tensors by $\mathbb{2}$, there is an analogue of Proposition 2.11 in the pseudo-limit (resp. lax-limit) case, whose proof may be derived by replacing 2-natural transformations by pseudo-natural ones (resp. lax-natural ones).

Proposition 4.8 *Suppose \mathcal{A} is a 2-category that admits tensors by $\mathbb{2}$, and let $F : I \rightarrow \mathcal{A}$ be a 2-functor. Then an object is 2-terminal in the strict-slice of pseudo-cones (resp. lax-cones) over F if and only if it is a pseudo-limit (resp. lax-limit) of F .*

We dedicate the rest of the section to exploring counter-examples which together refute all the remaining conjectures relating pseudo- and lax-limits to the lax- and pseudo-slices of appropriate cones. We begin by recalling Counter-example 3.7, which also shows that not every pseudo-limit is a 2-terminal object in the lax-slice of pseudo-cones:

Reduction 4.9 Since there are no invertible 2-morphisms in Counter-example 3.7, this is also an example of a pseudo-limit that is not 2-terminal in the lax-slice of pseudo-cones.

Let us recall that Counter-example 3.7 has a 2-morphism α . This 2-morphism was an obstruction to (L, λ) being 2-terminal in the lax-slice, but not to being a 2-limit. In the move to lax-cones, however, this 2-morphism introduces an additional lax-cone over F with summit X that is not in the image of $\lambda_* \circ \Delta$. Therefore, (L, λ) cannot be a lax-limit of F , and we need a new counter-example for the lax-limit case.

Our new counter-example should have a non-trivial 2-morphism to serve as an obstruction to 2-terminality in the lax-slice. But, to ensure that this 2-morphism is in the image of post-composition by the to-be lax-limit cone, we must also introduce new relations.

Counter-example 4.10 Let $I = \mathbb{2}$, and let \mathcal{A} be the 2-category generated by the data

$$\begin{array}{ccc}
 & \alpha_0 & \\
 & \curvearrowright & \\
 X & \Downarrow \alpha & A \xrightarrow{f} B \\
 & \curvearrowleft & \\
 & \alpha_1 &
 \end{array}$$

subject to the relations $f\alpha_0 = f\alpha_1$ and $f * \alpha = \text{id}_{f\alpha_0}$. Consider the 2-functor $f : \mathbb{2} \rightarrow \mathcal{A}$ given by the morphism $f : A \rightarrow B$.

Claim *The object $(A, \text{id}_f : \Delta A \Rightarrow f)$ is the lax-limit of f in \mathcal{A} , but it is not 2-terminal in the lax-slice $\Delta \downarrow_x^x f$ of lax-cones over f .*

Proof Let us begin by enumerating all the lax-cones over f :

$$(A, \text{id}_f), \quad (X, \text{id}_{f\alpha_0}), \quad \text{and} \quad (X, \text{id}_{f\alpha_1}).$$

Note that the last two above-listed objects differ in their cone components to A : in the first case the leg is α_0 , and in the second case it is α_1 .

We can see that (A, id_f) is a lax-limit of f , since we have

$$\mathcal{A}(X, A) = \{\alpha_0 \rightrightarrows \alpha_1\} \text{ and } \text{Lax}[\mathfrak{D}, \mathcal{A}](\Delta X, f) = \{\text{id}_{f\alpha_0} \xrightarrow{f*\alpha} \text{id}_{f\alpha_1}\}$$

and $\mathcal{A}(A, A) = \{\text{id}_A\}$ and $[\mathfrak{D}, \mathcal{A}](\Delta A, f) = \{\text{id}_f\}$.

However, there are two distinct morphisms from $(X, \text{id}_{f\alpha_1})$ to (A, id_f) in the lax-slice $\Delta \downarrow_I^{\text{lx}} f$ of lax-cones, given by

$$\begin{aligned} (\alpha_1, (\text{id}_{\alpha_1}, \text{id}_{f\alpha_1})) &: (X, \text{id}_{f\alpha_1}) \rightarrow (A, \text{id}_f) \\ (\alpha_0, (\alpha, \text{id}_{f\alpha_1})) &: (X, \text{id}_{f\alpha_1}) \rightarrow (A, \text{id}_f). \end{aligned}$$

where we have used the fact $f * \alpha = \text{id}_{f\alpha_1}$ in displaying the latter morphism. Therefore, (A, id_f) is not 2-terminal in the lax-slice of lax-cones over f . □

Remark 4.11 Note that the object (A, id_f) is also a pseudo-limit of f . This gives a second example of a pseudo-limit that is not 2-terminal in the lax-slice of pseudo-cones.

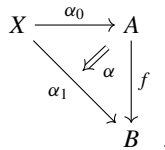
Reduction 4.12 By requiring α to be invertible in Counter-example 4.10, we can similarly show that (A, id_f) is a lax-limit (resp. pseudo-limit) of f , which is not 2-terminal in the pseudo-slice of lax-cones (resp. pseudo-cones).

Moreover, one can derive from Counter-example 3.9 that not every 2-terminal object in the lax-slice of pseudo-cones is a pseudo-limit:

Reduction 4.13 Since there are no invertible 2-morphisms in Counter-example 3.9, this is also an example of a 2-terminal object in the lax-slice of pseudo-cones that is not a pseudo-limit.

Counter-example 3.9 in fact also applies in the lax-cone case. Although the computations are more involved, one can check this is also an example of a 2-terminal object in the lax-slice of lax-cones that it is not a lax-limit. However, there is a more striking example for this case. When considering diagrams of shape \mathfrak{D} , it turns out that even a single non-trivial lax-cone over a morphism exhibits a 2-terminal object in the lax-slice of lax-cones that is not a lax-limit of the morphism.

Counter-example 4.14 Let $I = \mathfrak{D}$, and let \mathcal{A} be the 2-category generated by the data



Consider the 2-functor $f: \mathfrak{D} \rightarrow \mathcal{A}$ given by the morphism $f: A \rightarrow B$.

Claim *The object $(A, \text{id}_f: \Delta A \Rightarrow f)$ is 2-terminal in the lax-slice $\Delta \downarrow_I^{\text{lx}} f$ of lax-cones over f , but the functor*

$$(\text{id}_f)_* \circ \Delta: \mathcal{A}(X, A) \rightarrow \text{Lax}[\mathfrak{D}, \mathcal{A}](\Delta X, f)$$

is not surjective on objects, thus (A, id_f) is not a lax-limit of f .

Proof The objects of the lax-slice $\Delta \downarrow_I^{\text{lx}} f$ are the following lax-cones over f :

$$(A, \text{id}_f), \quad (X, \text{id}_{f\alpha_0}), \quad \text{and} \quad (X, \alpha).$$

Each of these objects admits precisely one morphism to (A, id_f) in $\Delta \downarrow_I^{\text{lx}} f$, given by

$$\begin{aligned} (\text{id}_A, (\text{id}_A, \text{id}_f)) &: (A, \text{id}_f) \rightarrow (A, \text{id}_f) \\ (\alpha_0, (\text{id}_{\alpha_0}, \text{id}_{f\alpha_0})) &: (X, \text{id}_{f\alpha_0}) \rightarrow (A, \text{id}_f) \\ (\alpha_0, (\text{id}_{\alpha_0}, \alpha)) &: (X, \alpha) \rightarrow (A, \text{id}_f). \end{aligned}$$

As there are no non-trivial 2-morphisms between X and A in \mathcal{A} , there are no non-trivial 2-morphisms to (A, id_f) in $\Delta \downarrow_I^{\text{lx}} f$. This shows that (A, id_f) is 2-terminal in the lax-slice $\Delta \downarrow_I^{\text{lx}} f$ of lax-cones.

Next, observe that the lax-cone $\alpha: f\alpha_0 \Rightarrow \alpha_1$ is an object of $\text{Lax}[I, \mathcal{A}](\Delta X, f)$, but it is not in the image of $(\text{id}_f)_* \circ \Delta$. Hence (A, id_f) is not a lax-limit of f . □

Reduction 4.15 By requiring α to be invertible in Counter-example 4.14, we can similarly show that (A, id_f) is 2-terminal in the pseudo-slice of lax-cones (resp. pseudo-cones) over f , but that it is not a lax-limit (resp. pseudo-limit) of f .

5 Bi-Type Limits for the Completionist

At this point we have seen that 2-terminal objects in all slices and 2-dimensional limits do not generally align. In this last section, which we present for completeness, we address one final weakening of the central definitions we have thus far considered. In defining 2-dimensional limits with various strengths of cones, we have always asked for an *isomorphism* of categories to govern the universal property. However, we might seek to relax this requirement by asking instead that the relevant functor induces only an *equivalence* of categories. This leads to the following definitions.

Definition 5.1 Let I and \mathcal{A} be 2-categories, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. A **bi-limit** of F comprises the data of an object $L \in \mathcal{A}$ together with a 2-natural transformation $\lambda: \Delta L \Rightarrow F$ which are such that, for each object $X \in \mathcal{A}$, the functor

$$\lambda_* \circ \Delta: \mathcal{A}(X, L) \rightarrow [I, \mathcal{A}](\Delta X, F)$$

given by post-composition with λ is an equivalence of categories.

Similarly, we can define **pseudo-bi-limit** (resp. **lax-bi-limit**) by replacing $[I, \mathcal{A}]$ in the above with $\text{Ps}[I, \mathcal{A}]$ (resp. $\text{Lax}[I, \mathcal{A}]$).¹

Remark 5.2 The two aspects of a universal property of a bi-limit may be reformulated more explicitly by expanding the content of the equivalence of categories above. For every $X \in \mathcal{A}$,

- (1) for every 2-cone $\mu: \Delta X \Rightarrow F$, there is a morphism $f: X \rightarrow L$ in \mathcal{A} and an invertible modification $\varphi: \lambda \Delta f \Rightarrow \mu$,
- (2) for all morphisms $f, g: X \rightarrow L$, and for every modification $\Theta: \lambda \Delta f \Rrightarrow \lambda \Delta g$, there is a unique 2-morphism $\alpha: f \Rightarrow g$ in \mathcal{A} such that $\lambda * \Delta \alpha = \Theta$.

Definition 5.3 Let \mathcal{A} be a 2-category. An object $L \in \mathcal{A}$ is **bi-terminal** if for all $X \in \mathcal{A}$ there is an equivalence of categories $\mathcal{A}(X, L) \simeq \mathbb{1}$.

¹ Note that what we call bi-limit here does not match the typical notion of bi-limit present in the literature, which is usually considered in a weaker context than that of (strict) 2-natural transformations. Pseudo-bi-limits therefore coincide with the usual notion of bi-limits.

Table 3 Bi-type limits which are not bi-terminal

\rightarrow implies bi-terminal in \downarrow	Bi-limit	Pseudo-bi-limit	Lax-bi-limit
Pseudo-slice	✓ Prop. 5.4	✓ Remark 5.6	✓ Remark 5.6
Lax-slice	Counter-ex. 3.7	Reduction 4.9	Counter-ex. 4.10

Table 4 Bi-terminal objects which are not bi-type limits

Bi-terminal in \downarrow implies \rightarrow	Bi-limit	Pseudo-bi-limit	Lax-bi-limit
Pseudo-slice	Counter-ex. 2.10	Reduction 4.4	Counter-ex. 4.5
Lax-slice	Counter-ex. 3.9	Reduction 4.13	Counter-ex. 4.14

In formulating analogous conjectures for the bi-limit and bi-terminal cases, we should pay careful attention to Remark 5.2 (1). Observe that, given a 2-cone $\mu: \Delta X \Rightarrow F$, the 1-dimensional aspect of the universal property of a bi-limit (L, λ) gives only a morphism $(X, \mu) \rightarrow (L, \lambda)$ of the pseudo-slice of 2-cones $\Delta \downarrow_p F$. It is thus inappropriate to look at bi-terminal objects in the strict-slice of 2-cones when attempting to recover a general bi-limit.

Much as was the case for 2-limits, bi-limits are in general bi-terminal objects in the pseudo-slice of 2-cones. In fact, all of the positive results of Sect. 2 follows in this context. We defer all proofs to the paper [3] which deals with such relationships in greater generality. The first result can be deduced from [3, Corollary 7.22] and the second is [3, Corollary 7.25].

Proposition 5.4 *Let I and \mathcal{A} be 2-categories, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. If $(L, \lambda: \Delta L \Rightarrow F)$ is a bi-limit of F , then (L, λ) is bi-terminal in the pseudo-slice $\Delta \downarrow_p F$ of 2-cones over F .*

Proposition 5.5 *Suppose \mathcal{A} is a 2-category that admits tensors by $\mathbb{2}$, and let $F: I \rightarrow \mathcal{A}$ be a 2-functor. Then an object is bi-terminal in the pseudo-slice $\Delta \downarrow_p F$ of 2-cones over F if and only if it is a bi-limit of F .*

Remark 5.6 Proposition 5.4, 5.5 also hold true when the 2-cones are replaced by pseudo- and lax-cones.

With every sunrise there is a sunset, and just as the positive results extended themselves to this weaker context, so too do the negative. Since isomorphisms of categories are, in particular, equivalences of categories, a 2-type limit or a 2-terminal object is, in particular, a bi-type limit or a bi-terminal object. Moreover, an examination of all of the counter-examples and reductions referenced in the tables below shows that there are no non-trivial invertible 2-morphisms in the 2-categories involved. This allows us to deduce the following facts. First, the notions of 2-type limits (resp. 2-terminal objects) and bi-type limits (resp. bi-terminal objects) in each of these counter-examples coincide. Second, for each of Counter-examples 2.10, 4.5, the strict-slice and pseudo-slice coincide. Therefore, all counter-examples and reductions for 2-type limits and 2-terminality seen in previous sections are also counter-examples and reductions for bi-type limits and bi-terminality, as summarised in the Tables 3 and 4.

Acknowledgements Both authors are indebted to Emily Riehl for her close readings of and thoughtful inputs on several early drafts of this paper. In addition, both authors are grateful to Jérôme Scherer for his careful input on an early draft. Finally, both authors also wish to extend their gratitude to Alexander Campbell and

Emily Riehl for their enthusiasm for what became Counter-example 4.14, which provided the impetus to write this paper.

Funding Open Access funding enabled and organized by Projekt DEAL. This work was realised while both authors were at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester. The first-named author benefited from support by the National Science Foundation under Grant No. DMS-1440140, while at residence in MSRI. The second-named author was supported by the Swiss National Science Foundation under the Project P1ELP2_188039. The first-named author was additionally supported by the National Science Foundation Grant DMS-1652600, as well as the JHU Catalyst Grant.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Auderset, C.: Adjonctions et monades au niveau des 2-catégories. *Cahiers Topologie Géom. Différentielle* **15**, 3–20 (1974)
2. Borceux, F., Kelly, G.M.: A notion of limit for enriched categories. *Bull. Aust. Math. Soc.* **12**, 49–72 (1975)
3. clingman, t., Moser, L.: Bi-initial objects and bi-representations are not so different. *Cahiers Topologie Géom. Différentielle Catég.* **63**(3), 259–330 (2022)
4. Grandis, M.: Higher dimensional categories. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2020). From double to multiple categories
5. Grandis, M., Paré, R.: Limits in double categories. *Cahiers Topologie Géom. Différentielle Catég.* **40**(3), 162–220 (1999)
6. Grandis, M., Paré, R.: Persistent double limits and flexible weighted limits. <https://www.mscs.dal.ca/~pare/DblPrs2.pdf> (2019)
7. Kelly, G.M.: Basic Concepts of Enriched Category Theory. London Mathematical Society Lecture Note Series, vol. 64. Cambridge University Press, Cambridge (1982)
8. Kelly, G.M.: Elementary observations on 2-categorical limits. *Bull. Aust. Math. Soc.* **39**(2), 301–317 (1989)
9. Lack, S.: A 2-categories companion. In *Towards higher categories*, volume 152 of IMA Vol. Math. Appl., pp. 105–191. Springer, New York (2010)
10. Street, R.: Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra* **8**(2), 149–181 (1976)
11. Verity, D.: Enriched categories, internal categories and change of base. *Repr. Theory Appl. Categ.* **20**, 1–266 (2011)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.