

## Erratum to: Analyzing restricted fragments of the theory of linear arithmetic

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Owing to an error in the production process, the following article has been published incorrectly online. The article including the illustrations is presented hereafter.

**Abstract** The theory of linear arithmetic (TLA), also known as the Theory of Rationals, is an extremely well-studied theory. It has been widely applied to a number of domains, including program verification and constraint specification. This paper discusses the computational complexities of two broad fragments of TLA, namely Quantified Linear Programs (QLPs) and Quantified Linear Implications (QLIs). These fragments are ideal for expressing specifications in real-time scheduling, and for modeling reactive systems. In this paper, we study the computational complexities of several variants of QLPs and QLIs. Our principal result shows that there exists a one-to-one correspondence between alternations in a class of

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QLIs and the complexity classes comprising the polynomial hierarchy **PH**. In other words, for each class of the **PH**, there exists a class of QLIs that is complete for that class. Our work mirrors the work in [L.J. Stockmeyer, The Polynomial-Time Hierarchy, *Theoretical Computer Science*, 3:1-22, 1977] which established the connection between the classes of **PH** and quantifier alternations in Quantified Boolean Formulas. Our results are surprising, since the variables in the fragments that we consider are continuous, as opposed to discrete.

**Keywords** Quantified linear programming · Quantified linear implication · Theory of real numbers with addition · Complexity classes

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## 1 Introduction

The Theory of Linear Arithmetic (TLA), also known as the Theory of Rationals (see Appendix A.1), has been well-studied in the literature, on account of its wide applicability to a number of practical domains. Indeed, this theory forms the backbone of most automated program verifiers [4]. This paper studies the computational complexities of some interesting fragments of TLA. In particular, it deals with Quantified Linear Programs (QLPs) and Quantified Linear Implications (QLIs). *Quantified linear programming* is the problem of checking whether a linear system is satisfiable with respect to a given quantifier string. The quantifier string specifies the sort of each variable (existential or universal) and the order in which they occur. Hence, quantified linear programming is a (non-trivial) generalization of linear programming. For each universally quantified variable, the quantifier string specifies a non-empty, continuous and closed interval on the real-line; this interval corresponds to the permissible values for the variable [33]. It is not hard to see that QLPs represent the conjunctive fragment of TLA with one major exception, viz., the universally quantified variables are bounded. Note that the bounds can be removed by rewriting the constraints implied by the QLP as an implication. The bounds on the universally quantified variables are placed in the Left Hand Side (LHS) of the implication, while the original constraints form the Right Hand Side (RHS). This is demonstrated in Example 1.

*Example 1* Let us consider QLP (1):

$$\forall y_1 \in [0, 1] \exists x_1 \quad x_1 + y_1 \leq 1. \quad (1)$$

This is equivalent to QLI (2):

$$\forall y_1 \exists x_1 \quad [0 \leq y_1 \leq 1 \rightarrow x_1 + y_1 \leq 1]. \quad (2)$$

Allowing an arbitrary linear system on the LHS of the implication results in a Quantified Linear Implication (QLI) [14]. Note that the abbreviations QLP and QLI and their variants PQLP, UQLP, PQLI and UQLI are used to denote both a class of problems and specific instances (see Section 2). The usage is clarified by the context.

In this paper, we examine some new variants of QLP and QLI, and establish their computational complexities. We also settle several open questions in the literature regarding QLI. We prove that the variants of QLP and QLI in which every universally quantified variable is partially bounded can be decided in polynomial time. Similarly, we show that the variants of these problems in which every universally quantified variable is unbounded can be decided in polynomial time. We also show that corresponding to each class of the polynomial hierarchy

**(PH)** [27, 28], there exists some instantiation of the QLI framework that is **complete** for this class. This is interesting due to the fact that QLI covers the **PH** using only continuous variables, thus providing a continuous analogue to the results of [32]. Finally, we examine the computational complexities of some classes of QLI with one quantifier alternation.

The rest of this paper is organized as follows: Section 2 formally describes the problems that we consider in this paper. Section 3 describes the motivation for our work. In Section 4, we survey related results in the literature. Sections 5 and 6 discuss variants of QLP and QLI respectively from the perspective of computational complexity. Section 7 proves that Quantified Linear Implication covers the polynomial hierarchy. Related complexity results are described in Section 8. We conclude in Section 9 by summarizing our contributions and outlining avenues for future research.

## 2 Statement of problems

We use the standard notation of linear algebra [29] to formally present the basic notions of this paper.  $\mathbb{R}$  is the set of real numbers. Let capital bold letters ( $\mathbf{A}, \mathbf{B}, \dots$ ) denote matrices and small bold letters ( $\mathbf{x}, \mathbf{y}, \mathbf{b}, \dots$ ) denote column vectors.  $\mathbf{0}$  is the column vector with all elements equal to 0. The product of  $\mathbf{A}$  and  $\mathbf{x}$  is denoted by  $\mathbf{A} \cdot \mathbf{x}$ . Likewise, let  $\mathbf{x} \cdot \mathbf{y}$  denote the inner product of  $\mathbf{x}$  and  $\mathbf{y}$ . Finally, we assume that the dimensions of vectors and matrices in products are compatible.

A QLP is a conjunction of linear constraints in which every variable is either existentially or universally quantified:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \in [\mathbf{l}_1, \mathbf{u}_1] \dots \exists \mathbf{x}_n \forall \mathbf{y}_n \in [\mathbf{l}_n, \mathbf{u}_n] \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}. \tag{3}$$

In QLP (3),

- (i)  $\mathbf{x}_1 \dots \mathbf{x}_n$  is a partition of  $\mathbf{x}$  with  $\mathbf{x}_i$  possibly empty.
- (ii)  $\mathbf{y}_1 \dots \mathbf{y}_n$  is a partition of  $\mathbf{y}$  with  $\mathbf{y}_n$  possibly empty.
- (iii)  $\mathbf{l}_i$  is the lower bound for  $\mathbf{y}_i$ ,  $i = 1, \dots, n$ .
- (iv)  $\mathbf{u}_i$  is the upper bound for  $\mathbf{y}_i$ ,  $i = 1, \dots, n$ .

Note that in a QLP every universally quantified variable is bounded from above and below. It is clear that a QLP that in which every variable is existentially quantified is a Linear Program (LP).

We now introduce two variants of Quantified Linear Programming, both of which alter the nature of the bounds on the universal variables.

A *Partially-bounded Quantified Linear Program* (PQLP) is a QLP in which every universally quantified variable is only bounded on one side.

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \in [\mathbf{l}_1, +\infty) \dots \exists \mathbf{x}_n \forall \mathbf{y}_n \in (-\infty, \mathbf{u}_n] \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$$

Without loss of generality, we can assume this single bound forces each such variable to be non-negative:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \in [\mathbf{0}, +\infty) \dots \exists \mathbf{x}_n \forall \mathbf{y}_n \in [\mathbf{0}, +\infty) \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$$

This assumption holds because we can always transform any PQLP into the desired form. This is demonstrated in Example 2.

*Example 2* Consider PQLP (4):

$$\forall y_1 \in (-\infty, -2] \exists x_1 \quad x_1 + y_1 \leq 1. \tag{4}$$

We can change the upper bound on  $y_1$  into a lower bound by negating all instances of  $y_1$  in PQLP (4). This results in PQLP (5):

$$\forall y_1 \in [2, +\infty) \exists x_1 \quad x_1 - y_1 \leq 1. \tag{5}$$

Then, we can change the lower bound on  $y_1$  from 2 to 0 by replacing all instances of  $y_1$  in PQLP (5) with  $(y_1 + 2)$ . This results in PQLP (6):

$$\forall y_1 \in [0, +\infty) \exists x_1 \quad x_1 - (y_1 + 2) \leq 1. \tag{6}$$

An *Unbounded Quantified Linear Program* (UQLP) is a QLP where there are no bounds on any universal variable:

$$\exists x_1 \forall y_1 \dots \exists x_n \forall y_n \quad \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$$

Note that UQLP represents the conjunctive fragment of TLA.

QLIs generalize the notion of polyhedral inclusion. The typical QLI has the following form:

$$\exists x_1 \forall y_1 \dots \exists x_n \forall y_n \quad [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}] \tag{7}$$

In [14], a nomenclature was presented to represent the various classes of QLI. We briefly describe the nomenclature below:

Every QLI is exactly characterized by an ordered 3-tuple  $\langle A, Q, T \rangle$ . In this 3-tuple,

- (i)  $A$  denotes the number of quantifier alternations in the quantifier string.
- (ii)  $Q$  denotes the first quantifier.
- (iii)  $T$  is an  $(A + 1)$ -character string that specifies whether each quantified set of variables in the quantifier string appears on the **Left**, on the **Right**, or on **Both** sides of the implication.

*Example 3* Consider the 3-tuple  $\langle 1, \forall, \mathbf{BL} \rangle$ . This indicates a problem described by QLI (8):

$$\forall y \exists x \quad [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}] \tag{8}$$

We extend the notion of partially bounded and unbounded universally quantified variables to QLIs. A *Partially-bounded Quantified Linear Implication* (PQLI) is a QLI in which every universally quantified variable is bounded only by a single absolute constraint. As with PQLPs, we can assume without loss of generality that this single constraint is a non-negativity constraint. This constraint is placed in the LHS of the implication. Thus, a PQLI has the following form:

$$\exists x_1 \forall y_1 \dots \exists x_n \forall y_n \quad [(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \wedge \mathbf{y} \geq \mathbf{0}) \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}]$$

An *Unbounded Quantified Linear Implication* (UQLI) is a QLI in which no universally quantified variable appears in the LHS of the implication. Thus, a UQLI has the following form:

$$\exists x_1 \forall y_1 \dots \exists x_n \forall y_n \quad [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}]$$

It is important to note that 2-person game semantics have been proposed both for QLP [33] and QLI [14]. These are described in detail in Appendix B.1 and Appendix B.2 respectively.

A boolean formula  $\phi$  is said to be in disjunctive normal form (DNF), if it consists of a disjunction of conjuncts, where each conjunct is a conjunction of literals. The formula is said to be in 3DNF, if each conjunct has exactly three literals.

A *Quantified 3DNF* (Q3DNF) formula is a quantified boolean expression of the form  $Q(\mathbf{x}, \mathbf{y}) \phi(\mathbf{x}, \mathbf{y})$ , where  $Q(\mathbf{x}, \mathbf{y})$  is a quantifier string and  $\phi(\mathbf{x}, \mathbf{y})$  is a boolean formula in 3DNF.

The following quantified boolean expression is an instance of Q3DNF:

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 \exists x_3 \forall y_3 (x_1 \wedge y_3 \wedge \neg x_2) \vee (x_3 \wedge y_2 \wedge \neg x_1) \vee (x_2 \wedge y_1 \wedge \neg x_3)$$

We will use Q3DNFs to establish the connection between Quantified Linear Implication and the Polynomial Hierarchy. A 2-person game semantics for this problem is provided in Appendix B.3.

If  $\mathbf{L}$  is a QLP, QLI or Q3DNF, the following terms are equivalent and used interchangeably:

1.  $\mathbf{L}$  holds.
2.  $\mathbf{L}$  is feasible.
3.  $\mathbf{L}$  is satisfiable.
4. The existential player has a *winning strategy* for  $\mathbf{L}$  (see Appendix B).

### 3 Motivation

Quantified Linear Programming represents a rich language that is ideal for expressing schedulability specifications in *real-time scheduling* [8, 16]. In a real-time scheduling instance, a dispatcher typically determines whether a set of ordered, non-preemptive jobs can be scheduled within given time frames. Associated with each job is a start time and an execution time. The execution time of a job is a range-bound variable. There exist timing constraints that constrain the execution of jobs. These requirements can be modeled using QLPs.

For example, consider a real-time scheduling application with two jobs  $J_1$  and  $J_2$  with start times  $s_1$  and  $s_2$ , and execution times  $e_1 \in [l_{e_1}, u_{e_1}]$  and  $e_2 \in [l_{e_2}, u_{e_2}]$ . Assume that  $J_1$  must start and finish before  $J_2$  starts (i.e.,  $s_1 + e_1 \leq s_2$ ), while  $J_2$  must start at most two time units after  $J_1$  finishes (i.e.,  $s_2 \leq s_1 + e_1 + 2$ ). The corresponding QLP is the following:

$$\exists s_1 \forall e_1 \in [l_{e_1}, u_{e_1}] \exists s_2 \forall e_2 \in [l_{e_2}, u_{e_2}] s_1 + e_1 - s_2 \leq 0, \\ s_2 - (s_1 + e_1) \leq 2$$

Now, consider the case where the dispatcher has already obtained a schedule (solution). If new constraints are added to the specification, then the dispatcher may have to recompute the schedule. Alternatively, if it can be concluded that the current schedule does not cause violation of the newly added constraints, then the dispatcher can use the existing schedule. QLIs can be utilized to model the above decision problem. Note that QLP subsumes the two-level programming model discussed in [3].

QLPs and QLIs can be used to model reactive systems. A system is called *reactive*, if it maintains an ongoing interaction with its environment. A reactive system changes its actions, outputs, and status in response to the input it receives from the environment. Reactive systems are used in several real-world important applications and in various fields (see e.g., [17–19, 21, 23, 24]). QLP and QLI are important modeling tools for the design and implementation of reactive systems. The universally quantified variables can be used to represent the environmental input, while the existentially quantified variables can be used to represent the system’s response.

## 4 Related work

In this section, we briefly survey related work in the literature.

A decision procedure for the *full elementary theory of real closed fields* (see Appendix A.2) with addition (+), multiplication ( $\cdot$ ) and order ( $<$ ,  $=$ ) was established in [34]. Several quantifier elimination methods [13, 35] and efficient-in-practice approaches have been proposed since then [5, 9, 11, 12, 25, 26]. The complexity of these quantifier elimination procedures is, in the worst case, doubly exponential in the number of quantifier alternations and exponential in the number of variables [10, 35].

Real numbers cannot be fully axiomatized by a first-order theory. Tarski's axiomatization of the reals requires a non-first-order axiom to express the Dedekind completeness of the real numbers (i.e., the property that asks all bounded subsets of real numbers to have a real least upper bound and a real greatest lower bound). The axiom in question involves universal quantification over subsets of the real numbers, which cannot be expressed in first-order logic.

Any field that satisfies all the same first-order properties as the real numbers is called a *real closed field*. Note that although the real algebraic numbers comprise a real closed field, they are not Dedekind complete. It is possible to construct a set of algebraic rational numbers, that has  $\pi$ , which is transcendental, as a supremum.

Several sub-classes of the full elementary theory of the reals have been studied.

The *existential theory of the reals* is obtained by restricting allowable expressions to existentially quantified formulas  $\exists \mathbf{x} F(\mathbf{x})$  where  $F(\mathbf{x})$  is a quantifier-free formula. There exists a decision procedure for this problem that is singly exponential in the number of quantified variables [1]. From the complexity perspective, it is known that this problem is **NP-hard** [30] and in **PSPACE** [7].

The *theory of reals with addition and order* (TLA) is obtained by restricting the set of function symbols to  $\{+\}$ . A quantifier elimination procedure for sentences in this theory that is singly exponential in space and doubly exponential in time is presented in [15]. An exponential time lower bound is shown in [2], where the time and space complexities at various levels of quantifier alternations are also determined.

Consider a formula in the theory of reals with addition and order in prenex normal form with  $(k - 1)$  quantifier alternations

$$\exists \mathbf{x}_1 \forall \mathbf{x}_2 \dots Q \mathbf{x}_k F(\mathbf{x}_1, \dots, \mathbf{x}_k)$$

where  $Q$  is  $\exists$  for  $k$  odd and  $\forall$  for  $k$  even, while  $F(\mathbf{x}_1, \dots, \mathbf{x}_k)$  is a quantifier-free formula. This class of formulas has been proven to be log-space complete for  $\Sigma_k \mathbf{P}$  [31].

In this paper, we examine two interesting fragments of TLA:

- (i) The Quantified Linear Programming fragment, where  $F(\mathbf{x}_1, \dots, \mathbf{x}_k)$  consists of conjunctions of linear inequalities, namely a linear system.
- (ii) The Quantified Linear Implication fragment, where  $F(\mathbf{x}_1, \dots, \mathbf{x}_k)$  consists of a linear system implication, i.e., an implication between two linear systems.

QLP decidability is in **PSPACE**, though hardness has not yet been established [33]. The sub-classes of QLP with one quantifier alternation have the following complexities:

- (i) If the first quantifier is existential, then the problem is in **P** [33].
- (ii) If the first quantifier is universal, then the problem is **coNP-complete** [33].

Decidability of QLI with an arbitrary number of quantifier alternations has been shown to be **PSPACE-hard** [14]. The computational complexities of several sub-classes of QLI decidability with a given number of quantifier alternations have also been established [14].

**Table 1** The computational complexity of deciding  $\mathbb{R}$ ,  $\mathbb{Q}$ , QLI, and QLP

Theory	Complexity
$\mathbb{R}$	<b>EXPTIME-hard</b> and <b>2-EXPTIME</b> [10]
$\mathbb{R}^{(+,\leq)} = \mathbb{Q} = \text{TLA}$	<b>EXPTIME-hard</b> [4] and <b>2-EXPTIME</b> [15, 31]
<b>QLI</b>	<b>PSPACE-hard</b> [14]
<b>QLP</b>	<b>coNP-hard</b> and <b>PSPACE</b> [33]

Table 1 captures the various theories described above and the complexity of deciding the same. The corresponding inclusion diagram is provided in Fig. 1.

The principal contributions of this paper are presented below. We establish that:

- (i) PQLP and UQLP decidability are in **P**.
- (ii) PQLI and UQLI decidability are in **P**.
- (iii) For each complexity class comprising the **PH**, there exists a class of QLI (i.e., an instantiation of the  $\langle A, Q, T \rangle$  triple) which is complete for that class.
- (iv)  $\langle 1, \forall, \mathbf{RB} \rangle$  is in **P**.
- (v)  $\langle 1, \exists, \mathbf{BL} \rangle$  is in **P**.
- (vi)  $\langle 1, \forall, \mathbf{BR} \rangle$  and  $\langle 1, \forall, \mathbf{BB} \rangle$  are **coNP-complete**.

### 5 Variants of QLP

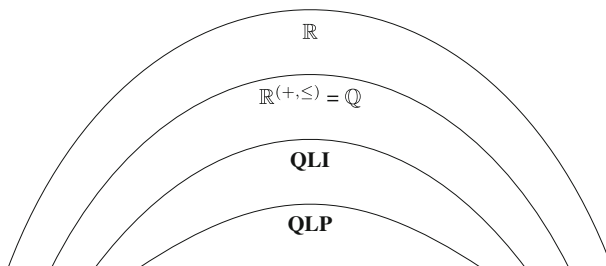
In this section, we examine the computational complexities of PQLP and UQLP. We commence our analysis by showing that PQLP decidability is in **P**.

**Theorem 1** *PQLP decidability is in P.*

*Proof* Let **L** be the following PQLP:

$$\begin{aligned} \exists x_n \forall y_n \in [0, +\infty) \exists x_{n-1} \forall y_{n-1} \in [0, +\infty) \dots \exists x_1 \\ \forall y_1 \in [0, +\infty) \exists x_0 \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} \end{aligned} \tag{9}$$

Let  $\mathbf{A} = (\mathbf{a}_n, \mathbf{a}_{n-1}, \dots, \mathbf{a}_0)$ ,  $\mathbf{B} = (\mathbf{b}_n, \mathbf{b}_{n-1}, \dots, \mathbf{b}_1)$ ,  $\mathbf{x} = (x_n, x_{n-1}, \dots, x_0)^T$ , and  $\mathbf{y} = (y_n, y_{n-1}, \dots, y_1)^T$ . Note that any PQLP can be reduced to the form specified by PQLP (9) through the addition of dummy variables. Enforcing this strict alternation of quantifiers will at most double the total number of program variables.



**Fig. 1** Inclusion diagram of  $\mathbb{R}$ ,  $\mathbb{Q}$ , QLI, and QLP

Consider the linear program:

$$LP_n : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{c}, \tag{10}$$

and, for  $i = 1, \dots, n$  consider the linear program:

$$\begin{aligned} LP_{i-1} : \mathbf{A} \cdot \mathbf{x} + \mathbf{b}_i &\leq \mathbf{0} \\ x_i, \dots, x_n &= 0. \end{aligned} \tag{11}$$

We will show that PQLP (9) is feasible if and only if  $LP_n$  is feasible, and  $LP_{i-1}$  is feasible for all  $i = 1 \dots n$ .

Assume that the linear program represented by System (10) and the linear programs represented by System (11) are feasible.

Let  $\hat{\mathbf{x}}^n$  denote a solution to  $LP_n$ . Furthermore, let  $\hat{\mathbf{x}}^{i-1}$  denote a solution to  $LP_{i-1}$  for  $i = 1, \dots, n$ .

Let  $\mathbf{x}^s$  denote the strategy of the existential player  $\mathbf{X}$  and let  $\mathbf{y}^s$  denote the strategy of the universal player  $\mathbf{Y}$ .

Consider the assignment

$$\mathbf{x}^s = \hat{\mathbf{x}}^n + \sum_{i=1}^n y_i \cdot \hat{\mathbf{x}}^{i-1} \tag{12}$$

Note that in  $LP_{i-1}$ , the constraints  $x_i, \dots, x_n = 0$  ensure that  $x_n^s$  through  $x_i^s$  do not depend on the values of  $y_i$  through  $y_1$ . Hence, the moves specified by the strategy  $\mathbf{x}^s$  depend only on the moves previously made by  $\mathbf{Y}$ , i.e., System (12) is a strategy for  $\mathbf{X}$ .

As per the 2-person game semantics of QLPs, the game corresponding to System (9) will be played as follows:

1.  $\mathbf{X}$  chooses  $x_n = x_n^s = \hat{x}_n^n$ .
2.  $\mathbf{Y}$  chooses  $y_n = y_n^s \in [0, \infty)$ ;  $\mathbf{X}$  chooses  $x_{n-1} = x_{n-1}^s = \hat{x}_{n-1}^n + y_n \cdot \hat{x}_{n-1}^{n-1}$ .
- $\vdots$
3.  $\mathbf{Y}$  chooses  $y_{i+1} = y_{i+1}^s \in [0, \infty)$ ;  $\mathbf{X}$  chooses  $x_i = x_i^s = \hat{x}_i^n + \sum_{j=i}^{n-1} y_{j+1} \cdot \hat{x}_i^j$ .
- $\vdots$
4.  $\mathbf{Y}$  chooses  $y_1 = y_1^s \in [0, \infty)$ ;  $\mathbf{X}$  chooses  $x_0 = x_0^s = \hat{x}_0^n + \sum_{j=0}^{n-1} y_{j+1} \cdot \hat{x}_0^j$ .

Observe that all the  $x_i^s, i = 0, \dots, n$  and the  $y_i^s, i = 1, \dots, n$  are assigned linear values. Let  $\mathbf{x}_T$  be the numeric vector assigned to  $\mathbf{x}$  by the strategy  $\mathbf{x}^s$ . It follows that  $\mathbf{x}_T = \hat{\mathbf{x}}^n + \sum_{i=1}^n y_i^s \cdot \hat{\mathbf{x}}^{i-1}$ . Likewise, let  $\mathbf{y}_T = (y_n^s, y_{n-1}^s, \dots, y_1^s)$ . Accordingly, the outcome of the play can be evaluated as:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{x}_T + \mathbf{B} \cdot \mathbf{y}_T &= \mathbf{A} \cdot \left( \hat{\mathbf{x}}^n + \sum_{i=1}^n y_i^s \cdot \hat{\mathbf{x}}^{i-1} \right) + \sum_{i=1}^n \mathbf{b}_i \cdot y_i^s \\ &= \mathbf{A} \cdot \hat{\mathbf{x}}^n + \sum_{i=1}^n y_i^s \cdot (\mathbf{A} \cdot \hat{\mathbf{x}}^{i-1} + \mathbf{b}_i) \\ &\leq \mathbf{c} + \sum_{i=1}^n y_i^s \cdot \mathbf{0} \\ &= \mathbf{c} \end{aligned}$$

Since  $\mathbf{Y}$ 's strategy was chosen arbitrarily, at the end of any play,  $\mathbf{X}$ 's strategy forces  $\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$ . Thus, PQLP (9) is feasible, proving our claim.



Now assume that PQLP (9) is feasible. Thus,  $\mathbf{X}$  has a winning strategy  $\mathbf{x}^s$ .

Consider the case when  $\mathbf{Y}$  plays according to the strategy  $\mathbf{y}^s = \mathbf{0}$ . Let  $\mathbf{x}^*$  be the numeric vector assigned to  $\mathbf{x}$  by the strategy  $\mathbf{x}^s$  in this situation. It is clear that  $\mathbf{x}^*$  satisfies  $LP_n$ .

Let us focus on a specific value of  $i$ . Let  $\bar{\mathbf{y}}^i = (0, 0, \dots, 1, \dots, 0)^T$ . Note that  $\bar{y}_i^i = 1$  is the only non-zero element of  $\bar{\mathbf{y}}^i$ .

Assume that  $LP_{i-1}$  is infeasible. Let  $\mathbf{x}^i = (x_{i-1}, \dots, x_0)$  and let  $\mathbf{A}^i$  be the matrix consisting of the corresponding columns of  $\mathbf{A}$ . Note that we can rewrite  $LP_{i-1}$  as  $\mathbf{A}^i \cdot \mathbf{x}^i \leq -\mathbf{b}_i$ . By Farkas' Lemma, there exists a vector  $\mathbf{z} \in \mathbb{R}_+^m$  such that  $\mathbf{z}^T \cdot \mathbf{A}^i = \mathbf{0}$  and  $-\mathbf{z}^T \cdot \mathbf{b}_i < 0$ . We can assume without loss of generality that  $\mathbf{z}^T \cdot \mathbf{b}_i = 1$ .

Let  $M \in \mathbb{Z}^+$  denote an arbitrary integer constant. Consider the case when  $\mathbf{Y}$  plays according to the strategy  $M \cdot \bar{\mathbf{y}}^i$ . Let  $\mathbf{x}'$  be the numeric vector assigned to  $\mathbf{x}$  by the strategy  $\mathbf{x}^s$  in this situation. Let  $\bar{\mathbf{x}}^{iM}$  be the numeric vector such that  $\mathbf{x}' = (\mathbf{x}^* + \bar{\mathbf{x}}^{iM})$ . Thus  $\bar{\mathbf{x}}^{iM}$  represents the change in  $\mathbf{X}$ 's response when  $\mathbf{Y}$ 's play changes by  $M \cdot \bar{\mathbf{y}}^i$ .

Until  $y_i$  is assigned a value,  $\mathbf{Y}$ 's plays (viz.,  $\mathbf{0}$  and  $M \cdot \bar{\mathbf{y}}^i$ ) are indistinguishable. Thus,  $\mathbf{X}$ 's response cannot change until after  $y_i$  is assigned a value. This means that  $\bar{x}_i^{iM}, \dots, \bar{x}_n^{iM} = 0$ .

We have  $\mathbf{A} \cdot \mathbf{x}^* \leq \mathbf{c}$  and  $\mathbf{A} \cdot (\mathbf{x}^* + \bar{\mathbf{x}}^{iM}) + \mathbf{B} \cdot (M \cdot \bar{\mathbf{y}}^i) \leq \mathbf{c}$ . Since  $M$  was chosen arbitrarily, we have:

$$(\forall M) (\mathbf{A} \cdot \mathbf{x}^*) + \mathbf{A} \cdot \bar{\mathbf{x}}^{iM} + M \cdot (\mathbf{B} \cdot \bar{\mathbf{y}}^i) \leq \mathbf{c}. \tag{13}$$

Since  $\bar{x}_i^{iM}, \dots, \bar{x}_n^{iM} = 0$ , we can rewrite  $\mathbf{A} \cdot \bar{\mathbf{x}}^{iM}$  as  $\mathbf{A}^i \cdot \bar{\mathbf{x}}^{iM}$ . Thus System (13) can be rewritten as:

$$(\forall M) (\mathbf{A} \cdot \mathbf{x}^*) + \mathbf{A}^i \cdot \bar{\mathbf{x}}^{iM} + M \cdot \mathbf{b}_i \leq \mathbf{c}. \tag{14}$$

We can now multiply System (14) by  $\mathbf{z}^T$ . Since  $\mathbf{z}$  is non-negative we get:

$$\begin{aligned} (\forall M) \mathbf{z}^T \cdot (\mathbf{A} \cdot \mathbf{x}^*) + \mathbf{z}^T \cdot (\mathbf{A}^i \cdot \bar{\mathbf{x}}^{iM} + M \cdot \mathbf{b}_i) &\leq \mathbf{z}^T \cdot \mathbf{c} \\ (\forall M) \mathbf{z}^T \cdot (\mathbf{A} \cdot \mathbf{x}^*) + ((\mathbf{z}^T \cdot \mathbf{A}^i) \cdot \bar{\mathbf{x}}^{iM} + M \cdot \mathbf{z}^T \cdot \mathbf{b}_i) &\leq \mathbf{z}^T \cdot \mathbf{c} \\ (\forall M) \mathbf{z}^T \cdot (\mathbf{A} \cdot \mathbf{x}^*) + (\mathbf{0} \cdot \bar{\mathbf{x}}^{iM} + M) &\leq \mathbf{z}^T \cdot \mathbf{c} \\ (\forall M) M &\leq \mathbf{z}^T \cdot (\mathbf{c} - \mathbf{A} \cdot \mathbf{x}^*). \end{aligned}$$

This is clearly infeasible. Thus,  $LP_{i-1}$  must be feasible. Since  $i$  was chosen arbitrarily, we have that this is true for every  $i = 1 \dots n$ .

Thus, we have successfully reduced the problem of deciding the feasibility of PQLP (9) to the problem of deciding the feasibility of  $LP_n$  and the feasibility of  $LP_{i-1}$  for  $i = 1 \dots n$ . Since the problem of deciding LP feasibility is in  $\mathbf{P}$  [20], it follows that the problem of deciding PQLP feasibility is also in  $\mathbf{P}$ .  $\square$

We now show that UQLP decidability is in  $\mathbf{P}$ .

**Theorem 2** *UQLP decidability is in  $\mathbf{P}$ .*

*Proof* Consider the following UQLP:

$$\exists \mathbf{x}_n \forall \mathbf{y}_n \exists \mathbf{x}_{n-1} \forall \mathbf{y}_{n-1} \dots \exists \mathbf{x}_1 \forall \mathbf{y}_1 \exists \mathbf{x}_0 \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} \tag{15}$$

To show that UQLP (15) can be solved in polynomial time, we reduce it to a PQLP.

Consider the following PQLP:

$$\begin{aligned}
 &\exists \mathbf{x}_n \forall \mathbf{y}'_n \in [0, +\infty) \forall \mathbf{y}''_n \in [0, +\infty) \exists \mathbf{y}_n \exists \mathbf{x}_{n-1} \\
 &\quad \forall \mathbf{y}'_{n-1} \in [0, +\infty) \forall \mathbf{y}''_{n-1} \in [0, +\infty) \exists \mathbf{y}_{n-1} \dots \\
 &\quad \exists \mathbf{x}_1 \forall \mathbf{y}'_1 \in [0, +\infty) \forall \mathbf{y}''_1 \in [0, +\infty) \exists \mathbf{y}_1 \exists \mathbf{x}_0 \\
 &\quad \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} \\
 &\quad \mathbf{y}_i = \mathbf{y}'_i - \mathbf{y}''_i, \quad i = 1, 2, \dots, n.
 \end{aligned} \tag{16}$$

In PQLP (16),  $\mathbf{y}'_1, \dots, \mathbf{y}'_n$  is a partition of  $\mathbf{y}'$  and  $\mathbf{y}''_1, \dots, \mathbf{y}''_n$  is a partition of  $\mathbf{y}''$ , such that  $|\mathbf{y}'_i| = |\mathbf{y}''_i| = |\mathbf{y}_i|$ , for  $i = 1 \dots n$ .

The key observation is that the universal player can win System (15) if and only if he can win System (16). To see this, note that the value of  $\mathbf{y}_i$  in System (16) is completely determined by the universal player’s choices for  $\mathbf{y}'_i$  and  $\mathbf{y}''_i$ . Since  $\mathbf{y}'_i \in [0, +\infty)$  and  $\mathbf{y}''_i \in [0, +\infty)$ ,  $\mathbf{y}$  can take any value in the interval  $(-\infty, +\infty)$ . Thus, the quantifier sequence  $\forall \mathbf{y}_i$  in System (15)  $i = 1, 2, \dots, n$ , can be replaced respectively by the quantifier sequence  $\forall \mathbf{y}'_i \in [0, +\infty) \forall \mathbf{y}''_i \in [0, +\infty) \exists \mathbf{y}_i$  to get System (16), without affecting its feasibility. In other words, UQLP (15) is feasible if and only if PQLP (16) is feasible. From Theorem 1, it follows that UQLP decidability is in **P**. □

By proving that UQLP is in **P**, we have essentially shown the following: The computational complexity of deciding a formula in TLA depends on both the number of alternations in the quantifier specification and the syntactic restriction. Observe that in the absence of any syntactic restriction, and in the presence of unbounded alternation, the TLA decidability problem is in 2-EXPTIME [15, 31] and EXPTIME-hard [4]. However, the conjunctive fragment of TLA, even with unbounded alternation, is decidable in polynomial time.

### 6 Variants of QLI

We utilize the results of Section 5 to show that PQLI and UQLI are also solvable in polynomial time.

**Corollary 1** *PQLI decidability is in P.*

*Proof* Consider the following PQLI:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \dots \exists \mathbf{x}_n \forall \mathbf{y}_n \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{y} \geq \mathbf{0} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{y} \leq \mathbf{f}] \tag{17}$$

We will focus on the following two cases:

1. There exists a point  $\mathbf{z}$ , such that  $\mathbf{A} \cdot \mathbf{z} \not\leq \mathbf{b}$  - It follows that  $\forall \mathbf{x} (\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$  does not hold. In this case, the PQLI is trivially satisfied, since the existential player can guess  $\mathbf{z}$  and cause the left hand side of the implication to be falsified.
2. There is no point  $\mathbf{z}$  such that  $\mathbf{A} \cdot \mathbf{z} \not\leq \mathbf{b}$  - In this case, the PQLI (17) reduces to the following PQLI.

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \dots \exists \mathbf{x}_n \forall \mathbf{y}_n \ [\mathbf{y} \geq \mathbf{0} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{y} \leq \mathbf{f}] \tag{18}$$

PQLI (18) can in turn be written as:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \in [0, +\infty) \dots \exists \mathbf{x}_n \forall \mathbf{y}_n \in [0, +\infty) \ \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{y} \leq \mathbf{f} \tag{19}$$

However, System (19) is a PQLP, and hence can be decided in polynomial time by Theorem 1. □

**Corollary 2** *UQLI decidability is in P.*

*Proof* Consider the following UQLI:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \dots \exists \mathbf{x}_n \forall \mathbf{y}_n [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{y} \leq \mathbf{f}] \tag{20}$$

As argued in the proof of Corollary 1, if  $\forall \mathbf{x} (\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$  does not hold, then UQLI (20) is trivially satisfied.

Otherwise, UQLI (20) is equivalent to:

$$\exists \mathbf{x}_1 \forall \mathbf{y}_1 \dots \exists \mathbf{x}_n \forall \mathbf{y}_n \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{y} \leq \mathbf{f}$$

which is a UQLP, and hence can be decided in polynomial time by Theorem 2. □

### 7 QLI and the polynomial hierarchy

In this section, we prove that for each class of the **PH**, there exists a class of QLI decidability that is complete for that class. This is interesting, since QLIs are comprised of continuous variables, as opposed to the discrete variables comprising QBFs. Hence, we provide a continuous analogue to the results in [32], where the **PH** is generated using QBFs.

Let  $\mathbf{B}^{k+1}$  denote a string of  $(k + 1)$  **B**s. The following results were obtained in [14].

**Theorem 3**  $\langle k, \exists, \mathbf{B}^{k+1} \rangle$  with  $k$  odd, is  $\Sigma_k$ **P-hard**.

**Theorem 4**  $\langle k, \forall, \mathbf{B}^{k+1} \rangle$  with  $k$  even, is  $\Pi_k$ **P-hard**.

To establish the computational complexities of  $\langle k, \exists, \mathbf{B}^{k+1} \rangle$  when  $k$  is even, and  $\langle k, \forall, \mathbf{B}^{k+1} \rangle$  when  $k$  is odd, we first provide a reduction from Q3DNF to QLI.

#### 7.1 Reduction from Q3DNF to QLI

Consider a Q3DNF instance  $\Phi : Q(\mathbf{x}, \mathbf{y})\phi(\mathbf{x}, \mathbf{y})$ , where  $Q(\mathbf{x}, \mathbf{y})$  represents the quantifier string,  $\mathbf{x}$  is the set of existentially quantified variables, and  $\mathbf{y}$  is the set of universally quantified variables. Note that  $\phi(\mathbf{x}, \mathbf{y}) = \phi_1 \wedge \phi_2 \wedge \dots \wedge \phi_m$  where each  $\phi_i$  is a *disjunctive* clause. Without loss of generality, we can assume that the innermost quantifier of  $Q(\mathbf{x}, \mathbf{y})$  is  $\forall$ , since if the innermost quantifier is  $\exists$ , then this variable can be eliminated in polynomial time. This is because Q3DNF is the complement of Q3CNF and an innermost quantifier of  $\forall$  can be eliminated from a Q3CNF formula in polynomial time [22].

We will produce a QLI  $\mathbf{I} : Q'(\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}, \mathbf{w}) E \rightarrow F$ , such that the existential player has a winning strategy for  $\mathbf{I}$ , if and only if the existential player has a winning strategy for  $\Phi$ . Note that  $E$  represents the set of constraints on the LHS of the constructed implication, and  $F$  represents the set of constraints on the RHS of the implication.

For each existentially quantified variable  $x_i$  in  $\Phi$ , we add an existentially quantified variable  $x_i$  and a universally quantified variable  $r_i$  to  $\mathbf{I}$ . We also add the constraints  $r_i \leq x_i$  and  $r_i \leq 1 - x_i$  to  $E$ , and the constraint  $r_i \leq 0$  to  $F$ . Note that these constraints are equivalent to  $r_i \leq \min(x_i, 1 - x_i) \rightarrow r_i \leq 0$ . Finally, we add the constraint  $0 \leq x_i \leq 1$  to  $F$ .

For each universally quantified variable  $y_i$  in  $\Phi$ , we add an existentially quantified variable  $s_i$  and a universally quantified variable  $y_i$  to  $\mathbf{I}$ . We also add the constraints  $0 \leq y_i \leq 1$  to  $E$ , and the constraints  $0 \leq s_i \leq 1, 2 \cdot y_i - 1 \leq s_i$ , and  $s_i \leq 2 \cdot y_i$  to  $F$ . Note that these are the only constraints that use  $y_i$  variables, since the clause constraints use only  $x_i$  and  $s_i$  variables.

For each clause  $\phi_j$  in  $\phi(\mathbf{x}, \mathbf{y})$ , we add the existentially quantified variable  $w_j$  to  $\mathbf{I}$ , and 3 constraints to  $F$ . These constraints ensure that  $w_j$  is less than or equal to the existential variables corresponding to the literals in  $\phi_j$ . Note that these constraints contain only existential variables. Even in the case of a universal variable  $y_i$  in  $\phi_j$ , the constraint contains the existential variable  $s_i$  of the QLI.

For example, if  $\phi_j = (x_i, y_k, \bar{x}_l)$ , we add  $w_j \leq x_i, w_j \leq s_k$ , and  $w_j \leq 1 - x_l$  to  $F$ , while if  $\phi_j = (x_i, \bar{y}_k, \bar{x}_l)$ , we add  $w_j \leq x_i, w_j \leq 1 - s_k$ , and  $w_j \leq 1 - x_l$  to  $F$ .

Finally, we add the linear constraint  $w_1 + w_2 + \dots + w_m \geq 1$  to  $F$ .

We inductively create the quantifier string  $Q'(\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}, \mathbf{w})$  of  $\mathbf{I}$  based on  $Q(\mathbf{x}, \mathbf{y})$ :  $Q'(\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}, \mathbf{w})$  of  $\mathbf{I}$  is initialized to  $\epsilon$  (the empty string). For each  $i = 1, 2, \dots, n$ ,

1. If the  $i^{th}$  quantifier of  $Q(\mathbf{x}, \mathbf{y})$  is  $\exists x_i$ , then we append  $\exists x_i \forall r_i$  to the end of  $Q'(\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}, \mathbf{w})$ .
2. If the  $i^{th}$  quantifier of  $Q(\mathbf{x}, \mathbf{y})$  is  $\forall y_i$ , then we append  $\forall y_i \exists s_i$  to the end of  $Q'(\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}, \mathbf{w})$ .

Finally, we add  $\exists \mathbf{w}$  to the end of  $Q'(\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}, \mathbf{w})$ .

Since the final quantifier of  $Q(\mathbf{x}, \mathbf{y})$  is  $\forall$ , the number of quantifier alternations in  $Q'(\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}, \mathbf{w})$  is one more than the number of quantifier alternations in  $Q(\mathbf{x}, \mathbf{y})$ .

Note that the final quantifier of  $Q'(\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}, \mathbf{w})$  is  $\exists$ .

*Example 4* Let us consider the Q3DNF instance

$$\Phi = \exists x_1 \forall y_1 \exists x_2 \forall y_2 \underbrace{(x_1, \bar{y}_1, y_2)}_{\phi_1} \vee \underbrace{(\bar{x}_1, x_2, y_2)}_{\phi_2}.$$

Under the above construction,  $\Phi$  becomes an instance of  $\langle 4, \exists, \mathbf{BBBBR} \rangle$  with the quantifier string

$$\exists x_1 \forall r_1 \forall y_1 \exists s_1 \exists x_2 \forall r_2 \forall y_2 \exists s_2 \exists w_1 \exists w_2.$$

Note that this is also an instance of  $\langle 4, \exists, \mathbf{BBBBB} \rangle$  [14].

The LHS of the implication ( $E$ ) consists of the following set of constraints:

$$\begin{aligned} r_1 &\leq x_1, & r_1 &\leq 1 - x_1, \\ r_2 &\leq x_2, & r_2 &\leq 1 - x_2, \\ 0 &\leq y_1 \leq 1, & 0 &\leq y_2 \leq 1, \end{aligned}$$

The RHS of the implication ( $F$ ) consists of the following set of constraints:

$$\phi_1 \left\{ \begin{array}{ll} r_1 \leq 0, & 0 \leq x_1 \leq 1, \\ r_2 \leq 0, & 0 \leq x_2 \leq 1, \\ 0 \leq s_1 \leq 1, & 0 \leq s_2 \leq 1, \\ 2 \cdot y_1 - 1 \leq s_1, & s_1 \leq 2 \cdot y_1, \\ 2 \cdot y_2 - 1 \leq s_2, & s_2 \leq 2 \cdot y_2, \\ w_1 \leq x_1, & w_2 \leq 1 - x_1, \\ w_1 \leq 1 - s_1, & w_2 \leq x_2, \\ w_1 \leq s_2, & w_2 \leq s_2, \\ w_1 + w_2 \geq 1. \end{array} \right\} \phi_2$$

We now establish that all the variables in  $\mathbf{I}$  can be restricted to  $\{0, 1\}$  without affecting its feasibility. To do so, we utilize the game semantics discussed in Appendix B.

**Lemma 1** *The existential player  $\mathbf{X}$  has a winning strategy, if and only if he has a winning strategy, when each  $x_i$  is chosen from  $\{0, 1\}$ .*

*Proof* The *if* part is obvious.

Accordingly, we focus on proving the *only if* part. Assume that the existential player  $\mathbf{X}$  has a winning strategy. First, observe that if  $x_i \notin [0, 1]$ , then at least one of the constraints in  $F$  is violated. In particular, the constraint  $0 \leq x_i \leq 1$  is violated. Now focus on the constraints in  $E$ . Since each  $r_i$  is chosen by the universal player  $\mathbf{Y}$  after the existential player chooses  $x_i$ , these constraints can only be violated by  $\mathbf{Y}$ . In other words  $\mathbf{Y}$  can always satisfy the RHS of the implication. Thus, the implication does not hold and  $\mathbf{X}$  does not have a winning strategy.

We now consider the case  $x_i \in (0, 1)$ . In this case,  $\mathbf{Y}$  could choose  $r_i = \min(x_i, 1 - x_i) > 0$ . Note that both  $x_i$  and  $r_i$  are positive with  $r_i \leq x_i$  and  $r_i \leq 1 - x_i$ . Thus the universal player  $\mathbf{Y}$  wins the game, since the constraints involving  $x_i$  and  $r_i$  in  $E$ , viz.,  $r_i \leq x_i$  and  $r_i \leq 1 - x_i$  are satisfied and the constraint  $r_i \leq 0$  in  $F$  is violated.

To sum up, any choice of  $x_i \notin \{0, 1\}$  would cause  $\mathbf{X}$  to lose the game.

It follows that if  $\mathbf{X}$  has a winning strategy, then he has a winning strategy when each  $x_i$  is chosen from  $\{0, 1\}$ . □

**Lemma 2** *The universal player  $\mathbf{Y}$  has a winning strategy, if and only if he has a winning strategy, when each  $y_i$  is chosen from  $\{0, 1\}$ .*

*Proof* The *if* part is obvious.

Accordingly, we focus on proving the *only-if* part. It is clear that in order to win, the universal player  $\mathbf{Y}$  must choose  $y_i \in [0, 1]$ . Otherwise, the constraint  $0 \leq y_i \leq 1$  in the LHS  $E$  of the implication is violated and the existential player  $\mathbf{X}$  wins the game.

Suppose now that  $\mathbf{Y}$  can win by choosing  $y_i \in (0, 1)$ . As per the construction of Section 7.1, the only constraints involving  $s_i$  and  $y_i$  are given by System (21).

$$0 \leq s_i \leq 1, 2 \cdot y_i - 1 \leq s_i, s_i \leq 2 \cdot y_i \tag{21}$$

If  $y_i \in (0, \frac{1}{2}]$ , then from System (21) it follows that  $s_i$  can take any value in  $[0, 2 \cdot y_i]$ . However, if instead  $\mathbf{Y}$  had chosen  $y_i = 0$ , then  $s_i$  is forced to equal 0. It follows that if  $\mathbf{Y}$  can win by choosing  $y_i \in (0, \frac{1}{2}]$ , then  $\mathbf{Y}$  can win by choosing  $y_i = 0$ . Similarly, if  $\mathbf{Y}$  can win by choosing  $y_i \in [\frac{1}{2}, 1)$ , then  $\mathbf{Y}$  can win by choosing  $y_i = 1$ .

To sum up, we can safely assume that the universal player only chooses  $y_i \in \{0, 1\}$ .

It follows that if  $\mathbf{Y}$  has a winning strategy, then he has a winning strategy when each  $y_i$  is chosen from  $\{0, 1\}$ . □

The import of Lemma 1 and Lemma 2 is that we can confine our analyses to games in which the existential player  $\mathbf{X}$  and universal player  $\mathbf{Y}$  make moves in  $\{0, 1\}$  for the  $x_i$  and  $y_i$  variables respectively.

An interesting observation is that if  $y_i$  is restricted to  $\{0, 1\}$ , then so is  $s_i$ . The constraints involving  $s_i$  and  $y_i$  in the RHS  $F$  of the implication are described by System (21). If  $y_i = 0$ , the constraints in System (21) force  $s_i$  to be 0; likewise, if  $y_i = 1$ , they force  $s_i$  to be 1. In other words,  $\mathbf{X}$  is forced to set  $s_i = y_i$ . Any other choice of  $s_i$  would violate at least one

constraint of  $F$ , causing  $\mathbf{X}$  to lose the game. Hence, the  $s_i$  variables are also restricted in the set  $\{0, 1\}$ .

**Theorem 5**  $\mathbf{I}$  is feasible if and only if  $\Phi$  is feasible.

*Proof* We show that the existential player  $\mathbf{X}$  has a winning strategy for  $\mathbf{I}$  if and only if he has a winning strategy for  $\Phi$ .

*Only if part:* Assume that the existential player  $\mathbf{X}$  has a winning strategy for  $\mathbf{I}$ . Consider a play of the game, in which  $\mathbf{X}$  chooses values for the existentially quantified variables and  $\mathbf{Y}$  chooses values for the universally quantified variables. After  $\mathbf{X}$  chooses a value for  $x_i$ ,  $\mathbf{Y}$  can choose  $r_i$ , such that the constraints  $r_i \leq x_i$  and  $r_i \leq 1 - x_i$  are both satisfied. By our 2-person game semantics, the universal player  $\mathbf{Y}$  will ensure that all the constraints in the LHS are satisfied. In order for  $\mathbf{X}$  to win the game, he has to ensure that all the constraints in the RHS are satisfied as well. In particular, the constraint  $w_1 + w_2 + \dots + w_m \geq 1$  in the RHS of the implication must be satisfied. Consequently,  $w_j > 0$ , for at least one  $j$ . This  $w_j$  corresponds to the clause  $\phi_j$  of  $\Phi$ . Assume that  $\phi_j$  has the form  $(x_i, y_k, x_l)$ . Since the  $x_i$ s and  $s_i$ s are restricted to the set  $\{0, 1\}$  (see Lemma 1 and Lemma 2), the constraints  $w_j \leq x_i$ ,  $w_j \leq s_k$ , and  $w_j \leq x_l$  force each variable  $(x_i, s_k$  and  $x_l)$  to be 1. It follows that  $\phi_j$  is satisfied in this play. Similar arguments can be made for other forms of  $\phi_j$ . Since the play was chosen arbitrarily, the same argument applies for all plays, i.e., at least one clause of  $\phi(\mathbf{x}, \mathbf{y})$  is satisfied in every play. Hence,  $\Phi$  is feasible.

*If part:* Assume that the existential player  $\mathbf{X}$  has a winning strategy for  $\Phi$ .

At the end of any play,  $\phi(\mathbf{x}, \mathbf{y})$  is satisfied. Thus, at least one clause, say  $\phi_j$ , must be satisfied. Assume that  $\phi_j$  is of the form  $(x_i, y_k, x_l)$ . Consider the constraints constructed from  $\phi_j$ , viz.,  $w_j \leq x_i$ ,  $w_j \leq s_k$ , and  $w_j \leq x_l$ . Since  $\phi_j$  is satisfied,  $x_i, y_k, x_l$  are all true. Assume that  $\mathbf{X}$  sets the variables  $x_i, s_k, x_l$  and  $w_j$  to 1 and the  $w$  variables associated with other clauses to 0. It is clear that the constraints corresponding to the other clauses are trivially satisfied. Likewise, the aggregate constraint  $w_1 + w_2 + \dots + w_m \geq 1$  is also satisfied. Similar arguments can be made for other forms of  $\phi_j$ .

From Lemma 1 and Lemma 2 and the subsequent discussion,  $x_i$  and  $y_i$  can be restricted to the set  $\{0, 1\}$ . Furthermore, the existential player must choose  $s_i = y_i$ . Observe that for each  $i$ ,  $r_i \leq x_i$  and  $r_i \leq 1 - x_i$  imply that  $r_i \leq 0$  (thus satisfying the corresponding constraint in  $F$ ). Similarly, for any  $i$ , the constraints  $0 \leq x_i \leq 1, 0 \leq y_i \leq 1, 0 \leq s_i \leq 1, 2 \cdot y_i - 1 \leq s_i$ , and  $s_i \leq 2 \cdot y_i$  are all satisfied, under this assignment. Hence,  $E \rightarrow F$  holds. Since the play was chosen arbitrarily, the same argument applies for all plays, i.e.,  $\mathbf{X}$  has a winning strategy for  $\mathbf{I}$  and  $\mathbf{I}$  is feasible. □

Theorem (5) allows us to obtain the following two results.

**Corollary 3**  $\langle k, \exists, \mathbf{B}^{k+1} \rangle$  with  $k$  even, is  $\Sigma_k\mathbf{P}$ -hard.

*Proof* Let  $\Phi$  denote a Q3DNF formula having  $k$  ( $k$  even) quantifiers ( $(k - 1)$  quantifier alternations) starting with  $\exists$ , i.e., with a quantifier string of the form  $\exists\forall \dots \exists\forall$ . Deciding the feasibility of  $\Phi$  is  $\Sigma_k\mathbf{P}$ -complete. This is because this problem is the complement of the problem  $\forall\exists \dots \forall\exists \phi(\mathbf{x}, \mathbf{y})$ , where  $\phi(\mathbf{x}, \mathbf{y})$  is a 3CNF formula, which is  $\Sigma_k\mathbf{P}$ -complete [22] (the assumption that  $k$  is even is essential). By Theorem 5, we can reduce  $\Phi$  to a QLI,  $\mathbf{I}$ . Note that the quantifier string of  $\mathbf{I}$  has  $k$  quantifier alternations, and that the first and last

quantifiers are  $\exists$ . Thus,  $\mathbf{I}$  is a  $\langle k, \exists, \mathbf{B}^k \mathbf{R} \rangle$  QLI. This can be trivially reduced to a  $\langle k, \exists, \mathbf{B}^{k+1} \rangle$  QLI. Hence, the result follows.  $\square$

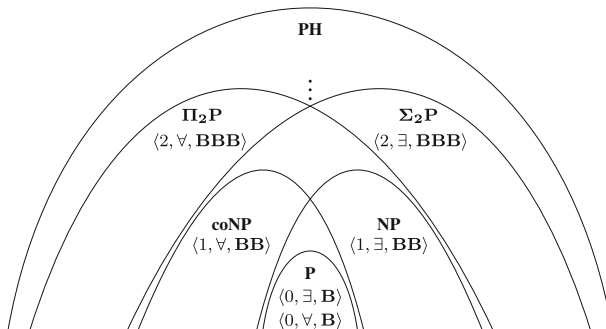
**Corollary 4**  $\langle k, \forall, \mathbf{B}^{k+1} \rangle$  with  $k$  odd, is  $\Pi_k \mathbf{P}$ -hard.

*Proof* Let  $\Phi$  denote a Q3DNF formula having  $k$  ( $k$  odd) quantifiers ( $(k - 1)$  quantifier alternations) starting with  $\forall$ , i.e., with a quantifier string of the form  $\forall \exists \dots \exists \forall$ . Deciding the feasibility  $\Phi$  belongs to is  $\Pi_k \mathbf{P}$ -complete. This is because this problem is the complement of  $\exists \forall \dots \exists \phi(x, y)$ , where  $\phi(x, y)$  is a 3CNF formula, which is  $\Sigma_k \mathbf{P}$ -complete [22] (the assumption that  $k$  is odd is essential). By Theorem 5, we can reduce  $\Phi$  to a QLI,  $\mathbf{I}$ . Note that the quantifier string of  $\mathbf{I}$  has  $k$  quantifier alternations, and that the last quantifiers is  $\exists$ . Thus,  $\mathbf{I}$  is a  $\langle k, \forall, \mathbf{B}^k \mathbf{R} \rangle$  QLI. This can be trivially reduced to a  $\langle k, \forall, \mathbf{B}^{k+1} \rangle$  QLI. Hence, the result follows.  $\square$

**Theorem 6**  $\langle k, \exists, \mathbf{B}^{k+1} \rangle$  is  $\Sigma_k \mathbf{P}$ -complete,  $\langle k, \forall, \mathbf{B}^{k+1} \rangle$  is  $\Pi_k \mathbf{P}$ -complete.

*Proof* Given Theorems 3-4 and Corollaries 3-4, it suffices to show that each of these problems is also contained within the corresponding complexity class of the polynomial hierarchy. Let  $\mathbf{L}$  denote the problem of deciding an arbitrary boolean combination of linear constraints under a quantifier string with a limited number of alternations. Sontag [31] showed that problem  $\mathbf{L}$  can be solved by an alternating algorithm in which the guesses made by both the  $\forall$  player and the  $\exists$  player are rational and at most polynomial in the size of the input. QLIs with a finite number of quantifier alternations are clearly sub-problems of  $\mathbf{L}$ , since they can be rewritten as a quantified disjunction of the RHS constraints and the negation of the LHS constraints. It follows that the feasibility of QLIs with a finite number of alternations is preserved, even if every variable is restricted to values that are polynomially-sized with respect to the input. Consider the problem  $\langle k, \exists, \mathbf{B}^{k+1} \rangle$ . After  $k$  rounds in which the  $\exists$  player and  $\forall$  player alternate and guess polynomially-sized values, the QLI reduces to either  $\langle 0, \exists, \mathbf{B} \rangle$  or  $\langle 0, \forall, \mathbf{B} \rangle$  (depending on whether  $k$  is even or odd respectively), both of which are in  $\mathbf{P}$  [14]. Thus,  $\langle k, \exists, \mathbf{B}^{k+1} \rangle$  is in  $\Sigma_k \mathbf{P}$ . Likewise,  $\langle k, \forall, \mathbf{B}^{k+1} \rangle$  is in  $\Pi_k \mathbf{P}$ .  $\square$

For example, QLIs of the form  $\langle 3, \forall, \mathbf{BBBB} \rangle$  are in  $\Pi_3 \mathbf{P}$ .



**Fig. 2** QLI and the Polynomial Hierarchy

The various forms of QLI cover the polynomial hierarchy as shown in Fig. 2. We make the following observations on the basis of the theorems derived above:

- (i) In case of QLPs, if the innermost variable is universally quantified, it can be removed using quantifier elimination techniques, in polynomial time. In case of QLIs, the complexity class to which the problem belongs, does in fact depend upon whether the innermost variable is existentially or universally quantified (see Fig. 2).
- (ii) There exists a class of QLI that is complete for each class in the PH. This is not true, if the underlying formula is in CNF form. For instance, there is no QCNF formula, which is complete for the class **coNP**.

### 8 Other complexity results

In this section, we establish the computational complexities of various classes of QLI with one quantifier alternation that were left open in [14].

**Lemma 3**  $\langle 1, \forall, \mathbf{RB} \rangle$ , i.e., deciding whether  $\forall \mathbf{x} \exists \mathbf{y} [\mathbf{M} \cdot \mathbf{x} \leq \mathbf{n} \rightarrow \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}]$  holds, is in **P**.

*Proof* We will focus on the following two cases:

1. There exists a point  $\mathbf{z}$ , such that  $\mathbf{M} \cdot \mathbf{z} \not\leq \mathbf{n}$  - It follows that  $\forall \mathbf{x} (\mathbf{M} \cdot \mathbf{x} \leq \mathbf{n})$  does not hold. In this case, the QLI is trivially satisfied, since the existential player can guess  $\mathbf{z}$  and cause the left hand side of the implication to be falsified.
2. There is no point  $\mathbf{z}$  such that  $\mathbf{M} \cdot \mathbf{z} \not\leq \mathbf{n}$  - In this case, the QLI reduces to  $\forall \mathbf{y} \exists \mathbf{x} \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$ , which is a UQLP and hence can be decided in polynomial time by Theorem 2.

□

The decision problem for formula  $\exists \mathbf{x} \forall \mathbf{y} [\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} \rightarrow \mathbf{M} \cdot \mathbf{y} \leq \mathbf{n}]$  is **NP-complete** [14]. However, if there are no universally quantified variables on the RHS, the problem becomes tractable.

**Lemma 4**  $\langle 1, \exists, \mathbf{BL} \rangle$ , i.e., deciding whether  $\exists \mathbf{x} \forall \mathbf{y} [\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} \rightarrow \mathbf{M} \cdot \mathbf{x} \leq \mathbf{n}]$  holds, is in **P**.

*Proof* First, we check whether  $\exists \mathbf{x} \mathbf{M} \cdot \mathbf{x} \leq \mathbf{n}$  holds, which can be done in polynomial time [20]. If  $\exists \mathbf{x} \mathbf{M} \cdot \mathbf{x} \leq \mathbf{n}$  holds, then  $\exists \mathbf{x} \forall \mathbf{y} [\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} \rightarrow \mathbf{M} \cdot \mathbf{x} \leq \mathbf{n}]$  trivially holds. If  $\exists \mathbf{x} \mathbf{M} \cdot \mathbf{x} \leq \mathbf{n}$  does not hold, then the only way in which  $\exists \mathbf{x} \forall \mathbf{y} [\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c} \rightarrow \mathbf{M} \cdot \mathbf{x} \leq \mathbf{n}]$  can hold is if  $\exists \mathbf{x} \forall \mathbf{y} \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$  does not hold. However, the latter formula is a UQLP and hence it can be checked to hold in polynomial time by Theorem 2.

□

Let us turn our attention to the class  $\langle 1, \forall, \mathbf{BR} \rangle$  and its super-class  $\langle 1, \forall, \mathbf{BB} \rangle$ . Both these classes were shown to be **coNP-hard** in [14]. Note that these classes are also in **coNP** by Theorem 6.

Lemma 3 and Lemma 4 settle some open problems on the computational complexities of QLIs with one quantifier alternation. A complete representation of all classes with one quantifier alternation QLIs is given in Fig. 3 (starting with an existential quantifier) and Fig. 4 (starting with a universal quantifier).



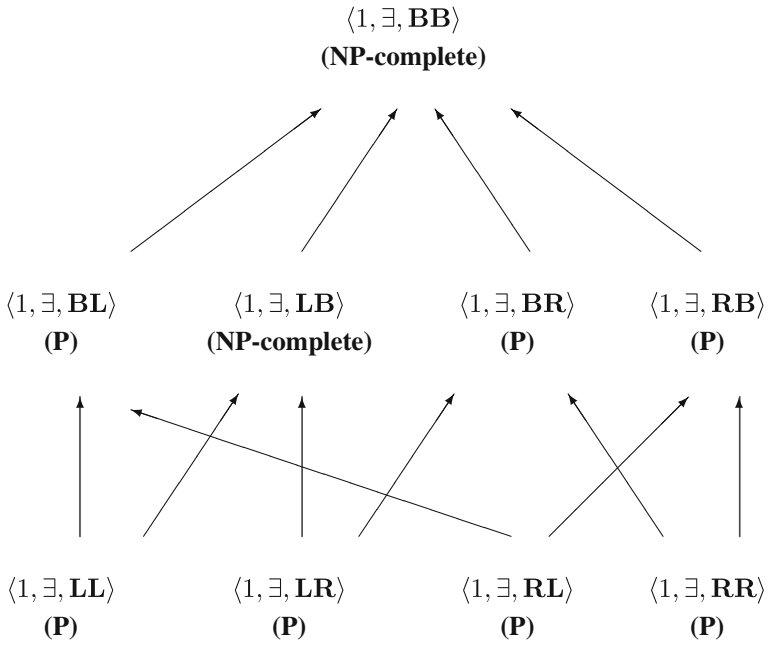


Fig. 3 Complexity of  $\exists\forall$  classes of QLI. Arrows denote inclusions

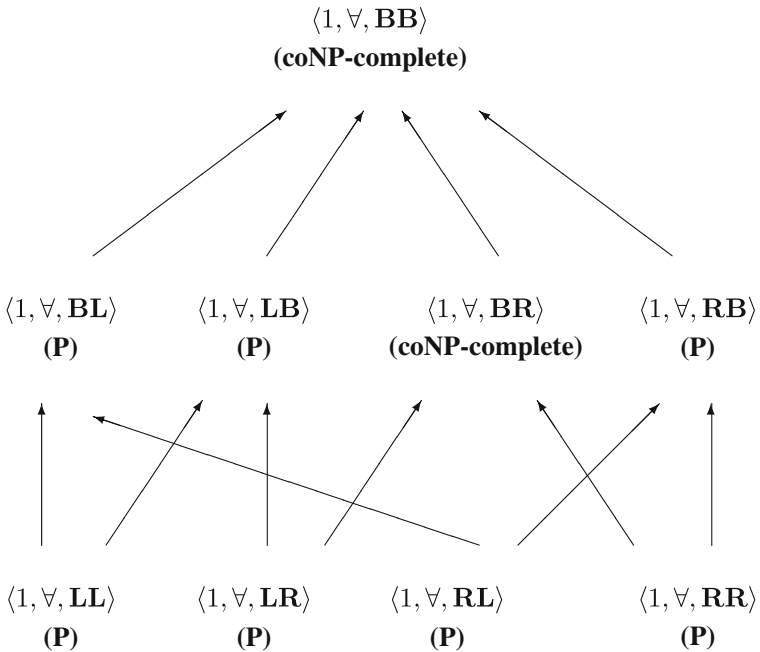


Fig. 4 Complexity of  $\forall\exists$  classes of QLI. Arrows denote inclusions

## 9 Conclusion

In this paper, we established the computational complexities of various fragments of the theory of linear arithmetic. More specifically, we examined several variants of QLP and QLI that arise when the universally quantified variables are partially bounded or unbounded and showed that all these variants are in  $\mathbf{P}$ . Furthermore, for QLIs, we proved that  $\langle k, \exists, \mathbf{B}^{k+1} \rangle$  is  $\Sigma_k \mathbf{P}$ -complete and  $\langle k, \forall, \mathbf{B}^{k+1} \rangle$  is  $\Pi_k \mathbf{P}$ -complete. Hence, we showed that any class of  $\mathbf{PH}$  can be represented by some instantiation to the QLI framework, i.e., using only continuous variables, thus providing a continuous analogue to the way QBFs cover the  $\mathbf{PH}$ . In particular, our work mirrors the work in [32], which established the connection between the classes of  $\mathbf{PH}$  and the quantifier alternations in Quantified Boolean Formulas. Our results are surprising, since the variables in the fragments that we consider are continuous, as opposed to discrete. Moreover, we answered several open questions on the computational complexities of classes of QLI with one quantifier alternation, thus completing the map of the computational complexities of all such classes of QLI (see Figs. 3 and 4).

From our perspective, the following problems are worth pursuing:

1. What is the complexity of Quantified Linear Programming? As discussed previously, we have shown in previous research that the problem is in  $\mathbf{PSPACE}$  and  $\mathbf{coNP}$ -hard. Note that our results are similar to the results obtained in [6] for Quantified Horn formulas.
2. What is the complexity of Quantified Linear Implication? It is known that the decidability of generic QLIs with unbounded alternations is  $\mathbf{PSPACE}$ -hard [14]. However, completeness is not yet shown. Note that the known decision methods for TLA with an arbitrary number of quantifier alternations run in double exponential time [15, 31].
3. In deriving our complexity results, we mostly instantiated the  $T$  parameter in  $\langle A, Q, T \rangle$  to  $\mathbf{B}$ . It remains to be seen if and how the complexity results change when  $T$  is assigned  $\mathbf{L}$  or  $\mathbf{R}$ . At this point, it is important to note that the assignments  $\mathbf{L}$  and  $\mathbf{R}$  are not symmetric from the perspective of computational complexity [14].

### Compliance with Ethical Standards

**Conflict of Interest** The authors declare that they have no conflict of interest.

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## Appendix

### A Axioms

We now provide the axioms of each theory mentioned in the paper.

## A.1 Theory of linear arithmetic

The Theory of Linear Arithmetic, also known as the Theory of Rationals, is a set of first order axioms which define the properties of rational numbers.

The signature of this theory is:

$$\Sigma_{\mathbb{Q}} = \{0, 1, +, -, =, \geq\}.$$

The axioms of this theory are as follows:

1.  $\forall x \forall y \ x \geq y \wedge y \geq x \rightarrow x = y.$
2.  $\forall x \forall y \forall z \ x \geq y \wedge y \geq z \rightarrow x \geq z.$
3.  $\forall x \forall y \ x \geq y \vee y \geq x.$
4.  $\forall x \forall y \forall z \ (x + y) + z = x + (y + z).$
5.  $\forall x \ x + 0 = x.$
6.  $\forall x \ x + (-x) = 0.$
7.  $\forall x \forall y \ x + y = y + x.$
8.  $\forall x \forall y \forall z \ x \geq y \rightarrow x + z \geq y + z.$
9.  $\forall x \ \forall n \in \mathbb{Z}^+ \ n \cdot x = 0 \rightarrow x = 0.$
10.  $\forall x \ \forall n \in \mathbb{Z}^+ \ \exists y \ x = n \cdot y.$

## A.2 Theory of real closed fields

The Theory of Real Closed Fields is a set of axioms which define the properties of real numbers.

The signature of this theory is:

$$\Sigma_{\mathbb{R}} = \{0, 1, +, -, \cdot, =, \geq\}.$$

The first order axioms of this theory are as follows:

1.  $\forall x \forall y \ x \geq y \wedge y \geq x \rightarrow x = y.$
2.  $\forall x \forall y \forall z \ x \geq y \wedge y \geq z \rightarrow x \geq z.$
3.  $\forall x \forall y \ x \geq y \vee y \geq x.$
4.  $\forall x \forall y \forall z \ (x + y) + z = x + (y + z).$
5.  $\forall x \ x + 0 = x.$
6.  $\forall x \ x + (-x) = 0.$
7.  $\forall x \forall y \ x + y = y + x.$
8.  $\forall x \forall y \forall z \ x \geq y \rightarrow x + z \geq y + z.$
9.  $\forall x \forall y \forall z \ (x \cdot y) \cdot z = x \cdot (y \cdot z).$
10.  $\forall x \ x \cdot 1 = x.$
11.  $\forall x \ x \neq 0 \rightarrow \exists y \ x \cdot y = 1.$
12.  $\forall x \forall y \ x \cdot y = y \cdot x.$
13.  $\forall x \forall y \ x \geq 0 \wedge y \geq 0 \rightarrow x \cdot y \geq 0.$
14.  $\forall x \forall y \forall z \ x \cdot (y + z) = x \cdot y + x \cdot z.$
15.  $0 \neq 1.$
16.  $\forall x \ \exists y \ x = y^2 \vee -x = y^2.$
17.  $\forall x \ \forall n \in \mathbb{Z}^+ \ \exists y \ n \bmod 2 \equiv 1 \rightarrow y^n + x_1 \cdot y^{n-1} + \dots + x_{n-1} \cdot y + x_n = 0.$

However these fail to fully describe the real numbers. To do this, we need the following second order axiom:

$$\forall S \subseteq \mathbb{R} \exists x_1 \exists y \forall x_2 \in S \forall x_3 > y \exists x_4 \in S \ x_1 \leq x_2 \rightarrow (y \leq x_2 \wedge x_4 < x_3).$$

This axiom represents the Dedekind completeness of real numbers. Namely that every set  $S$  that has a lower bound  $x_1$  has a greatest lower bound  $y$ .

## B Game semantics

We provide 2-person game semantics for QLP, QLI, and Q3DNF.

### B.1 QLP semantics

We interpret QLP decidability as a 2-person game. Such a game includes an existential player  $\mathbf{X}$ , who chooses values for the existentially quantified variables, and a universal player  $\mathbf{Y}$ , who chooses values for the universally quantified variables. Our analysis focuses on QLPs in general form [33], but also holds for the partially bounded and unbounded variants, discussed in this paper.

Consider the generic form of QLP (i.e., QLP (3)) and assume, without loss of generality, that  $\mathbf{x}_1$  and  $\mathbf{y}_n$  are not empty (dummy variables can be added, if necessary). The *initial board configuration* of the game is:

$$\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \tag{22}$$

The game is played in a sequence of  $2 \cdot n$  rounds. Let  $i = 1, \dots, n$ . In round  $(2 \cdot i - 1)$ ,  $\mathbf{X}$  makes his  $i^{th}$  move (by choosing values for the variables in the set  $\mathbf{x}_i$ ). Then,  $\mathbf{Y}$  makes his  $i^{th}$  move (by choosing values for the variables in the set  $\mathbf{y}_i$ ) in round  $2 \cdot i$ . Hence,  $\mathbf{X}$  and  $\mathbf{Y}$  make their moves by selecting values for their respective variable sets. The moves are strictly alternating:  $\mathbf{X}$  makes his  $i^{th}$  move, which is followed by  $\mathbf{Y}$ 's  $i^{th}$  move, after which  $\mathbf{X}$  makes his  $(i + 1)^{th}$  move and so on. After all the moves have been made in the order specified by the quantifier string, if  $\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$  holds, we say that  $\mathbf{X}$  wins the game; otherwise, we say that  $\mathbf{Y}$  wins the game.

### B.2 QLI semantics

We now interpret a QLI as a 2-person game. As with the QLP semantics discussed above, such a game includes an existential player  $\mathbf{X}$ , who chooses values for the existentially quantified variables, and a universal player  $\mathbf{Y}$ , who chooses values for the universally quantified variables. Again, our analysis focuses on QLIs of generic form but also holds for the partially bounded and unbounded variants.

Consider the generic form of QLI (i.e., QLI (7)), and assume, without loss of generality, that  $\mathbf{x}_1$  and  $\mathbf{y}_n$  are not empty (dummy variables can be added, if necessary). The *initial board configuration* of the game is:

$$[\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}] \tag{23}$$

The game is played in a sequence of  $2 \cdot n$  rounds. Let  $i = 1, \dots, n$ . In round  $(2 \cdot i - 1)$ ,  $\mathbf{X}$  makes his  $i^{th}$  move (by choosing values for the variables in the set  $\mathbf{x}_i$ ). Then,  $\mathbf{Y}$  makes his  $i^{th}$  move (by choosing values for the variables in the set  $\mathbf{y}_i$ ) in round  $2 \cdot i$ . Hence,  $\mathbf{X}$  and

**Y** make their moves by selecting values for their respective variable sets. The moves are strictly alternating: **X** makes his  $i^{\text{th}}$  move, which is followed by **Y**'s  $i^{\text{th}}$  move, after which **X** will make  $(i + 1)^{\text{th}}$  move and so on. Assume that all the moves have been made by both players in the order specified by the quantifier string. Let  $\mathbf{u} = \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y}$  and let  $\mathbf{v} = \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y}$ . If  $\mathbf{u} \leq \mathbf{b}$  and  $\mathbf{v} \not\leq \mathbf{d}$ , we say that **Y** wins the game. Otherwise, **X** wins the game.

### B.3 Q3DNF semantics

We now discuss game semantics for Q3DNF. Again, the existential and the universal player make moves according to the quantifier string. The existential player wins a (disjunctive) clause, if all the literals in the clause are set to **true**. Conversely the universal player wins the clause, if at least one literal is set to **false**. The existential player wins the game, if he wins at least one clause in the disjunction, thus causing the expression to evaluate to **true**. The universal player wins the game, if he wins every term in the disjunction, thus causing the expression to evaluate to **false**. This can be considered as a reversal of the players' objectives in an instance of Q3SAT, where the existential player needs to win *all* the conjunctive clauses, and the universal player only needs to win a single clause. This occurs because the underlying boolean formula of a Q3DNF instance is in DNF, while the underlying boolean formula of a Q3SAT instance is in CNF.

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