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Notes on the existence of a solution in the pairwise comparisons method using the heuristic rating estimation approach

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Abstract Pairwise comparisons (PC) is a well-known method for modeling the subjective preferences of a decision maker. The method is very often used in the models of voting systems, social choice theory, decision techniques (such as *AHP - Analytic Hierarchy Process*) or multi-agent AI systems. In this approach, a set of paired comparisons is transformed into one overall ranking of alternatives. Very often, only the results of individual comparisons are given, whilst the weights (indicators of significance) of the alternatives need to be computed. According to Heuristic Rating Estimation *(HRE),* the new approach discussed in the article, besides the results of comparisons, the weights of some alternatives can also be a priori known. Although *HRE* uses a similar method to the popular *AHP* technique to compute the weights of individual alternatives, the solution obtained is not always positive and real. This article tries to answer the question of when such a correct solution exists. Hence, the sufficient condition for the existence of a positive and real solution in the *HRE* approach is formulated and proven. The influence of inconsistency in the paired comparisons set for the existence of a solution is also discussed.

Keywords Decision support systems · Pairwise comparisons · AHP · Heuristic rating estimation · Data inconsistency

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1 Introduction

The ability to compare things is one of the most useful human skills [8]. When comparing the quality of products in a grocery store, the taste of foods in a restaurant or the fuel prices at a gas station, people are able to make the best choices. The problem starts when the list of possible options is too long and none of the options is clearly better than others. Probably anyone who has visited an electronics store felt slightly uncomfortable looking at several TVs, at the same time wondering which one is the best. In such a case, the pairwise comparisons *(PC)* method [23] may help. According to the method, instead of comparing all the alternatives (hereinafter referred to as concepts) at once, it is better to compare them in pairs. Then, knowing the results of each paired comparison, the final priority values for the considered concept can be calculated. The process of aggregating the results of individual comparisons into the common ranking list of compared concepts hereinafter will be referred to as the priority deriving procedure.

The first written evidence of the use of paired comparisons dates back to the thirteenth century and *RamonLlull's* binary election systems [11]. According to *Llull*, during several voting rounds, every set of two candidates is compared in a pair and the winner is the one who wins by a majority in the greatest number of binary[1] paired comparisons. The method was later repeatedly reinvented. *Condorcet* [12] and *Borda* [14] proposed it in the second half of the eighteen century in their voting systems. *Thurstone* uses the generalized[2] pairwise comparisons in experimental psychology [44]. *Llull's* basic system was then reinvented (with some minor modifications) by *Copeland* in the context of welfare economics [11, 13]. The pairwise comparisons method is a cornerstone of *AHP (Analytic Hierarchy Process)* - a multi-criteria decision technique [38]. According to *AHP*, each concept is compared with each other with respect to different criteria. In this way, each concept receives some ranking value associated with the given criterion. Then, the criteria are compared with each other. Hence, every criterion also gets a priority value. The final priority value assigned to the concept is the weighted sum of its criterion's specific priorities multiplied by priorities of the criteria [40].

Comparing alternatives in pairs is also widely used in other than *AHP* multi-criteria decision making methods such as *ELECTRE* [16, 19], *PROMETHEE* [7] or *MACBETH* [2]. In this study, however, pairwise comparisons have two equally important meanings: ordinal and cardinal. This makes the presented approach similar to *AHP* rather than *ELECTRE* or *PROMETHEE*. Thus, in this approach, unlike in some models known from the social choice theory [34, 43] the result of the comparison is a real number representing the relative value (strength) of the preference. In this sense the *PC* method as proposed by *Thurstone* [44], and then developed by *Saaty* [38] (hereinafter referred to as the *PC* method) seems to be closer to the generalized *Arrow's* model proposed by *Sen* [34, 42] than the earlier works. On the other hand in the *PC* method an expert (or a group of experts) is obligated to provide a matrix containing the results of the comparisons of any two alternatives. This makes it similar to the paired-comparisons voting rules such as the *Kemeny-Young* method or *Simpson-Kramer Min-Max* rule [33]. Although the *PC* method usually does not appear in the debate on the social choice theory it can be useful in this context [41].

[1] The result of a single paired comparison was binary: 0 or 1. Each element of the pair could be either a winner or a loser

[2] In contrast to the binary paired comparisons, the result of a generalized paired comparison was a number determining the ratio between the relative intensity of two stimuli

Despite their long history, paired comparisons are still an inspiration and a challenge to researchers. Examples of their exploration are the approaches based on using *rough sets* [22], fuzzy PC relation [35], incomplete *PC* relation [5, 18, 26], reduction of data inconsistency [28], non-numerical rankings [24], the social choice theory [33], additive *PC* [27] and weight effectiveness [3, 4].

A recent contribution to the *PC* method includes the *Heuristic Rating Estimation (HRE)* approach [30, 31] that allows the user to explicitly define a reference set of concepts, for which the utilities (the ranking values) are known a priori. In *HRE*, the relative value of a single non-reference concept is determined as the weighted average of all the other concepts. Such a proposition leads to the formulation of a linear equation system whose solution, a vector of weights, determines the desired ordering of concepts. The vector need to be strictly positive and real. The presented article is the first one which tries to provide the answer to the question when this vector is positive and real. The resulting outcome (Section 4) is an intuitive and easy to check criterion ensuring the existence of an admissible solution. Although the presented criterion is sufficient (but not necessary), it may be useful for a wide class of problems for which the reference concepts are roughly a bit more than half of all the objects (see Section 4, Remark 3).

Basic information about the *PC* method, the *M-matrix* theory and the *HRE* method can be found in Sections 1, 2.3 and 3 correspondingly. The main results of the work including an existence condition (Theorem 2) and three additional Remarks on its properties are presented in Section 4. A brief summary is provided in Section 5.

2 Preliminaries

2.1 The pairwise comparisons method

The pairwise comparisons method is very often used as a technique that allows an expert (or a group of experts) to synthesize individual pairwise judgments into one, common ranking. The subject of the rankings can be any tangible or intangible entities (anything that experts can assess), hereinafter referred to as concepts or alternatives. Let $C \overset{df}{=} \{c_1, \ldots, c_n\}$ be a finite set of concepts to be judged and/or analyzed, and $\{m_{1,1}, \ldots, m_{1,n}, m_{2,1}, \ldots, m_{2,n}, m_{3,1}, \ldots, m_{n,n}\}$ be the set of expert judgments about each pair of concepts $c_i, c_j \in C$. The judgments (preferences) of experts are represented in the form of real, positive numbers. Thus, assigning a particular value v to m_{ij}, expresses an opinion[3] that c_i is v times more important than c_j. A set of judgments can be conveniently represented as a *PC* matrix $M = (m_{ij})$. Because a comparison of a given concept to itself may not indicate a predominance of any of the two compared elements (since they are identical), the diagonal of M contains all ones.

Let us define the function that assigns the value of importance (also referred to as priority, preference or rank) to every $c \in C$.

Definition 1 The ranking function for C (the ranking of C) is a function $\mu : C \to \mathbb{R}_+$ that assigns a positive value from $\mathbb{R}_+$ to every compared concept.

[3] Sometimes, to help experts to express their verbal opinions in the form of numbers, different measurement scales are used. For example, in *AHP* the judgment values must lie between $1/9$ and 9, and each of the values $1/9, 1/8, \ldots, 8, 9$ has its own well-defined textual representation [38]

The μ function is usually defined as a vector of weights:

$$\mu \stackrel{df}{=} [\mu(c_1), \ldots, \mu(c_n)]^T \tag{1}$$

The values m_{ij} and m_{ji} represent subjective expert judgments as to the relative importance, utility or quality indicators of the concepts c_i and c_j. Thus, according to the best knowledge of experts, it should hold that $\mu(c_i) = m_{ij}\mu(c_j)$. This observation allows us to define the two properties of the matrix M: *reciprocity* and *consistency*.

Definition 2 A matrix M is said to be reciprocal if for every i, j such that $1 \leq i, j \leq n$ it holds that $m_{ij} = 1/m_{ji}$, and M is said to be consistent if for every i, j, k where $1 \leq i, j, k \leq n$ it holds that $m_{ij} \cdot m_{jk} \cdot m_{ki} = 1$.

Although the matrix M may be neither reciprocal not consistent [21], still, in most cases is it assumed that reciprocity is satisfied. Unfortunately, usually M is not consistent. Since the data in the *PC* matrix represents the subjective opinions of the experts, they might be inconsistent. Hence, there may be a triad m_{ij}, m_{jk}, m_{ki} of entries in M for which $m_{ik} \cdot m_{kj} \neq m_{ij}$. This leads to a situation in which the relative importance of c_i with respect to c_j can be determined either as $m_{ik} \cdot m_{kj}$ or m_{ij} and both ways lead to two different results. In other words, either $\mu(c_i) = m_{ik}\mu(c_k) = m_{ik}\left(m_{kj}\mu(c_j)\right)$ or $\mu(c_i) = m_{ij}\mu(c_j)$, where $m_{ik}\left(m_{kj}\mu(c_j)\right) \neq m_{ij}\mu(c_j)$. In such a situation it seems natural to adopt the weighted mean of priorities of all the other concepts as a desired value of $\mu(c_i)$, i.e.,

$$\mu(c_i) = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} m_{ij}\mu(c_j) \tag{2}$$

AHP - one of the most popular decision making methods [38] is in line with the above postulate. It advocates users to adopt the principal eigenvector of M as the priority vector, rescaled so that the sum of all its entries is 1, i.e.

$$\mu_{ev} = \left[\frac{\mu_{max}(c_1)}{s_{ev}}, \ldots, \frac{\mu_{max}(c_n)}{s_{ev}}\right]^T \quad \text{and} \quad s_{ev} = \sum_{i=1}^{n} \mu_{max}(c_i) \tag{3}$$

where μ_{ev} is the ranking function, μ_{max} is the principal eigenvector of M. Hence, it holds that:

$$\mu_{max}(c_i) = \frac{1}{\lambda_{max} - 1} \sum_{j=1, j \neq i}^{n} m_{ij}\mu_{max}(c_j) \tag{4}$$

where λ_{max} is a principal eigenvalue of M. Due to the *Peron-Frobenius* theory, when M is positive, such a real and positive λ_{max} exists [38]. In particular, if M is consistent then λ_{max} equals n, hence (4) is a weighted arithmetic mean as postulated in (2).

Let us see how the pairwise comparisons method works in practice by providing the following simple example in which four candidates $c_1, \ldots, c_4$ apply for the position of chancellor of some university. In the adopted election scheme, the university senate shall discuss the submitted applications and then proceed to vote. Then, taking into account the outcome of voting, the Rector shall select a candidate for the position of chancellor[4].

For the purpose of the example, let us assume that during the vote senators evaluate each of the six pairs $(c_1, c_2), (c_1, c_2), \ldots, (c_3, c_4)$ by assigning 1,2 or 3 either to the first or to

[4]The presented election scheme is quite popular in Poland. See for example (in Polish) Statute of AGH UST (in Polish), art. 19, par. 2.8, http://regent2.uci.agh.edu.pl/statut/statut-agh.pdf

the second candidate within the pair. By assigning 1 (to any out of the two in a pair) they indicate that both candidates are *equally preferred*. Assigning 2 to a candidate will mean that he/she is *more preferred* than the opponent in a pair and, finally, assigning 3 to the given candidate will mean that he/she is *much more preferred* than his/her opponent in the pair. To express intermediate judgments, voters are allowed to use intermediate values. For example, in order to express the opinion that c_i is *slightly more preferred* than c_j a voter may assign 1.5 to c_i. Voter assignments easily translate to the entries of the *PC* matrix. Whenever, considering the pair (c_i, c_j), the voter v_r assigns x to the concept c_i, the value m_{ij} is set to x, and correspondingly, m_{ji} is set to $1/x$.

Judgments expressed during the vote can be stored in the set of *PC* matrices $M^{(1)}, \ldots, M^{(q)}$ where every matrix $M^{(r)} = (m_{ij}^{(r)})$ corresponds to the opinion of one out of the q voters. The resulting matrices $M^{(1)}, \ldots, M^{(q)}$ can be aggregated into one *PC* matrix $\widehat{M} = (\widehat{m}_{ij})$ with the help of a geometric mean[5] [1], where:

$$\widehat{m}_{ij} = \left(\prod_{r=1}^{q} m_{ij}^{(r)} \right)^{1/q} \tag{5}$$

According to (3) the final ranking $\widehat{\mu}_{ev}$ is obtained as the rescaled principal eigenvector of $\widehat{M}$.

For the sake of the simplicity of calculations, let us assume that the voting was held by the senate committee consisting of three persons s_1, s_2 and s_3. Their votes were written down in the form of the following three matrices[6]:

$$M^{(1)} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \mathbf{\frac{5}{2}} \\ \mathbf{2} & 1 & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \frac{1}{2} & 1 & \mathbf{2} \\ \frac{2}{5} & \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}, \quad M^{(2)} = \begin{pmatrix} 1 & \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \frac{1}{2} & 1 & \frac{1}{2} & \mathbf{2} \\ \frac{1}{2} & \mathbf{2} & 1 & \mathbf{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \quad M^{(3)} = \begin{pmatrix} 1 & \mathbf{3} & \frac{1}{2} & \mathbf{2} \\ \frac{1}{3} & 1 & \frac{1}{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & 1 & \mathbf{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \tag{6}$$

Thus, according to the (5) the matrix of aggregated results is prepared as follows:

$$\widehat{M} = \begin{pmatrix} 1. & 1.44 & 0.794 & 2.47 \\ 0.693 & 1. & 0.794 & 2.29 \\ 1.26 & 1.26 & 1. & 2.29 \\ 0.405 & 0.437 & 0.437 & 1. \end{pmatrix} \tag{7}$$

The appropriately rescaled eigenvector of $\widehat{M}$ provides the desired ranking vector $\widehat{\mu}_{ev}$ (see 3).

$$\widehat{\mu}_{ev} = [\, 0.304 \;\; 0.248 \;\; 0.324 \;\; 0.123 \,]^T \tag{8}$$

The winner is the third candidate, whose application gets the highest rank $\widehat{\mu}_{ev}(c_3) = 0.324$. Thus, the senators recommend c_3 for the position of chancellor.

2.2 Matrix inconsistency

When the matrix M is inconsistent, it is difficult to unambiguously determine the relative importance of one concept with respect to the other. In particular, it may turn out that

[5] There are also other ways of aggregating the results in the multiple expert pairwise comparisons method [15, 17, 20], but the use of a geometric mean of judgments appears to be the most popular

[6] The values indicated by voters are written in bold

$m_{ij}m_{jk} \neq m_{ik}$ although both m_{ik} and $m_{ij}m_{jk}$ are equally well suited to determine the relative importance of c_i with reference to c_k. Thus, even if we compute the ranking value $\mu(c_i)$, the question arises regarding the extent to which it reflects the expert's actual opinion [39]? This question prompted researchers to define inconsistency (consistency) indices as methods for measuring the inconsistency of M.

There is a number of different consistency/inconsistency indices [6, 10, 23, 32]. Despite a few attempts of axiomatization [10, 29], there is no single commonly accepted definition of an inconsistency index. However, probably all known indices equal 0 for a fully consistent matrix M, and grow (or at least do not decrease) along with the increase in disturbances of triads (m_{ij}, m_{jk}, m_{ki}) (see Definition 2). Therefore, it is widely accepted that the lower the inconsistency index, the more consistent the *PC* matrix, and hence, the more reliable and trustworthy the results.

In his seminal work [38] Saaty defined the consistency index *(CI)* with the help of λ_{max} (the principal eigenvalue of M).

Definition 3 Given a $n \times n$ *PC* matrix M, *Saaty's CI* is defined as:

$$CI(M) = \frac{\lambda_{max} - n}{n-1} \tag{9}$$

Indeed, since it holds that $\lambda_{max} = n$ for the fully consistent M, then for such a matrix $CI(M) = 0$. Similarly, the more products in the form $m_{ij}m_{jk}m_{ki}$ that differ from 1, the higher[7] the $CI(M)$. In most cases, $CI(M) < 0.1$ is considered as an acceptable level of inconsistency[8]. When the inconsistency is too high, the result of the ranking is regarded as inconclusive.

For the purpose of the rest of the article, the more restrictive *Koczkodaj's inconsistency index* is adopted [25]. Let us denote:

$$\kappa_{i,j,k} \overset{df}{=} \min\left\{\left|1 - \frac{m_{ij}}{m_{ik}m_{kj}}\right|, \left|1 - \frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\} \tag{10}$$

Definition 4 Koczkodaj's inconsistency index $\mathscr{K}$ of $n \times n$ and $(n > 2)$ reciprocal matrix M is equal to

$$\mathscr{K}(M) = \max_{1 \leq i,j,k \leq n} \left\{\kappa_{i,j,k}\right\} \tag{11}$$

where $i \neq j$, $j \neq k$ and $i \neq k$.

It is easy to see that Koczkodaj's inconsistency index also equals 0 when M is consistent. Similarly, the increase in disturbances of triads (m_{ij}, m_{jk}, m_{ki}) ultimately leads to the increase of $\mathscr{K}(M)$. It is assumed that the acceptable threshold of inconsistency, for most practical applications, is $\mathcal{K}(M) < 1/3$ [28]. A more complete overview of different indices, including a comparison of these two, can be found in the literature [6, 9].

In the context of the considered example (Section 2.1), the high inconsistency of matrices $M^{(1)}, M^{(2)}, M^{(3)}$ or $\widehat{M}$ may induce the Rector to make a decision contrary to the

[7] Some authors argue that the increase is too slow [29]

[8] The exact procedure for determining the acceptable value of inconsistency can be found the article [38]

recommendation of the senate committee. Conversely, the low inconsistency of these matrices is an argument for proceeding in accordance with the indication of the committee.

2.3 M-matrices

The analysis of the HRE method presented in the article requires knowledge of the concept of the *M-matrix* [37]. In order to introduce the notion of the *M-matrix* and its properties, let us denote $\mathcal{M}_{\mathbb{R}}(n)$ - the set of $n \times n$ matrices over $\mathbb{R}$, $\mathcal{M}_Z(n)$ - the set of all $A = (a_{ij}) \in \mathcal{M}_{\mathbb{R}}(n)$ with $a_{ij} \leq 0$ if $i \neq j$ and $1 \leq i, j \leq n$. Moreover, for every matrix $A \in \mathcal{M}_{\mathbb{R}}(n)$ and vector $b \in \mathbb{R}^n$ the notation $A \geq 0$ and $b \geq 0$ will mean that each entry of A and b is non-negative and neither A nor b equals 0. The spectral radius of A is defined as $\rho(A) = \max\{|\lambda| : \det(\lambda I - A) = 0\}$.

Definition 5 An $n \times n$ matrix that can be expressed in the form $A = sI - B$ where $B = [b_{ij}]$ with $b_{ij} \geq 0$ for $1 \leq i, j \leq n$, and $s \geq \rho(B)$, the maximum of the absolute value of the eigenvalues of B (i.e., $\rho(B) = \max_i |\lambda_i|$, where λ_i is an eigenvalue of B), is called an *M-matrix*.

In practice, solving many problems in the biological sciences and in the social sciences can be reduced to problems involving *M-matrices* [36]. For this reason, *M-matrices* have been of interest to researchers for a long time and many of their properties are known. Following the work of Plemmons [36] some of them are recalled below in the form of Theorem 1.

Theorem 1 (M-matrix properties) *For every $A \in \mathcal{M}_Z(n)$ the following conditions are equivalent:*

1. *A is inverse positive. That is, A^{-1} exists and $A^{-1} \geq 0$*
2. *A is semi-positive. That is, there exists vector $x > 0$ with $Ax > 0$*
3. *There exists a positive diagonal matrix D such that AD has all positive row sums.*
4. *A is a non-singular M-matrix*

Note that if A is non-singular then A^{-1} is also non-singular. Thus, the solution of $A\mu = b$ is $A^{-1}b$. Moreover for $b > 0$ and A - *M-matrix*, due to the theorem above $A^{-1} \geq 0$, the vector μ also must be strictly positive, i.e., $\mu = A^{-1}b > 0$.

3 Heuristic rating estimation approach

In the eigenvalue based approach [38], the ranking function μ for all the concepts $c \in C$ is initially unknown. Hence, every $\mu(c)$ needs to be determined by the priority deriving procedure. In real life, however, it may turn out that for some concepts the priority values are known. Sometimes decision makers have extra knowledge about the group of elements $C_K \subseteq C$ that allows them to determine $\mu(c)$ for all $c \in C_K$ in advance.

For example, let c_1, c_2 and c_3 be goods that company X intends to place on the market, whilst c_4 and c_5 have been available for some time in stores. In order to choose the most profitable and promising product out of $c_1, \ldots, c_3$, company X wants to calculate the function μ for c_1, c_2 and c_3. Due to some similarities between $c_1, \ldots, c_3$ and the pair c_4, c_5, company X wants to include them in the ranking, treating them as a reference. Of course, it makes no sense to ask experts about how profitable c_4 and c_5 are. The values $\mu(c_4)$ and $\mu(c_5)$ can be easily determined based on sales reports.

The situation as outlined in this simple example leads to the *Heuristic Rating Estimation* method *(HRE)* [30, 31]. The main heuristic of the *HRE* method assumes that the set of concepts C is composed of the unknown concepts $C_U = \{c_1, \ldots, c_k\}$ and the known (reference) concepts $C_K = \{c_{k+1}, \ldots, c_n\}$. Of course, only the values μ_j for $c \in C_U$ need to be estimated, whilst the values $\mu(c_i)$ for $c_i \in C_K$ are considered to be known. The idea behind the adopted heuristic (2), the same as for the eigenvalue based priority deriving method (3, 4) with the fully consistent *PC* matrix, is that for every unknown $c_j \in C_U$ the value $\mu(c_j)$ should be estimated as the arithmetic mean of all the other values $\mu(c_i)$ multiplied by the factor m_{ji}. Thus, the values $\mu(c_i)$ for each unknown concept $c_j \in C_U$ are calculated according to the following formulas:

$$
\begin{array}{rcl}
\mu(c_1) & = & \frac{1}{n-1}(m_{2,1}\mu(c_2) + \ldots\ldots\ldots\ldots\ldots + m_{n,1}\mu(c_n)) \\
\mu(c_2) & = & \frac{1}{n-1}(m_{1,2}\mu(c_1) + m_{3,2}\mu(c_3) + \ldots + m_{n,2}\mu(c_n)) \\
\ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\mu(c_k) & = & \frac{1}{n-1}\big(m_{1,k}\mu(c_1) + \ldots\ldots + m_{k-1,k}\mu(c_{k-1}) + \\
 & & + m_{k+1,k}\mu(c_{k+1}) + \ldots\ldots + m_{n,k}\mu(c_n)\big)
\end{array}
\tag{12}
$$

Since the values $\mu(c_{k+1}), \ldots, \mu(c_n)$ are known and constant ($c_{k+1}, \ldots, c_n$ are the reference concepts), they can be grouped together. Let us denote:

$$
b_j = \frac{1}{n-1} m_{k+1,j}\mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{n,j}\mu(c_n) \tag{13}
$$

Thus (12) could be written as the linear equation system $A\mu = b$ where the matrix A is:

$$
A = \begin{bmatrix}
1 & \cdots & -\frac{1}{n-1}m_{1,k} \\
-\frac{1}{n-1}m_{2,1} & \cdots & -\frac{1}{n-1}m_{2,k} \\
\vdots & \ddots & \vdots \\
-\frac{1}{n-1}m_{k,1} & \cdots & 1
\end{bmatrix}, \tag{14}
$$

and the vectors b and μ are:

$$
b = \begin{bmatrix}
\frac{1}{n-1}\sum_{i=k+1}^{n} m_{1,i}\mu(c_i) \\
\frac{1}{n-1}\sum_{i=k+1}^{n} m_{2,i}\mu(c_i) \\
\vdots \\
\frac{1}{n-1}\sum_{i=k+1}^{n} m_{k,i}\mu(c_i)
\end{bmatrix}, \quad
\mu = \begin{bmatrix}
\mu(c_1) \\
\mu(c_2) \\
\vdots \\
\mu(c_2)
\end{bmatrix} \tag{15}
$$

It is worth noting that $b > 0$, since every b_i for $i = 1, \ldots, k$ is the sum of strictly positive components.

Let us consider the following numerical example. In some local election there are 7 seats to be filled in a district council. Elections are conducted in accordance with the rule "7 best wins", hence only 7 people with the best election results are chosen to become members of the district council. Each party (election committee) may nominate any number of candidates. However, due to the cost of the election campaign, it is important that the nominated candidates actually have a chance of entering the council. On the other hand, based on the results of the previous election, it is known that the result of more than 2000 votes per candidate guaranteed a place in the council. Therefore, the parties are faced with the difficult task of identifying candidates who have a real chance of gathering at least 2000 votes.

One of the political parties participating in the elections plans to support at most five persons. As, during the inner-party meeting, seven candidates $c_1, \ldots, c_7$ have been put forward (including three current members c_5, c_6 and c_7 of the council), the party leadership has to decide whom to support. For this purpose, the party has hired a group of experts, whose task is to assess the chance of each candidate by comparing their election chances in pairs. During the meeting, the experts have prepared[9] the *PC* matrix $M = (m_{ij})$ such that every m_{ij} corresponds to the relative popularity (attractiveness to voters) of c_i candidate with respect to c_j.

$$M = \begin{pmatrix} 1 & 2 & 4.5 & 1.5 & 0.75 & 1.2 & 0.9 \\ \frac{1}{2} & 1 & 2 & 0.7 & 0.35 & 0.5 & 0.4 \\ 0.22 & \frac{1}{2} & 1 & 0.4 & 0.2 & 0.3 & 0.2 \\ 0.67 & 1.43 & 2.5 & 1 & 0.4 & 0.7 & 0.5 \\ 1.33 & 2.86 & 5. & 2.5 & 1 & \frac{3042}{2511} & \frac{3042}{3220} \\ 0.833 & 2. & 3.33 & 1.43 & \frac{2511}{3042} & 1 & \frac{2511}{3220} \\ 1.11 & 2.5 & 5. & 2. & \frac{3220}{3042} & \frac{3220}{2511} & 1 \end{pmatrix} \tag{16}$$

Since the current popularity of the party is similar to that during the previous elections, and it is known that previously c_5, c_6 and c_7 received $\mu(c_5) = 3042$, $\mu(c_6) = 2511$ and $\mu(c_7) = 3220$ votes correspondingly, then experts do not evaluate the pairs (c_5, c_6), (c_5, c_7) and (c_6, c_7). Instead, for each $i, j = 5, 6, 7$ the value $\mu(c_i)/\mu(c_j)$ as $m_{i,j}$ has been adopted.

To estimate the expected number of votes for other candidates the *HRE* method is used, where $C_U = \{c_1, \ldots, c_4\}$ and $C_K = \{c_5, c_6, c_7\}$. The matrix A and the vector b formed from the matrix M are as follows:

$$A = \begin{pmatrix} 1 & -0.33 & -0.75 & -0.25 \\ -0.083 & 1 & -0.33 & -0.12 \\ -0.037 & -0.083 & 1 & -0.067 \\ -0.11 & -0.24 & -0.42 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1400 \\ 600 \\ 330 \\ 760 \end{pmatrix} \tag{17}$$

The ranking vector μ as a solution of the equation $A\mu = b$ is as follows:

$$\mu = \begin{pmatrix} 2661.5 & 1226.48 & 643.048 & 1619.76 \end{pmatrix}^T \tag{18}$$

Hence, according to the experts, only c_1 (with approximately $\mu(c_1) = 2661$ votes) may count on the support of more than 2000 voters. Based on the results of the ranking, the party leadership decide to nominate the three current members of the district council c_5, c_6 and c_7 who can count on the support of 3042,2511 and 3220 votes correspondingly, and one new person, c_1, who expects to gain about 2661 votes.

4 Inconsistency based condition for the existence of a solution

To receive the ranking estimates in the *HRE* approach it is enough to solve the linear equation system $A\mu = b$. Therefore, on one hand, it is easy to find a solution by using almost

[9]For the purpose of the example, there is no need to specify how the experts obtained the matrix M. One of several possible methods [20] involving geometric averaging results provided by each expert has been presented in the previous example (Section 2.1)

any mathematical software including *Excel*®. On the other hand, a solution may not always exist, as the calculated μ may not always be positive and real. The following reasoning is an attempt to find an inconsistency related criterion that helps to decide on the existence of a solution in the *HRE* approach.

Let us note that the entries of $M = (m_{ij})$ are always positive as they represent the comparative opinions of experts. Thus, it holds that $M > 0$. For the same reason, the matrix A (14), formed on the basis of M, has positive entries only on the diagonal, i.e., $A \in \mathcal{M}_Z(n)$ (see Section 2.3). Therefore, proving that A satisfies any of the conditions of Theorem 1, implies that A is an *M-matrix*.

The sufficient condition for A to be an *M-matrix* is formulated with the help of the inconsistency index $\mathcal{K}(M)$ (Definition 4). The paired rankings for which the inconsistency index is too high are considered as unreliable [38]. Using an inconsistency index simplifies the evaluation of $A\mu = b$ and enables linking the reliability of expert assessments with the solution existence problem.

Theorem 2 (On the existence of a solution) *The linear equation system $A\mu = b$ introduced in the HRE approach has exactly one strictly positive solution if*

$$\mathcal{K}(M) < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \tag{19}$$

where $n = |C_U \cup C_K|$ is the number of all the estimated concepts, whilst $r = |C_K|$ - is the number of known concepts and $0 < r \leq n - 2$.

Proof Following Definition 4, the value of *Koczkodaj's inconsistency index* $\mathscr{K}(M)$, in short $\mathscr{K}$, means that the maximal inconsistency for some triad m_{pq}, m_{qr} and m_{pr} is $\mathscr{K}$. Thus, in the case of an arbitrarily chosen triad m_{ik}, m_{kj}, m_{ij} it must hold that:

$$\mathscr{K} \geq \kappa_{i,j,k} = \min\left\{\left|1 - \frac{m_{ij}}{m_{ik}m_{kj}}\right|, \left|1 - \frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\} \tag{20}$$

This means that either: $m_{ij} \leq m_{ik}m_{kj}$ implies that $\mathscr{K} \geq 1 - \frac{m_{ij}}{m_{ik}m_{kj}}$, or $m_{ik}m_{kj} \leq m_{ij}$ implies that $\mathscr{K} \geq 1 - \frac{m_{ik}m_{kj}}{m_{ij}}$. Denoting

$$\alpha \overset{df}{=} 1 - \mathscr{K} \tag{21}$$

we obtain that $m_{ij} \leq m_{ik}m_{kj}$ implies $m_{ij} \geq \alpha \cdot m_{ik}m_{kj}$, and $m_{ik}m_{kj} \leq m_{ij}$ implies $\frac{1}{\alpha} \cdot m_{ik}m_{kj} \geq m_{ij}$. It is easy to see that $0 \leq \mathcal{K} < 1$, thus $0 < \alpha \leq 1$. Thus, both these assertions lead to the common conclusion:

$$\alpha \cdot m_{ik}m_{kj} \leq m_{ij} \leq \frac{1}{\alpha} m_{ik}m_{kj} \tag{22}$$

for every i, j, k such that $1 \leq i, j, k \leq n$. This mutual relationship between the entries of M can be written as the parametric equation $m_{ij} = t \cdot m_{ik}m_{kj}$ where $\alpha \leq t \leq \frac{1}{\alpha}$. Using this equation, the matrix A (see 14) can be written as:

$$A = \begin{bmatrix} t_{1,1}m_{1,k}m_{k,1} & \cdots & -\frac{t_{1,k-1}m_{1,k}m_{k,k-1}}{n-1} & -\frac{m_{1,k}}{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{t_{k-1,1}m_{k-1,k}m_{k,1}}{n-1} & \ddots & t_{k-1,k-1}m_{k-1,k}m_{k,k-1} & -\frac{m_{k-1,k}}{n-1} \\ -\frac{t_{k,1}m_{k,1}}{n-1} & \cdots & -\frac{t_{k,k-1}m_{k,k-1}}{n-1} & 1 \end{bmatrix} \tag{23}$$

where $\alpha \le t_{ij} \le \frac{1}{\alpha}$, for i, j such that $1 \le i, j \le k-1$ (please note that the last column remained unchanged). Hence, finally the matrix A can be written as the matrix product $A = BC$ where:

$$B = \begin{bmatrix} t_{1,1}m_{1,k} & \cdots & -\frac{t_{1,k-1}m_{k-1,k}}{n-1} & -\frac{m_{1,k}}{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{t_{k-1,1}m_{k-1,k}}{n-1} & \vdots & t_{k-1,k-1}m_{k-1,k} & -\frac{m_{k-1,k}}{n-1} \\ -\frac{t_{k,1}}{n-1} & \cdots & -\frac{t_{k,k-1}}{n-1} & 1 \end{bmatrix} \tag{24}$$

and

$$C = \begin{bmatrix} m_{k,1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & m_{k,k-1} & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{25}$$

Since both t_{ij} and m_{ij} are strictly positive, it holds that $B \in \mathcal{M}_Z(n)$. Therefore, due to the third condition of Theorem 1 with $D \overset{df}{=} I$, B is a non-singular *M-matrix* if (and only if) the sums of the rows of $(n-1)B$ are positive. In other words, B is *an M-matrix* if all of the following inequalities (26) are true:

$$\begin{aligned} m_{1,k}(n-1)t_{1,1} - m_{1,k}(t_{1,2} + t_{1,3} + \ldots + t_{1,k-1} + 1) &> 0 \\ m_{2,k}(n-1)t_{2,2} - m_{2,k}(t_{2,1} + t_{2,3} + \ldots + t_{2,k-1} + 1) &> 0 \\ &\vdots \\ (n-1) - (t_{k,1} + t_{k,2} + \ldots\ldots\ldots\ldots\ldots\ldots + t_{k,k-1}) &> 0 \end{aligned} \tag{26}$$

Due to the constraints introduced by the inconsistency $\mathcal{K}(M)$, the minimal and the maximal value of every t_{ij} is α and $\frac{1}{\alpha}$ correspondingly. Thus, the inequalities (26) are true if the following two inequalities are satisfied[10]:

$$(n-1)\alpha > (\underbrace{\frac{1}{\alpha} + \ldots + \frac{1}{\alpha}}_{n-r-2} + 1) \quad \text{and} \quad (n-1) > (\underbrace{\frac{1}{\alpha} + \ldots + \frac{1}{\alpha}}_{n-r-1}) \tag{27}$$

where $r = n - k$ is the number of elements in C_K. In other words, B is *an M-matrix* if the following two conditions are met:

$$f(\alpha) > 0, \quad \text{where} \quad f(\alpha) \overset{df}{=} (n-1)\alpha^2 - \alpha - (n-r-2) \tag{28}$$

and

$$g(\alpha) > 0, \quad \text{where} \quad g(\alpha) \overset{df}{=} (n-1)\alpha - (n-r-1) \tag{29}$$

[10]Let us denote $p_i(t) \overset{df}{=} m_{i,k}(n-1)t$ and $q_i(t_1, \ldots, t_{k-1}) \overset{df}{=} m_{i,k}(t_1 + t_2 + \ldots + t_{k-1} + 1)$ for $0 < i < k$. Since every $\alpha < t, t_i < \frac{1}{\alpha}$, and $0 < \alpha < 1$, all but the last inequalities of (26) have a form $p_i(t) - q_i(t_1, \ldots, t_{k-1}) > 0$. It is easy to see that $p_i(t)$ reaches the minimum in the interval $\alpha < t < \frac{1}{\alpha}$ for $t = \alpha$, and similarly, $q_i(t_1, \ldots, t_{k-1})$ is maximal for every t_i such that $\alpha < t_i < \frac{1}{\alpha}$ when $t_1 = \ldots = t_{k-1} = \frac{1}{\alpha}$. Thus, the function $p_i(t) - q_i(t_1, \ldots, t_{k-1})$ reaches its minimum for $t = \alpha$ and $t_1 = \ldots = t_{k-1} = \frac{1}{\alpha}$. Since every $m_{i,k} > 0$, thus to decide the truth of all but the last inequalities of (26) it is enough to examine $\frac{1}{m_{i,k}}\left(p_i(\alpha) - q_i(\frac{1}{\alpha}, \ldots, \frac{1}{\alpha})\right) > 0$, i.e., $(n-1)\alpha - \left((k-2)\frac{1}{\alpha} + 1\right) > 0$. The same applies to the last inequality of (26)

Table 1 The upper bounds for $\mathcal{K}(M)$ for which there is a guarantee that A is an *M-matrix*

$0 \leq \mathcal{K}(M) <$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
$n = 3$	0.5	-	-	-	-
$n = 4$	0.232	0.666	-	-	-
$n = 5$	0.156	0.359	0.75	-	-
$n = 6$	0.118	0.259	0.441	0.8	-
$n = 7$	0.095	0.204	0.333	0.5	0.833

By solving $f(\alpha) = 0$ and choosing the larger root[11] we obtain that:

$$\mathcal{K}(M) < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \tag{30}$$

whilst the right, linear, inequality $g(\alpha) > 0$ leads to

$$\mathcal{K}(M) < 1 - \frac{(n-r-1)}{(n-1)} \tag{31}$$

In order to decide which of these criteria are more restrictive and which should therefore be chosen, the following two cases need to be considered:

(a) $r = n - 2$
(b) $0 < r \leq n - 3$

When $r = n - 2$, it is easy to see that $f(\alpha) = \alpha g(\alpha)$. Thus, both functions $f(\alpha)$ and $g(\alpha)$ take the 0 value for the same values of argument α. Hence, both criteria are equivalent.

If $0 < r \leq n-3$, it is easy to prove (see Appendix A) that the first condition (30) is more restrictive than the second one, i.e., whenever (30) holds, the inequality (31) is also true. In other words, to provide a guarantee that B is an *M-matrix,* it is enough to consider the more restrictive condition (30).

The fact that B is *an M-matrix* implies that there is an inverse matrix $B^{-1} \geq 0$ (Theorem 1). Hence, due to the form of the matrix C, it is easy to see that the inverse matrix C^{-1} exists, thus A^{-1} exists and $A^{-1} = C^{-1}B^{-1} \geq 0$. Thus, due to the first condition of Theorem 1, A is *an M-matrix*, which means that the equation $A\mu = b$ has a unique strictly positive solution. This conclusion completes the proof of the theorem. □

Of course, the theorem proven above does not address the case $r = n-1$. This is because $r = n-1$ implies A is a scalar, hence solving $A\mu = b$ is trivial. When M is fully consistent, i.e., $\mathcal{K}(M) = 0$ and $\alpha = 1$, it is easy to see that both conditions (27) are satisfied. Thus, in such a case A is an *M-matrix*, and therefore $A\mu = b$ always has a strictly positive solution. Several upper bounds for $\mathcal{K}(M)$ related to the parameters n and r arising from the above theorem are gathered in Table 1. In a broader range, the relationship between n, r and $\mathcal{K}(M)$ is shown in (Fig. 1).

Remark 1 Let us note that for any combination of $r, n \in \mathbb{N}_{+}$ where $0 < r \leq n - 2$, the right side of (30) is greater than 0. In other words, for a sufficiently low inconsistency, the equation $A\mu = b$ always has a feasible solution.

[11] The smaller root $\frac{1-\sqrt{1+4(n-1)(n-r-2)}}{2(n-1)} \leq 0$ for any $n = 3, 4 \ldots$ and $0 < r \leq n - 2$, so it does not need to be taken into account

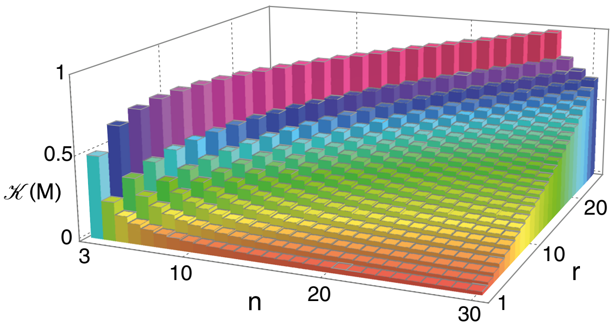

Fig. 1 Limit values of $\mathcal{K}(M)$ below which there is a guarantee that the *HRE* method has a solution

To prove this (see 30) it is enough to show that for $n = 3, 4, \ldots$ it holds that:

$$\frac{\left(1 + \sqrt{1 + 4(n-1)(n-r-2)}\right)}{2(n-1)} < 1 \tag{32}$$

Since $\sqrt{1 + 4(n-1)(n-r-2)} \leq \sqrt{1 + 4(n-1)(n-3)}$, it is enough to show

$$\frac{\left(1 + \sqrt{1 + 4(n-1)(n-3)}\right)}{2(n-1)} < 1 \tag{33}$$

Thus,

$$\sqrt{1 + 4(n-1)(n-3)} < 2n - 3 \tag{34}$$

and

$$4(n-1)(n-3) < (2n-3)^2 - 1 \tag{35}$$

which is equivalent to

$$4(n-1)(n-3) < 4(n-1)(n-2) \tag{36}$$

Thus, for every $n > 1$ the above equation reduces to:

$$n - 3 < n - 2 \tag{37}$$

The last inequality is always satisfied, which proves that (32) is true for $n \geq 3$.

Remark 2 Another interesting observation is that the proof of Theorem 2 takes into account only those entries of the matrix M that form the matrix A. Hence, there is no need to analyze the inconsistency for the whole matrix M. Instead, it is enough to analyze $\widetilde{M}$ - the matrix obtained from M by removing rows and columns corresponding to the elements from the set of known concepts C_K. It also holds[12] that $\mathcal{K}(\widetilde{M}) \leq \mathcal{K}(M)$. Thus, it may turn out that the inconsistency of $\widetilde{M}$ meets the condition (30), whilst the inconsistency of M is too high.

[12] By definition of the *Koczkodaj inconsistency index*, $\mathcal{K}(M)$ is the maximum of $T_M = \{\kappa_{i,j,r}$ such that $1 \leq i, j, r \leq n\}$. Similarly, $\mathcal{K}(\widetilde{M})$ is the maximum of $T_{\widetilde{M}} = \{\kappa_{i,j,r}$ such that $1 \leq i, j, r \leq k\}$, where k is the number of elements in C_U. Since $k < n$ thus, also $T_{\widetilde{M}} \subseteq T_M$. This implies that $\max T_{\widetilde{M}} \leq \max T_M$, which leads to the observation that $\mathcal{K}(\widetilde{M}) \leq \mathcal{K}(M)$

Table 2 The values of r that guarantee the existence of a solution in the HRE approach providing that $\mathcal{K}(M) < 1/3$

n	4	5	6	7	8	9	10	11	12
$r \geq$	2	2	3	3	4	5	5	6	6

Assuming that $C_U = \{c_1, \ldots, c_k\}$, the matrix $\widetilde{M}$ is as follows:

$$\widetilde{M} = \begin{bmatrix} 1 & \cdots & m_{1,k} \\ \vdots & \vdots & \vdots \\ m_{k-1,1} & \ddots & m_{k-1,k} \\ m_{k,1} & \cdots & 1 \end{bmatrix} \tag{38}$$

It might be noticed that, assuming $\alpha \stackrel{df}{=} 1 - \mathscr{K}(\widetilde{M})$ in (21), the proof of Theorem 2 does not change. Hence, instead of exploring the inconsistency of M it is sufficient to examine the inconsistency of the reduced matrix $\widetilde{M}$. Thereby, the upper bounds given in the Table 1 can be applied to $\mathscr{K}(\widetilde{M})$ instead of $\mathscr{K}(M)$.

Remark 3 For most practical applications, *Koczkodaj's* inconsistency lower than $1/3$ is recommended as acceptable [28]. Assuming $\mathcal{K}(M) = 1/3$ the condition (19) can be written as:

$$\frac{1}{3} < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \tag{39}$$

which is equivalent to

$$r > h(n), \quad \text{where} \quad h(n) \stackrel{df}{=} n - 2 - \frac{(4(n-1)-3)^2 - 9}{36(n-1)} \tag{40}$$

Hence, for the given $n \geq 3$ and $\mathcal{K}(M) < 1/3$, it is easy to compute how many known concepts guarantee the existence of a solution (Table 2).

In particular, in the example of the district council elections in Section 3, the inconsistency of the *PC* matrix (16) is $\mathcal{K}(M) \approx 0.308 < 1/3$. Since three out of the seven considered candidates have known ranking values, thus, according to the criterion (19), the solution must exist see Table 2. Moreover, it holds that:

$$\lim_{n \to \infty} \frac{h(n)}{n} = \frac{5}{9} \approx 0.5556 \tag{41}$$

Thus, whenever *Koczkodaj's* inconsistency index $\mathcal{K}(M)$ is lower than $1/3$ (i.e., inconsistency is considered as acceptable) the solution always exists if the known concepts are at least 55.56% of all the ranked concepts.

5 Summary

The reliability of the results achieved in the *PC* models are inseparably linked to the degree of inconsistency of the input data [38]. The lower the inconsistency the better and more reliable the results might be expected to be. Therefore, most practical applications of the *PC* method method seek to construct the *PC* matrix with the smallest possible inconsistency. The theorem proven in this article is in line with the tendency to seek *PC* solutions with low inconsistency. It shows that for an appropriately small inconsistency $\mathcal{K}(M)$ the recently proposed *HRE* method always has an admissible solution. Moreover, given that the inconsistency is acceptable, i.e., $\mathcal{K}(M) < 1/3$, 55.56 % or more of the known concepts guarantees the existence of a solution. This observation makes the provided criterion especially useful in situations where there are many known concepts.

The *HRE* approach is a relatively new estimation method of the relative order of concepts when a non-empty reference subset of concepts exists. The properties of this method are not yet fully understood. Thus, it requires further studies. The presented considerations are accompanied by two numerical examples demonstrating how the *PC* method and *HRE* can be used in different situations. Due to the possibility of applying the *HRE* method to solve various decision problems, it may be of interest to a wide range of researchers and practitioners.

Appendix A: Remark about the Restrictiveness of the Criteria

Assuming that $n - 3 \geq r > 0$ (i.e., $n - 2 > r > 0$) the criterion given as (30) is said to be more restrictive than the criterion given as (31), when the right side of (30) is smaller than the right side of (31), i.e., when:

$$1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} < 1 - \frac{(n-r-1)}{(n-1)} \tag{42}$$

The above inequality is true if and only if

$$\frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} > \frac{(n-r-1)}{(n-1)} \tag{43}$$

Let us denote $m \stackrel{df}{=} n - 1$, then (43) is equivalent to:

$$\frac{1 + \sqrt{1 + 4m(m-r-1)}}{2m} > \frac{(m-r)}{m} \tag{44}$$

Since $m > 0$, thus the above expression is equivalent to:

$$1 + \sqrt{1 + 4m(m-r-1)} > 2(m-r) \tag{45}$$

and consequently

$$\sqrt{1 + 4m(m-r-1)} > 2\left(m - r - \frac{1}{2}\right) \tag{46}$$

The above inequality holds if and only if

$$1 + 4m(m-r-1) > 4\left(m - \left(r + \frac{1}{2}\right)\right)^2 \tag{47}$$

thus,

$$1 + 4m^2 - 4mr - 4m > 4m^2 - 8m\left(r + \frac{1}{2}\right) + 4\left(r + \frac{1}{2}\right)^2 \tag{48}$$

and consequently

$$1 + 4m^2 - 4mr - 4m > 4m^2 - 8mr - 4m + 4r^2 + 4r + 1 \tag{49}$$

The above expression is equivalent to:

$$mr > (r + 1)r \tag{50}$$

which corresponds to

$$m - 1 > r \tag{51}$$

In the light of the assumptions that $n - 2 > r$ and the fact that $m = n - 1$ the inequality (51) is always met. Since, all the expressions (42) - (51) are equivalent the truth of (51) means that the criterion (30) is more restrictive than the criterion (31).

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