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A DEFINITIVE IMPROVEMENT OF A GAME-THEORETIC BOUND AND THE LONG TIGHTNESS GAME

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Abstract. The main goal of the paper is the full proof of a cardinal inequality for a space with points G_δ, obtained with the help of a long version of the Menger game. This result, which improves a similar one of Scheepers and Tall, was already established by the authors under the Continuum Hypothesis. The paper is completed by few remarks on a long version of the tightness game.

1. Introduction

As usual, for notation and undefined notions we refer to [3]. In this paper we consider the long version of two well-known topological games. In particular, we study the influence of the existence of a winning strategy for the second player in both games on certain cardinality properties of the space.

The main result (Theorem 5) shows that a cardinality bound, obtained by Scheepers and Tall with the help of the Rothberger game, continues to hold with the much weaker help of the Menger game. Our generalization works in the class of regular spaces and we will remark that some separation axiom is definitely needed for it (see Example 7).

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The second part of the paper deals with a long version of the tightness game. Although this game is very different from the Menger game, the main result here, Theorem 9, looks quite similar to Theorem 5.

2. Long Menger game and cardinality

After Arhangel'skiĭ's cardinal inequality $|X| \leq 2^\omega$, for any first countable Lindelöf T_2 space X, a lot of attention has been paid to the possibility of extending this theorem to the whole class of spaces with G_δ points (see e.g. [6]). The problem turned out to be very non-trivial and the first negative consistent answer was given by Shelah. Later on, a simpler construction of a Lindelöf T_3 space with points G_δ whose cardinality is bigger than the continuum was obtained by Gorelic [4]. Somewhat related to the Lindelöf property are the Rothberger and Menger games (see e.g. [8]). Indeed, by working in this direction, Scheepers and Tall [10] proved a cardinality bound for a topological space with points G_δ by means of a long version of the Rothberger game. The natural question to extend this result to the much weaker Menger game was studied in [1]. There, a partial answer was obtained under the Continuum Hypothesis. The main purpose of this note is to provide the full solution to the question in ZFC. The proof we present here uses elementary submodels and looks much simpler and direct.

We follow the standard notation for games: we will denote by $\mathsf{G}_1^\kappa(\mathcal{A}, \mathcal{B})$ the game played by players Alice and Bob such that, at each inning $\xi < \kappa$, Alice chooses $A_\xi \in \mathcal{A}$. Then Bob chooses $a_\xi \in A_\xi$. Bob wins if $\{a_\xi : \xi < \kappa\} \in \mathcal{B}$.

We will denote by O the family of all open covers for a given space. Thus, $\mathsf{G}_1^\kappa(\mathrm{O}, \mathrm{O})$ means that at each inning Alice chooses an open cover and Bob chooses one of its members. Bob wins if the collection of the chosen sets covers the space.

According to this notation, $\mathsf{G}_1^\omega(\mathrm{O}, \mathrm{O}) = \mathsf{G}_1(\mathrm{O}, \mathrm{O})$ is the classical Rothberger game.

As usual, $\mathfrak{c} = 2^\omega$.

The starting point of our investigation is in the following:

Theorem 1 (Scheepers–Tall [10]). *If X is a space with points G_δ and* Bob *has a winning strategy in the game* $\mathsf{G}_1^{\omega_1}(\mathrm{O}, \mathrm{O})$, *then* $|X| \leq 2^\omega$.

To appreciate the strength of the above result and consequently of Theorem 5 below, notice that the example of Gorelic [4] provides a space X with points G_δ in which Alice does not have a winning strategy in $\mathsf{G}_1^{\omega_1}(\mathrm{O}, \mathrm{O})$ and $|X| > 2^\omega$ (see [10] for a justification of this fact).

A very natural question arises on whether the Scheepers–Tall's inequality can be improved by replacing "G_1" with "$\mathsf{G}_{\mathrm{fin}}$", i.e., the game where Bob chooses finitely many sets per inning, instead of only one. In other words, we wonder whether the long Menger game can suffice in the above cardinal inequality.

We already obtained a positive partial answer under the continuum hypothesis in [1]. Our goal here is to present a proof of this statement in ZFC.

In [1] the duality between $\mathsf{G}^{\omega_1}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ and the compact-open game of length ω_1 is used. This duality is true under CH but we do not know if it is true in general. The proof presented here does not use any duality.

From now on, let σ be a fixed winning strategy for Bob in the game $\mathsf{G}^{\kappa}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ played on the space X. Recall that a strategy for Bob in $\mathsf{G}^{\kappa}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ is a function $\sigma\colon \mathrm{O}^{<\kappa} = \bigcup\{{}^{\alpha+1}\mathrm{O} : \alpha < \kappa\} \to \left[\bigcup \mathrm{O}\right]^{<\omega}$ and for any $s \in {}^{\alpha+1}\mathrm{O}$ we have $\sigma(s) \subset s(\alpha)$.

We will say that $K \subset X$ is *good* if there is an $s \in \mathrm{O}^{<\kappa}$ such that $K = \bigcap_{\mathcal{C}\in\mathrm{O}} \overline{\bigcup \sigma(s^\frown\mathcal{C})}$.

Lemma 2. *Every good subset of a regular space is compact.*

Proof. Let $K = \bigcap_{\mathcal{C}\in\mathrm{O}} \overline{\bigcup \sigma(s^\frown\mathcal{C})}$ and take a collection $\mathcal{V}$ of open sets such that $K \subset \bigcup\mathcal{V}$. Fix a neighbourhood assignment $\mathcal{V} = \{V_x : x \in X\}$ in such a way that $\overline{V_x} \subset U_x \in \mathcal{V}$ if $x \in K$ and $\overline{V_x} \cap K = \emptyset$ if $x \in X \setminus K$. If $\sigma(s^\frown\mathcal{V}) = \{V_x : x \in F \in [X]^{<\omega}\}$, then we clearly have $K \subset \bigcup\{\overline{V_x} : x \in F \cap K\} \subset \bigcup\{U_x : x \in F \cap K\}$. □

Lemma 3. *Let X be a space. If K is good, i.e. there is an $s \in \mathrm{O}^{<\kappa}$ such that $K = \bigcap_{\mathcal{C}\in\mathrm{O}} \overline{\bigcup \sigma(s^\frown\mathcal{C})}$, and $K = \bigcap_{\xi<\lambda} V_\xi$ where each V_ξ is open, then there is an $\mathrm{O}' \subset \mathrm{O}$ such that $K = \bigcap_{\mathcal{C}\in\mathrm{O}'} \overline{\bigcup \sigma(s^\frown\mathcal{C})}$ and $|\mathrm{O}'| < \lambda + \kappa$.*

Proof. As we are assuming that Bob has a winning strategy in $\mathsf{G}^{\kappa}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, the Lindelöf degree of X is at most κ. Consequently, we have $L(X \setminus K) \le \lambda + \kappa$. Since the family $\left\{ X \setminus \overline{\bigcup \sigma(s^\frown\mathcal{C})} : \mathcal{C} \in \mathrm{O} \right\}$ is an open cover of $X \setminus K$, there exists $\mathrm{O}' \subset \mathrm{O}$ such that $|\mathrm{O}'| \le \lambda + \kappa$ and $\left\{ X \setminus \overline{\bigcup \sigma(s^\frown\mathcal{C})} \colon \mathcal{C} \in \mathrm{O}' \right\}$ covers $X \setminus K$. Therefore, $K = \bigcap_{\mathcal{C}\in\mathrm{O}'} \overline{\bigcup \sigma(s^\frown\mathcal{C})}$ and we are done. □

Lemma 4. *Let X be a space with points G_δ. Then for every compact subset K there is a sequence $\langle V_\xi : \xi < 2^\omega \rangle$ of open sets such that $K = \bigcap_{\xi<2^\omega} V_\xi$.*

Proof. First note that each compact $K \subset X$ satisfies $|K| \le 2^\omega$. This is a consequence of a theorem of Gryzlov [5]. For every $x \in K$, let $\{V^x_n : n \in \omega\}$ be a family of open subsets of X satisfying $\bigcap_{n<\omega} V^x_n = \{x\}$.

Let $\mathcal{B} = \left\{ \bigcup_{i=0}^{k} V^{x_i}_{n_i} \supset K : x_0, \dots, x_k \in K,\ n_0, \dots, n_k \in \omega \right\}$. Note that $\bigcap \mathcal{B} = K$ and $|\mathcal{B}| \le 2^\omega$. □

Now, we have everything to prove the announced result.

Theorem 5. *Let X be a regular space with points G_δ such that* Bob *has a winning strategy for the* $\mathsf{G}_{\mathrm{fin}}^{\omega_1}(\mathrm{O},\mathrm{O})$ *game. Then* $|X| \leq 2^\omega$.

Proof. Let μ be a large enough regular cardinal and M be an elementary submodel of $H(\mu)$ such that $|M| = 2^\omega$, $X, \sigma, \mathrm{O} \in M$, $\mathfrak{c}+1 \subset M$ and $[M]^\omega \subset M$. Let $\mathcal{K} = \{K \subset X : K \in M \text{ and } K \text{ is good}\}$. It is enough to show that $X = \bigcup \mathcal{K}$, since each $K \in \mathcal{K}$ is such that $|K| \leq 2^\omega$.

Assuming the contrary, there is an $x \in X \setminus \bigcup \mathcal{K}$. Let $K_0 = \bigcap_{\mathcal{C}\in\mathrm{O}} \overline{\bigcup \sigma(\mathcal{C})}$. Note that K_0 is definable in M and so $K_0 \in \mathcal{K}$. Working inside of M, K_0 is compact by Lemma 2, therefore, by Lemma 4, we can apply Lemma 3 and obtain that there is an $\mathrm{O}' \in M$ such that $|\mathrm{O}'| \leq 2^\omega$ and $K_0 = \bigcap_{\mathcal{C}\in\mathrm{O}'} \overline{\bigcup \sigma(\mathcal{C})}$. Since $\mathfrak{c}+1 \subset M$, we actually have $\mathrm{O}' \subset M$ and so there is a $\mathcal{C}_0 \in M \cap \mathrm{O}$ such that $x \notin \bigcup \sigma(\mathcal{C}_0)$. We now proceed by induction. Assume to have already defined open covers $\{\mathcal{C}_\alpha : \alpha < \xi\} \subset M$ and define $s\colon \xi \to \mathrm{O}$ by letting $s(\alpha) = \mathcal{C}_\alpha$ for $\alpha < \xi$. Since M is ω-closed, we actually have $s \in M$, since the κ in Lemma 3 is now ω_1. Therefore, $K_\xi = \bigcap_{\mathcal{C}\in\mathrm{O}} \overline{\bigcup \sigma(s^\frown\mathcal{C})}$ is definable in M and so it is again an element of $\mathcal{K}$. Then, as before we can obtain a $\mathcal{C}_\xi \in M \cap \mathrm{O}$ such that $x \notin \bigcup \sigma(s^\frown\mathcal{C}_\xi)$.

But note that doing like this, we find a play of the game where Bob loses although using a winning strategy. $\square$

Note that we actually proved that under the hypothesis of Theorem 5, $X = \bigcup_{\xi<\mathfrak{c}} K_\xi$, where each K_ξ is compact. However, this is not enough to guarantee that the first player wins in the long compact-open game without CH (see [1]).

Furthermore, note that with a simple modification in the previous argument, using a countable submodel we obtain the Telgarsky's result (reproved by Scheepers in [9]):

Corollary 6. *If X is a regular space where every compact set is a G_δ and* Bob *has a winning strategy for the usual Menger game* $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, *then X is σ-compact.*

Since Theorem 1 is actually true for T_1 spaces, we could suppose that the same happens to Theorem 5. But, Theorem 5 drastically fails for T_1 spaces. Indeed, even under the stronger assumption that Bob has a winning strategy in the "short" Menger game, the cardinality of a space with points G_δ can be very big.

Example 7. If κ is less than the first measurable cardinal, then there exists a T_1 space X with points G_δ such that Bob has a winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ and $|X| \geq \kappa$.

Proof. The example we need is just the space X constructed by Juhász in [7, Example 7.2]. Following the notation in [7], we have $X = \bigcup\{X_n : n < \omega\}$, where $X_0 = \kappa$. In [7] it is pointed out that for a given $n < \omega$ every open family covering X_{n+1} has a finite subfamily covering all but finitely many members of X_n. The latter assertion clearly implies that every open cover of X has a finite subfamily which covers the whole X_n and this in turn guarantees an easy winning strategy to Bob in $\mathsf{G}_{\text{fin}}(\mathrm{O}, \mathrm{O})$. □

The original cardinality bound of Arhangel'skiĭ as well as most of its variations work for T_2 spaces. So, it is reasonable to ask:

QUESTION 8. *Does Theorem* 5 *continue to hold for T_2 spaces?*

Recall that, given a space X, the symbol X_δ denotes the space with the same underlying set X with the topology generated by the G_δ subsets of X. In [1] it was shown that Theorem 1 is actually a consequence of the more general statement that a winning strategy for Bob in $\mathsf{G}_1^{\omega_1}(\mathrm{O}, \mathrm{O})$ implies that the Lindelöf degree of X_δ is at most 2^ω. This seems to suggest a possible further strengthening of Theorem 5 as follows: if X is a regular space where Bob has a winning strategy in $\mathsf{G}_{\text{fin}}^{\omega_1}(\mathrm{O}, \mathrm{O})$, then $L(X_\delta) \le 2^\omega$. However, this conjecture drastically fails because there are compact T_2 spaces such that the Lindelöf degree of the G_δ-modification is much bigger than the continuum (see e.g. [11] or [12]), while for every compact space Bob may win in $\mathsf{G}_{\text{fin}}^{\omega_1}(\mathrm{O}, \mathrm{O})$ at the first inning!

3. Few remarks on the long tightness game

We conclude the paper by looking at a long version of the tightness game. One reason is in the similarity of Theorem 5 and Theorem 9 below.

Given a space X and a point $x \in X$, Ω_x denotes the collection of all sets $A \subseteq X$ satisfying $x \in \overline{A}$. The tightness game $\mathsf{G}_1(\Omega_x, \Omega_x)$ is played between players Alice and Bob in such a way that, at every inning $n \in \omega$, Alice chooses a member $A_n \in \Omega_x$, and then Bob chooses $a_n \in A_n$. Bob is declared the winner if, and only if, $\{a_n : n \in \omega\} \in \Omega_x$ (see [2] for much more).

If the previous game consists of ω_1-many innings, then we have the long tightness game $\mathsf{G}_1^{\omega_1}(\Omega_x, \Omega_x)$.

THEOREM 9. *If X is a regular space that has a dense subset E with $|E| \le 2^\omega$ and* Bob *has a winning strategy in the game $\mathsf{G}_1^{\omega_1}(\Omega_p, \Omega_p)$ for some $p \in X$, then $\chi(p, X) \le 2^\omega$.*

PROOF. Let σ be a winning strategy for Bob. Let μ be a large enough regular cardinal and M be an elementary submodel of $H(\mu)$ such that $E \subset M$, $X, \sigma, p, \Omega_p \in M$, $[M]^\omega \subset M$ and $|M| = 2^\omega$. For every sequence $s \in \Omega_p^{<\omega_1}$, there is a neighbourhood V_s of p such that for every $x \in V_s$, there is a $D \in \Omega_p$ such that $x = \sigma(s^\frown D)$. We will call such a neighbourhood *good*. To verify the existence of V_s, assume the contrary and let D be the set of all $x \in X$ such that $\sigma(s^\frown A) \neq x$ for each $A \in \Omega_p$. But then $D \in \Omega_p$ and so $\sigma(s^\frown D) \in D$, in contrast with the definition of D.

Now, to prove the theorem it is enough to show that $\mathcal{V} = \{V \subset X : V \in M \text{ and } V \text{ is good}\}$ is a local base at p. Assume the contrary. Then, by regularity, there is an open neighborhood W of p such that $V \not\subset \overline{W}$ for every $V \in \mathcal{V}$. Let V_0 be an open set such that for every $x \in V_0$ there is a $D \in \Omega_p$ such that $x = \sigma(D)$. V_0 is definable in M and so $V_0 \in \mathcal{V}$. Besides, by density, there is an $e_0 \in (V_0 \setminus \overline{W}) \cap E$. Note that e_0 is in M, therefore there is a D_0 such that $\sigma(D_0) = e_0$. Now, we proceed by induction, by assuming to have already defined points $e_\alpha \in E$ and sets $D_\alpha \in \Omega_p$ for $\alpha < \xi$. let $s = \{(e_\alpha, D_\alpha) : \alpha < \xi\} \in (\Omega_p \cap M)^{<\omega_1}$. Since M is countably closed, $s \in M$. Therefore, there is an open neighborhood V_s of p such that for every $x \in V_s$, $\sigma(s^\frown D) = x$. Again, $V_s \in M$. As before, we can take $e_\xi \in (V_s \setminus \overline{W}) \cap E$ and then choose D_ξ such that $e_\xi = \sigma(s^\frown D_\xi)$. Note that $D_\xi \in M$. But, playing like this, at the end Bob would loose the game — a contradiction. □

One may wonder if the above theorem is the best possible, namely if we could get $\chi(p, X) \leq \omega_1$. This obviously happens by assuming $2^\omega = \omega_1$, but the next example shows it is no longer true without the Continuum Hypothesis.

Example 10. There exist a regular space X with a dense set E of size 2^ω and a point p such that Bob has a winning strategy in $\mathsf{G}_1^{\omega_1}(\Omega_p, \Omega_p)$ and $\chi(p, X) = 2^\omega$.

Proof. Let E be a set of cardinality 2^ω with the discrete topology and let $X = E \cup \{p\}$ be the one-point Lindelöfication of E. Observe that U is a neighbourhood of p in X if and only if $p \in U$ and $|X \setminus U| \leq \omega$. We have $\chi(p, X) = 2^\omega$. Indeed, if $\mathcal{U}$ is a collection of neighborhoods of p satisfying $|\mathcal{V}| < 2^\omega$, then $|E \setminus \bigcap \mathcal{V}| \leq |\mathcal{V}|\omega < 2^\omega$ and so $|\bigcap \mathcal{V}| = 2^\omega$, which in turn implies that $\mathcal{V}$ cannot be a local base. On the other hand, Bob has an easy winning strategy in $\mathsf{G}_1^{\omega_1}(\Omega_p, \Omega_p)$: fix $\xi < \omega_1$ and suppose that e_α is the point Bob has chosen at the inning $\alpha < \xi$. If at the ξ-inning Alice plays $A_\xi \in \Omega_p$, then Bob simply takes a point $e_\xi \in A_\xi \setminus \{e_\alpha : \alpha < \xi\}$. This can be done because A_ξ is uncountable. Now, at the end of the game Bob has chosen an uncountable set of points and so he wins. □

Let us denote by D the collection of all dense subsets of a given topological space. Note that if Bob has a winning strategy for the game $\mathsf{G}_1^{\omega_1}(\mathrm{D}, \mathrm{D})$, then the density of the space is less than or equal to ω_1. Therefore, the next result can be proved with almost the same argument as that in Theorem 9:

Theorem 11. *If X is a regular space where* Bob *has a winning strategy in the game $\mathsf{G}_1^{\omega_1}(\mathrm{D}, \mathrm{D})$ then $\pi w(X) \leq 2^\omega$.*

Comparing Theorems 1 and 5, one may be tempted to conjecture that a result similar to Theorem 9 continues to hold for $\mathsf{G}_{\mathrm{fin}}$ instead of G_1. But, it turns out that even the difference between G_2 and G_1 can be very big —

here G_2 is the game where BOB is allowed to take at most 2 points instead of just one. Indeed, even the fact that BOB always wins the "short" game $\mathsf{G}_2(\Omega_p, \Omega_p)$ does not guarantee that BOB has a winning strategy in the long tightness game, as the following example from [2] shows:

EXAMPLE 12. A zero-dimensional T_1 space where BOB has a winning strategy in $\mathsf{G}_2(\Omega_p, \Omega_p)$ and ALICE has a winning strategy in $\mathsf{G}_1^{\omega_1}(\Omega_p, \Omega_p)$.

PROOF. Let $X = \{p\} \cup \omega^{<\omega_1}$ with the following topology: every point other than p is isolated. The basic neighborhoods at p are of the form

$$\{p\} \cup \omega^{<\omega_1} \setminus F$$

where F is the union of finitely many branches in the tree $\omega^{<\omega_1}$. Let us show that ALICE has a winning strategy in $\mathsf{G}_1^{\omega_1}(\Omega_p, \Omega_p)$. ALICE starts with $D_0 = \{\langle n \rangle : n \in \omega\}$. Let s be the choice of BOB. Note that then ALICE can play $D_1 = \{s^\frown n : n \in \omega\}$. Indeed, by playing in this way, at a certain inning the set of all choices of BOB is a function $s\colon \alpha + 1 \to \omega$. Then ALICE simply can play $D = \{s^\frown n : n \in \omega\}$. Note that playing like this, at the end all of the choices of BOB forms a branch thus ALICE wins.

Now let us see that BOB has a winning strategy for the $\mathsf{G}_2(\Omega_p, \Omega_p)$ game. It is enough to show that, for each $n \in \omega$, the set of all the answers played by BOB in the first n innings includes a set $\{s_1, \dots, s_n\}$ with the property that no branch contains two elements of it.

Let us proceed by induction. If, in the first inning, ALICE plays A_1, then BOB chooses $\{s_1, s_2\} \subset A_1$ such that s_1 and s_2 are not in the same branch. Suppose that at the end of the n-th inning, the set of all answers of BOB contains a set $\{s_1, \dots, s_n\}$ satisfying our assumption. Let A_{n+1} be the play of ALICE at the inning $n+1$. If there is a point in A_{n+1} that lies in a branch missing $\{s_1, \dots, s_n\}$, then BOB chooses this point together with some other one. In the remaining case, since p is in the closure of A_{n+1}, there is at least one s_i and two incompatible elements $a_1, a_2 \in A_{n+1}$ such that $s_i \subset a_1$ and $s_i \subset a_2$. The answer of BOB in the $(n+1)$-th inning will be just $\{a_1, a_2\}$. Observe that every branch meets the set $\{s_j : j \neq i\} \cup \{a_1, a_2\}$ in at most one point. □

In the previous proof, we did not use that much information about the height of the tree. Therefore, we can easily modify the example to obtain the following:

PROPOSITION 13. *There is a zero-dimensional T_1 space X and a point $p \in X$ such that* BOB *has a winning strategy in $\mathsf{G}_2(\Omega_p, \Omega_p)$, $|X| = 2^\omega$ and $\chi(p, X) > 2^\omega$.*

In particular, this shows that we cannot generalize Theorem 9 for the version where BOB is allowed to pick two points instead of one!

Finally, a simplified version of the above construction gives:

PROPOSITION 14. *There is a countable zero-dimensional space X where* BOB *has a winning strategy in* $\mathsf{G}_2(\Omega_p, \Omega_p)$ *but* $\chi(p, X) > \omega$.

Inspired by Example 12, we finish with a similar construction that may serve as an example of the ideas used here. Let T be an uncountable tree with no uncountable chains (e.g. an ω_1-Aronszajn three) and consider $X = T \cup \{p\}$ with the following topology: every point of T is isolated and the neighborhoods of p are of the form $X \setminus \bigcup F$ where F is a finite collection of branches of T. Note that ALICE cannot repeat the analogous strategy made in Example 12, since that would imply the existence of an uncountable branch, which is impossible. Moreover, it is very easy for BOB to guarantee his own victory. Indeed, it is enough to him to play in a manner where he ends up by playing uncountably many distinct points. This is enough since in a tree any uncountable set contains either an uncountable branch or an infinite antichain.

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