



Crossed Modules of Monoids II: Relative Crossed Modules

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Abstract

This is the second part of a series of three strongly related papers in which three equivalent structures are studied:

- Internal categories in categories of monoids; defined in terms of pullbacks relative to a chosen class of spans.
- Crossed modules of monoids relative to this class of spans.
- Simplicial monoids of so-called Moore length 1 relative to this class of spans.

The most important examples of monoids that are covered are small categories (treated as monoids in categories of spans) and bimonoids in symmetric monoidal categories (regarded as monoids in categories of comonoids). In this second part we define relative crossed modules of monoids and prove their equivalence with the relative categories of Part I.

Keywords Crossed module · Internal category · Bimonoid

Introduction

Since their appearance in [25], crossed modules of groups have been intensively studied and applied in various contexts; see e.g. the reviews [19,21,22] and the references in them. They admit several different descriptions: a *simplicial group whose Moore complex is concentrated in degrees 1 and 2* turns out to be the internal nerve of an internal category in the category of groups (which is necessarily an internal groupoid, a.k.a. *strict 2-group* or *Cat¹-group*) and the Moore complex yields a *crossed module*. These constructions establish, in fact, equivalences between these three notions.

The first (to our knowledge) proofs of the equivalence between crossed modules and strict 2-groups can be found in [6]—where it is referred also to the unpublished proof [7]—and in [17]. Based on the fact that groups constitute a semi-abelian category in the sense of [15], another short and deeply conceptual proof is due to Janelidze [14]. It also leads to a broad generalization describing the equivalent notions of crossed modules and internal categories

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of all of Lie algebras, Ω -groups in the sense of [13], Heyting semi-lattices, the dual of the category of pointed sets and much more. Thus by working in an arbitrary semi-abelian category, not only a more transparent proof is obtained, but also a much wider generality, also unifying earlier results in [16,23].

More recently, however, some results on, and certain applications of crossed modules of groups were extended to crossed modules of ordinary monoids [20], groupoids [5] and of Hopf algebras [1,8,9,18,25]. From this list only cocommutative Hopf algebras over a field are known to constitute a semi-abelian category [11,12]. Hence Janelidze’s proof can not be applied directly to the rest of these generalizations. Our aim is therefore to develop a wider theory of crossed modules of monoids in more general monoidal categories which are not expected to have all pullbacks (not even along split epimorphisms). We have the above two main examples in mind:

- Categories of spans whose monoids are small categories, including groupoids in particular.
- Categories of comonoids in symmetric monoidal categories whose monoids are bimonoids including Hopf monoids in particular.

In the first part [3] of this series of papers we discussed classes of spans satisfying appropriate conditions; and relative pullbacks with respect to them. Assuming that such pullbacks exist—as they do in our key examples—we introduced a monoidal category with monoidal product provided by these pullbacks. We defined a relative (to the chosen class of spans) category as a monoid in this monoidal category. It is given by the usual data

$$\begin{array}{ccc}
 B & \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{array} & A \xleftarrow{d} A \square_B A
 \end{array} \tag{*}$$

where \square_B is now a relative pullback.

In the current article we make the next step and prove the equivalence of the following categories for a fixed class of suitable spans in a monoidal category:

- The category of relative categories in the category of monoids,
- The category of relative crossed modules of monoids.

Our methodology is inspired by Janelidze’s paper [14]. In Sect. 1 we investigate first some category of the category of split epimorphisms of monoids. We obtain an equivalent description of a split epimorphism of monoids $B \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{s} \end{array} A$ in terms of a distributive law which allows for handy characterizations of possible morphisms t and d in (*). This is used in Sects. 2 and 3, respectively, to present equivalent descriptions of some reflexive graphs of monoids in terms of relative pre-crossed modules of monoids; and of relative category objects (*) in categories of monoids in terms of relative crossed modules of monoids. Applying our results to categories of spans and to categories of comonoids, respectively, we re-obtain the definitions of crossed modules of groupoids in [5] and of crossed modules of Hopf monoids in [25], respectively.

Our next aim is to extend to our setting the equivalence of the category of strict 2-groups (that is, internal groupoids in the category of groups) and the category of crossed modules of groups to the further category of simplicial groups whose Moore complex has length 1. This will be achieved in Part III of this series [2].

1 Split Epimorphisms of Monoids Versus Distributive Laws

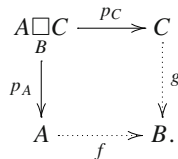
We freely use definitions, notation and results from [3]. Throughout, the composition of some morphisms $A \xrightarrow{g} B$ and $B \xrightarrow{f} C$ in an arbitrary category will be denoted by $A \xrightarrow{f \cdot g} C$. Identity morphisms will be denoted by 1 (without any reference to the (co)domain object if it causes no confusion). In any monoidal category \mathcal{C} the monoidal product will be denoted by juxtaposition and the monoidal unit will be I . For the monoidal product of n copies of the same object A also the power notation A^n will be used. For any monoid A in \mathcal{C} , the multiplication and the unit morphisms will be denoted by $A^2 \xrightarrow{m} A$ and $I \xrightarrow{u} A$, respectively. If \mathcal{C} is also braided, then for the braiding the symbol c will be used.

Recall that a class \mathcal{S} of spans in an arbitrary category is said to be *admissible* if it satisfies the following two properties in [3, Definition 2.1].

- (POST) If $X \xleftarrow{f} A \xrightarrow{g} Y \in \mathcal{S}$ then $X' \xleftarrow{f'} X \xleftarrow{f} A \xrightarrow{g} Y \xrightarrow{g'} Y' \in \mathcal{S}$ too, for any morphisms $X \xrightarrow{f'} X'$ and $Y \xrightarrow{g'} Y'$.
- (PRE) If $X \xleftarrow{f} A \xrightarrow{g} Y \in \mathcal{S}$ then $X \xleftarrow{f} A \xleftarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \in \mathcal{S}$, for any morphism $B \xrightarrow{h} A$.

The *relative pullback* of a cospan $A \xrightarrow{f} B \xleftarrow{g} C$ with respect to such a class \mathcal{S} was introduced in [3, Definition 3.1] as a span $A \xleftarrow{p_A} A \square_B C \xrightarrow{p_C} C$ in \mathcal{S} satisfying the following properties.

- *Commutativity* of the diagram



- *Universality*. For any $A \xleftarrow{f'} X \xrightarrow{g'} C \in \mathcal{S}$ such that $f \cdot f' = g \cdot g'$, there is a unique morphism $X \xrightarrow{h} A \square_B C$ which satisfies $p_A \cdot h = f'$ and $p_C \cdot h = g'$.
- *Reflection*. If both

$$A \xleftarrow{p_A} A \square_B C \xleftarrow{k} D \xrightarrow{l} E \quad \text{and} \quad C \xleftarrow{p_C} A \square_B C \xleftarrow{k} D \xrightarrow{l} E$$

belong to \mathcal{S} then also $A \square_B C \xleftarrow{k} D \xrightarrow{l} E$ belongs to \mathcal{S} ; and symmetrically, if

$$E \xleftarrow{l} D \xrightarrow{k} A \square_B C \xrightarrow{p_A} A \quad \text{and} \quad E \xleftarrow{l} D \xrightarrow{k} A \square_B C \xrightarrow{p_C} C$$

belong to \mathcal{S} then also $E \xleftarrow{l} D \xrightarrow{k} A \square_B C$ belongs to \mathcal{S} .

As in [3, Definition 2.10], we say that a cospan $A \xrightarrow{f} B \xleftarrow{g} C$ has legs in a class \mathcal{S} of spans if $A = A \xrightarrow{f} B$ and $B \xleftarrow{g} C = C$ belong to \mathcal{S} . [3, Assumption 4.1] asserts that there exists the \mathcal{S} -relative pullback of those cospans whose legs are in an admissible class \mathcal{S} . Under this assumption it was proven in [3, Corollary 4.6] that for any object B for which $B = B = B \in \mathcal{S}$, those spans $B \xleftarrow{t} A \xrightarrow{s} B$ over B whose legs $B \xleftarrow{t} A = A$ and $A = A \xrightarrow{s} B$ are in \mathcal{S} , constitute a monoidal category with the monoidal product given by the \mathcal{S} -relative pullback over B . An \mathcal{S} -relative category (with object of objects B) is defined in [3, Definition 4.9] as a monoid therein. Explicitly, this means morphisms

$$B \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{array} A \xleftarrow{d} A \square_B A$$

subject to the following axioms.

- The legs of $B \xleftarrow{t} A \xrightarrow{s} B$ are in \mathcal{S} (so that its \mathcal{S} -relative pullback denoted by $A \xleftarrow{p_1} A \square_B A \xrightarrow{p_2} A$ exists).
- i is a common section of s and t (that is, $B \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{array} A$ is a reflexive graph).
- $t \cdot d = t \cdot p_1$ and $s \cdot d = s \cdot p_2$.
- $d \cdot (i \square 1) = 1 = d \cdot (1 \square i)$.
- $d \cdot (d \square 1) = d \cdot (1 \square d)$.

A class \mathcal{S} of spans in a monoidal category is said to be monoidal if it satisfies the following two conditions in [3, Definition 2.5].

(UNITAL) For any morphisms f and g whose domain is the monoidal unit I , $X \xleftarrow{f} I \xrightarrow{g} Y \in \mathcal{S}$.

(MULTIPLICATIVE) If both $X \xleftarrow{f} A \xrightarrow{g} Y \in \mathcal{S}$ and $X' \xleftarrow{f'} A' \xrightarrow{g'} Y' \in \mathcal{S}$ then also $XX' \xleftarrow{ff'} AA' \xrightarrow{gg'} YY' \in \mathcal{S}$.

It is discussed in [3, Example 2.8] that a monoidal admissible class \mathcal{S} of spans in a braided monoidal category \mathcal{C} induces a monoidal admissible class of spans in the category of monoids in \mathcal{C} ; and it is shown in [3, Example 4.4] that if \mathcal{S} satisfies [3, Assumption 4.1] then so does the induced class in the category of monoids. This allows for the discussion of \mathcal{S} -relative categories in the category of monoids.

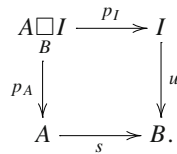
In this paper we will be interested mainly in these relative categories of monoids. They contain, in particular, a split epimorphism of monoids (consisting of the morphisms i and s of $(*)$ in the Introduction). So we start with the analysis of the following category of split epimorphisms of monoids.

Theorem 1.1 Consider a monoidal admissible class \mathcal{S} of spans in a monoidal category \mathcal{C} for which [3, Assumption 4.1] holds. The following categories are equivalent.

SplitEpiMon $_{\mathcal{S}}(\mathcal{C})$ whose

objects are split epimorphisms $B \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{i} \end{array} A$ of monoids in \mathcal{C} subject to the following conditions.

(a) $A \xrightarrow{s} B \in \mathcal{S}$; so that by the unitality of \mathcal{S} and [3, Assumption 4.1], there exists the \mathcal{S} -relative pullback



(b) $q := (A \square_B I) B \xrightarrow{p_A i} A^2 \xrightarrow{m} A$ is invertible.

morphisms are pairs of monoid morphisms $(B \xrightarrow{b} B', A \xrightarrow{a} A')$ such that $s' \cdot a = b \cdot s$ and $i' \cdot b = a \cdot i$.

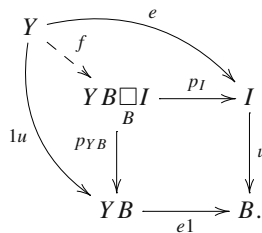
$\text{DistLaw}_{\mathcal{S}}(\mathcal{C})$ whose

objects consist of monoids B and Y , a monoid morphism $Y \xrightarrow{e} I$ and a distributive law $BY \xrightarrow{x} YB$ subject to the following conditions.

(a') $Y \xrightarrow{e} I \in \mathcal{S}$ and $B \xrightarrow{e} B \xrightarrow{e} B \in \mathcal{S}$. Then by the monoidality of \mathcal{S} also $YB \xrightarrow{e_1} YB \xrightarrow{e_1} B \in \mathcal{S}$ so by [3, Assumption 4.1] there exists the \mathcal{S} -relative pullback $YB \square_B I$ in the diagram below.

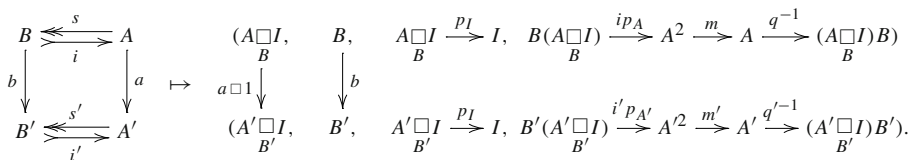
(b') $e_1 \cdot x = 1e$.

(c') The morphism f occurring in the diagram below is invertible. (It is well-defined since by (a') and condition (POST) in [3, Definition 2.1], $YB \xleftarrow{1u} Y \xrightarrow{e} I \in \mathcal{S}$.)



morphisms are pairs of monoid morphisms $(B \xrightarrow{b} B', Y \xrightarrow{y} Y')$ such that $e' \cdot y = e$ and $x' \cdot by = yb \cdot x$.

Proof We prove the theorem by constructing mutually inverse equivalence functors. The first one $\text{SplitEpiMon}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{DistLaw}_{\mathcal{S}}(\mathcal{C})$ sends



Let us see that the object map is meaningful. By construction B is a monoid and $B \xrightarrow{b} B'$ is a monoid morphism. By [3, Proposition 3.7 (1)] $A \square_B I$ is a monoid and $A \square_B I \xrightarrow{p_I} I$ is a

monoid morphism. By [3, Lemma 1.5] $B(A \square_B I) \xrightarrow{ip_A} A^2 \xrightarrow{m} A \xrightarrow{q^{-1}} (A \square_B I)B$ is a distributive law. Concerning property (a'), $I = I = I \in \mathcal{S}$ by the unitality of \mathcal{S} ; hence by [3, Lemma 3.4 (2)] $A \square_B I = A \square_B I \xrightarrow{p_I} I \in \mathcal{S}$. By [3, Lemma 2.4 (1)]¹ also $B = B = B$ belongs to \mathcal{S} . Condition (b') holds since commutativity of the first diagram of

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & q & & \\
 & & \curvearrowright & & \\
 (A \square_B I)B & \xrightarrow{p_A i} & A^2 & \xrightarrow{m} & A \\
 \downarrow p_I 1 & & \downarrow ss & & \downarrow s \\
 B & \xrightarrow{u 1} & B^2 & \xrightarrow{m} & B \\
 & \curvearrowleft & & \curvearrowright & \\
 & & & &
 \end{array} &
 \begin{array}{ccccc}
 B(A \square_B I) & \xrightarrow{ip_A} & A^2 & \xrightarrow{m} & A & \xrightarrow{q^{-1}} & (A \square_B I)B \\
 \downarrow 1 p_I & & \downarrow ss & & \downarrow s & & \downarrow p_I 1 \\
 B & \xrightarrow{1u} & B^2 & \xrightarrow{m} & B & = & B \\
 & \curvearrowleft & & \curvearrowright & & &
 \end{array}
 \end{array} \tag{1.1}$$

implies the commutativity of the second diagram. For condition (c') observe that by the unitality of the monoid morphism i the equality $q \cdot 1u = p_A$ holds, equivalently, $q^{-1} \cdot p_A = 1u$. With this identity in mind we see that the morphism f of condition (c') is equal to $q^{-1} \square 1$ in the first diagram of

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 A \square_B I & & & & \\
 \downarrow p_A & \searrow q^{-1} \square 1 & \searrow p_I & & \\
 A & \xrightarrow{q^{-1}} & (A \square_B I)B & \xrightarrow{p_I 1} & B \\
 & & \downarrow p_{(A \square_B I)B} & & \downarrow u \\
 & & (A \square_B I)B \square_B I & \xrightarrow{p_I} & I
 \end{array} &
 \begin{array}{ccccc}
 (A \square_B I)B \square_B I & & & & \\
 \downarrow p_{(A \square_B I)B} & \searrow q \square 1 & \searrow p_I & & \\
 (A \square_B I)B & \xrightarrow{q} & A & \xrightarrow{s} & B \\
 & & \downarrow p_A & & \downarrow u \\
 & & (A \square_B I)B \square_B I & \xrightarrow{p_I} & I
 \end{array}
 \end{array}$$

Then by [3, Proposition 3.5 (2)] it is invertible with the inverse $q \square 1$ in the second diagram. Both morphisms $q^{-1} \square 1$ and $q \square 1$ are well-defined by the commutativity of the first diagram of (1.1); see [3, Proposition 3.5 (1)]. This proves that the object map of our candidate functor is meaningful.

Concerning the morphism map, $a \square 1$ is a well-defined morphism in \mathcal{C} by the assumption that $b \cdot s = s' \cdot a$ (see [3, Proposition 3.5 (1)]) and it is a monoid morphism by [3, Proposition 3.7 (2)]. Condition $p_I \cdot (a \square 1) = p_I$ holds by construction and the other equality holds since the commutativity of the first diagram of

¹ Apologies about a regrettable typo in the first line of [3, Lemma 2.4], interchanging the symbols A and B .

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & q & & \\
 & & \curvearrowright & & \\
 (A \square_I) B & \xrightarrow{p_A i} & A^2 & \xrightarrow{m} & A \\
 \downarrow (a \square 1) b & & \downarrow aa & & \downarrow a \\
 (A' \square_I) B' & \xrightarrow{p_{A'} i'} & A'^2 & \xrightarrow{m'} & A' \\
 & & \curvearrowleft & & \\
 & & q' & &
 \end{array} &
 \begin{array}{ccccc}
 B(A \square_I) & \xrightarrow{i p_A} & A^2 & \xrightarrow{m} & A & \xrightarrow{q^{-1}} & (A \square_I) B \\
 \downarrow b(a \square 1) & & \downarrow aa & & \downarrow a & & \downarrow (a \square 1) b \\
 B'(A' \square_I) & \xrightarrow{i' p_{A'}} & A'^2 & \xrightarrow{m'} & A' & \xrightarrow{q'^{-1}} & (A' \square_I) B'
 \end{array} & (1.2)
 \end{array}$$

implies the commutativity of the second diagram.

In the opposite direction $\text{DistLaw}_S(\mathcal{C}) \rightarrow \text{SplitEpiMon}_S(\mathcal{C})$ we define a functor sending

$$\begin{array}{ccc}
 (Y, B, Y \xrightarrow{e} I, BY \xrightarrow{x} YB) & & \begin{array}{ccc} B & \xrightleftharpoons[e1]{u1} & YB \\ b \downarrow & & \downarrow yb \\ B' & \xrightleftharpoons[e'1]{u'1} & Y'B' \end{array} \\
 \begin{array}{ccc} y \downarrow & & \downarrow b \\ (Y', B', Y' \xrightarrow{e'} I, B'Y' \xrightarrow{x'} Y'B') \end{array} & \mapsto &
 \end{array}$$

Here YB is considered with the monoid structure induced by the distributive law x , see [3, Lemma 1.4]. Then $B \xrightarrow{u1} YB$ is a monoid morphism by [3, Lemma 1.4] again. By [3, Lemma 1.6] condition (b') implies that $YB \xrightarrow{e1} B$ is a monoid morphism too. (For a more direct proof consider the commutative diagrams

$$\begin{array}{ccc}
 YBYB \xrightarrow{1x1} Y^2B^2 \xrightarrow{mm} YB & & I \xrightarrow{uu} YB \\
 \begin{array}{ccc} e1e1 \downarrow & \swarrow (b') & \downarrow e1 \\ B^2 & \xrightarrow{m} & B \end{array} & & \begin{array}{ccc} \parallel & & \downarrow e1 \\ I & \xrightarrow{u} & B \end{array}
 \end{array}$$

whose unlabelled regions commute since $e : Y \rightarrow I$ is a monoid morphism.) The rows are split epimorphisms (of monoids) by the unitality of the monoid morphism e . By (a') and the multiplicativity of S , $YB \equiv YB \xrightarrow{e1} B \in S$ so that condition (a) holds. For condition (b) note that the commutativity of

$$\begin{array}{ccccccc}
 YB & \xlongequal{\quad} & YB & & & & \\
 \downarrow f1 & \searrow 1u1 & & \nearrow 1m & & & \\
 & & YB^2 & \xlongequal{\quad} & YB^2 & & \\
 & & \downarrow 11u1 & & \downarrow 1u11 & & \\
 (YB \square_I) B & \xrightarrow{p_{YB} u1} & (YB)^2 & \xrightarrow{1x1} & Y^2 B^2 & \xrightarrow{mm} & YB
 \end{array}$$

implies that the bottom row is the inverse of the isomorphism $f1$ in the left column hence it is invertible. This proves that the object map is well defined.

Concerning the morphism map, it follows by the assumption $yb \cdot x = x' \cdot by$ that yb is a monoid morphism, see

$$\begin{array}{ccc}
 YBYB & \xrightarrow{1x1} & Y^2B^2 \xrightarrow{mm} YB \\
 \downarrow ybyb & & \downarrow yybb \quad \downarrow yb \\
 Y'B'Y'B' & \xrightarrow{1x'1} & Y'^2B'^2 \xrightarrow{m'm'} Y'B'
 \end{array}
 \qquad
 \begin{array}{ccc}
 I & \xrightarrow{uu} & YB \\
 \parallel & & \downarrow yb \\
 I & \xrightarrow{u'u'} & Y'B'.
 \end{array}$$

The monoid morphisms (b, yb) are compatible with the monomorphisms $B \xrightarrow{u1} YB$ and $B' \xrightarrow{u'1} Y'B'$ by the unitality of y , and they are compatible with the epimorphisms $YB \xrightarrow{e1} B$ and $Y'B' \xrightarrow{e'1} B'$ by the assumption that $e' \cdot y = e$.

So we have well-defined functors in both directions, it remains to see that their composites are naturally isomorphic to the identity functors. The composite

$$\text{SplitEpiMon}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{DistLaw}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{SplitEpiMon}_{\mathcal{S}}(\mathcal{C})$$

acts as

$$\begin{array}{ccc}
 \begin{array}{ccc}
 B & \xrightleftharpoons[s]{i} & A \\
 \downarrow b & & \downarrow a \\
 B' & \xrightleftharpoons[s']{i'} & A
 \end{array}
 & \mapsto &
 \begin{array}{ccc}
 B & \xrightleftharpoons[u1]{p_I 1} & (A \square_I) B \\
 \downarrow b & & \downarrow (a \square_I) b \\
 B' & \xrightleftharpoons[u'1]{p_{I'} 1} & (A' \square_{I'}) B'.
 \end{array}
 \end{array}$$

We claim that a natural isomorphism from this to the identity functor has the components $(B = B, (A \square_I) B \xrightarrow{q} A)$. Since p_A is a monoid morphism by [3, Proposition 3.7 (1)], so is q by [3, Lemma 1.5]. The stated pair $(1, q)$ is a morphism in $\text{SplitEpiMon}_{\mathcal{S}}(\mathcal{C})$ by the commutativity of the first diagram of (1.1) and by the fact that the unitality of p_A implies $q \cdot 1u = i$. Naturality with respect to any morphism $(B \xrightarrow{b} B', A \xrightarrow{a} A')$ in $\text{SplitEpiMon}_{\mathcal{S}}(\mathcal{C})$ follows by the commutativity of the first diagram of (1.2).

Composing our functors in the opposite order

$$\text{DistLaw}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{SplitEpiMon}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{DistLaw}_{\mathcal{S}}(\mathcal{C})$$

we obtain the functor sending

$$\begin{array}{ccc}
 (Y, & B, & Y \xrightarrow{e} I, BY \xrightarrow{x} YB) \\
 \downarrow y & \downarrow b & \\
 (Y', & B', & Y' \xrightarrow{e'} I, B'Y' \xrightarrow{x'} Y'B')
 \end{array}$$

to

$$\begin{array}{ccc}
 (YB \square_I B, & B, & YB \square_I B \xrightarrow{p_I} I, B(YB \square_I B) \xrightarrow{u^1 p_{YB}} (YB)^2 \xrightarrow{1x1} Y^2 B^2 \xrightarrow{mm} YB \xrightarrow{f1} (YB \square_I) B) \\
 \downarrow yb \square 1 & \downarrow b & \\
 (Y'B' \square_{I'} B', & B', & Y'B' \square_{I'} B' \xrightarrow{p_{I'}} I, B'(Y'B' \square_{I'} B') \xrightarrow{u'^1 p_{Y'B'}} (Y'B')^2 \xrightarrow{1x'1} Y'^2 B'^2 \xrightarrow{m'm'} Y'B' \xrightarrow{f'1} (Y'B' \square_{I'}) B').
 \end{array}$$

We claim that a natural isomorphism from this to the identity functor has the invertible components $(B = B , Y \xrightarrow{f} YB \square_B I)$. By construction f is a monoid morphism, see [3, Proposition 3.7 (2)]. The compatibility of the monoid morphisms $(1, f)$ with $Y \xrightarrow{e} I$ and $YB \square_B I \xrightarrow{p_I} I$ holds by the definition of f and the compatibility with the distributive laws $BY \rightarrow YB$ and $B(YB \square_B I) \rightarrow (YB \square_B I)B$ holds by the commutativity of

$$\begin{array}{ccccccc}
 BY & \xlongequal{\quad} & BY & \xrightarrow{x} & YB & & \\
 \downarrow 1f & & \downarrow 11u & \nearrow 11u & \parallel & \searrow f1 & \\
 B(YB \square_B I) & \xrightarrow{1p_{YB}} & BYB & \xrightarrow{x1} & YB^2 & \xrightarrow{1m} & YB & \xrightarrow{f1} & (YB \square_B I)B.
 \end{array}$$

Finally, the naturality with respect to an arbitrary morphism $(B \xrightarrow{b} B' , Y \xrightarrow{y} Y')$ in $\text{DistLaw}_S(\mathcal{C})$ follows by the commutativity of the diagrams

$$\begin{array}{ccc}
 YB \square_B I & \xrightarrow{yb \square 1} & Y' B' \square_{B'} I \\
 \uparrow f & \searrow p_{YB} & \downarrow p_{Y' B'} \\
 Y & \xrightarrow{1u} & YB & \xrightarrow{yb} & Y' B' \\
 \downarrow y & \nearrow 1u' & & & \uparrow p_{Y' B'} \\
 Y' & \xrightarrow{f'} & Y' B' \square_{B'} I & & \\
 & & & & \\
 YB \square_B I & \xrightarrow{yb \square 1} & Y' B' \square_{B'} I & & \\
 \uparrow f & \searrow p_I & \downarrow p_I & & \\
 Y & \xrightarrow{e} & I & & \\
 \downarrow y & \nearrow e' & & & \uparrow p_I \\
 Y' & \xrightarrow{f'} & Y' B' \square_{B'} I & &
 \end{array}$$

using that the morphisms $YB \xleftarrow{p_{YB}} YB \square_B I \xrightarrow{p_I} I$ are jointly monic in \mathcal{C} . □

Example 1.2 For any fixed set X , the category \mathcal{C} of spans over X is monoidal via the pullback over X . A monoid in \mathcal{C} is a small category with the object set X and a monoid morphism is a functor acting on the objects as the identity map. Moreover, \mathcal{C} has all pullbacks (computed in the underlying category of sets). So taking as \mathcal{S} the class of all spans in \mathcal{C} , from Theorem 1.1 we obtain the equivalence of the following categories (from now on we shall denote by s the source map and by t the target map of any category).

$\text{SplitEpiMon}(\mathcal{C})$ whose

objects are pairs of identity-on-objects functors $B \xrightleftharpoons[\iota]{\sigma} A$ between categories of the common object set X such that the composite $\sigma \iota$ is the identity functor, and the map

$$q : (A \square_B X) \square_X B = \{(a, x, b) | \sigma(a) = 1_x, x = t(b)\} \rightarrow A \quad (a, x, b) \mapsto a \cdot \iota(b) \quad (1.3)$$

is invertible. (The map of (1.3) is invertible e.g. if B is a groupoid; then its inverse takes a morphism a to $(a \cdot \iota(\sigma(a)^{-1}), t(a), \sigma(a))$.)

morphisms are pairs of identity-on-objects functors $(A \xrightarrow{\alpha} A' , B \xrightarrow{\beta} B')$ for which $\alpha \iota = \iota' \beta$ and $\beta \sigma = \sigma' \alpha$.

DistLaw(C) whose

objects consist of categories B and Y with the common object set X such that Y has no morphisms between non-equal objects (that is, its source map s and target map t coincide; using the terminology of [4] this means that Y is a *totally disconnected* category); and an action $B \square_X Y = \{(b, y) | s(b) = t(y)\} \xrightarrow{\triangleright} Y$ in the sense of [5, Definition 1.1]; meaning the following axioms for all morphisms b, b' in B and y, y' in Y for which $s(b') = t(b)$ and $s(b) = t(y) = s(y) = t(y') = s(y')$.

- (i) $t(b \triangleright y) = t(b)$
- (ii) $b \triangleright (y \cdot y') = (b \triangleright y) \cdot (b \triangleright y')$ and $b \triangleright 1_{s(b)} = 1_{t(b)}$
- (iii) $(b' \cdot b) \triangleright y = b' \triangleright (b \triangleright y)$ and $1_{t(y)} \triangleright y = y$.

morphisms are pairs of identity-on-objects functors $(Y \xrightarrow{\nu} Y', B \xrightarrow{\beta} B')$ for which $\nu(b \triangleright y) = \beta(b) \triangleright \nu(y)$ for all morphisms b in B and y in Y for which $s(b) = t(y)$.

Only the above description of an object in DistLaw(C) requires some explanation.

The monoidal unit of C is the trivial span $X = X = X$. Its trivial monoid structure yields the discrete category $D(X)$. An identity-on-objects functor $Y \xrightarrow{e} D(X)$ as in Theorem 1.1 exists if and only if Y is totally disconnected. Then there is precisely one such functor sending any morphism to the identity morphism on its equal source and target objects. For this functor e , precisely those maps $B \square_X Y \xrightarrow{x} Y \square_X B$ satisfy $(e \square 1) \cdot x = 1 \square e$ which are of the form $(b, y) \mapsto (b \triangleright y, b)$ in terms of some map \triangleright obeying condition (i). It is straightforward to see that x is then a distributive law if and only if conditions (ii) and (iii) hold.

The morphism f of Theorem 1.1 (c') is invertible because the diagram

$$\begin{array}{ccc}
 Y \square_X C & \xrightarrow{e \square 1} & C \\
 \downarrow 1 \square g & & \downarrow g \\
 Y \square_X B & \xrightarrow{e \square 1} & B
 \end{array} \tag{1.4}$$

is clearly a pullback of X -spans for any span morphism g .

Example 1.3 Let M be a symmetric monoidal category in which equalizers exist and are preserved by taking the monoidal product with any object.

Take C to be the category of comonoids in M with the monoidal admissible class \mathcal{S} in [3, Example 2.3] of spans in C . This means that $X \xleftarrow{f} A \xrightarrow{g} Y$ belongs to \mathcal{S} if and only if $A \xrightarrow{\delta} A^2 \xrightarrow{fg} XY$ is a comonoid morphism, which holds if and only if $c \cdot fg \cdot \delta = gf \cdot \delta$. Thanks to the symmetry of M , its monoidal structure is inherited by C . A monoid A in C is known as a *bimonoid* in M . Recall that the monoidal structure of M is lifted to the category of (left or right) modules over the monoid A in M . A monoid (respectively, a comonoid) in the category of A -modules is known as an *A-module monoid* (respectively, *A-module comonoid*).

Recall from [3, Example 3.3] that for a cospan $A \xrightarrow{f} B \xleftarrow{g} C$ of comonoids whose legs are in \mathcal{S} , the \mathcal{S} -relative pullback is given by the so-called *cotensor product*, defined as the equalizer

$$A \square_B C \xrightarrow{j} AC \begin{matrix} \xrightarrow{1f1 \cdot \delta 1} \\ \xrightarrow{1g1 \cdot 1\delta} \end{matrix} \rightrightarrows ABC \tag{1.5}$$

in M (where δ denotes both comultiplications of the comonoids A and C .)

Below we describe the equivalent categories of Theorem 1.1 in this context.

$\text{SplitEpiMon}_{\mathcal{S}}(C)$ whose

objects are split epimorphisms $B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \end{matrix} A$ of bimonoids in M subject to the following conditions.

- (a) The comultiplication δ of A satisfies $c \cdot 1s \cdot \delta = s1 \cdot \delta$.
- (b) In terms of the morphism j of (1.5), $q := (A \square_B C)B \xrightarrow{ji} A^2 \xrightarrow{m} A$ is invertible.

morphisms are pairs of bimonoid morphisms which are compatible with the epimorphisms s as well as their sections i .

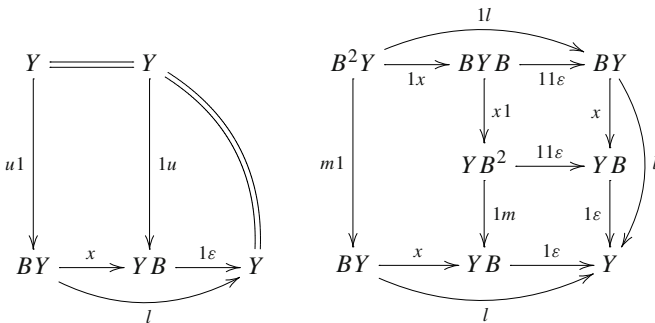
$\text{DistLaw}_{\mathcal{S}}(C)$ whose

objects consist of a cocommutative bimonoid B and a bimonoid Y in M , together with a left B -action on Y which makes Y both a left B -module monoid and a left B -module comonoid.

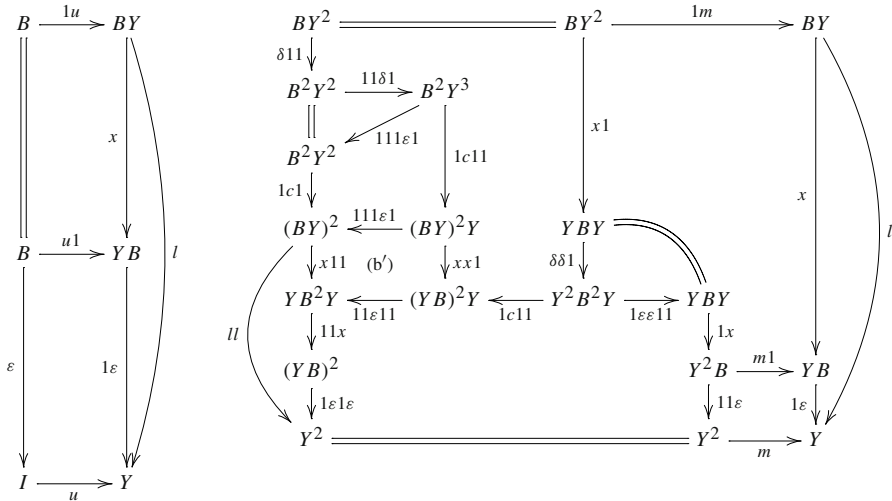
morphisms are pairs of bimonoid morphisms $(B \xrightarrow{b} B', Y \xrightarrow{y} Y')$ which are compatible with the actions $BY \xrightarrow{l} Y$ and $B'Y' \xrightarrow{l'} Y'$ in the sense that $l' \cdot by = y \cdot l$.

This concise description of $\text{DistLaw}_{\mathcal{S}}(C)$ requires a proof. Note that the monoidal unit I is now a terminal object in C ; the unique morphism $Y \rightarrow I$ is the counit ε . It obviously satisfies $Y = Y \xrightarrow{\varepsilon} I \in \mathcal{S}$. The other condition $B = B = B \in \mathcal{S}$ in (a') of Theorem 1.1 reduces to the requirement that the comonoid B is cocommutative.

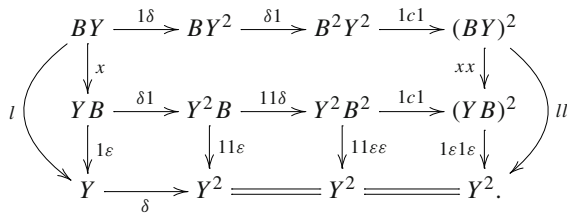
Next we establish a bijective correspondence between distributive laws $BY \rightarrow YB$ satisfying property (b') of Theorem 1.1 and left actions $BY \rightarrow Y$ as in the description above. Starting with a distributive law $BY \xrightarrow{x} YB$, put $l := 1\varepsilon \cdot x$. It is a unital action by the left unitality of x and it is associative by the left multiplicativity of x :



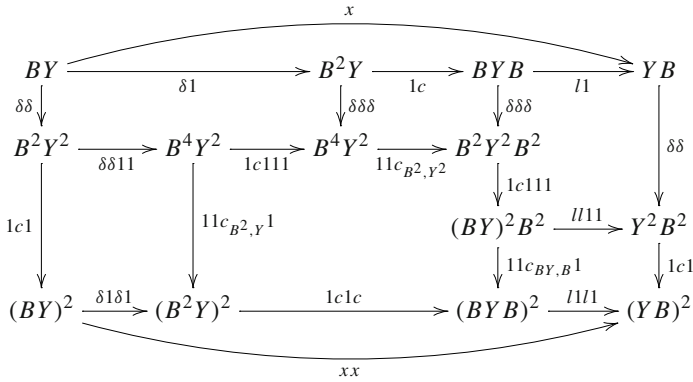
By the right unitality of x the unit $I \xrightarrow{u} Y$ is a morphism of B -modules and by the right multiplicativity of x the multiplication $Y^2 \xrightarrow{m} Y$ is a morphism of B -modules:



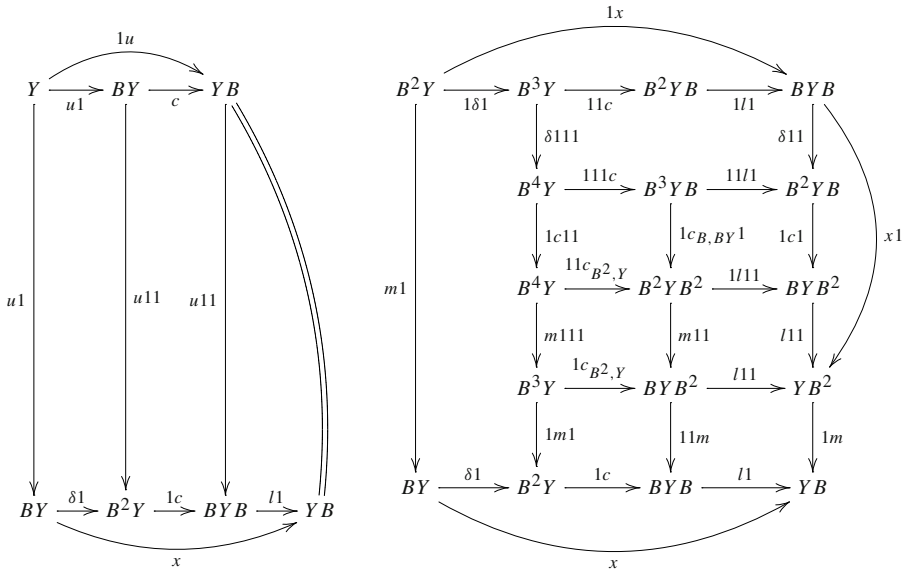
(note that here we also used the comultiplicativity of x). The condition that the counit $Y \xrightarrow{\varepsilon} I$ is a morphism of B -modules coincides with the counitality of l and also with the counitality of x . The comultiplication $Y \xrightarrow{\delta} Y^2$ is a morphism of B -modules, equivalently, l is comultiplicative by the comultiplicativity of x :



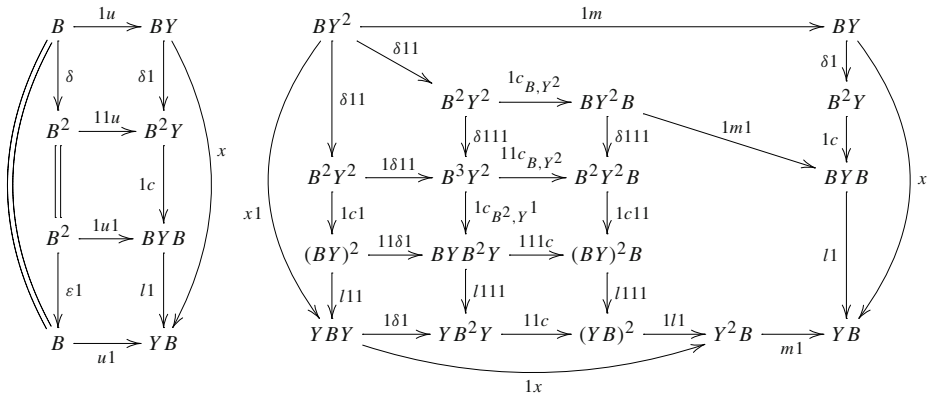
Conversely, given an action l as above, put $x := BY \xrightarrow{\delta 1} B^2Y \xrightarrow{1c} BYB \xrightarrow{l} YB$. It clearly satisfies (b') by the counitality of l hence it is counital. It is comultiplicative by the comultiplicativity of l :



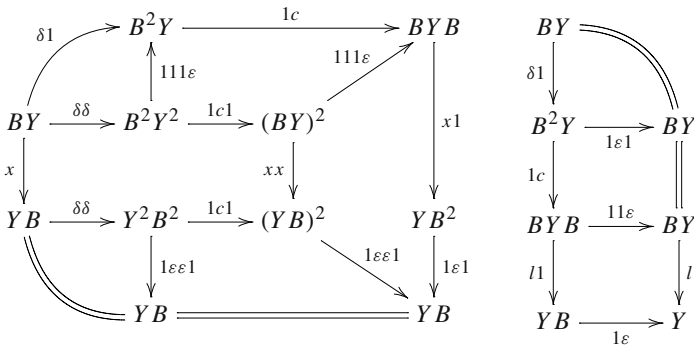
where the top-left region commutes by the coassociativity and cocommutativity of the comonoid B . This morphism x is a distributive law. Indeed, the left unitality and the left multiplicativity follow by the unitality and the associativity of the action l , respectively:



and the right unitality and the right multiplicativity of x follow using that the unit and the multiplication of Y are B -module morphisms:



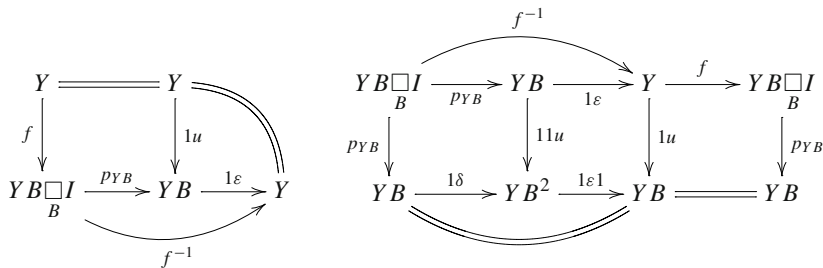
The above correspondences between l and x are bijective by the commutativity of



for a comultiplicative morphism x satisfying (b') and any morphism l .

Finally, we show that the morphism $Y \xrightarrow{f} YB \square_B I$ in part (c') of Theorem 1.1 is invertible without any further assumption; its inverse is constructed as $f^{-1} := YB \square_B I \xrightarrow{p_{YB}} YB \xrightarrow{1_\epsilon} Y$. In order to see that it is the inverse, indeed, recall that by [3,

Example 3.3] the morphism p_{YB} is the equalizer of $YB \xrightarrow{1_\delta} YB^2$ and $YB \xrightarrow{1_{1u}} YB^2$. Hence the following diagrams commute.



This completes the characterization of the objects of $\text{DistLaw}_S(\mathcal{C})$. Concerning the morphisms $(B \xrightarrow{b} B', Y \xrightarrow{y} Y')$, the first condition in Theorem 1.1 is the counitality of the

bimonoid morphism y hence it identically holds. The second condition in Theorem 1.1 is equivalent to $y \cdot l = l' \cdot by$ by the commutativity of

$$\begin{array}{ccc}
 \begin{array}{ccccccc}
 & & x & & & & \\
 & & \curvearrowright & & & & \\
 BY & \xrightarrow{\delta 1} & B^2Y & \xrightarrow{1c} & BYB & \xrightarrow{1l} & YB \\
 by \downarrow & & \downarrow bby & & \downarrow byb & & \downarrow yb \\
 B'Y' & \xrightarrow{\delta' 1} & B'^2Y' & \xrightarrow{1c} & B'Y'B' & \xrightarrow{l' 1} & Y'B' \\
 & & \curvearrowleft & & & & \\
 & & x' & & & &
 \end{array}
 &
 \begin{array}{ccc}
 & & l & & \\
 & & \curvearrowright & & \\
 BY & \xrightarrow{x} & YB & \xrightarrow{1\varepsilon} & Y \\
 by \downarrow & & \downarrow yb & & \downarrow y \\
 B'Y' & \xrightarrow{x'} & Y'B' & \xrightarrow{1\varepsilon'} & Y' \\
 & & \curvearrowleft & & \\
 & & l' & &
 \end{array}
 \end{array}$$

Example 1.4 We can apply Example 1.3 to the particular case of a finitely complete category M regarded with the cartesian monoidal structure. Then the category C of comonoids in M is isomorphic to M .

Since in this case the monoidal unit I of M is a terminal object, with the trivial monoid structure it becomes the zero object in the category of monoids in M . Then for any morphism $A \xrightarrow{s} B$ of monoids in M , we may identify $p_A : A \square_I B \rightarrow A$ with the kernel of s .

The equivalent categories of Theorem 1.1 reduce to the following ones.

SplitEpiMon_S(M) whose

objects are split epimorphisms $B \overset{s}{\underset{i}{\rightrightarrows}} A$ of monoids in M such that in terms of the

morphism j of (1.5), $q := (A \square_I B) \xrightarrow{j} A^2 \xrightarrow{m} A$ is invertible.

morphisms are pairs of monoid morphisms which are compatible with the epimorphisms s as well as their sections i .

DistLaw_S(M) whose

objects consist of monoids B and Y in M , together with a left B -action on Y which makes the multiplication and the unit of the monoid Y left B -linear.

morphisms are pairs of monoid morphisms $(B \xrightarrow{b} B', Y \xrightarrow{y} Y')$ which are compatible with the actions $BY \xrightarrow{l} Y$ and $B'Y' \xrightarrow{l'} Y'$ in the sense that $l' \cdot by = y \cdot l$.

In particular, if M is the cartesian monoidal category of sets, then the morphism q in the above description of the objects of **SplitEpiMon_S(M)** is invertible if and only if $B \overset{s}{\underset{i}{\rightrightarrows}} A$

satisfies the *Schreier property* of [20]. That is, for each element a of A , there is a unique element z_a in the kernel of s such that $a = z_a \cdot is(a)$. (Indeed, this condition clearly implies the surjectivity of q . For its injectivity assume $z \cdot i(b) = z' \cdot i(b')$ for some z, z' in the kernel of s and b, b' in B . Applying the monoid morphism s we obtain $b = b'$. From

$$z \cdot is(z \cdot i(b)) = z \cdot i(b) = z' \cdot i(b) = z' \cdot is(z \cdot i(b))$$

and the uniqueness part of the Schreier property we infer $z = z'$. Conversely, if q is invertible then its inverse $a \mapsto (z_a, s(a))$ defines the required element z_a of the kernel.) On the other hand, in this case an object of **DistLaw_S(M)** reduces to a monoid morphism from B to the monoid of monoid endomorphisms of Y .

Recall that a bimonoid B —with monoid structure (m, u) and comonoid structure (δ, ε) —is a *Hopf monoid* provided that there exists a morphism $B \xrightarrow{z} B$ —the so-called *antipode*—

which renders commutative the diagram

$$\begin{array}{ccccc}
 B & \xrightarrow{\delta} & B^2 & \xrightarrow{z1} & B^2 \\
 \delta \downarrow & \searrow \varepsilon & & & \downarrow m \\
 B^2 & \xrightarrow{1z} & B^2 & \xrightarrow{m} & B.
 \end{array}$$

If the antipode exists then it is unique. It is a monoid morphism from B to the monoid with the opposite multiplication $m \cdot c$ and comonoid morphism from B to the comonoid with the opposite comultiplication $c \cdot \delta$.

Proposition 1.5 (1) *The equivalent categories of Example 1.3 have equivalent full subcategories as follows.*

- *The category whose*

objects are split epimorphisms $B \overset{s}{\underset{i}{\rightrightarrows}} A$ of bimonoids in \mathbb{M} subject to the following conditions.

(a) *The comultiplication δ of A satisfies $c \cdot 1s \cdot \delta = s1 \cdot \delta$.*

(b) *B is a Hopf monoid.*

morphisms are pairs of bimonoid morphisms which are compatible with the epimorphisms s as well as their sections i .

- *The category whose*

objects consist of a cocommutative Hopf monoid B and a bimonoid Y in \mathbb{M} , together with a left B -action on Y which makes Y both a left B -module monoid and a left B -module comonoid.

morphisms are pairs of bimonoid morphisms $(B \xrightarrow{b} B', Y \xrightarrow{y} Y')$ which are compatible with the actions $BY \xrightarrow{l} Y$ and $B'Y' \xrightarrow{l'} Y'$ in the sense that $l' \cdot by = y \cdot l$.

(2) *The equivalent categories of part (1) have equivalent full subcategories as follows.*

- *The category whose*

objects are split epimorphisms $B \overset{s}{\underset{i}{\rightrightarrows}} A$ of cocommutative Hopf monoids.

morphisms are pairs of bimonoid morphisms which are compatible with the epimorphisms s as well as their sections i .

- *The category whose*

objects consist of cocommutative Hopf monoids B and Y in \mathbb{M} , together with a left B -action on Y which makes Y both a left B -module monoid and a left B -module comonoid.

morphisms are pairs of bimonoid morphisms $(B \xrightarrow{b} B', Y \xrightarrow{y} Y')$ which are compatible with the actions $BY \xrightarrow{l} Y$ and $B'Y' \xrightarrow{l'} Y'$ in the sense that $l' \cdot by = y \cdot l$.

Proof (1) The second listed category is obviously a full subcategory of $\text{DistLaw}_S(\mathbb{C})$ of Example 1.3; thus via the equivalence of Theorem 1.1 it is equivalent to some full subcategory of $\text{SplitEpiMon}_S(\mathbb{C})$ of Example 1.3. Our task is to show that it is the first listed category above. For that we only need to show that it is a subcategory of $\text{SplitEpiMon}_S(\mathbb{C})$; that is, that for any object $B \overset{s}{\underset{i}{\rightrightarrows}} A$ of it, the morphism q in part (b) of Example 1.3 is invertible.

Following ideas in [24], we use the antipode z of B and the image of the equalizer (1.5) under the functor $-B$ to construct the inverse:

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & \xrightarrow{q^{-1}} & (A \square_I B) \\
 & & & & & & & & & & & \downarrow j_1 \\
 A & \xrightarrow{\delta} & A^2 & \xrightarrow{1s} & AB & \xrightarrow{1\delta} & AB^2 & \xrightarrow{1z1} & AB^2 & \xrightarrow{1i1} & A^2B & \xrightarrow{m1} & AB \\
 & & & & & & & & & & & & \downarrow 1s1 \cdot \delta 1 \\
 & & & & & & & & & & & & AB^2 \\
 & & & & & & & & & & & & \downarrow 1u1
 \end{array}$$

This definition works because the horizontal morphism equalizes the parallel morphisms of the fork on the right; see Fig. 1. The so constructed morphism q^{-1} is the inverse of q by the commutativity of the diagrams of Fig. 2 (in the second case we also need to use that the columns are equal monomorphisms).

(2) If both Y and B are cocommutative comonoids then clearly so is YB ; and if both Y and B have antipodes z then $YB \xrightarrow{zz} YB$ is the antipode of the Hopf monoid YB .

Conversely, if A is cocommutative then evidently so is its sub-comonoid $A \square_I B$. If furthermore A has an antipode z then it restricts to $A \square_I B$ by the commutativity of the following diagram.

$$\begin{array}{ccccc}
 A \square_I B & \xrightarrow{j} & A & \xrightarrow{z} & A \\
 \downarrow j & & \downarrow \delta & & \downarrow \delta \\
 & & A^2 & \xrightarrow{zz} & A^2 \\
 & & \downarrow 1s & & \downarrow 1s \\
 A & \xrightarrow{1u} & AB & & AB \\
 \downarrow z & & \searrow zz & & \downarrow 1s \\
 A & \xrightarrow{1u} & AB & & AB
 \end{array}$$

The top right region commutes by the Hopf monoid identity $\delta \cdot z = zz \cdot c \cdot \delta$ and the assumed cocommutativity of A . The bottom right region commutes since any bimonoid morphism s commutes with the antipodes. \square

Example 1.6 Proposition 1.5 can be applied in particular to a finitely complete category M , regarded as a cartesian monoidal category. From Proposition 1.5 we obtain equivalences between the following pairs of categories.

- (1) • The category whose
 - objects are split epimorphisms $B \begin{smallmatrix} \xrightarrow{s} \\ \xleftarrow{i} \end{smallmatrix} A$ of monoids in M such that B is a group object.
 - morphisms are pairs of monoid morphisms which are compatible with the epimorphisms s as well as their sections i .
- The category whose
 - objects consist of a group object B and a monoid Y in M , together with a left B -action on Y which makes Y a left B -module monoid.

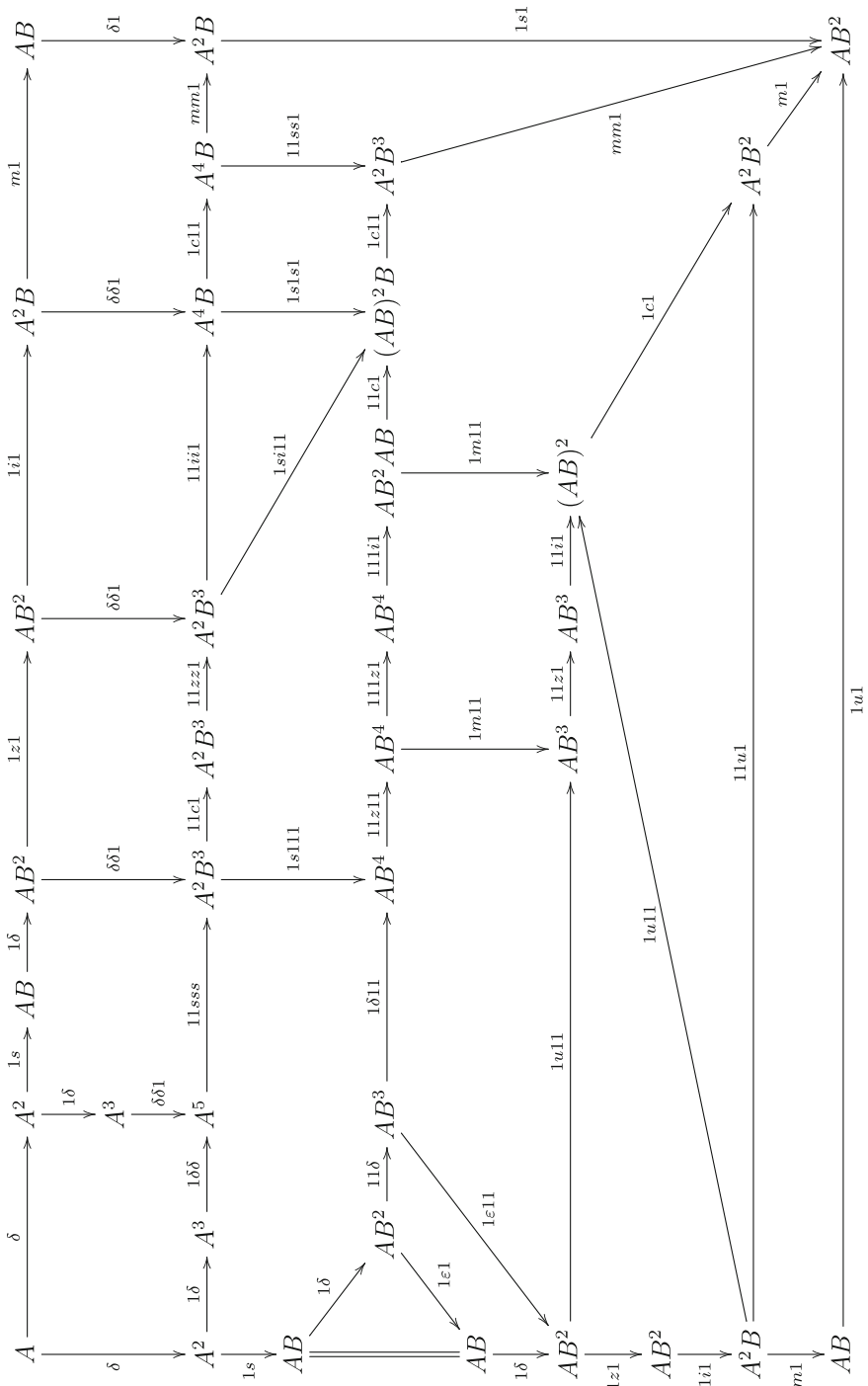


Fig. 1 Construction of q^{-1}

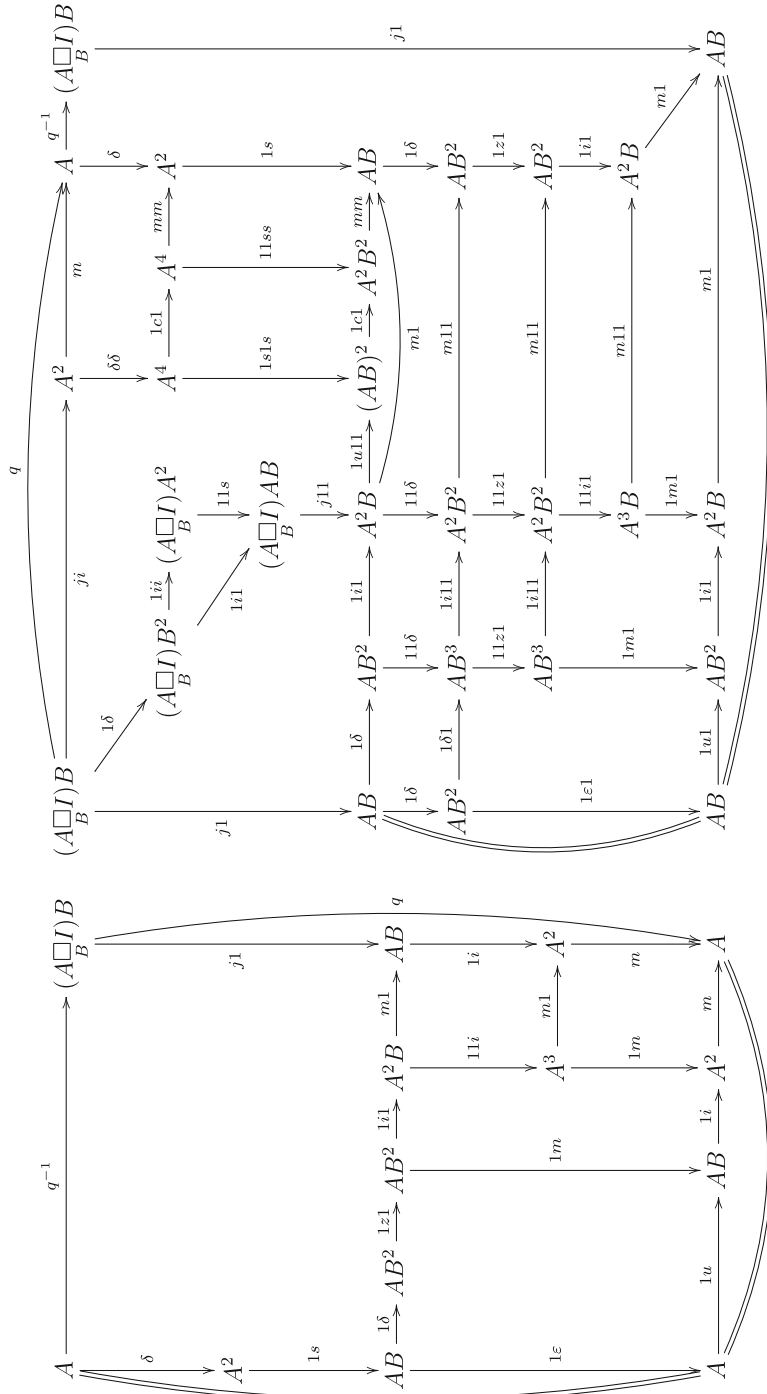


Fig. 2 Invertibility of q

morphisms are pairs of monoid morphisms $(B \xrightarrow{b} B' , Y \xrightarrow{y} Y')$ which are compatible with the actions $BY \xrightarrow{l} Y$ and $B'Y' \xrightarrow{l'} Y'$ in the sense that $l' \cdot by = y \cdot l$.

(2) • The category whose

objects are split epimorphisms $B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \end{matrix} A$ of group objects.

morphisms are pairs of monoid morphisms which are compatible with the epimorphisms s as well as their sections i .

- The category whose

objects consist of group objects B and Y in M , together with a left B -action on Y which makes Y a left B -module group.

morphisms are pairs of monoid morphisms $(B \xrightarrow{b} B' , Y \xrightarrow{y} Y')$ which are compatible with the actions $BY \xrightarrow{l} Y$ and $B'Y' \xrightarrow{l'} Y'$ in the sense that $l' \cdot by = y \cdot l$.

Remark 1.7 There are particular symmetric monoidal categories M whose cocommutative Hopf monoids constitute semi-abelian categories $\text{Hopf}(M)$; e.g. the category of sets (which is cartesian monoidal hence the Hopf monoids are the groups, all of them cocommutative) or the category of vector spaces over an arbitrary field [12] (for the particular case of an algebraically closed field see [11]). In such cases the equivalence of Proposition 1.5 (2) is in fact the equivalence $\text{SplitEpi}(\text{Hopf}(M)) \cong \text{Act}(\text{Hopf}(M))$ discussed in [14, Section 1], see [14, Example 3.10].

2 Reflexive Graphs of Monoids Versus Pre-crossed Modules

Consider a monoidal admissible class S of spans in a monoidal category C for which [3, Assumption 4.1] holds. Take an object $B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \end{matrix} A$ in the category $\text{SplitEpiMon}_S(C)$ of Theorem 1.1. Then by property (b) in Theorem 1.1, the induced morphism $q := (A \square_B I)B \xrightarrow{pA^i} A^2 \xrightarrow{m} A$ is invertible. Therefore by [3, Corollary 1.7] there is a bijective correspondence between the retractions t of the monoid morphism i and the monoid morphisms $A \square_B I \xrightarrow{k} B$ rendering commutative the diagram

$$\begin{array}{ccccc}
 B(A \square_B I) & \xrightarrow{i p A} & A^2 & \xrightarrow{m} & A & \xrightarrow{q^{-1}} & (A \square_B I)B \\
 \downarrow 1k & & & & & & \downarrow k1 \\
 B^2 & \xrightarrow{m} & B & \xleftarrow{m} & B^2 & &
 \end{array}$$

The correspondence is given by

$$t \mapsto k := A \square_B I \xrightarrow{pA} A \xrightarrow{t} B \quad k \mapsto t := A \xrightarrow{q^{-1}} (A \square_B I)B \xrightarrow{k1} B^2 \xrightarrow{m} B . \quad (2.1)$$

Combining this observation with the equivalence of Theorem 1.1, next we present an equivalent description of a suitable category of reflexive graphs of monoids. This leads to the notion of pre-crossed module over a monoid.

Theorem 2.1 Consider a monoidal admissible class \mathcal{S} of spans in a monoidal category \mathcal{C} for which [3, Assumption 4.1] holds. The following categories are equivalent.

$\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$ whose

objects are reflexive graphs $B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{matrix} A$ of monoids in \mathcal{C} subject to the following conditions.

- (a) $A \equiv A \xrightarrow{s} B \in \mathcal{S}$ (hence the \mathcal{S} -relative pullback $A \square_B I$ in Theorem 1.1 exists).
- (b) $q := (A \square_B I) B \xrightarrow{p_A i} A^2 \xrightarrow{m} A$ is invertible.

morphisms are pairs of monoid morphisms $(B \xrightarrow{b} B' , A \xrightarrow{a} A')$ such that $s' \cdot a = b \cdot s$, $t' \cdot a = b \cdot t$ and $i' \cdot b = a \cdot i$.

$\text{PreX}_{\mathcal{S}}(\mathcal{C})$ whose

objects consist of monoids B and Y , monoid morphisms $Y \xrightarrow{e} I$ and $Y \xrightarrow{k} B$ and a distributive law $BY \xrightarrow{x} YB$ subject to the following conditions.

- (a') $Y \equiv Y \xrightarrow{e} I \in \mathcal{S}$ and $B \equiv B \equiv B \in \mathcal{S}$.
- (b') $e1 \cdot x = 1e$ and $m \cdot k1 \cdot x = m \cdot 1k$.
- (c') The morphism f of Theorem 1.1 (c') is invertible.

morphisms are pairs of monoid morphisms $(B \xrightarrow{b} B' , Y \xrightarrow{y} Y')$ such that $e' \cdot y = e$, $k' \cdot y = b \cdot k$ and $x' \cdot by = yb \cdot x$.

Proof We show that the equivalence functors of Theorem 1.1 lift to the equivalence of the claim. In the direction $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{PreX}_{\mathcal{S}}(\mathcal{C})$ we send

$$\begin{array}{ccc}
 B & \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{matrix} & A \\
 \downarrow b & & \downarrow a \\
 B' & \begin{matrix} \xleftarrow{s'} \\ \xrightarrow{i'} \\ \xleftarrow{t'} \end{matrix} & A'
 \end{array}$$

to

$$\begin{array}{ccccccc}
 (A \square_B I, & B, & A \square_B I \xrightarrow{p_I} I, & A \square_B I \xrightarrow{p_A} A \xrightarrow{t} B, & B(A \square_B I) \xrightarrow{i p_A} A^2 \xrightarrow{m} A \xrightarrow{q^{-1}} (A \square_B I) B) \\
 \downarrow a \square 1 & \downarrow b & & & & & \\
 (A' \square_{B'} I, & B', & A' \square_{B'} I \xrightarrow{p_I} I, & A' \square_{B'} I \xrightarrow{p_{A'}} A' \xrightarrow{t'} B', & B'(A' \square_{B'} I) \xrightarrow{i' p_{A'}} A'^2 \xrightarrow{m'} A' \xrightarrow{q'^{-1}} (A' \square_{B'} I) B')
 \end{array}$$

By [3, Proposition 3.7 (1)], p_A is a monoid morphism hence so is $t \cdot p_A$. The second condition in (b') holds by the considerations preceding the theorem. Hence in light of the proof of Theorem 1.1 the object map is well-defined. Concerning the morphisms, the second condition

holds by the commutativity of

$$\begin{array}{ccccc}
 A \square_I & \xrightarrow{p_A} & A & \xrightarrow{t} & B \\
 \downarrow a \square 1 & & \downarrow a & & \downarrow b \\
 A' \square_{B'} & \xrightarrow{p_{A'}} & A' & \xrightarrow{t'} & B'
 \end{array}$$

Thus using again the proof of Theorem 1.1 we conclude that this functor is well-defined.

In the opposite direction the functor $\text{PreX}_{\mathcal{S}}(\mathcal{C}) \rightarrow \text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$ is defined by

$$\begin{array}{ccc}
 (Y, B, Y \xrightarrow{e} I, Y \xrightarrow{k} B, BY \xrightarrow{x} YB) & & B \xleftarrow{e1} YB \\
 \downarrow y & \downarrow b & \xleftarrow{u1} \\
 (Y', B', Y' \xrightarrow{e'} I, Y' \xrightarrow{k'} B', B'Y' \xrightarrow{x'} Y'B') & \mapsto & B' \xleftarrow{e'1} Y'B' \\
 & & \xleftarrow{u'1}
 \end{array}$$

By [3, Lemma 1.6], it follows from the second equality of (b') that $m \cdot k1$ is a monoid morphism. It is a retraction of $B \xrightarrow{u1} AB$ by the unitality of k . The monoid morphisms (b, yb) are compatible with $m \cdot k1$ by the compatibility of (b, y) with k and the multiplicativity of b . So using again the proof of Theorem 1.1 we conclude that this functor is well-defined too.

By the commutativity of

$$\begin{array}{ccc}
 (A \square_I)B \xrightarrow{p_{A^1}} AB \xrightarrow{t1} B^2 \xrightarrow{m} B & & Y \xrightarrow{k} B \\
 \downarrow p_{A^i} & \searrow tt & \downarrow 1u \\
 A^2 \xrightarrow{m} A & & YB \square_I \xrightarrow{p_{YB}} YB \xrightarrow{k1} B^2 \xrightarrow{m} B \\
 \downarrow q & \searrow t & \downarrow f \\
 A & & YB \square_B
 \end{array}$$

the components $(1, q)$ and $(1, f)$ of the natural isomorphisms in the proof of Theorem 1.1 are morphisms in the appropriate category. This proves that the stated functors are mutually inverse equivalences. \square

Example 2.2 As in Example 1.2, take the (evidently admissible and monoidal) class of all spans in the monoidal category \mathcal{C} of spans over a fixed set X . Then the equivalent categories of Theorem 2.1 take the following forms.

$\text{ReflGraphMon}(\mathcal{C})$ whose

- objects are reflexive graphs $B \begin{array}{c} \xleftarrow{\sigma} \\ \xrightarrow{\tau} \end{array} A$ of categories with the common object set X and identity-on-objects functors between them, such that the map (1.3) in Example 1.2 is invertible (recall that this holds e.g. if B is a groupoid).
- morphisms are pairs of compatible identity-on-objects functors.

$\text{PreX}(\mathcal{C})$ whose

- objects consist of categories B and Y of the common object set X such that Y is totally

disconnected (in the sense of [4]); an action (cf. Example 1.2) $B \square_X Y \xrightarrow{\triangleright} Y$ and an identity-on-objects functor $Y \xrightarrow{\kappa} B$ such that

$$\kappa(b \triangleright y) \cdot b = b \cdot \kappa(y) \tag{2.2}$$

for all morphisms b in B and y in Y for which $s(b) = t(y)$. (If B is a groupoid then (2.2) has the equivalent form $\kappa(b \triangleright y) = b \cdot \kappa(y) \cdot b^{-1}$; so when both B and Y are groupoids we recover the notion of *pre-crossed module* of groupoids in [5, Definition 1.2].)

morphisms are pairs of identity-on-objects functors $(B \xrightarrow{\beta} B', Y \xrightarrow{\nu} Y')$ such that $\kappa' \nu = \beta \kappa$ and $\nu(b \triangleright y) = \beta(b) \triangleright \nu(y)$ for all morphisms b in B and y in Y for which $s(b) = t(y)$.

Example 2.3 In the setting of Example 1.3, the equivalent categories of Theorem 2.1 take the following explicit forms.

$\text{RefGraphMon}_{\mathcal{S}}(\mathcal{C})$ whose

objects are reflexive graphs $B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \\ \xrightarrow{t} \end{matrix} A$ of bimonoids in \mathcal{M} subject to the following conditions.

- (a) The comultiplication δ of A satisfies $c \cdot 1s \cdot \delta = s1 \cdot \delta$.
- (b) In terms of the morphism j of (1.5), $q := (A \square B) \xrightarrow{j} A^2 \xrightarrow{m} A$ is invertible.

morphisms are pairs of bimonoid morphisms $(B \xrightarrow{b} B', A \xrightarrow{a} A')$ such that $s' \cdot a = b \cdot s, t' \cdot a = b \cdot t$ and $i' \cdot b = a \cdot i$.

$\text{PreX}_{\mathcal{S}}(\mathcal{C})$ whose

objects consist of a cocommutative bimonoid B and a bimonoid Y in \mathcal{M} , together with a left B -action l on Y which makes Y both a left B -module monoid and a left B -module comonoid, and a bimonoid morphism $Y \xrightarrow{k} B$ for which the following diagram commutes.

$$\begin{array}{ccccc}
 BY & \xrightarrow{\delta 1} & B^2 Y & \xrightarrow{1c} & BYB & \xrightarrow{l1} & YB & \tag{2.3} \\
 \downarrow 1k & & & & & & \downarrow k1 & \\
 B^2 & \xrightarrow{m} & B & \xleftarrow{m} & B^2 & & &
 \end{array}$$

morphisms are pairs of bimonoid morphisms $(B \xrightarrow{b} B', Y \xrightarrow{y} Y')$ which are compatible with the actions $BY \xrightarrow{l} Y$ and $B'Y' \xrightarrow{l'} Y'$ in the sense that $l' \cdot by = y \cdot l$ and which satisfy $k' \cdot y = b \cdot k$.

Proposition 2.4 *The equivalent categories of Example 2.3 have equivalent full subcategories as follows.*

- The category whose

objects are reflexive graphs $B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \\ \xrightarrow{t} \end{matrix} A$ of bimonoids in \mathcal{M} such that the comultiplication δ of A satisfies $c \cdot 1s \cdot \delta = s1 \cdot \delta$ and B is a Hopf monoid.

morphisms are pairs of bimonoid morphisms $(B \xrightarrow{b} B', A \xrightarrow{a} A')$ such that $s' \cdot a = \bar{b} \cdot s$, $t' \cdot a = b \cdot t$ and $i' \cdot b = a \cdot i$.

- The category whose *objects* consist of a cocommutative Hopf monoid B and a bimonoid Y in \mathbf{M} , together with a left action $BY \xrightarrow{l} Y$ which makes Y both a left B -module monoid and a left B -module comonoid, and a bimonoid morphism $Y \xrightarrow{k} B$ for which the diagram

$$\begin{array}{ccccc}
 BY & \xrightarrow{\delta k} & B^3 & \xrightarrow{1c} & B^3 & \xrightarrow{mz} & B^2 & & (2.4) \\
 \downarrow l & & & & & & \downarrow m & & \\
 Y & \xrightarrow{\quad\quad\quad k \quad\quad\quad} & & & & & B, & &
 \end{array}$$

occurring in [25, Definition 12 (iv)], commutes.

morphisms are pairs of bimonoid morphisms $(B \xrightarrow{b} B', Y \xrightarrow{y} Y')$ for which $l' \cdot by = y \cdot l$ and $k' \cdot y = b \cdot k$.

Proof For a reflexive graph $B \overset{s}{\underset{i}{\rightleftarrows}} A$ of bimonoids such that B is a Hopf monoid,

$(A \square_B I)B \xrightarrow{q} A$ in part (b) of Example 2.3 is invertible by Example 1.5.

Whenever B has an antipode z , the commutative diagram (2.3) has an equivalent form (2.4). Their equivalence follows by the commutativity of the diagrams of Fig. 3. □

Lemma 2.5 Consider a monoidal admissible class \mathcal{S} of spans in a monoidal category \mathbf{C} for which [3, Assumption 4.1] holds. For any object $B \overset{s}{\underset{i}{\rightleftarrows}} A$ of the category $\text{ReflGraphMon}_{\mathcal{S}}(\mathbf{C})$ of Theorem 2.1, the following assertions are equivalent.

- (i) $B \xleftarrow{t} A = A \in \mathcal{S}$.
- (ii) $B \xleftarrow{k:=t \cdot p_A} A \square_B I = A \square_B I \in \mathcal{S}$.

Proof Assertion (i) implies (ii) by [3, Lemma 3.4]. Conversely, since $B \overset{s}{\underset{i}{\rightleftarrows}} A$ is an object of $\text{ReflGraphMon}_{\mathcal{S}}(\mathbf{C})$, we have $A = A \xrightarrow{s} B \in \mathcal{S}$. Then it follows by [3, Lemma 2.4] ² that $B = B = B \in \mathcal{S}$, whence (ii) implies $B^2 \xleftarrow{k^1} (A \square_B I)B = (A \square_B I)B \in \mathcal{S}$ by the multiplicativity of \mathcal{S} . Then by (PRE) also $B^2 \xleftarrow{k^1} (A \square_B I)B \xleftarrow{q^{-1}} A \xrightarrow{q^{-1}} (A \square_B I)B \in \mathcal{S}$. So using the identity $t = m \cdot k^1 \cdot q^{-1}$ from (2.1), (i) follows by (POST) (composing by m on the left and by q on the right). □

² Apologies about a regrettable typo in the first line of [3, Lemma 2.4], interchanging the symbols A and B .

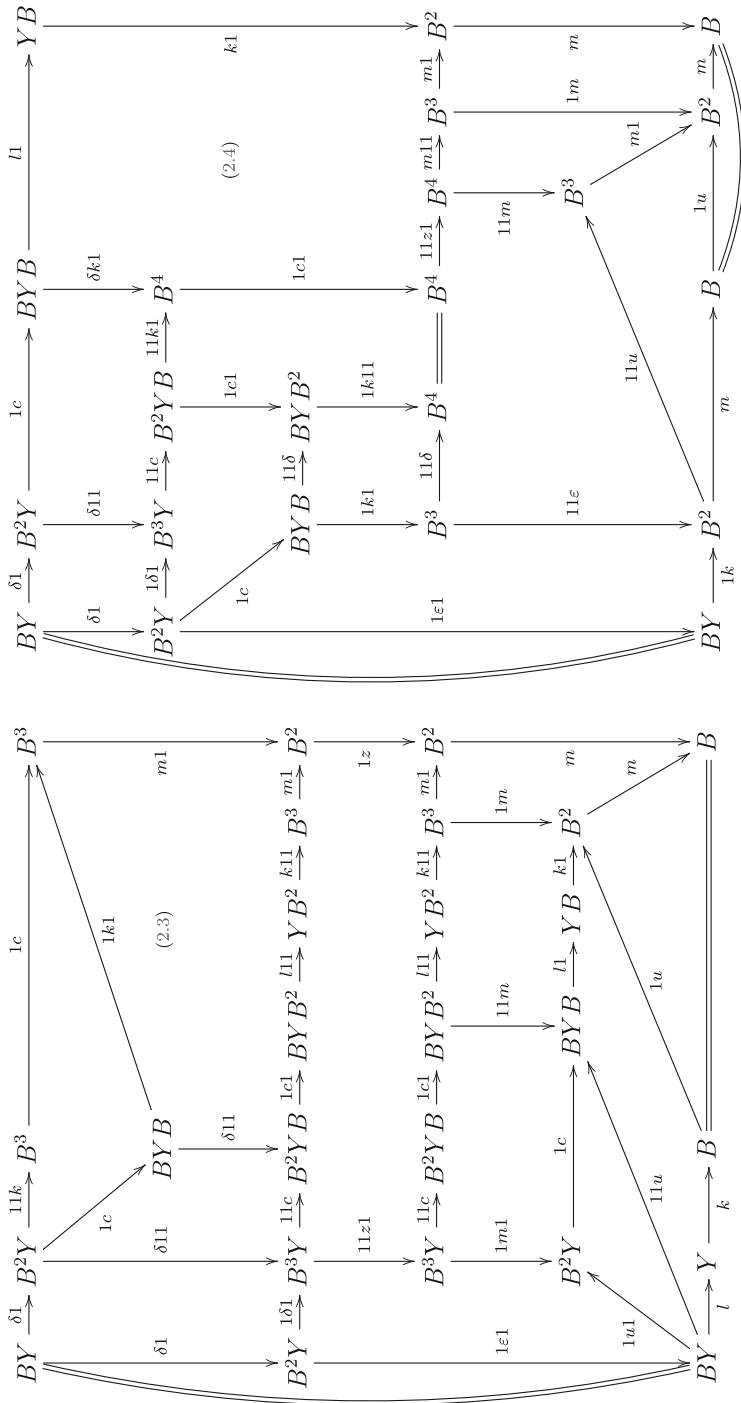


Fig. 3 Equivalence of (2.3) and (2.4) for Hopf monoids B

3 Relative Categories of Monoids Versus Crossed Modules

Consider again a monoidal admissible class \mathcal{S} of spans in a monoidal category \mathcal{C} for which [3, Assumption 4.1] holds. Take an object $B \overset{s}{\leftarrow} \overset{i}{\rightleftarrows} A \overset{t}{\rightarrow}$ of the category $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$ of Theorem 2.1 such that also $B \overset{t}{\leftarrow} A \overset{s}{\rightarrow} = A \in \mathcal{S}$; that is, the legs of the cospan $A \overset{s}{\rightarrow} B \overset{t}{\leftarrow} A$ are in \mathcal{S} (hence there exists its \mathcal{S} -relative pullback $A \overset{p_1}{\leftarrow} A \square_B A \overset{p_2}{\rightarrow} A$). Whenever the morphism

$$q_2 := (A \square_B I)A \xrightarrow{p_A 1} A^2 \xrightarrow{(1 \square i)(i \square 1)} (A \square_B A)^2 \xrightarrow{m} A \square_B A \quad (3.1)$$

is invertible, we infer from [3, Corollary 1.7] that there exists at most one monoid morphism d rendering commutative

$$\begin{array}{ccccc} A \square_B I & \xrightarrow{p_A} & A & \xrightarrow{1 \square i} & A \square_B A & \xleftarrow{i \square 1} & A \\ & \searrow p_A & & & \downarrow d & & \\ & & & & A & & \end{array} \quad (3.2)$$

which is our candidate to serve as the composition morphism of a relative category. Note that if there is a monoid morphism d rendering commutative the diagram of (3.2), then it satisfies

$$d \cdot q_2 = d \cdot m \cdot (1 \square i)(i \square 1) \cdot p_A 1 = m \cdot d \cdot (1 \square i)(i \square 1) \cdot p_A 1 = m \cdot p_A 1. \quad (3.3)$$

So if q_2 is invertible, then the only candidate is $d = m \cdot p_A 1 \cdot q_2^{-1}$. By this motivation, in this section first we investigate the condition that q_2 of (3.1) is invertible. Assuming so, next we show that whenever the morphism d of (3.2) exists, it makes the object $B \overset{s}{\leftarrow} \overset{i}{\rightleftarrows} A \overset{t}{\rightarrow}$ of $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$ to an \mathcal{S} -relative category. Finally, based on Theorem 2.1, we give an equivalent description of the category of \mathcal{S} -relative categories in the category of monoids in \mathcal{C} , in terms of crossed modules introduced hereby.

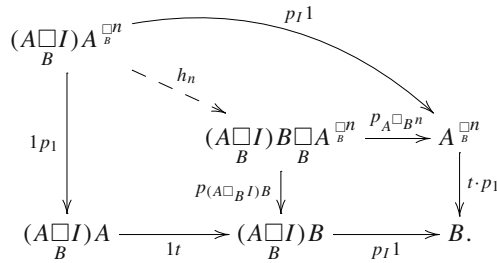
3.1 Invertibility of Some Canonical Morphisms

Lemma 3.1 *Consider a monoidal admissible class \mathcal{S} of spans in a monoidal category \mathcal{C} for which [3, Assumption 4.1] holds. For any monoid B in \mathcal{C} for which $B = B = B$ is in \mathcal{S} , for any span of monoids $B \overset{t}{\leftarrow} A \overset{s}{\rightarrow} B$ with legs in \mathcal{S} , and for any natural number n , the following assertions hold. (Recall the convention $A \square_B^0 := B$ from [3, Corollary 4.6].)*

(1) *There exists the \mathcal{S} -relative pullback*

$$\begin{array}{ccc} (A \square_B I)B \square_B A \square_B^n & \xrightarrow{p \square_B^n} & A \square_B^n \\ \downarrow p(A \square_B I)B & & \downarrow t \cdot p_1 \\ (A \square_B I)B & \xrightarrow{p_1 1} & B. \end{array}$$

(2) There is a unique morphism h_n rendering commutative the diagram



(3) For a common section i of s and t , consider the morphism

$$q_{n+1} := (A \square_B I) A_B^{\square n} \xrightarrow{p_A 1} A A_B^{\square n} \xrightarrow{(1 \square i \dots \square i)(i \square 1 \square \dots \square 1)} (A \square_B^{\square n+1})^2 \xrightarrow{m} A_B^{\square n+1} \tag{3.4}$$

(it is well-defined by [3, Proposition 3.5] and q_1 is equal to q in Theorem 1.1 (b)). If q_{n+1} is invertible for some n , then q_k is invertible for all $0 < k \leq n$.

(4) For a common section i of s and t the following are equivalent.

- (i) h_n in part (2) and q_1 in part (3) are invertible.
- (ii) q_{n+1} in part (3) is invertible.

Proof (1) By assumption $B \xleftarrow{t} A = A \in \mathcal{S}$ and by the unitality of \mathcal{S} , $I = I = I \in \mathcal{S}$. Then by [3, Lemma 3.4],

$$B \xleftarrow{t} A \xleftarrow{p_1} A_B^{\square n} = A_B^{\square n} \in \mathcal{S} \text{ and } A \square_B I = A \square_B I \xrightarrow{p_1} I \in \mathcal{S}. \tag{3.5}$$

By assumption also $B = B = B \in \mathcal{S}$ hence by the second assertion in (3.5) and the multiplicativity of \mathcal{S}

$$(A \square_B I) B = (A \square_B I) B \xrightarrow{p_1 1} B \in \mathcal{S}. \tag{3.6}$$

The first assertion of (3.5) and (3.6) say that the legs of $(A \square_B I) B \xrightarrow{p_1 1} B \xleftarrow{t \cdot p_1} A_B^{\square n}$ are in \mathcal{S} hence their \mathcal{S} -relative pullback exists by assumption.

(2) By (3.5) and the multiplicativity of \mathcal{S} ,

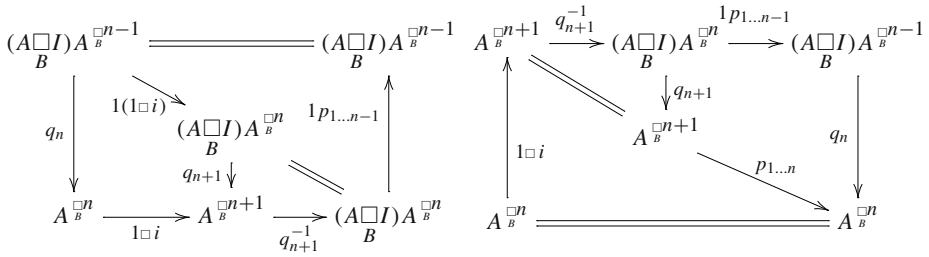
$$(A \square_B I) B \xleftarrow{1_t} (A \square_B I) A \xleftarrow{1_{p_1}} (A \square_B I) A_B^{\square n} \xrightarrow{p_1 1} A_B^{\square n} \in \mathcal{S}.$$

Hence by the evident commutativity of the exterior of the diagram in part (2), universality of the \mathcal{S} -relative pullback in its codomain implies the existence of the unique morphism h_n .

(3) For some positive integer n assume that q_{n+1} is invertible. Then so is q_n with the inverse

$$A_B^{\square n} \xrightarrow{1 \square i} A_B^{\square n+1} \xrightarrow{q_{n+1}^{-1}} (A \square_B I) A_B^{\square n} \xrightarrow{1_{p_1 \dots n-1}} (A \square_B I) A_B^{\square n-1}. \tag{3.7}$$

Indeed, (3.7) renders commutative both diagrams



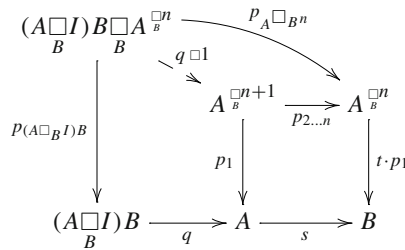
The leftmost region of the first diagram commutes by the explicit expression (3.4) of q_n and q_{n+1} , multiplicativity of $1_{\square i}$ and the functoriality of \square , see [3, Proposition 3.5 (2)]. The rightmost region of the second diagram commutes again by the explicit expression (3.4) of q_n and q_{n+1} and the multiplicativity of $p_{1\dots n}$.

(4) Our strategy is to prove that q_{n+1} can be rewritten as

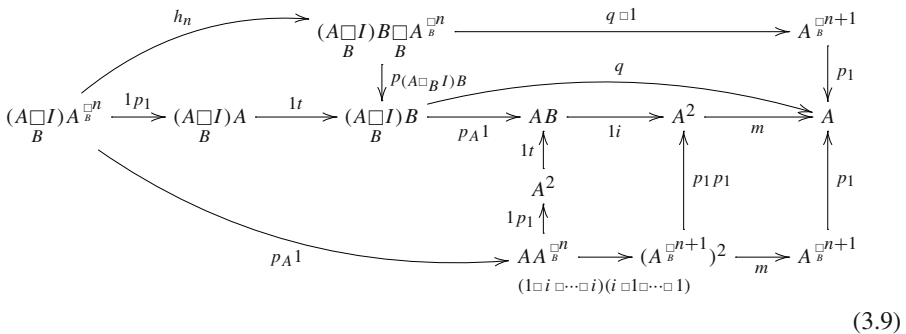
$$(A \square I)_B A_B^{\square n} \xrightarrow{h_n} (A \square I)_B B \square A_B^{\square n} \xrightarrow{q \square 1} A_B^{\square n+1}. \tag{3.8}$$

Then (i) obviously implies (ii) and in view of part (3) also the opposite implication holds.

The occurring morphism $q \square 1$ is defined as the unique morphism rendering commutative the diagram



It is well-defined by the commutativity of the first diagram of (1.1); see [3, Proposition 3.5 (2)]. The morphism of (3.8) is equal to q_{n+1} by the commutativity of both diagrams



(3.9)

$$\begin{array}{ccc}
 (A \square I) A \square^n & \xrightarrow{q \square 1} & A \square^{n+1} \\
 \downarrow p \square B^n & & \downarrow p_{2 \dots n} \\
 (A \square I) A \square^n & \xrightarrow{u1} & BA \square^n \xrightarrow{(i \square \dots \square i)1} (A \square^n)^2 \xrightarrow{m} A \square^n \\
 \downarrow p_{A \square B^n} & \uparrow s1 & \uparrow p_{2 \dots n} p_{2 \dots n} \\
 (A \square I) A \square^n & \xrightarrow{pA^1} & AA \square^n \xrightarrow{(1 \square i \square \dots \square i)(i \square 1 \square \dots \square 1)} (A \square^{n+1})^2 \xrightarrow{m} A \square^{n+1} \\
 & & \uparrow p_{2 \dots n}
 \end{array}$$

(3.10)

whose right vertical arrows are jointly monomorphic. □

Example 3.2 In the category \mathcal{C} of spans over a given set X from Example 1.2, the morphisms h_n of Lemma 3.1 (2) are isomorphisms, see the pullback (1.4). Hence for any reflexive graph $B \begin{smallmatrix} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{smallmatrix} A$ of categories with common object set X and identity-on-objects functors between them, all morphisms $\{q_n\}_{n>0}$ in Lemma 3.1 (3) are invertible if and only if q_1 is so; see Lemma 3.1 (4). The latter condition holds e.g. if B is a groupoid, see Example 1.2.

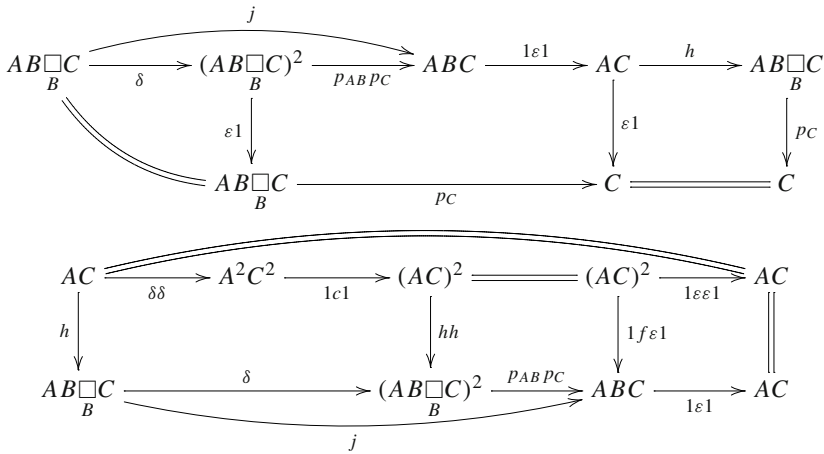
Example 3.3 In the context of Example 1.3 we know from [3, Example 4.3] that [3, Assumption 4.1] holds for the monoidal admissible class \mathcal{S} in [3, Example 2.3] and [3, Example 2.7] of spans in \mathcal{C} .

In this situation, for any cocommutative comonoid B in M and any comonoid morphism $C \xrightarrow{f} B$ such that the comultiplication δ of C satisfies $f1 \cdot \delta = f1 \cdot c \cdot \delta$, there is a unique isomorphism h rendering commutative the diagram

$$\begin{array}{ccc}
 AC & \xrightarrow{\varepsilon 1} & C \\
 \downarrow h & \searrow p_C & \downarrow f \\
 AB \square_B C & \xrightarrow{p_C} & C \\
 \downarrow p_{AB} & & \downarrow f \\
 AB & \xrightarrow{\varepsilon 1} & B
 \end{array}$$

with the inverse $AB \square_B C \xrightarrow{j} ABC \xrightarrow{1\varepsilon 1} AC$ (where $j = p_{AB} p_C \cdot \delta$ is the equalizer of $1\delta 1$ and $11f1 \cdot 11\delta$ as in (1.5); and ε stands for both counits of A and B). Indeed, the following diagrams commute.

$$\begin{array}{ccccccc}
 AB \square_B C & \xrightarrow{j} & (AB \square_B C)^2 & \xrightarrow{1\varepsilon 1} & ABC & \xrightarrow{h} & AB \square_B C \\
 \downarrow p_{AB} & \delta & \downarrow p_{AB} p_{AB} & \downarrow 11f & \downarrow 1f & \downarrow p_{AB} & \\
 AB & \xrightarrow{\delta \delta} & A^2 B^2 & \xrightarrow{11\varepsilon 1} & AB^2 & \xrightarrow{1\varepsilon 1} & AB \\
 & & \downarrow 1c1 & & & & \\
 & & (AB)^2 & & & &
 \end{array}$$



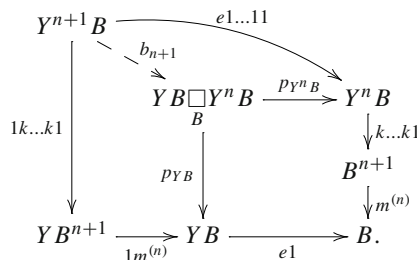
By [3, Example 2.8] there is an induced monoidal admissible class (also denoted by \mathcal{S}) in the category of monoids in \mathcal{C} (that is, the category of bimonoids in \mathcal{M}) also satisfying [3, Assumption 4.1] by [3, Example 4.4]. So whenever the above morphism f is a monoid morphism as well, there is a bimonoid isomorphism h in the diagram, see [3, Proposition 3.7]. Consequently, in the category of bimonoids in \mathcal{M} , the morphisms h_n of Lemma 3.1 (2) are isomorphisms. Therefore q_n in Lemma 3.1 (3) is an isomorphism for all positive integer n if and only if it is invertible for $n = 1$; and this holds whenever B is a Hopf monoid, see Proposition 1.5.

Lemma 3.4 *Let \mathcal{S} be a monoidal admissible class of spans in a monoidal category \mathcal{C} for which [3, Assumption 4.1] holds and let $(B, Y, Y \xrightarrow{e} I, Y \xrightarrow{k} B, BY \xrightarrow{x} YB)$ be an object of the category $\text{PreX}_{\mathcal{S}}(\mathcal{C})$ in Theorem 2.1 such that $B \xleftarrow{k} Y = Y \in \mathcal{S}$. For any natural number n denote by $B^{n+1} \xrightarrow{m^{(n)}} B$ the n -times iterated multiplication (unique by the associativity of m ; by definition the identity morphism for $n = 0$) and consider the span*

$$B \xleftarrow{m^{(n)}} B^{n+1} \xleftarrow{k \dots k1} Y^n B \xrightarrow{e \dots e1} B. \tag{3.11}$$

For any natural number n the following assertions hold.

- (1) The cospan $YB \xrightarrow{e1} B \xleftarrow{m^{(n)}} B^{n+1} \xleftarrow{k \dots k1} Y^n B$ has its legs in \mathcal{S} (hence there exists its \mathcal{S} -relative pullback $YB \square_B Y^n B$).
- (2) There exists a unique morphism b_{n+1} of spans (for the spans (3.11)) rendering commutative



- (3) If b_{n+1} in part (2) is an isomorphism then also b_k is an isomorphism for all $0 < k \leq n$.
- (4) For the morphism

$$q_{n+1} := (Y B \square_I) (Y B) \square_B^{n+1} \xrightarrow{p_{YB}^1} Y B (Y B) \square_B^n \xrightarrow{(1 \square u 1 \square \dots \square u 1)(u 1 \square 1)} ((Y B) \square_B^{n+1})^2 \xrightarrow{m} (Y B) \square_B^{n+1}$$

the following diagram commutes

$$\begin{array}{ccccccc} Y^{n+1} B & \xrightarrow{b_{n+1}} & Y B \square_B Y^n B & \xrightarrow{1 \square b_n} & \dots & \xrightarrow{1 \square b_1} & (Y B) \square_B^{n+1} \square_B B \\ \downarrow f \dots f 1 & & & & & & \downarrow p_{(Y B) \square_B^{n+1}} \\ (Y B \square_I)^{n+1} B & \xrightarrow{1 \dots 1 q_1} & (Y B \square_I)^n Y B & \xrightarrow{1 \dots 1 q_2} & \dots & \xrightarrow{q_{n+1}} & (Y B) \square_B^{n+1} \end{array}$$

where f is the isomorphism in Theorem 1.1 (c').

- (5) b_{n+1} in part (2) is an isomorphism if and only if q_{n+1} in part (4) is an isomorphism.

Proof (1) By definition the first two spans in

$$Y = Y \xrightarrow{e} I \quad B = B = B \quad B \xleftarrow{k} Y = Y \quad Y B = Y B \xrightarrow{e^1} B \quad (3.12)$$

belong to \mathcal{S} hence so does the last one by the multiplicativity of \mathcal{S} . Again, by definition the second and the third spans of (3.12) belong to \mathcal{S} hence by the multiplicativity of \mathcal{S} so does the first one in

$$B^{n+1} \xleftarrow{k \dots k 1} Y^n B = Y^n B \quad B \xleftarrow{m^{(n)}} B^{n+1} \xleftarrow{k \dots k 1} Y^n B = Y^n B. \quad (3.13)$$

Then the second span of (3.13) is in \mathcal{S} by (POST).

(2) Since the first span of (3.12) and the second span of (3.13) are in \mathcal{S} , the multiplicativity of \mathcal{S} implies that so is

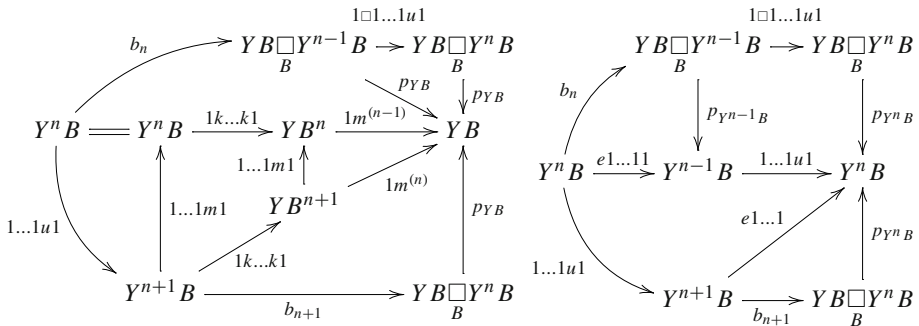
$$Y B \xleftarrow{1 m^{(n)}} Y B^{n+1} \xleftarrow{1 k \dots k 1} Y^{n+1} B \xrightarrow{e^1 \dots 1} Y^n B.$$

So by the evident commutativity of the exterior of the diagram of part (2) the stated morphism b_{n+1} exists. It is a morphism of spans (for the spans (3.11)) by the commutativity of the following diagrams.

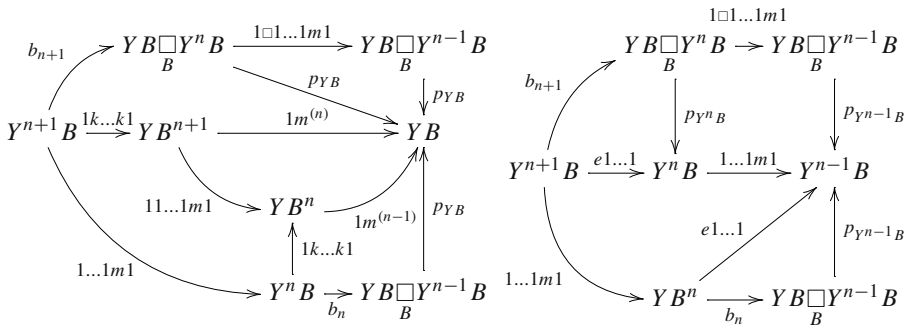
$$\begin{array}{ccc} Y^{n+1} B & \xrightarrow{e \dots e 1} & B \\ \downarrow b_{n+1} & \searrow e^1 \dots 1 & \parallel \\ Y B \square_B Y^n B & \xrightarrow{p_{Y^n B}} & Y^n B \xrightarrow{e \dots e 1} B \end{array} \quad \begin{array}{ccccccc} Y^{n+1} B & \xrightarrow{1 k \dots k 1} & Y B^{n+1} & \xrightarrow{k 1 \dots 1} & B^{n+2} & \xrightarrow{m^{(n+1)}} & B \\ \downarrow b_{n+1} & & \downarrow 1 m^{(n)} & & \downarrow 1 m^{(n)} & & \parallel \\ Y B \square_B Y^n B & \xrightarrow{p_{Y B}} & Y B & \xrightarrow{k 1} & B^2 & \xrightarrow{m} & B \end{array}$$

- (3) Since for a positive integer n , $Y^{n-1} B \xrightarrow{1 \dots 1 u 1} Y^n B$ is a morphism between the spans of (3.11), the morphism in the top row of the following diagram is well-defined by [3,

Proposition 3.5].



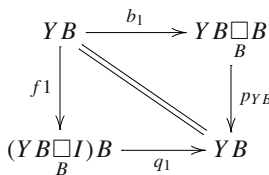
By their commutativity we infer $b_{n+1} \cdot 1 \dots 1u1 = (1 \square 1 \dots 1u1) \cdot b_n$. Similarly, since for $n > 0$ also $Y^n B \xrightarrow{1 \dots 1m1} Y^{n-1} B$ is a morphism between the spans of (3.11), the morphism in the top row of the following diagram is well-defined by [3, Proposition 3.5].



By their commutativity, $b_n \cdot 1 \dots 1m1 = (1 \square 1 \dots 1m1) \cdot b_{n+1}$. It follows from these identities and the unitality of the monoid Y that whenever b_{n+1} is invertible then so is b_n with the inverse

$$Y B \square_{B} Y^{n-1} B \xrightarrow{1 \square 1 \dots 1u1} Y B \square_{B} Y^n B \xrightarrow{b_{n+1}^{-1}} Y^{n+1} B \xrightarrow{1 \dots 1m1} Y^n B.$$

(4) We proceed by induction in n . For $n = 0$ the diagram in the claim reduces to the diagram



whose upper half part commutes by construction (see part (2)) and the lower half part commutes since $f1$ and $q1$ are mutual inverses (see the proof of Theorem 1.1).

For any positive value of n , denote the top-right path in the diagram of the claim by \tilde{b}_{n+1} and the bottom row by \tilde{q}_{n+1} . Then the diagram takes the form

$$\begin{array}{ccc}
 Y^{n+1} B & \xrightarrow{\tilde{b}_{n+1}} & (Y B)_B^{\square n+1} \\
 \downarrow f1 \dots 11 & & \parallel \\
 (Y B \square_B I) Y^n B & \searrow \tilde{b}_n & \\
 \downarrow 1f \dots f1 & & \\
 (Y B \square_B I)^{n+1} B & \xrightarrow{1\tilde{q}_n} (Y B \square_B I) (Y B)_B^{\square n} \xrightarrow{q_{n+1}} (Y B)_B^{\square n+1} & \\
 & \xrightarrow{\tilde{q}_{n+1}} &
 \end{array}$$

The region at the bottom left corner commutes if the claim holds for $n - 1$; and the commutativity of the large region is proven in Fig. 4.

(5) By Theorem 1.1 q_1 is an isomorphism without any further assumption; it is the inverse of the isomorphism $Y B \xrightarrow{f1} (Y B \square_B I) B$. Also b_1 is an isomorphism; the inverse of the isomorphism $Y B \square_B B \xrightarrow{p_{YB}} Y B$ in [3, Proposition 3.6 (1)].

Assume that b_l is iso for some $l > 1$. Take the diagram of part (4) for $n = 1$; it says $b_2 = q_2 \cdot f11$. Since f is an isomorphism by definition and b_2 is an isomorphism by part (3), also q_2 is an isomorphism. If $l = 2$ then this completes the proof. If $l > 2$ then take next the diagram of part (4) for $n = 2$; it says $(1 \square b_2) \cdot b_3 = q_3 \cdot 1q_2 \cdot ff11$. All of the occurring morphisms but q_3 are known to be isomorphisms proving that so is q_3 . Repeating this reasoning for all $n \leq l$ we conclude that q_n is an isomorphism for all $0 < n \leq l$.

The opposite implication is proven by the same steps. Assume that q_l is iso for some $l > 1$. Take the diagram of part (4) for $n = 1$; it says $b_2 = q_2 \cdot f11$. Since f is an isomorphism by definition and q_2 is an isomorphism by Lemma 3.1 (3), also b_2 is an isomorphism. If $l = 2$ then this completes the proof. If $l > 2$ then take next the diagram of part (4) for $n = 2$; it says $(1 \square b_2) \cdot b_3 = q_3 \cdot 1q_2 \cdot ff11$. All of the occurring morphisms but b_3 are known to be isomorphisms proving that so is b_3 . Repeating this reasoning for all $n \leq l$ we conclude that b_n is an isomorphism for all $0 < n \leq l$. □

Example 3.5 Take \mathcal{S} to be the (monoidal and admissible) class of all spans in the monoidal category \mathcal{C} of spans over a given set. For any object of the category $\text{ReflGraphMon}(\mathcal{C})$ of Example 2.2 and for any positive integer n , the morphism b_n in Lemma 3.4 (2) is invertible, see the pullback (1.4).

Example 3.6 In the setting of Example 1.3 we know from Example 3.3 that the morphism q_n of Lemma 3.1 (3) is invertible for any positive integer n and for any object of $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$. By the isomorphism of Theorem 2.1 this means that the morphism q_n

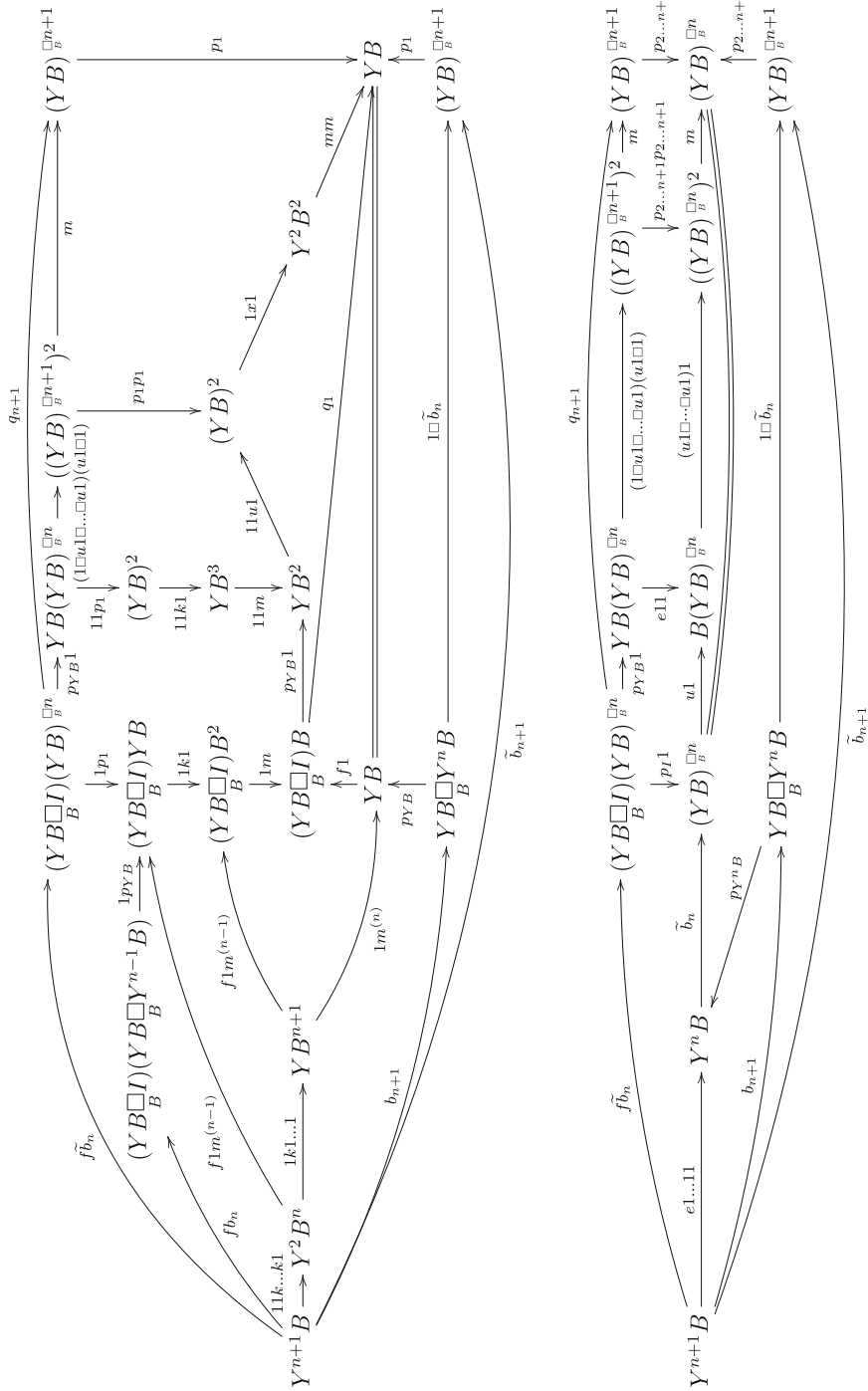
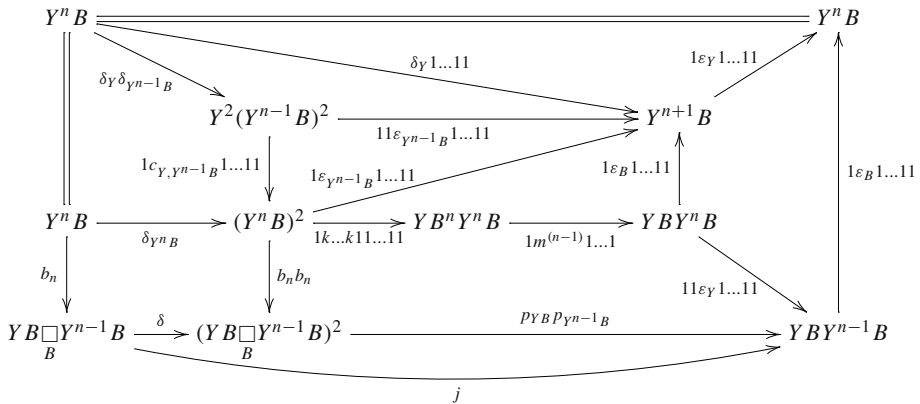


Fig. 4 Proof of $\tilde{b}_{n+1} = q_{n+1} \cdot f_{\tilde{b}_n}$

of Lemma 3.4 (4) is invertible for any object of $\text{PreX}_S(\mathcal{C})$. Then also the morphism b_n of Lemma 3.4 (2) is invertible by Lemma 3.4 (5). Since the diagram



commutes, we conclude that the morphism in its bottom-right path—involving the equalizer j as in (1.5)—is the inverse of b_n .

Let \mathcal{S} be a monoidal admissible class of spans in a monoidal category \mathcal{C} for which [3, Assumption 4.1] holds. For any object $(B, Y, Y \xrightarrow{e} I, Y \xrightarrow{k} B, BY \xrightarrow{x} YB)$ of the category $\text{PreX}_S(\mathcal{C})$ in Theorem 2.1, it follows by the conditions in (a') and the multiplicativity of \mathcal{S} that the span $YB = YB \xrightarrow{e1} B$ belongs to \mathcal{S} . Then by [3, Proposition 3.6] the \mathcal{S} -relative pullback in the first diagram of

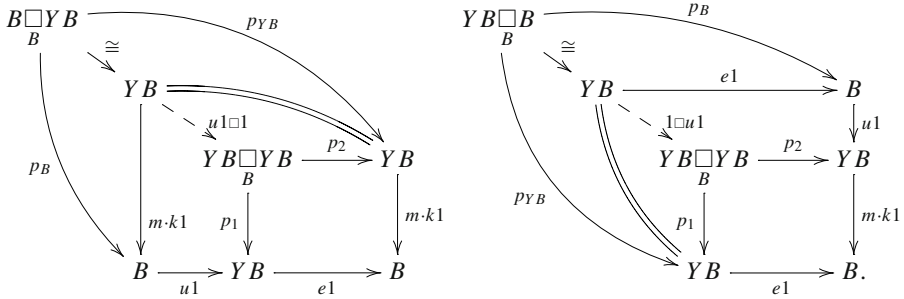
$$\begin{array}{ccc}
 YB \square_B & \xrightarrow{p_B} & B \\
 p_{YB} \downarrow & & \parallel \\
 YB & \xrightarrow{e1} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 B \square_B YB & \xrightarrow{p_{YB}} & YB \\
 p_B \downarrow & & \downarrow m \cdot k1 \\
 B & \xlongequal{\quad} & B
 \end{array}
 \tag{3.14}$$

exists and the left vertical of the first diagram is an isomorphism. If in addition $B \xleftarrow{k} Y = Y \in \mathcal{S}$, then by (a') and the multiplicativity of \mathcal{S} also $B^2 \xleftarrow{k1} YB = YB \in \mathcal{S}$. Hence by (POST) $B \xleftarrow{m} B^2 \xleftarrow{k1} YB = YB \in \mathcal{S}$. We infer again by [3, Proposition 3.6] that the \mathcal{S} -relative pullback in the second diagram of (3.14) exists and the top row of of the second diagram is an isomorphism.

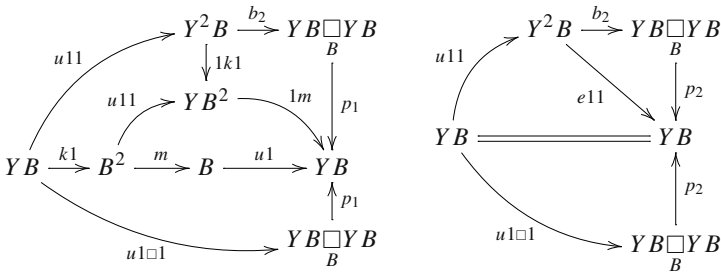
Lemma 3.7 *Let \mathcal{S} be a monoidal admissible class of spans in a monoidal category \mathcal{C} for which [3, Assumption 4.1] holds and let $(B, Y, Y \xrightarrow{e} I, Y \xrightarrow{k} B, BY \xrightarrow{x} YB)$ be an object of the category $\text{PreX}_S(\mathcal{C})$ in Theorem 2.1 such that $B \xleftarrow{k} Y = Y \in \mathcal{S}$. The morphism b_2 in Lemma 3.4 (2) satisfies the identities*

- (1) $b_2 \cdot u11 = u1 \square 1$ and
- (2) $b_2 \cdot 1u1 = 1 \square u1$,

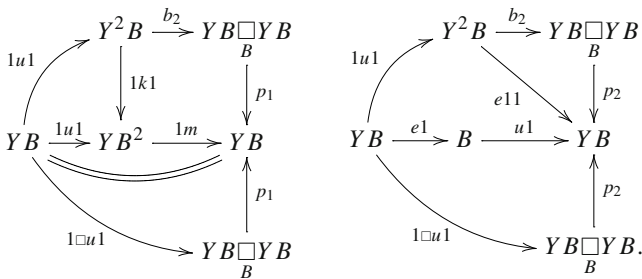
whose right hand sides are defined as the unique fillers of the respective diagrams



Proof Assertion (1) follows by the commutativity of the diagrams



and part (2) follows by the commutativity of the diagrams



□

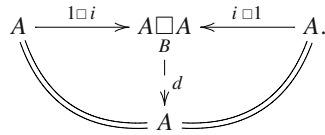
3.2 The Composition Morphism of a Relative Category of Monoids

Proposition 3.8 Consider a monoidal admissible class \mathcal{S} of spans in a monoidal category \mathcal{C} such that [3, Assumption 4.1] holds. Take an object $B \overset{s}{\leftarrow} \overset{i}{\rightleftarrows} \overset{t}{\rightarrow} A$ of the category $\text{RefGraphMon}_{\mathcal{S}}(\mathcal{C})$ of Theorem 2.1 such that the following properties hold.

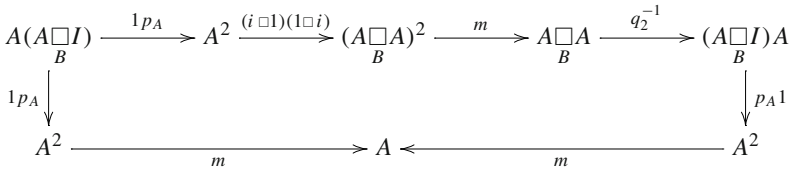
- $B \overset{t}{\leftarrow} A \equiv A$ belongs to \mathcal{S}
- the morphism q_3 of Lemma 3.1 (3) is invertible.

The following assertions hold.

(1) *There is at most one monoid morphism d rendering commutative*



(2) *The monoid morphism d of part (1) exists if and only if the following diagram commutes (recall that q_2 is invertible by Lemma 3.1 (3)).*

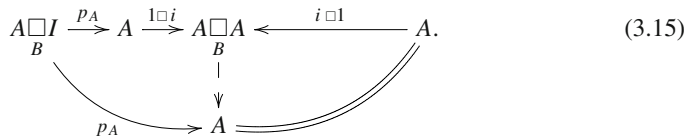


Moreover, in this case d is equal to $A \square_B A \xrightarrow{q_2^{-1}} (A \square_B I)A \xrightarrow{p_A 1} A^2 \xrightarrow{m} A$.

(3) *Whenever the monoid morphism d of part (1) exists, $B \begin{smallmatrix} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{smallmatrix} A \xleftarrow{d} A \square_B A$ is an \mathcal{S} -relative category in the category of monoids in \mathcal{C} .*

Proof The proof is built on [3, Corollary 1.7].

(1) Since the morphism q_2 in Lemma 3.1 (3) is invertible, we know from [3, Corollary 1.7] that there is at most one monoid morphism rendering commutative



Since a monoid morphism d as in part (1) obviously renders commutative (3.15), this proves its uniqueness.

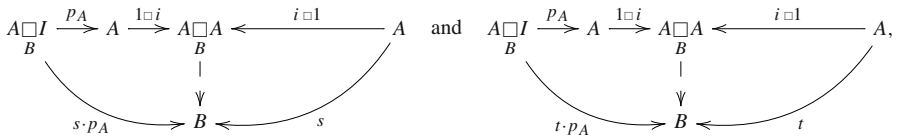
(2) By [3, Corollary 1.7] commutativity of the diagram of part (2) is equivalent to the existence of a (unique) monoid morphism making (3.15) commute. Since a monoid morphism d in part (1) provides such a morphism, its existence implies commutativity of the diagram of part (2).

In order to prove the converse implication, we show that any monoid morphism d making (3.15) commute renders commutative also the diagram of part (1). Recall from [3, Lemma 1.2] that the invertibility of q in Theorem 2.1 (b) implies that p_A and i are jointly epic morphisms of monoids. Hence if d makes (3.15) commute then it does so the left hand side of the diagram of part (1) by $d \cdot (1 \square i) \cdot i = d \cdot (i \square 1) \cdot i = i$.

The stated expression of d immediately follows from [3, Corollary 1.7] (see also (3.3)).

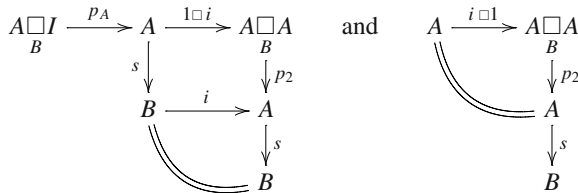
(3) In order to see that the monoid morphism d in part (1) is a morphism of spans, we use that by the invertibility of q_2 there are unique morphisms rendering commutative the

respective diagrams



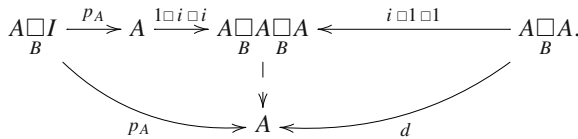
see [3, Corollary 1.7]. Now $s \cdot d$ obviously makes the first diagram commute and so does

$A \square A \xrightarrow{p_2} A \xrightarrow{s} B$ by the commutativity of

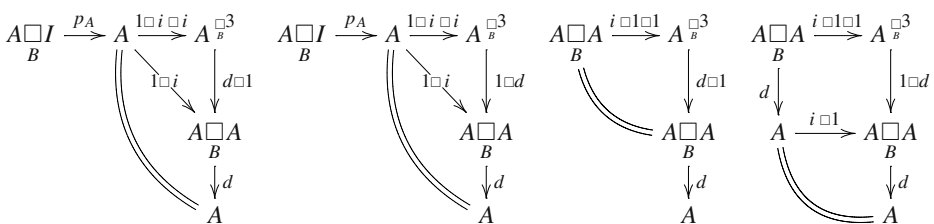


Thus they are equal. Similarly, both $t \cdot d$ and $A \square A \xrightarrow{p_1} A \xrightarrow{t} B$ render commutative the second diagram proving that they are equal.

The to-be composition morphism d in part (1) admits the unit i by construction. Its associativity follows again by [3, Corollary 1.7] since by the invertibility of q_3 there is at most one morphism rendering commutative



Since both $d \cdot (d \square 1)$ and $d \cdot (1 \square d)$ do so by the commutativity of



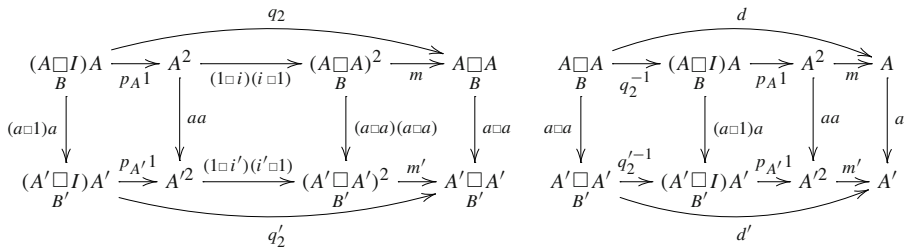
this proves their equality (modulo the omitted associativity isomorphism in [3, Proposition 3.6]). □

Proposition 3.9 Consider a monoidal admissible class \mathcal{S} of spans in a monoidal category \mathcal{C} such that [3, Assumption 4.1] holds. Between \mathcal{S} -relative categories in the category of monoids in \mathcal{C} for which the morphisms q_2 in Lemma 3.1 (3) are invertible, any morphism of reflexive graphs of monoids is in fact an \mathcal{S} -relative functor.

Proof Take \mathcal{S} -relative categories $B \xrightleftharpoons[s]{s} A \xleftarrow{d} A \square A$ and $B' \xrightleftharpoons[s']{s'} A' \xleftarrow{d'} A' \square A'$

as in the claim. We need to check the compatibility of any morphism of reflexive graphs

$(B \xrightarrow{b} B' , A \xrightarrow{a} A')$ with the composition morphisms d and d' . The first diagram of



commutes since $a \square a$ is multiplicative by [3, Proposition 3.7 (2)] and by the functoriality of \square ; see [3, Proposition 3.5 (2)]. It is used to prove the commutativity of the second diagram. □

3.3 The Equivalence Between Relative Categories and Crossed Modules of Monoids

Theorem 3.10 *Consider a monoidal admissible class \mathcal{S} of spans in a monoidal category \mathcal{C} such that [3, Assumption 4.1] holds. Use the same notation \mathcal{S} for the induced admissible class of spans in the category of monoids in \mathcal{C} from [3, Example 2.8] (also satisfying [3, Assumption 4.1] by [3, Example 4.4]). The following categories are equivalent.*

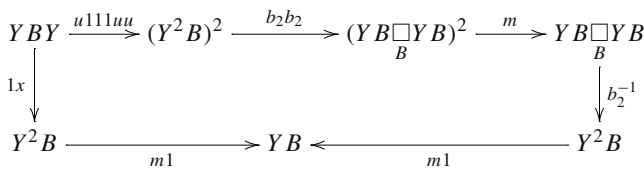
$\text{CatMon}_{\mathcal{S}}(\mathcal{C})$ whose

- objects are \mathcal{S} -relative categories $B \xrightleftharpoons[i]{s} A \xleftarrow{d} A \square A$ in the category of monoids in \mathcal{C} such that the morphisms q_n of (3.4) are invertible for any positive integer n .
- morphisms are \mathcal{S} -relative functors in the category of monoids in \mathcal{C} .

$\text{Xmod}_{\mathcal{S}}(\mathcal{C})$ whose

- objects consist of monoids B and Y , monoid morphisms $Y \xrightarrow{e} I$ and $Y \xrightarrow{k} B$ and a distributive law $BY \xrightarrow{x} YB$ subject to the following conditions.

- (a') $B \xleftarrow{k} Y = Y \in \mathcal{S}$, $Y = Y \xrightarrow{e} I \in \mathcal{S}$ and $B = B = B \in \mathcal{S}$.
- (b') $e1 \cdot x = 1e$ and $m \cdot k1 \cdot x = m \cdot 1k$.
- (c') The morphism f of Theorem 1.1 (c') is invertible and the morphisms b_n of Lemma 3.4 (2) are invertible for all positive integers n .
- (d') Regarding YB as a monoid via the structure induced by the distributive law x , the following diagram commutes.



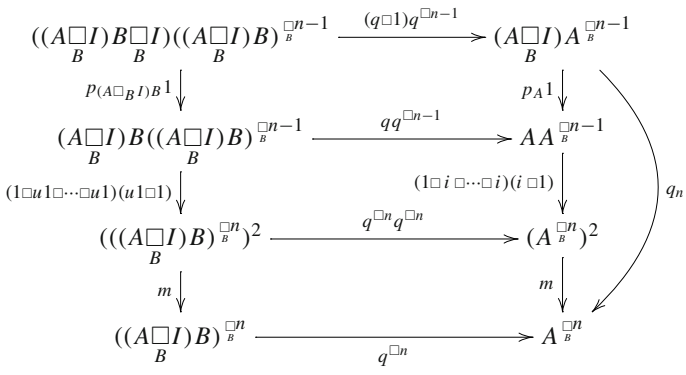
morphisms are pairs of monoid morphisms $(B \xrightarrow{b} B' , Y \xrightarrow{y} Y')$ such that $e' \cdot y = e$, $k' \cdot y = b \cdot k$ and $x' \cdot by = yb \cdot x$.

Proof It follows by Propositions 3.8 and 3.9 that $\text{CatMon}_{\mathcal{S}}(\mathcal{C})$ is a full subcategory of $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$ and obviously $\text{Xmod}_{\mathcal{S}}(\mathcal{C})$ is a full subcategory of $\text{PreX}_{\mathcal{S}}(\mathcal{C})$. Below we show that the mutually inverse functors of Theorem 2.1 restrict to functors between these subcategories thus establishing the stated equivalence.

Regarding an object $B \overset{s}{\underset{t}{\rightrightarrows}} A \xleftarrow{d} A \square_B A$ of $\text{CatMon}_{\mathcal{S}}(\mathcal{C})$ as an object $B \overset{s}{\underset{t}{\rightrightarrows}} A$ of $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$, the functor in the proof of Theorem 2.1 takes it to the object $(B, A \square_B I, A \square_B I \xrightarrow{p_I} I, A \square_B I \xrightarrow{p_A} A \xrightarrow{t} B, B(A \square_B I) \xrightarrow{i p_A} A^2 \xrightarrow{m} A \xrightarrow{q^{-1}} (A \square_B I) B)$ of the category $\text{PreX}_{\mathcal{S}}(\mathcal{C})$; we claim that it is in fact an object of $\text{Xmod}_{\mathcal{S}}(\mathcal{C})$.

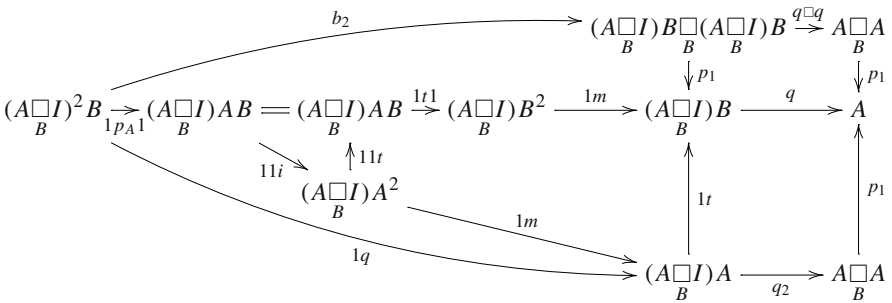
It satisfies the condition $B \xleftarrow{t} A \xleftarrow{p_A} A \square_B I = A \in \mathcal{S}$ by Lemma 2.5.

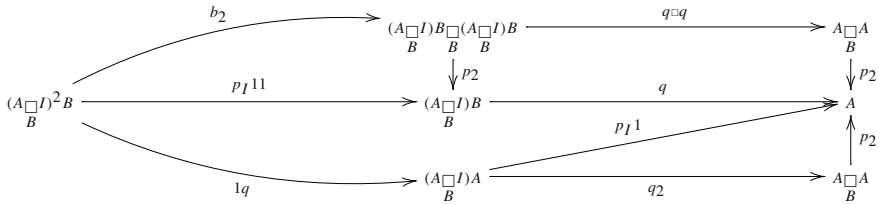
From Lemma 3.4 (5) we know that the morphism b_n of Lemma 3.4 (2) is invertible if and only if the left column of the commutative diagram



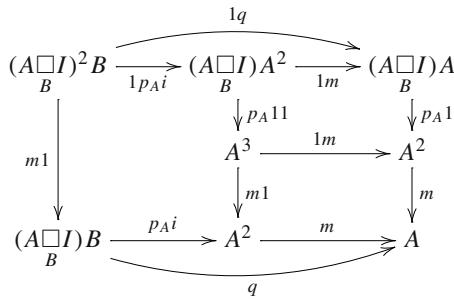
is invertible. Recognize the isomorphism q_n of Lemma 3.1 (3) in the right column. Since also the rows are isomorphisms by assumption, so is the left column and hence b_n . This also shows that the morphism of Lemma 3.1 (3) for the reflexive graph $B \overset{s}{\underset{t}{\rightrightarrows}} A$, and the morphism of Lemma 3.4 (4) for the corresponding object $(B, A \square_B I, p_I, t \cdot p_A, q^{-1} \cdot m \cdot i p_A)$ of $\text{PreX}_{\mathcal{S}}(\mathcal{C})$, differ by the isomorphisms of the top and bottom rows (justifying our use of the same symbol q_n for them).

The proof of the commutativity of the diagram in part (d') requires some preparation. The commutativity of





proves $(q \square q) \cdot b_2 = q_2 \cdot 1q$. (Here the bottom-right region of the first diagram commutes since the lower half of the diagram of (3.9) commutes and the bottom-right region of the second diagram commutes since the lower half of the diagram of (3.10) commutes.) By the associativity of the monoid A and the multiplicativity of $A \square_I \xrightarrow{p_A} A$ also the following diagram commutes.



With the help of these identities and Lemma 3.7, and using that the region marked by (*) commutes by Proposition 3.8 (2), the diagram of Fig. 5 is seen to commute. This proves that the stated object belongs to $X\text{mod}_S(\mathcal{C})$ indeed.

In the opposite direction, consider an object $(B, Y, Y \xrightarrow{e} I, Y \xrightarrow{k} B, BY \xrightarrow{x} YB)$ of $X\text{mod}_S(\mathcal{C})$ as an object of $\text{Pre}X_S(\mathcal{C})$. The functor in the proof of Theorem 2.1 takes it to the object $B \xrightleftharpoons[m \cdot k1]{e1} YB$ of $\text{Ref}Graph\text{Mon}_S(\mathcal{C})$; we claim that it can be seen as an object of $\text{Cat}Mon_S(\mathcal{C})$.

By Lemma 3.4 (1) the span $B \xleftarrow{m} B^2 \xleftarrow{k1} YB \equiv YB$ belongs to S .

The morphism q_n of Lemma 3.4 (4) is invertible for all positive integers n by Lemma 3.4 (5).

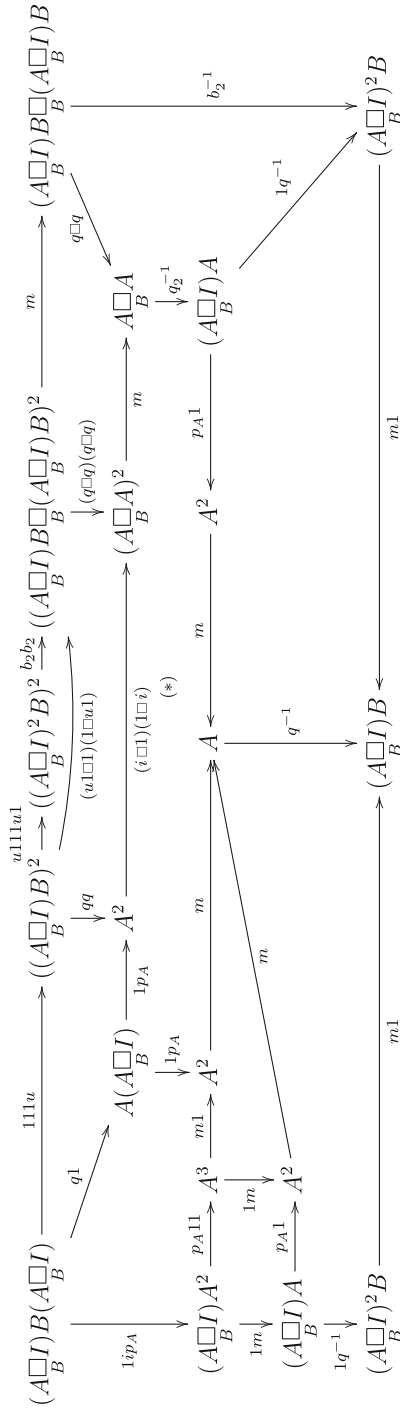
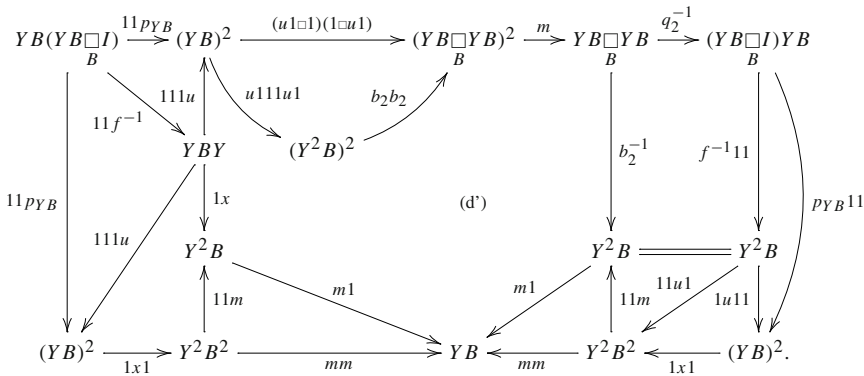


Fig. 5 Commutativity of the diagram in part (d')

By Proposition 3.8 (2) and (3), the reflexive graph of monoids $B \overset{e1}{\rightleftarrows} YB \overset{m \cdot k1}{\rightleftarrows}$ extends to an S -relative category in the category of monoids in \mathbb{C} by the commutativity of



The region at the top-right corner is the commutative diagram of Lemma 3.4 (4) for $n = 1$. The region bounded from below by the curved arrows commutes by Lemma 3.7. The region marked by (d') coincides with the diagram of part (d') hence it commutes. □

Example 3.11 As in Example 1.2, take the (evidently admissible and monoidal) class of all spans in the category \mathbb{C} of spans over a given set X . Then the equivalent categories of Theorem 3.10 take the following forms.

CatMon(C) whose

objects are the double categories with the object set X and only identity horizontal morphisms and such that the morphism (1.3) is invertible. (This last condition holds e.g. if the vertical edge category is a groupoid.)

morphisms are the double functors which are identities on the objects (and hence on the horizontal morphisms).

Xmod(C) whose

objects consist of categories B and Y with the common object set X such that Y is totally disconnected (in the sense of [4]); an action (see Example 1.2) $B \square_X Y \xrightarrow{p} Y$ and an

identity-on-objects functor $Y \xrightarrow{\kappa} B$ such that

$$\kappa(b \triangleright y) \cdot b = b \cdot \kappa(y) \quad \text{and} \quad (\kappa(y) \triangleright y') \cdot y = y \cdot y'$$

for all morphisms b in B and y, y' in Y for which $s(b) = t(y) = t(y')$.

morphisms are the same as the morphisms in PreXMon(C), see Example 2.2.

Note that these equivalent categories have equivalent full subcategories whose objects are such that the category B is a groupoid; and other equivalent full subcategories whose objects are such that both of the occurring categories are groupoids. In the latter case these are the category of categories in the category of groupoids; and the category of *crossed modules of groupoids* in [5, Definition 1.2], respectively.

Example 3.12 In the setting of Example 1.3, the equivalent categories of Theorem 3.10 take the following explicit forms.

$\text{CatMon}_{\mathcal{S}}(\mathcal{C})$ whose

objects are \mathcal{S} -relative categories $B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \\ \xrightarrow{t} \end{matrix} A \xleftarrow{d} A \square_B A$ in the category of monoids in \mathcal{C} —that is, in the category of bimonoids in \mathcal{M} —such that the morphism q of Theorem 1.1 (b) is invertible.

morphisms are \mathcal{S} -relative functors in the category of monoids in \mathcal{C} —that is, in the category of bimonoids in \mathcal{M} .

$\text{Xmod}_{\mathcal{S}}(\mathcal{C})$ whose

objects consist of a bimonoid Y and a cocommutative bimonoid B together with a left action $BY \xrightarrow{l} Y$ which makes Y both a B -module monoid and a B -module comonoid and a bimonoid morphism $Y \xrightarrow{k} B$ for which the following diagrams commute.

$$\begin{array}{ccccc}
 Y & \xrightarrow{\delta} & Y^2 & \xrightarrow{c} & Y^2 \\
 \delta \downarrow & & & & \downarrow k_1 \\
 Y^2 & \xrightarrow{k_1} & BY & & \\
 & & & & \\
 BY & \xrightarrow{\delta 1} & B^2 Y & \xrightarrow{1c} & BYB \xrightarrow{l1} YB \\
 1k \downarrow & & & & \downarrow k_1 \\
 B^2 & \xrightarrow{m} & B & \xleftarrow{m} & B^2 \\
 & & & & \\
 Y^2 & \xrightarrow{\delta 1} & Y^3 & \xrightarrow{1c} & Y^3 \xrightarrow{k11} BY^2 \\
 \parallel & & & & \downarrow l1 \\
 Y^2 & \xrightarrow{m} & Y & \xleftarrow{m} & Y^2
 \end{array}$$

The third condition appears in [25, Definition 12 (v)] under the name *Peiffer condition* (motivated by the terminology for groups).

morphisms are pairs of bimonoid morphisms $(B \xrightarrow{b} B' , Y \xrightarrow{y} Y')$ such that $k' \cdot y = b \cdot k$ and $l' \cdot by = y \cdot l$.

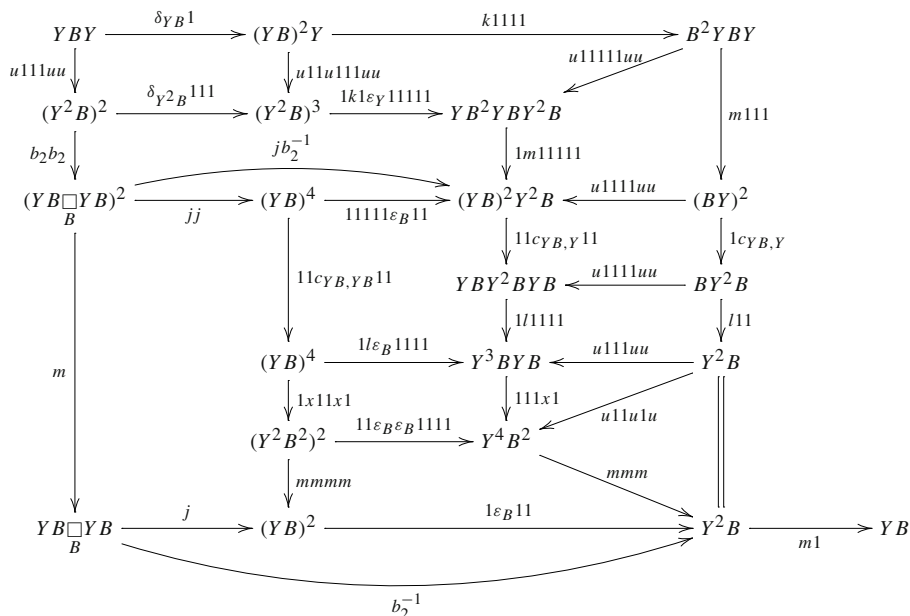
These equivalent categories are equivalent furthermore to the full subcategory of $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$ of Example 2.3 for whose objects $B \begin{matrix} \xleftarrow{s} \\ \xrightarrow{i} \\ \xrightarrow{t} \end{matrix} A$ the following diagrams commute.

$$\begin{array}{ccccc}
 A & \xrightarrow{\delta} & A^2 & \xrightarrow{c} & A^2 \\
 \delta \downarrow & & & & \downarrow r_1 \\
 A^2 & \xrightarrow{r_1} & & & BA
 \end{array} \tag{3.16}$$

$$\begin{array}{ccccccc}
 A(A \square_B I) & \xrightarrow{\delta 1} & A^2(A \square_B I) & \xrightarrow{1c} & B(A \square_B I)A & \xrightarrow{ip_A 1} & A^3 \xrightarrow{m_1} A^2 \xrightarrow{q^{-1} 1} (A \square_B I)BA \\
 1p_A \downarrow & & & & & & \downarrow p_A \varepsilon_B 1 \\
 A^2 & \xrightarrow{m} & A & \xleftarrow{m} & A^2 & &
 \end{array} \tag{3.17}$$

The above description of $\text{CatMon}_{\mathcal{S}}(\mathcal{C})$ requires no further explanation. In the description of $\text{Xmod}_{\mathcal{S}}(\mathcal{C})$ we need to show that the third diagram (the Peiffer condition) is equivalent to the diagram of Theorem 3.10 (d') in the current setting. The path on the right hand side of the

diagram of Theorem 3.10 (d') appears as the left bottom path of the commutative diagram



(in which x stands for the distributive law $BY \xrightarrow{\delta_1} B^2Y \xrightarrow{1c} BYB \xrightarrow{1l} YB$ of Example 1.3). Hence it can be replaced by the top right path yielding the equivalent form

$$\begin{array}{ccc}
 YBY & \xrightarrow{\delta_{YB^1}} & (YB)^2 Y & \xrightarrow{k1111} & B^2 YBY & \xrightarrow{m111} & (BY)^2 & (3.18) \\
 1\delta_B 1 \downarrow & & & & & & \downarrow 1c_{YB,Y} \\
 YB^2 Y & & & & & & BY^2 B \\
 11c \downarrow & & & & & & \downarrow 111 \\
 (YB)^2 & & & & & & Y^2 B \\
 111 \downarrow & & & & & & \downarrow \\
 Y^2 B & \xrightarrow{m1} & YB & \xleftarrow{m1} & Y^2 B & &
 \end{array}$$

of the diagram of Theorem 3.10 (d'). The first diagram of Fig. 6 shows that if the diagram of (3.18) commutes then the Peiffer condition in the above presentation of $X\text{mod}_{\mathcal{S}}(\mathcal{C})$ holds. The opposite implication is proven by the second diagram of Fig. 6.

In order to justify the further equivalent characterization of these categories as a full subcategory of $\text{ReflGraphMon}_{\mathcal{S}}(\mathcal{C})$, we need to see the equivalence of the diagram of Proposition 3.8 (2) in the current setting and the diagram of (3.17). This follows by noting that the top row of the diagram of Proposition 3.8 (2) in the current setting appears in the left-bottom

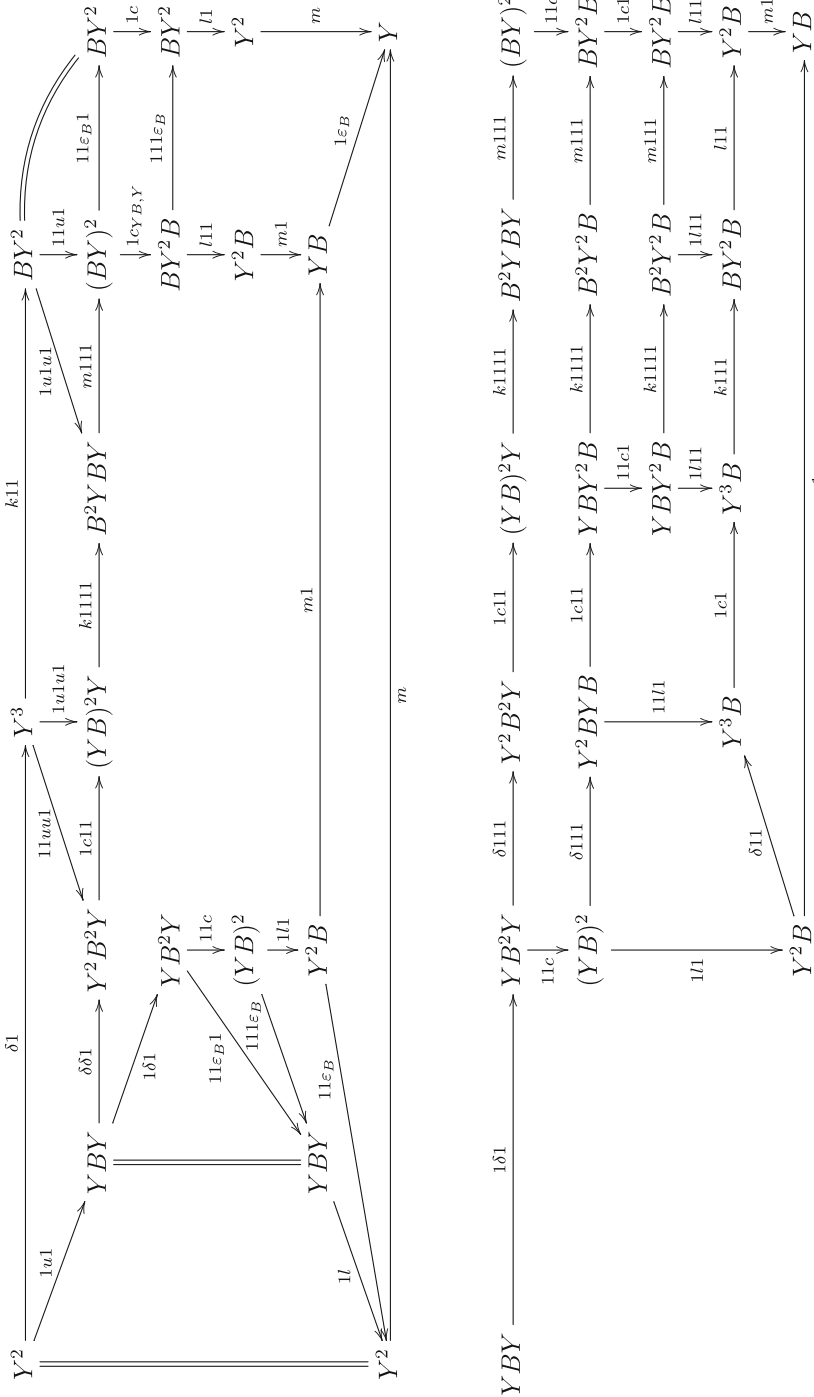
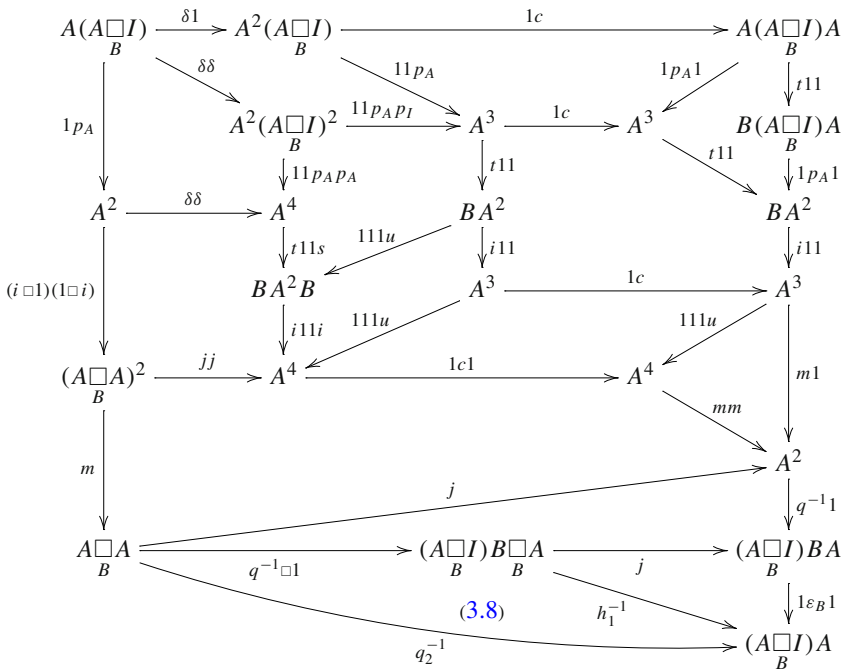


Fig. 6 Derivation of the Peiffer condition for crossed modules of bimonoids

path of the commutative diagram



hence it can be replaced by the top-right path. (The expression of h_1^{-1} in the bottom-right corner was computed in Example 3.3.)

In the particular case when M is the cartesian monoidal category of sets, the current example yields an equivalence between the category of those internal categories

$$B \begin{matrix} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{matrix} A \xleftarrow{d} A \square_B A$$

in the category of ordinary monoids for which the split epimorphism (i, s) satisfies the Schreier property of [20] (see Example 1.3), and the category of *crossed semimodules* in [20]. Hence it extends the main result of [20].

Proposition 3.13 *The equivalent categories of Example 3.12 have equivalent full subcategories as follows.*

- The full subcategory of $\text{CatMon}_S(\mathcal{C})$ for whose objects $B \begin{matrix} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{matrix} A \xleftarrow{d} A \square_B A$ the bimonoid B in M is a Hopf monoid.
- The full subcategory of $\text{Xmod}_S(\mathcal{C})$ for whose objects $(B, Y, BY \xrightarrow{l} Y, Y \xrightarrow{k} B)$ the bimonoid B in M is a Hopf monoid.
- The full subcategory of $\text{ReflGraphMon}_S(\mathcal{C})$ for whose objects $B \begin{matrix} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{matrix} A$ the following conditions hold.
 - B is a Hopf monoid (with antipode z)

- $t1 \cdot \delta = t1 \cdot c \cdot \delta$
- for the morphisms

$$\begin{aligned} \vec{s} &:= A \xrightarrow{\delta} A^2 \xrightarrow{1s} AB \xrightarrow{1z} AB \xrightarrow{1i} A^2 \xrightarrow{m} A, \\ \overleftarrow{t} &:= A \xrightarrow{\delta} A^2 \xrightarrow{t1} BA \xrightarrow{z1} BA \xrightarrow{i1} A^2 \xrightarrow{m} A \end{aligned}$$

the following diagram commutes.

$$\begin{array}{ccccc} A^2 & \xrightarrow{\vec{s} \overleftarrow{t}} & A^2 & \xrightarrow{c} & A^2 \\ \vec{s} \overleftarrow{t} \downarrow & & & & \downarrow m \\ A^2 & \xrightarrow{m} & & & A \end{array} \tag{3.19}$$

Proof The only ingredient that requires a proof is the equivalence of diagrams (3.17) and (3.19) in the case when B has an antipode z . The proof will repeatedly use the identity on \vec{s} encoded in the following commutative diagram.

Recall from [24] that if B has an antipode z then $A \square_B I \xrightarrow{p_A} A$ is a split monomorphism

in M ; a retraction is provided by $g_A := A \xrightarrow{q^{-1}} (A \square_B I)B \xrightarrow{1\varepsilon} A \square_B I$. Indeed,

$$g_A \cdot p_A = 1\varepsilon \cdot q^{-1} \cdot q \cdot 1u = 1\varepsilon \cdot 1u = 1.$$

On the other hand, since in Proposition 1.5 the inverse q^{-1} was constructed as the unique solution of $p_A 1 \cdot q^{-1} = \vec{s} s \cdot \delta$, also the equality

$$p_A \cdot g_A = p_A \cdot 1\varepsilon \cdot q^{-1} = 1\varepsilon \cdot p_A 1 \cdot q^{-1} = 1\varepsilon \cdot \vec{s} s \cdot \delta = \vec{s}$$

holds, proving that \vec{s} is idempotent.

Pre-composing both paths around (3.17) with the split epimorphism $1g_A$, we obtain the equivalent diagram

$$\begin{array}{ccccccccccccccc}
 A^2 & \xrightarrow{\delta 1} & A^3 & \xrightarrow{1c} & A^3 & \xrightarrow{t11} & BA^2 & \xrightarrow{i11} & A^3 & \xrightarrow{m1} & A^2 \\
 \downarrow 1\vec{s} & \searrow 1g_A & \downarrow 11g_A & & & & \downarrow 1g_{A^1} & & \downarrow 1\vec{s}1 & & \downarrow \vec{s}1 \\
 A^2 & \xrightarrow{\delta 1} & A^2(A \square_B I) & \xrightarrow{1c} & A(A \square_B I)A & \xrightarrow{t11} & B(A \square_B I)A & \xrightarrow{ip_{A^1}} & A^3 & \xrightarrow{m1} & A^2 \\
 \downarrow 1p_A & & & & & & & & \downarrow m1 & & \downarrow \vec{s}1 \\
 A^2 & \xrightarrow{m} & A & \xleftarrow{m} & A^2
 \end{array} \tag{3.21}$$

Its rightmost region commutes by (3.20) and the fact that \vec{s} is idempotent.

The morphism around the right hand side of (3.21) occurs as the left-bottom path of the commutative diagram

$$\begin{array}{ccccccc}
 A^2 & \xrightarrow{\delta 1} & A^3 & \xrightarrow{1c} & A^3 & \xrightarrow{t11} & BA^2 \\
 \delta 1 \downarrow & & \downarrow 1\delta 1 & & \downarrow 11\delta & & \downarrow i11 \\
 A^3 & \xrightarrow{\delta 11} & A^4 & & A^4 & & A^3 \\
 1c \downarrow & & \downarrow 11c & & \downarrow t1t1 & & \downarrow i11 \\
 A^3 & \xrightarrow{\delta 11} & A^4 & \xrightarrow{1c1} & A^4 & & A^3 \\
 t11 \downarrow & & \downarrow tt11 & & \downarrow ti11 & & \downarrow i11 \\
 BA^2 & & B^2A^2 & & (BA)^2 & & A^3 \\
 i11 \downarrow & & \downarrow ii11 & & \downarrow i1i1 & & \downarrow i11 \\
 A^3 & \xrightarrow{\delta 11} & A^4 & \xrightarrow{1c1} & A^4 & & A^3 \\
 m1 \downarrow & & & & \downarrow 11s1 & & \downarrow i11 \\
 A^2 & & & & A^2BA & \xleftarrow{11t1} & A^4 \xleftarrow{11\delta} A^3 \\
 \downarrow 1\vec{s}1 & & & & \downarrow 11z1 & & \downarrow 11\vec{t} \\
 A^2 & & & & A^2BA & \xrightarrow{11i1} & A^4 \xrightarrow{11m} A^3 \\
 & & & & & & \downarrow 1\vec{s}11 & \downarrow 1\vec{s}1 \\
 & & & & & & A^4 & \xrightarrow{11m} A^3 \\
 & & & & & & \swarrow m11 & \downarrow 1m \\
 & & & & & & A^3 & A^2
 \end{array}$$

(3.20)

Hence it can be replaced by the top-right path yielding the equivalent form

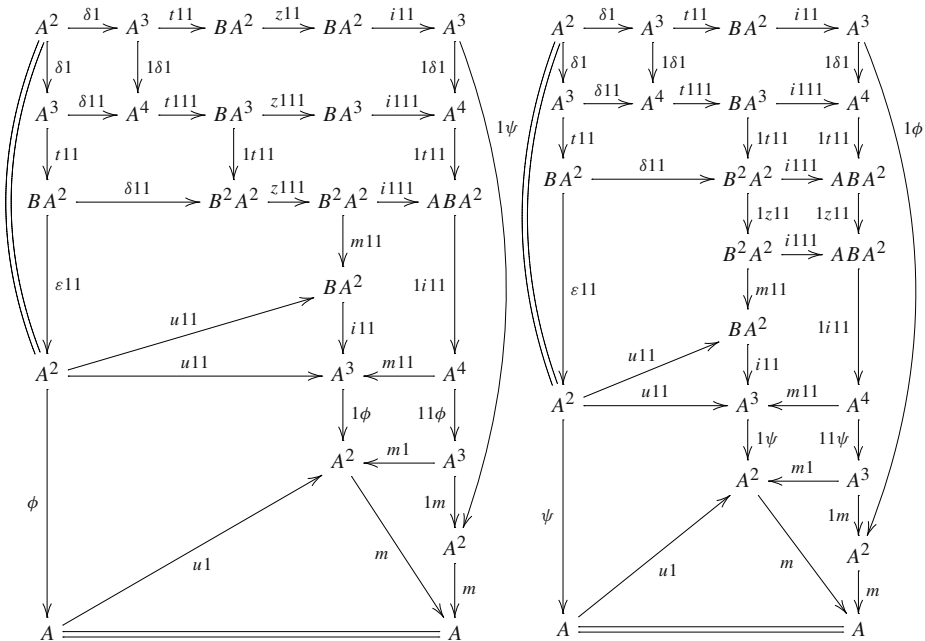
$$\begin{array}{ccccccc}
 A^2 & \xrightarrow{\delta_1} & A^3 & \xrightarrow{r_{11}} & BA^2 & \xrightarrow{i_{11}} & A^3 & \xrightarrow{lc} & A^3 & \xrightarrow{1\overrightarrow{s} \overleftarrow{t}} & A^3 \\
 \downarrow 1\overrightarrow{s} & & & & & & & & & & \downarrow 1m \\
 A^2 & \xrightarrow{m} & A & \xleftarrow{m} & A^2 & & & & & &
 \end{array} \tag{3.22}$$

of (3.21).

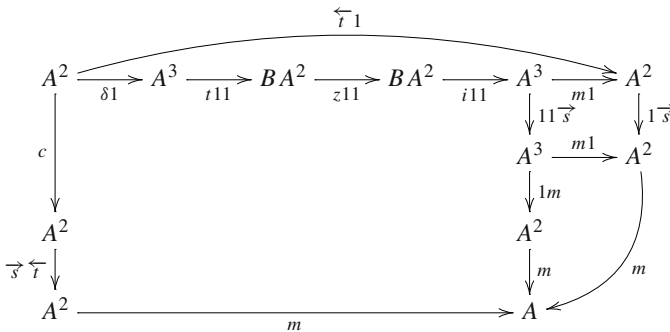
Finally, observe that for any morphisms $A^2 \xrightarrow{\phi, \psi} A$ the following diagrams are equivalent:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 A^2 & \xrightarrow{\delta_1} & A^3 & \xrightarrow{r_{11}} & BA^2 & \xrightarrow{i_{11}} & A^3 \\
 \downarrow \psi & & & & & & \downarrow 1\phi \\
 A & \xleftarrow{m} & A^2 & & & &
 \end{array} & \text{equivalent to} & \begin{array}{ccccc}
 A^2 & \xrightarrow{\delta_1} & A^3 & \xrightarrow{r_{11}} & BA^2 & \xrightarrow{z_{11}} & BA^2 & \xrightarrow{i_{11}} & A^3 \\
 \downarrow \phi & & & & & & & & \downarrow 1\psi \\
 A & \xleftarrow{m} & A^2 & & & & & &
 \end{array}
 \end{array} \tag{3.23}$$

Indeed, the first diagram below shows that if the first diagram of (3.23) commutes then so does the second one; and the opposite implication follows by the second diagram below.



Applying the equivalence of the diagrams of (3.23) to $\phi := A^2 \xrightarrow{c} A^2 \xrightarrow{\overrightarrow{s} \overleftarrow{t}} A^2 \xrightarrow{m} A$ and $\psi := A^2 \xrightarrow{1\overrightarrow{s}} A^2 \xrightarrow{m} A$, we obtain from (3.22) the equivalent form

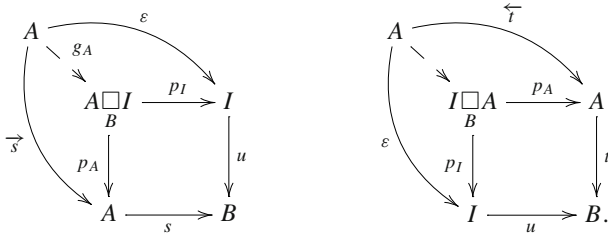


which is equivalent to (3.19) by the naturality of the symmetry c .

Note that by the equalities

$$p_I \cdot g_A = \varepsilon \cdot u \cdot p_I \cdot g_A = \varepsilon \cdot s \cdot p_A \cdot g_A = \varepsilon \cdot s = \varepsilon,$$

the above morphism g_A in the splitting of the idempotent morphism \overrightarrow{s} can also be obtained as the unique filler of the first diagram in



The symmetrically constructed filler of the second diagram above yields a splitting of the idempotent morphism \overleftarrow{t} in the ambient category M . So commutativity of the diagram of (3.19) is equivalent also to the commutativity of

$$\begin{array}{ccc} (A \square_I I) (I \square_A A) & \xrightarrow{p_A p_A} & A^2 \xrightarrow{c} A^2 \\ p_A p_A \downarrow & & \downarrow m \\ A^2 & \xrightarrow{m} & A. \end{array} \tag{3.24}$$

□

In the particular case of Proposition 3.13 when M is cartesian monoidal (e.g. it is the category of sets), $A \square_I \xrightarrow{p_A} A$ is the kernel of the monoid morphism s and $I \square_A \xrightarrow{p_A} A$ is the kernel of the monoid morphism t , see Example 1.4. Hence in this case the equivalent diagrams of (3.19) and (3.24) say precisely that the kernels of s and t commute in the monoid A .

The equivalent categories of Proposition 3.13 have equivalent full subcategories in whose objects both occurring bimonoids are Hopf monoids, and other equivalent full subcategories

in whose objects they are both cocommutative Hopf monoids. In this way, Proposition 3.13 includes [25, Proposition 11] and [25, Theorem 14] about the equivalence between the category of so-called Cat^1 -Hopf algebras and the category of crossed modules over Hopf algebras; hence in particular the equivalence between the category of Cat^1 -groups and the category of crossed modules over groups in [6, Theorem 1]—where it is also referred to unpublished works by Verdier and Duskin—, [17, Lemma 2.2] and [14, Section 3.9] (whose language is most similar to ours).

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