

Higher Tetrahedral Algebras

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Abstract We introduce and study the higher tetrahedral algebras, an exotic family of finite-dimensional tame symmetric algebras over an algebraically closed field. The Gabriel quiver of such an algebra is the triangulation quiver associated to the coherent orientation of the tetrahedron. Surprisingly, these algebras occurred in the classification of all algebras of generalized quaternion type, but are not weighted surface algebras. We prove that a higher tetrahedral algebra is periodic if and only if it is non-singular.

Keywords Syzygy · Periodic algebra · Symmetric algebra · Tame algebra

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1 Introduction and the Main Results

Throughout this paper, K will denote a fixed algebraically closed field. By an algebra we mean an associative finite-dimensional K -algebra with an identity. For an algebra A , we

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denote by $\text{mod } A$ the category of finite-dimensional right A -modules and by D the standard duality $\text{Hom}_K(-, K)$ on $\text{mod } A$. An algebra A is called *self-injective* if A_A is injective in $\text{mod } A$, or equivalently, the projective modules in $\text{mod } A$ are injective. A prominent class of self-injective algebras is formed by the *symmetric algebras* A for which there exists an associative, non-degenerate symmetric K -bilinear form $(-, -) : A \times A \rightarrow K$. Classical examples of symmetric algebras are provided by the blocks of group algebras of finite groups and the Hecke algebras of finite Coxeter groups. In fact, any algebra A is a quotient algebra of its trivial extension algebra $T(A) = A \ltimes D(A)$, which is a symmetric algebra.

From the remarkable Tame and Wild Theorem of Drozd (see [4, 8]) the class of algebras over K may be divided into two disjoint classes. The first class consists of the *tame algebras* for which the indecomposable modules occur in each dimension d in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the *wild algebras* whose representation theory comprises the representation theories of all algebras over K . Accordingly, we may realistically hope to classify the indecomposable finite-dimensional modules only for the tame algebras. Among the tame algebras we may distinguish the *algebras of polynomial growth* for which the number of one-parameter families of indecomposable modules in each dimension d is bounded by d^m for some positive integer m (depending only on the algebra) whose representation theory is usually well understood (see [2, 23, 24] for some general results). On the other hand, the representation theory of tame algebras of non-polynomial growth is still only emerging.

Let A be an algebra. Given a module M in $\text{mod } A$, its syzygy is defined to be the kernel $\Omega_A(M)$ of a minimal projective cover of M in $\text{mod } A$. The syzygy operator Ω_A is a very important tool to construct modules in $\text{mod } A$ and relate them. For A self-injective, it induces an equivalence of the stable module category $\underline{\text{mod}} A$, and its inverse is the shift of a triangulated structure on $\underline{\text{mod}} A$ [15]. A module M in $\text{mod } A$ is said to be *periodic* if $\Omega_A^n(M) \cong M$ for some $n \geq 1$, and if so the minimal such n is called the *period* of M . The action of Ω_A on $\text{mod } A$ can effect the algebra structure of A . For example, if all simple modules in $\text{mod } A$ are periodic, then A is a self-injective algebra. An algebra A is defined to be *periodic* if it is periodic viewed as a module over the enveloping algebra $A^e = A^{\text{op}} \otimes_K A$, or equivalently, as an A - A -bimodule. It is known that if A is a periodic algebra of period n then for any indecomposable non-projective module M in $\text{mod } A$ the syzygy $\Omega_A^n(M)$ is isomorphic to M (see [26, Theorem IV.11.19]).

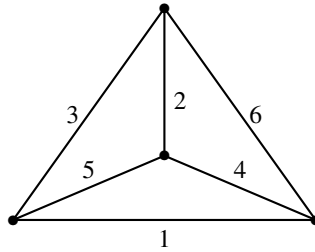
Finding or possibly classifying periodic algebras is an important problem. It is very interesting because of connections with group theory, topology, singularity theory and cluster algebras. Periodicity of an algebra, and its period, are invariant under derived equivalences [20] (see also [10]). Therefore, to study periodic algebras we may assume that the algebras are basic and indecomposable.

We are concerned with the classification of all periodic tame symmetric algebras. In [9] Dugas proved that every representation-finite self-injective algebra, without simple blocks, is a periodic algebra. We note that, by general theory (see [24, Section 3]), a basic, indecomposable, non-simple, symmetric algebra A is representation-finite if and only if A is socle equivalent to an algebra $T(B)^G$ of invariants of the trivial extension algebra $T(B)$ of a tilted algebra B of Dynkin type with respect to free action of a finite cyclic group G . The representation-infinite, indecomposable, periodic algebras of polynomial growth were classified by Białkowski, Erdmann and Skowroński in [2] (see also [23, 24]). In particular, it follows from [2] that every basic, indecomposable, periodic, representation-infinite symmetric tame algebra of polynomial growth is socle equivalent to an algebra $T(B)^G$ of invariants of the trivial extension algebra $T(B)$ of a tubular algebra B of tubular type

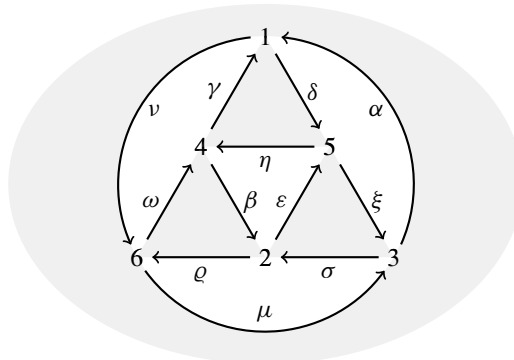
(2, 2, 2, 2), (3, 3, 3), (2, 4, 4), (2, 3, 6) (introduced by Ringel [21]) with respect to free action of a finite cyclic group G .

Recently we introduced in [11] the weighted surface algebras of triangulated surfaces with arbitrarily oriented triangles and proved that all these algebras, except the singular tetrahedral algebras, are periodic tame symmetric algebras of period 4. Here, we investigate the periodicity of higher tetrahedral algebras, being “higher analogs” of the tetrahedral algebras studied in [11]. These algebras occurred in the authors work concerning the classification of all representation-infinite tame symmetric periodic algebras of period 4.

Consider the tetrahedron



with the coherent orientation of triangles: (1 5 4), (2 5 3), (2 6 4), (1 6 3). Then, following [11], we have the associated triangulation quiver (Q, f) of the form



where f is the permutation of arrows of order 3 described by the shaded subquivers. We denote by g the permutation on the set of arrows of Q whose g -orbits are the four white 3-cycles.

Let $m \geq 1$ be a natural number and $\lambda \in K$. We denote by $\Lambda(m, \lambda)$ the algebra given by the above quiver and the relations:

$$\begin{aligned} \gamma\delta &= \beta\varepsilon + \lambda(\beta\rho\omega)^{m-1}\beta\varepsilon, & \delta\eta &= \nu\omega, & \eta\gamma &= \xi\alpha, & \nu\mu &= \delta\xi, \\ \rho\omega &= \varepsilon\eta + \lambda(\varepsilon\xi\sigma)^{m-1}\varepsilon\eta, & \omega\beta &= \mu\sigma, & \beta\rho &= \gamma\nu, & \mu\alpha &= \omega\gamma, \\ \xi\sigma &= \eta\beta + \lambda(\eta\gamma\delta)^{m-1}\eta\beta, & \sigma\varepsilon &= \alpha\delta, & \varepsilon\xi &= \rho\mu, & \alpha\nu &= \sigma\rho, \end{aligned}$$

$$(\theta f(\theta) f^2(\theta))^{m-1} \theta f(\theta) g(f(\theta)) = 0 \text{ for any arrow } \theta \text{ in } Q.$$

We call $\Lambda(m, \lambda)$ a *tetrahedral algebra of degree m* . Moreover, an algebra $\Lambda(m, \lambda)$ with $\lambda \in K^* = K \setminus \{0\}$ is said to be a *non-singular tetrahedral algebra of degree m* . We note that for $m = 1$, $\Lambda(1, \lambda)$ is the *tetrahedral algebra* $\Lambda(\mathbb{S}, \lambda + 1)$ investigated in [11, Section 6],

and is a weighted surface algebra. Further, the algebra $\Lambda(1, 0) = \Lambda(\mathbb{S}, 1)$ is called in [11] the *singular tetrahedral algebra*, because it is a unique algebra among all weighted surface algebras without periodic simple modules. On the other hand, a *non-singular tetrahedral algebra* $\Lambda(1, \lambda)$, $\lambda \in K^*$, is a periodic algebra of period 4, with all simple modules being periodic of period 4.

We call a tetrahedral algebra $\Lambda(m, \lambda)$ of degree $m \geq 2$ a *higher tetrahedral algebra*. The aim of this article is to describe basic properties of the higher tetrahedral algebras, which extend results on the tetrahedral algebras (of degree 1), established in [11], to the algebras given by the same quiver but much more complicated higher degree relations.

The following two theorems describe some properties of higher tetrahedral algebras.

Theorem 1 *Let $\Lambda = \Lambda(m, \lambda)$ be a higher tetrahedral algebra. Then Λ is a finite-dimensional symmetric algebra with $\dim_K \Lambda = 36m$.*

Theorem 2 *Let $\Lambda = \Lambda(m, \lambda)$ be a higher tetrahedral algebra. Then Λ is a tame algebra of non-polynomial growth.*

The following theorem is the main result of the paper.

Theorem 3 *Let $\Lambda = \Lambda(m, \lambda)$ be a higher tetrahedral algebra. Then the following statements are equivalent:*

- (i) *$\text{mod } \Lambda$ admits a periodic simple module.*
- (ii) *All simple modules in $\text{mod } \Lambda$ are periodic of period 4.*
- (iii) *Λ is a periodic algebra of period 4.*
- (iv) *Λ is non-singular.*

Following [12], an algebra A is called an algebra of generalized quaternion type if A is representation-infinite tame symmetric and every simple module in $\text{mod } A$ is periodic of period 4. Theorem 3 is one of the essential ingredients for proving in [12, Main Theorem] that an algebra A is of generalized quaternion type with 2-regular Gabriel quiver if and only if A is a socle deformation of a weighted surface algebra, different from the singular tetrahedral algebra, or is a non-singular higher tetrahedral algebra. On the other hand, it is expected that the singular higher tetrahedral algebras $\Lambda(m, 0)$, $m \geq 2$, will form an exotic family of algebras of generalized semidihedral type.

This paper is organized as follows. In Section 2 we recall background on special biserial algebras and degenerations of algebras. In Section 3 we describe our general approach and results for constructing a minimal projective bimodule resolution of an algebra with periodic simple modules. Section 4 is devoted to basic properties of the higher tetrahedral algebras and the proof of Theorem 1. Sections 5 and 6 contain the proofs of Theorems 2 and 3, respectively.

For general background on the relevant representation theory we refer to the books [1, 22, 26].

2 Preliminary Results

A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of a finite set Q_0 of vertices, a finite set Q_1 of arrows, and two maps $s, t : Q_1 \rightarrow Q_0$ which associate to each arrow $\alpha \in Q_1$

its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$. We denote by KQ the path algebra of Q over K whose underlying K -vector space has as its basis the set of all paths in Q of length ≥ 0 , and by R_Q the arrow ideal of KQ generated by all paths in Q of length ≥ 1 . An ideal I in KQ is said to be admissible if there exists $m \geq 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$. If I is an admissible ideal in KQ , then the quotient algebra KQ/I is called a bound quiver algebra, and is a finite-dimensional basic K -algebra. Moreover, KQ/I is indecomposable if and only if Q is connected. Every basic, indecomposable, finite-dimensional K -algebra A has a bound quiver presentation $A \cong KQ/I$, where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in KQ . For a bound quiver algebra $A = KQ/I$, we denote by $e_i, i \in Q_0$, the associated complete set of pairwise orthogonal primitive idempotents of A , and by $S_i = e_i A / e_i \text{rad } A$ (respectively, $P_i = e_i A$), $i \in Q_0$, the associated complete family of pairwise non-isomorphic simple modules (respectively, indecomposable projective modules) in $\text{mod } A$.

Following [25], an algebra A is said to be special biserial if A is isomorphic to a bound quiver algebra KQ/I , where the bound quiver (Q, I) satisfies the following conditions:

- (a) each vertex of Q is a source and target of at most two arrows,
- (b) for any arrow α in Q there are at most one arrow β and at most one arrow γ with $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

Moreover, if in addition I is generated by paths of Q , then $A = KQ/I$ is said to be a string algebra [3]. It was proved in [19] that the class of special biserial algebras coincides with the class of biserial algebras (indecomposable projective modules have biserial structure) which admit simply connected Galois coverings. Furthermore, by [27, Theorem 1.4] we know that every special biserial algebra is a quotient algebra of a symmetric special biserial algebra. We also mention that, if A is a self-injective special biserial algebra, then $A/\text{soc}(A)$ is a string algebra.

The following has been proved by Wald and Waschbüsch in [27] (see also [3, 7] for alternative proofs).

Proposition 2.1 *Every special biserial algebra is tame.*

For a positive integer d , we denote by $\text{alg}_d(K)$ the affine variety of associative K -algebra structures with identity on the affine space K^d . Then the general linear group $\text{GL}_d(K)$ acts on $\text{alg}_d(K)$ by transport of the structures, and the $\text{GL}_d(K)$ -orbits in $\text{alg}_d(K)$ correspond to the isomorphism classes of d -dimensional algebras (see [17] for details). We identify a d -dimensional algebra A with the point of $\text{alg}_d(K)$ corresponding to it. For two d -dimensional algebras A and B , we say that B is a degeneration of A (A is a deformation of B) if B belongs to the closure of the $\text{GL}_d(K)$ -orbit of A in the Zariski topology of $\text{alg}_d(K)$.

Geiss' Theorem [13] shows that if A and B are two d -dimensional algebras, A degenerates to B and B is a tame algebra, then A is also a tame algebra (see also [5]). We will apply this theorem in the following special situation.

Proposition 2.2 *Let d be a positive integer, and $A(t), t \in K$, be an algebraic family in $\text{alg}_d(K)$ such that $A(t) \cong A(1)$ for all $t \in K \setminus \{0\}$. Then $A(1)$ degenerates to $A(0)$. In particular, if $A(0)$ is tame, then $A(1)$ is tame.*

A family of algebras $A(t), t \in K$, in $\text{alg}_d(K)$ is said to be algebraic if the induced map $A(-) : K \rightarrow \text{alg}_d(K)$ is a regular map of affine varieties.

3 Bimodule Resolutions of Self-Injective Algebras

In this section we describe a general approach for proving that an algebra A with periodic simple modules is a periodic algebra.

Let $A = KQ/I$ be a bound quiver algebra, and $e_i, i \in Q_0$, be the primitive idempotents of A associated to the vertices of Q . Then $e_i \otimes e_j, i, j \in Q_0$, form a set of pairwise orthogonal primitive idempotents of the enveloping algebra $A^e = A^{op} \otimes_K A$ whose sum is the identity of A^e . Hence, $P(i, j) = (e_i \otimes e_j)A^e = Ae_i \otimes e_jA$, for $i, j \in Q_0$, form a complete set of pairwise non-isomorphic indecomposable projective modules in $\text{mod } A^e$ (see [26, Proposition IV.11.3]).

The following result by Happel [16, Lemma 1.5] describes the terms of a minimal projective resolution of A in $\text{mod } A^e$.

Proposition 3.1 *Let $A = KQ/I$ be a bound quiver algebra. Then there is in $\text{mod } A^e$ a minimal projective resolution of A of the form*

$$\dots \rightarrow \mathbb{P}_n \xrightarrow{d_n} \mathbb{P}_{n-1} \rightarrow \dots \rightarrow \mathbb{P}_1 \xrightarrow{d_1} \mathbb{P}_0 \xrightarrow{d_0} A \rightarrow 0,$$

where

$$\mathbb{P}_n = \bigoplus_{i, j \in Q_0} P(i, j)^{\dim_K \text{Ext}_A^n(S_i, S_j)}$$

for any $n \in \mathbb{N}$.

The syzygy modules have an important property, a proof for the next Lemma may be found in [26, Lemma IV.11.16].

Lemma 3.2 *Let A be an algebra. For any positive integer n , the module $\Omega_{A^e}^n(A)$ is projective as a left A -module and also as a right A -module.*

There is no general recipe for the differentials d_n in Proposition 3.1, except for the first three which we will now describe. This will be one of the main techniques we use in our proof of the implication (iv) \Rightarrow (iii) in Theorem 3.

We have

$$\mathbb{P}_0 = \bigoplus_{i \in Q_0} P(i, i) = \bigoplus_{i \in Q_0} Ae_i \otimes e_iA.$$

The homomorphism $d_0 : \mathbb{P}_0 \rightarrow A$ in $\text{mod } A^e$ defined by $d_0(e_i \otimes e_i) = e_i$ for all $i \in Q_0$ is a minimal projective cover of A in $\text{mod } A^e$. Recall that, for two vertices i and j in Q , the number of arrows from i to j in Q is equal to $\dim_K \text{Ext}_A^1(S_i, S_j)$ (see [1, Lemma III.2.12]). Hence we have

$$\mathbb{P}_1 = \bigoplus_{\alpha \in Q_1} P(s(\alpha), t(\alpha)) = \bigoplus_{\alpha \in Q_1} Ae_{s(\alpha)} \otimes e_{t(\alpha)}A.$$

Then we have the following known fact (see [2, Lemma 3.3] for a proof).

Lemma 3.3 *Let $A = KQ/I$ be a bound quiver algebra, and $d_1 : \mathbb{P}_1 \rightarrow \mathbb{P}_0$ the homomorphism in $\text{mod } A^e$ defined by*

$$d_1(e_{s(\alpha)} \otimes e_{t(\alpha)}) = \alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha$$

for any arrow α in Q . Then d_1 induces a minimal projective cover $d_1 : \mathbb{P}_1 \rightarrow \Omega_{A^e}^1(A)$ of $\Omega_{A^e}^1(A) = \text{Ker } d_0$ in $\text{mod } A^e$. In particular, we have $\Omega_{A^e}^2(A) \cong \text{Ker } d_1$ in $\text{mod } A^e$.

We will denote the homomorphism $d_1 : \mathbb{P}_1 \rightarrow \mathbb{P}_0$ by d . For the algebras A we will consider, the kernel $\Omega_{A^e}^2(A)$ of d will be generated, as an A - A -bimodule, by some elements of \mathbb{P}_1 associated to a set of relations generating the admissible ideal I . Recall that a relation in the path algebra KQ is an element of the form

$$\mu = \sum_{r=1}^n c_r \mu_r,$$

where c_1, \dots, c_r are non-zero elements of K and $\mu_r = \alpha_1^{(r)} \alpha_2^{(r)} \dots \alpha_{m_r}^{(r)}$ are paths in Q of length $m_r \geq 2, r \in \{1, \dots, n\}$, having a common source and a common target. The admissible ideal I can be generated by a finite set of relations in KQ (see [1, Corollary II.2.9]). In particular, the bound quiver algebra $A = KQ/I$ is given by the path algebra KQ and a finite number of identities $\sum_{r=1}^n c_r \mu_r = 0$ given by a finite set of generators of the ideal I . Consider the K -linear homomorphism $\pi : KQ \rightarrow \mathbb{P}_1$ which assigns to a path $\alpha_1 \alpha_2 \dots \alpha_m$ in Q the element

$$\pi(\alpha_1 \alpha_2 \dots \alpha_m) = \sum_{k=1}^m \alpha_1 \alpha_2 \dots \alpha_{k-1} \otimes \alpha_{k+1} \dots \alpha_m$$

in \mathbb{P}_1 , where $\alpha_0 = e_{s(\alpha_1)}$ and $\alpha_{m+1} = e_{t(\alpha_m)}$. Observe that $\pi(\alpha_1 \alpha_2 \dots \alpha_m) \in e_{s(\alpha_1)} \mathbb{P}_1 e_{t(\alpha_m)}$. Then, for a relation $\mu = \sum_{r=1}^n c_r \mu_r$ in KQ lying in I , we have an element

$$\pi(\mu) = \sum_{r=1}^n c_r \pi(\mu_r) \in e_i \mathbb{P}_1 e_j,$$

where i is the common source and j is the common target of the paths μ_1, \dots, μ_r . The following lemma shows that relations always produce elements in the kernel of d_1 ; the proof is straightforward.

Lemma 3.4 *Let $A = KQ/I$ be a bound quiver algebra and $d_1 : \mathbb{P}_1 \rightarrow \mathbb{P}_0$ the homomorphism in mod A^e defined in Lemma 3.3. Then for any relation μ in KQ lying in I , we have $d_1(\pi(\mu)) = 0$.*

For an algebra $A = KQ/I$ in our context, we will see that there exists a family of relations $\mu^{(1)}, \dots, \mu^{(q)}$ generating the ideal I such that the associated elements $\pi(\mu^{(1)}), \dots, \pi(\mu^{(q)})$ generate the A - A -bimodule $\Omega_{A^e}^2(A) = \text{Ker } d_1$. In fact, using Lemma 3.2, we will be able to show that

$$\mathbb{P}_2 = \bigoplus_{j=1}^q P(s(\mu^{(j)}), t(\mu^{(j)})) = \bigoplus_{j=1}^q A e_{s(\mu^{(j)})} \otimes e_{t(\mu^{(j)})} A,$$

and the homomorphism $d_2 : \mathbb{P}_2 \rightarrow \mathbb{P}_1$ in mod A^e such that

$$d_2(e_{s(\mu^{(j)})} \otimes e_{t(\mu^{(j)})}) = \pi(\mu^{(j)}),$$

for $j \in \{1, \dots, q\}$, defines a projective cover of $\Omega_{A^e}^2(A)$ in mod A^e . In particular, we have $\Omega_{A^e}^3(A) \cong \text{Ker } d_2$ in mod A^e . We will denote this homomorphism d_2 by R .

For the next map $d_3 : \mathbb{P}_3 \rightarrow \mathbb{P}_2$, which we will call $S := d_3$ later, we do not have a general recipe. To define it, we need a set of minimal generators for $\Omega_{A^e}^3(A)$, and Proposition 3.1 tells us where we should look for them.

4 Proof of Theorem 1

Let $\Lambda = \Lambda(m, \lambda)$ for some $m \geq 2$ and $\lambda \in K$. In this section we will study algebra properties of Λ , and in particular prove Theorem 1. The first results will be used to reduce calculations, and should also be of independent interest.

In order to construct a basis of Λ with good properties, we analyze the images of paths in Λ , they have very unusual properties. We introduce some notation. It follows from the relations defining Λ that we may define the elements

$$\begin{aligned} X_1 &= \delta\eta\gamma = \nu\mu\alpha, & X_2 &= \rho\omega\beta = \varepsilon\xi\sigma, & X_3 &= \alpha\nu\mu = \sigma\varepsilon\xi, \\ X_4 &= \gamma\delta\eta = \beta\rho\omega, & X_5 &= \eta\gamma\delta = \xi\sigma\varepsilon, & X_6 &= \omega\beta\rho = \mu\alpha\nu, \end{aligned}$$

given by products of the arrows around the shaded triangles. Moreover, we define the elements

$$\tilde{X}_2 = \varepsilon\eta\beta, \quad \tilde{X}_4 = \beta\varepsilon\eta, \quad \tilde{X}_5 = \eta\beta\varepsilon.$$

The quiver Q of Λ has an automorphism φ of order 3, defined as follows. Its action on vertices is given by the cycles

$$(5\ 4\ 2)(1\ 6\ 3)$$

and the action on arrows is

$$(\delta\ \omega\ \sigma)(\eta\ \beta\ \varepsilon)(\gamma\ \rho\ \xi)(\nu\ \mu\ \alpha).$$

Lemma 4.1 *The action of φ extends to an algebra automorphism of Λ .*

Proof We extend φ to an algebra map of KQ . Then we must check that φ preserves the relations, which is direct calculation. For example,

$$\varphi(\gamma\delta) = \varphi(\gamma)\varphi(\delta) = \rho\omega \quad \text{and} \quad \varphi(\beta\varepsilon) = \varphi(\beta)\varphi(\varepsilon) = \varepsilon\eta,$$

and

$$\varphi(X_4) = \varphi(\gamma)\varphi(\delta)\varphi(\eta) = \rho\omega\beta = X_2.$$

Hence, φ takes the relation for $\gamma\delta$ to the relation for $\rho\omega$. □

Lemma 4.2 *For each vertex i of Q , the element X_i^m belongs to the right socle of Λ .*

Proof It follows from the relations that, for each arrow θ in Q , we have $X_{s(\theta)}^m\theta = 0$. For example, we have

$$X_1^m\delta = (\nu\mu\alpha)^m\delta = \nu(\mu\alpha\nu)^{m-1}\mu\alpha\delta = \nu(X_{s(\mu)})^{m-1}\mu f(\mu)g(f(\mu)) = 0.$$

□

Lemma 4.3 *We have the following equalities in Λ .*

- (i) $X_1 = \nu\omega\gamma = \delta\xi\alpha, X_3 = \sigma\rho\mu = \alpha\delta\xi, X_6 = \mu\sigma\rho = \omega\gamma\nu.$
- (ii) $X_2 = \rho\rho\sigma, X_4 = \gamma\nu\omega, X_5 = \xi\alpha\delta.$
- (iii) $X_2 = \tilde{X}_2 + \lambda X_2^m, X_4 = \tilde{X}_4 + \lambda X_4^m, X_5 = \tilde{X}_5 + \lambda X_5^m.$
- (iv) $X_2^m = (\tilde{X}_2)^m, X_4^m = (\tilde{X}_4)^m, X_5^m = (\tilde{X}_5)^m.$

Proof The equalities in (i) and (ii) follow directly from the relations defining Λ . For (iii), observe that the vertices 2, 4, 5 are in one orbit of the automorphism φ . Hence, it is enough to show that $X_2 = \tilde{X}_2 + \lambda X_2^m$. We have

$$X_2 = \varepsilon\xi\sigma = \rho\mu\sigma = \rho\omega\beta.$$

Moreover

$$\rho\omega\beta = (\varepsilon\eta + \lambda X_2^{m-1}\varepsilon\eta)\beta = \tilde{X}_2 + \lambda X_2^{m-1}\varepsilon\eta\beta$$

and

$$X_2^{m-1}\varepsilon\eta\beta = X_2^{m-1}\varepsilon(\xi\sigma - \lambda X_5^{m-1}\eta\beta) = X_2^{m-1}\varepsilon\xi\sigma = X_2^m.$$

The equalities in (iv) follow from the equalities in (iii) and the fact that X_2^m, X_4^m, X_5^m are in the socle of Λ . □

Lemma 4.4 *For vertices $i \neq j$ in Q , any two paths of length 3 from i to j are equal and non-zero in Λ .*

Proof Consider paths of length three between different vertices i and j in Q . Such paths only exist if the vertices are “opposite”, and because of the automorphism φ , we may assume that $\{i, j\} = \{1, 2\}$. Concerning paths from 1 to 2 we have

$$\delta\eta\beta = \delta\xi\sigma - \lambda\delta X_5^{m-1}\eta\beta.$$

Now, $\delta X_5 = \delta\eta\gamma\delta = X_1\delta$ and therefore

$$\delta X_5^{m-1}\eta\beta = X_1^{m-1}\delta\eta\beta = X_{s(\delta)}^{m-1}\delta f(\delta)g(f(\delta)) = 0.$$

With this, we have

$$\delta\eta\beta = \delta\xi\sigma = \nu\mu\sigma = \nu\omega\beta,$$

as required. A similar calculation shows that all paths from 2 to 1 of length three are equal in Λ . □

Lemma 4.5 *The following statements hold:*

- (i) *For $4 \leq k \leq 3m - 1$, any two paths of length k between two vertices in Q are equal and non-zero in Λ .*
- (ii) *For $k = 3m$, any path of length k between two different vertices is zero in Λ .*
- (iii) *For $k = 3m$, any cycle of length k around a vertex i is equal to X_i^m .*
- (iv) *For $k > 3m$, any path of length k is zero in Λ .*

Proof For the following, we write X_{ij} for a path of length three between vertices $i \neq j$. We first show that any two paths of length four between two fixed vertices are equal. For this, it suffices to consider paths starting at 1 and paths starting at 2.

(i1) Paths from 1 of length four must end at vertex 5 or vertex 6. Consider paths ending at 5. Such a path either ends with arrow δ or it ends with arrow ε . If it ends with δ then it is the product of a cyclic path of length three from 1 to 1 with δ , hence by Lemma 4.3, is equal to $X_1\delta$. Similarly, any path of length four from 1 ending with ε is the product of a path of length three from 1 to 2 with ε , hence is equal in Λ to $X_{12}\varepsilon$. We must show that $X_1\delta = X_{12}\varepsilon$. We have

$$X_1\delta = \nu\mu\alpha\delta = \nu\mu\sigma\varepsilon = X_{12}\varepsilon.$$

Similarly, any path of length four from 1 to 6 ends with arrow ρ or with arrow ν , and one shows as above that all are equal in Λ .

(i2) Consider paths of length four starting at vertex 2, any such path ends at vertex 6 or vertex 5. Consider paths ending at vertex 6, the last arrow in such a path is ν or ρ . If it ends with ν then the path is of the form $X_{12}\nu$, and if it ends with ρ then it is either $X_2\rho$, or it is $\tilde{X}_2\rho$. We have

$$\tilde{X}_2\rho - (X_2 - \lambda X_2^m)\rho = X_2\rho$$

(noting that X_2^m is in the right socle of Λ). Moreover,

$$X_2\rho = \rho\omega\beta\rho = \rho\omega\gamma\nu = X_{12}\nu.$$

For paths ending at vertex 5 the proof is similar.

We finish the proof of (i) by induction on k , using arguments as for the case $k = 4$. Note that all paths of length $\leq 3m - 1$ in Q are non-zero in Λ since all zero relations of Λ have length $3m$ (and since the relations as listed are minimal).

We prove now the statements (ii) and (iii). It suffices again to consider paths starting at 1 and paths starting at 2. A cyclic path starting at 1 of length $3m$ is of the form $Y\gamma$ or $Y'\alpha$, where Y ends at vertex 4 and Y' ends at vertex 3. By part (i) we can take $Y = X_1^{m-1}\delta\eta$ and then $Y\gamma = X_1^m$. As well we can take $Y' = X_1^{m-1}\nu\mu$ and get $Y'\alpha = X_1^m$. Similarly, any path of length $3m$ from 2 to 2 is equal to X_2^m . Now consider a path from vertex 1 of length $3m$ which does not end at vertex 1, then it must end at vertex 2. It is of the form $Y\beta$ with Y from 1 to 4, or of the form $Y'\sigma$ with Y' from 1 to 3. By part (i) we can take $Y = X_1^{m-1}\delta\eta$ and then

$$Y\beta = X_{s(\delta)}^{m-1}\delta f(\delta)g(f(\delta)) = 0.$$

We also can take $Y' = X_1^{m-1}\nu\mu$ and then again, by the defining relations, $Y'\sigma = 0$. Finally, consider a path from vertex 2 of length $3m$ which does not end at vertex 2, then it must end at vertex 1. Such a path is either of the form $Y\gamma$, or of the form $Y'\alpha$, where Y and Y' are paths of length $3m - 1$. We can take $Y = X_2^{m-1}\rho\omega$ and then $Y\gamma = 0$, by the defining relations. Similarly, we can take $Y' = X_2^{m-1}\varepsilon\xi$ and then $Y'\alpha = 0$, by the defining relations.

The statement (iv) follows because X_i^m is in the right socle of Λ , for any vertex i of Q . □

We present now a basis of Λ with good properties. We fix a vertex i , and define a basis \mathcal{B}_i of $e_i\Lambda$ as follows. Choose a version of X_i , then suppose X_i starts with τ , then let $\bar{\tau}$ be the other arrow starting at i . Now let $\mathcal{B}_i :=$ the set of all initial subwords of X_i^m together with the set

$$\left\{ X_i^k \bar{\tau}, X_i^k \bar{\tau} f(\bar{\tau}) : 0 \leq k \leq m - 1 \right\} \cup \left\{ X_i^k \tau f(\tau) g(f(\tau)) : 0 \leq k < m - 1 \right\}.$$

Then \mathcal{B}_i is a basis for $e_i\Lambda$, and we take $\mathcal{B} := \cup_{i \in Q_0} \mathcal{B}_i$. For each vertex i , let $\omega_i := X_i^m$, this spans the socle of $e_i\Lambda$, by Lemma 4.5, and it lies in \mathcal{B} . The basis \mathcal{B} has the following properties:

- (a) For each k with $1 \leq k \leq 3m - 1$ the set \mathcal{B}_i contains precisely two elements of length k . The end vertices are determined by the congruence of k modulo 3.
- (b) Any path of length k for $4 \leq k \leq 3m - 1$ is equal to precisely one basis element, as well any path of length three, except the cyclic paths between vertices 2, 4, 5.
- (c) The product of two elements b, b' from \mathcal{B} is either zero, or is again an element in \mathcal{B} . It is non-zero if and only if $t(b) = s(b')$ and bb' has length $\leq 3m$, and if the length is $3m$ then $s(b) = t(b')$. (For this, note that the cyclic paths of length three through the vertices 2, 5, 4 are not products of basis elements.)

- (d) For each $b \in \mathcal{B}_i$ there is a unique $\hat{b} \in \mathcal{B}$ such that $b\hat{b} = \omega_i$: Say $b = be_j$ is of length k , then \mathcal{B}_j must contain a unique element \hat{b} of length $3m - k$ and moreover which ends at i . This is seen by checking through each congruence. Then $b\hat{b}$ is a path of length $3m$ from i to i and it must therefore be equal to ω_i , by Lemma 4.5. It must be unique with $bb' = \omega_i$ and $b' \in \mathcal{B}$.

Corollary 4.6 Λ has dimension $36m$.

The next theorem completes the proof of Theorem 1.

Theorem 4.7 Λ is a symmetric algebra.

Proof We use the above basis to define a symmetrizing bilinear form. If $b, b' \in \mathcal{B}$, define

$$(b, b') := \text{the coefficient of } \omega_i \text{ when } bb' \text{ is expanded in terms of } \mathcal{B}.$$

This extends to a bilinear form, and it is clearly associative. By (c) and (d) above, the Gram matrix of the bilinear form is non-singular, hence the form is non-degenerate. We show that the form is symmetric.

Let $b, b' \in \mathcal{B}$, where $b = e_i b e_j$ and $b' = e_k b' e_l$. Then we have

$$(b, b') = \begin{cases} 1 & j = k, i = l, \ell(bb') = 3m, \\ 0 & \text{else,} \end{cases}$$

and (b', b) is the same. □

5 Proof of Theorem 2

Let (Q, f) be the triangulation quiver associated to the tetrahedron. Then we have the involution $\bar{\cdot} : Q_1 \rightarrow Q_1$ on the set Q_1 of arrows of Q which assigns to an arrow $\theta \in Q_1$ the arrow $\bar{\theta}$ with $s(\theta) = s(\bar{\theta})$ and $\theta \neq \bar{\theta}$. With this, we obtain another permutation $g : Q_1 \rightarrow Q_1$ such that $g(\theta) = \overline{f(\bar{\theta})}$ for any $\theta \in Q_1$, as indicated in the introduction.

Let $m \geq 2$ be a natural number, $\lambda \in K$, and $\Lambda(m, \lambda)$ the associated higher tetrahedral algebra. We will prove first that $\Lambda(m, \lambda)$ is a tame algebra. We divide the proof into several steps.

Proposition 5.1 For each $\lambda \in K \setminus \{0\}$, $\Lambda(m, \lambda)$ degenerates to $\Lambda(m, 0)$.

Proof For each $t \in K$, consider the algebra $\Lambda(t)$ given by the quiver Q and the relations:

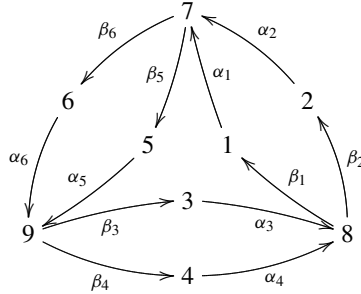
$$\begin{aligned} \gamma\delta &= \beta\varepsilon + t\lambda(\beta\rho\omega)^{m-1}\beta\varepsilon, & \delta\eta &= \nu\omega, & \eta\gamma &= \xi\alpha, & \nu\mu &= \delta\xi, \\ \rho\omega &= \varepsilon\eta + t\lambda(\varepsilon\xi\sigma)^{m-1}\varepsilon\eta, & \omega\beta &= \mu\sigma, & \beta\rho &= \gamma\nu, & \mu\alpha &= \omega\gamma, \\ \xi\sigma &= \eta\beta + t\lambda(\eta\gamma\delta)^{m-1}\eta\beta, & \sigma\varepsilon &= \alpha\delta, & \varepsilon\xi &= \rho\mu, & \alpha\nu &= \sigma\rho, \end{aligned}$$

$$(\theta f(\theta) f^2(\bar{\theta}))^{m-1} \theta f(\theta) g(f(\theta)) = 0 \text{ for any arrow } \theta \text{ in } Q.$$

Then $\Lambda(t)$, $t \in K$, is an algebraic family in the variety $\text{alg}_d(K)$, with $d = 36m$. Observe that $\Lambda(0) \cong \Lambda(m, 0)$ and $\Lambda(1) \cong \Lambda(m, \lambda)$. Fix $t \in K \setminus \{0\}$, and take an element $a_t \in K$ with $a_t^{3(m-1)} = t$. Then there is an isomorphism of algebras $\varphi_t : \Lambda(1) \rightarrow \Lambda(t)$ such that

$\varphi_t(\theta) = a_t\theta$ for any arrow θ in Q . This shows that $\Lambda(t) \cong \Lambda(1)$ for all $t \in K \setminus \{0\}$. Then it follows from Proposition 2.2 that $\Lambda(m, \lambda)$ degenerates to $\Lambda(m, 0) = \Lambda(0)$. \square

Let $\Omega(m)$ be the algebra given by quiver Δ of the form



and the relations:

$$\beta_1\alpha_1 = \beta_2\alpha_2, \quad \beta_3\alpha_3 = \beta_4\alpha_4, \quad \beta_5\alpha_5 = \beta_6\alpha_6,$$

$$\begin{aligned} \alpha_1(\beta_5\alpha_5\beta_3\alpha_3\beta_1\alpha_1)^{m-1}\beta_5\alpha_5\beta_3\alpha_3\beta_2 &= 0, & \alpha_2(\beta_6\alpha_6\beta_4\alpha_4\beta_2\alpha_2)^{m-1}\beta_6\alpha_6\beta_4\alpha_4\beta_1 &= 0, \\ \alpha_3(\beta_1\alpha_1\beta_5\alpha_5\beta_3\alpha_3)^{m-1}\beta_1\alpha_1\beta_5\alpha_5\beta_4 &= 0, & \alpha_4(\beta_2\alpha_2\beta_6\alpha_6\beta_4\alpha_4)^{m-1}\beta_2\alpha_2\beta_6\alpha_6\beta_3 &= 0, \\ \alpha_5(\beta_3\alpha_3\beta_1\alpha_1\beta_5\alpha_5)^{m-1}\beta_3\alpha_3\beta_1\alpha_1\beta_6 &= 0, & \alpha_6(\beta_4\alpha_4\beta_2\alpha_2\beta_6\alpha_6)^{m-1}\beta_4\alpha_4\beta_2\alpha_2\beta_5 &= 0. \end{aligned}$$

For each vertex i of Δ , we denote by e_i the primitive idempotent of $\Omega(m)$ associated to i . Moreover, let $e = e_1 + e_2 + e_3 + e_4 + e_5 + e_6$.

Lemma 5.2 *The following statements hold:*

- (i) $\Omega(m)$ is a finite-dimensional algebra with $\dim_K \Omega(m) = 81m + 3$.
- (ii) $\Lambda(m, 0)$ is isomorphic to the idempotent algebra $e\Omega(m)e$.

Proof (i) A direct checking shows that $\dim_K e_i\Omega(m) = 9m$ for $i \in \{1, 2, 3, 4, 5, 6\}$, and $\dim_K e_j\Omega(m) = 9m + 1$ for $j \in \{7, 8, 9\}$. Therefore, we obtain $\dim_K \Omega(m) = 81m + 3$.

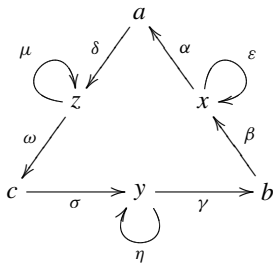
(ii) Consider the paths of length 2 in Δ

$$\begin{aligned} \delta &= \alpha_1\beta_5, & \nu &= \alpha_1\beta_6, & \varepsilon &= \alpha_2\beta_5, & \varrho &= \alpha_2\beta_6, & \alpha &= \alpha_3\beta_1, & \sigma &= \alpha_3\beta_2, \\ \gamma &= \alpha_4\beta_1, & \beta &= \alpha_4\beta_2, & \xi &= \alpha_5\beta_3, & \eta &= \alpha_5\beta_4, & \mu &= \alpha_6\beta_3, & \omega &= \alpha_6\beta_4. \end{aligned}$$

Then these paths satisfy the relations defining the algebra $\Lambda(m, 0)$. Therefore, $e\Omega(m)e$ is isomorphic to $\Lambda(m, 0)$. \square

The algebra $\Omega(m)$ can be viewed as a blowup of the algebra $\Lambda(m, 0)$. The reason to consider it here is as follows. The higher tetrahedral algebras $\Lambda(m, \lambda)$ have no visible degenerations to special biserial algebras. But the algebra $\Omega(m)$ admits a degeneration to a special biserial algebra, as we will show below. Then Proposition 2.1 will imply that $\Omega(m)$ is a tame algebra, and consequently $\Lambda(m, 0)$ is a tame algebra (see [6, Theorem]).

For each $t \in K$, let $\Sigma(m, t)$ be the algebra given by the quiver Σ of the form



and the relations:

$$\beta\alpha = 0, \quad \sigma\gamma = 0, \quad \delta\omega = 0, \quad \varepsilon^2 = t\varepsilon, \quad \eta^2 = t\eta, \quad \mu^2 = t\mu, \quad (1)$$

$$\begin{aligned} t(\alpha\delta\mu\omega\sigma\eta\gamma\beta\varepsilon)^m &= \varepsilon(\alpha\delta\mu\omega\sigma\eta\gamma\beta\varepsilon)^m, & t(\varepsilon\alpha\delta\mu\omega\sigma\eta\gamma\beta)^m &= (\varepsilon\alpha\delta\mu\omega\sigma\eta\gamma\beta)^m\varepsilon, \\ t(\gamma\beta\varepsilon\alpha\delta\mu\omega\sigma\eta)^m &= \eta(\gamma\beta\varepsilon\alpha\delta\mu\omega\sigma\eta)^m, & t(\eta\gamma\beta\varepsilon\alpha\delta\mu\omega\sigma)^m &= (\eta\gamma\beta\varepsilon\alpha\delta\mu\omega\sigma)^m\eta, \\ t(\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta\mu)^m &= \mu(\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta\mu)^m, & t(\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta)^m &= (\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta)^m\mu, \end{aligned} \quad (2)$$

$$\begin{aligned} (\alpha\delta\mu\omega\sigma\eta\gamma\beta\varepsilon)^m &= (\varepsilon\alpha\delta\mu\omega\sigma\eta\gamma\beta)^m, & (\gamma\beta\varepsilon\alpha\delta\mu\omega\sigma\eta)^m &= (\eta\gamma\beta\varepsilon\alpha\delta\mu\omega\sigma)^m, \\ (\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta\mu)^m &= (\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta)^m, \end{aligned} \quad (3)$$

$$\begin{aligned} (\delta\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha)^m\delta &= 0, & \alpha(\delta\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha)^m &= 0, & (\beta\varepsilon\alpha\delta\mu\omega\sigma\eta\gamma)^m\beta &= 0, \\ \gamma(\beta\varepsilon\alpha\delta\mu\omega\sigma\eta\gamma)^m &= 0, & (\sigma\eta\gamma\beta\varepsilon\alpha\delta\mu\omega)^m\sigma &= 0, & \omega(\sigma\eta\gamma\beta\varepsilon\alpha\delta\mu\omega)^m &= 0. \end{aligned} \quad (4)$$

We note that for $t \in K \setminus \{0\}$ the relations (3) follow from the relations (2), and the relations (4) from the relations (1) and (2). For example, we have the equalities

$$\begin{aligned} t(\delta\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha)^m\delta &= t\delta(\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta)^m = \delta(\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta)^m\mu \\ &= \delta\mu(\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta\mu)^m = t\delta(\omega\sigma\eta\gamma\beta\varepsilon\alpha\delta\mu)^m = 0, \end{aligned}$$

because $\delta\omega = 0$, and hence $(\delta\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha)^m\delta = 0$, for $t \in K \setminus \{0\}$. For each vertex i of Σ , we denote by f_i the primitive idempotent of $\Sigma(m, t)$ associated to i .

Lemma 5.3 *The following statements hold:*

- (i) For each $t \in K$, $\Sigma(m, t)$ is a finite-dimensional algebra with $\dim_K \Sigma(m, t) = 81m + 3$.
- (ii) $\Sigma(m, t) \cong \Sigma(m, 1)$ for any $t \in K \setminus \{0\}$.
- (iii) $\Sigma(m, 0)$ is a special biserial algebra.

Proof (i) It follows from the relations defining $\Sigma(m, t)$ that $\dim_K f_i \Sigma(m, t) = 9m + 1$ for $i \in \{a, b, c\}$, and $\dim_K f_j \Sigma(m, t) = 18m$ for $j \in \{x, y, z\}$. Hence, we obtain $\dim_K \Sigma(m, t) = 81m + 3$.

(ii) Fix $t \in K \setminus \{0\}$, and take an element $b_t \in K$ with $b_t^8 = t$. Then there exists an isomorphism of algebras $\psi_t : \Sigma(m, 1) \rightarrow \Sigma(m, t)$ such that $\psi_t(\varepsilon) = t^{-1}\varepsilon$, $\psi_t(\eta) = t^{-1}\eta$, $\psi_t(\mu) = t^{-1}\mu$, and $\psi_t(\theta) = b_t\theta$ for any arrow $\theta \in \{\alpha, \beta, \gamma, \sigma, \omega, \delta\}$.

(iii) Follows from the relations defining $\Sigma(m, 0)$. □

Lemma 5.4 *The algebras $\Omega(m)$ and $\Sigma(m, 1)$ are isomorphic.*

Proof We shall prove that there is a well defined isomorphism of algebras $\varphi : \Omega(m) \rightarrow \Sigma(m, 1)$ such that

$$\begin{aligned} \varphi(e_1) &= \varepsilon, & \varphi(e_2) &= f_x - \varepsilon, & \varphi(e_3) &= \eta, & \varphi(e_4) &= f_y - \eta, \\ \varphi(e_5) &= \mu, & \varphi(e_6) &= f_z - \mu, & \varphi(e_7) &= f_a, & \varphi(e_8) &= f_b, & \varphi(e_9) &= f_c, \\ \varphi(\alpha_1) &= \varepsilon\alpha, & \varphi(\alpha_2) &= \alpha - \varepsilon\alpha, & \varphi(\beta_1) &= \beta\varepsilon, & \varphi(\beta_2) &= -\beta + \beta\varepsilon, \\ \varphi(\alpha_3) &= \eta\gamma, & \varphi(\alpha_4) &= \gamma - \eta\gamma, & \varphi(\beta_3) &= \sigma\eta, & \varphi(\beta_4) &= -\sigma + \sigma\eta, \\ \varphi(\alpha_5) &= \mu\omega, & \varphi(\alpha_6) &= \omega - \mu\omega, & \varphi(\beta_5) &= \delta\mu, & \varphi(\beta_6) &= -\delta + \delta\mu. \end{aligned}$$

Observe that

$$\begin{aligned} \varphi(e_1 + e_2) &= f_x, & \varphi(e_3 + e_4) &= f_y, & \varphi(e_5 + e_6) &= f_z, \\ \varphi(\alpha_1 + \alpha_2) &= \alpha, & \varphi(\alpha_3 + \alpha_4) &= \gamma, & \varphi(\alpha_5 + \alpha_6) &= \omega, \\ \varphi(\beta_1 - \beta_2) &= \beta, & \varphi(\beta_3 - \beta_4) &= \sigma, & \varphi(\beta_5 - \beta_6) &= \delta. \end{aligned}$$

We have in $\Sigma(m, 1)$ the following equalities

$$\begin{aligned} \varphi(e_1^2) &= \varphi(e_1) = \varepsilon = \varepsilon^2 = \varphi(e_1)^2, \\ \varphi(e_2^2) &= \varphi(e_2) = f_x - \varepsilon = (f_x - \varepsilon)^2 = \varphi(e_2)^2, \\ \varphi(e_3^2) &= \varphi(e_3) = \eta = \eta^2 = \varphi(e_3)^2, \\ \varphi(e_4^2) &= \varphi(e_4) = f_y - \eta = (f_y - \eta)^2 = \varphi(e_4)^2, \\ \varphi(e_5^2) &= \varphi(e_5) = \mu = \mu^2 = \varphi(e_5)^2, \\ \varphi(e_6^2) &= \varphi(e_6) = f_z - \mu = (f_z - \mu)^2 = \varphi(e_6)^2, \\ \varphi(\beta_1)\varphi(\alpha_1) &= \beta\varepsilon^2\alpha = \beta\varepsilon\alpha = (-\beta + \beta\varepsilon)(\alpha - \varepsilon\alpha) = \varphi(\beta_2)\varphi(\alpha_2), \\ \varphi(\beta_3)\varphi(\alpha_3) &= \sigma\eta^2\gamma = \sigma\eta\gamma = (-\sigma + \sigma\eta)(\gamma - \eta\gamma) = \varphi(\beta_4)\varphi(\alpha_4), \\ \varphi(\beta_5)\varphi(\alpha_5) &= \delta\mu^2\omega = \delta\mu\omega = (-\delta + \delta\mu)(\omega - \mu\omega) = \varphi(\beta_6)\varphi(\alpha_6). \end{aligned}$$

It remains to show that the six zero relations defining $\Omega(m)$ correspond via φ to the six commutativity relations (2), with $t = 1$, defining $\Sigma(m, 1)$. We will show this for the first two relations, because the proof for the other four is similar.

We have the equalities

$$\begin{aligned} &\varphi(\alpha_1)(\varphi(\beta_5)\varphi(\alpha_5)\varphi(\beta_3)\varphi(\alpha_3)\varphi(\beta_1)\varphi(\alpha_1))^{m-1}\varphi(\beta_5)\varphi(\alpha_5)\varphi(\beta_3)\varphi(\alpha_3)\varphi(\beta_2) \\ &= \varepsilon\alpha(\delta\mu^2\omega\sigma\eta^2\gamma\beta\varepsilon^2\alpha)^{m-1}\delta\mu^2\omega\sigma\eta^2\gamma(-\beta + \beta\varepsilon) \\ &= -\varepsilon\alpha(\delta\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha)^{m-1}\delta\mu\omega\sigma\eta\gamma\beta + \varepsilon\alpha(\delta\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha)^{m-1}\delta\mu\omega\sigma\eta\gamma\beta\varepsilon \\ &= -(\varepsilon\alpha\delta\mu\omega\sigma\eta\gamma\beta)^m + (\varepsilon\alpha\delta\mu\omega\sigma\eta\gamma\beta)^m\varepsilon = 0, \\ &\varphi(\alpha_2)(\varphi(\beta_6)\varphi(\alpha_6)\varphi(\beta_4)\varphi(\alpha_4)\varphi(\beta_2)\varphi(\alpha_2))^{m-1}\varphi(\beta_6)\varphi(\alpha_6)\varphi(\beta_4)\varphi(\alpha_4)\varphi(\beta_1) \\ &= \varphi(\alpha_2)(\varphi(\beta_5)\varphi(\alpha_5)\varphi(\beta_3)\varphi(\alpha_3)\varphi(\beta_1)\varphi(\alpha_1))^{m-1}\varphi(\beta_5)\varphi(\alpha_5)\varphi(\beta_3)\varphi(\alpha_3)\varphi(\beta_1) \\ &= (\alpha - \varepsilon\alpha)(\delta\mu\omega\sigma\eta\gamma\beta\varepsilon\alpha)^{m-1}\delta\mu\omega\sigma\eta\gamma\beta\varepsilon \\ &= (\alpha\delta\mu\omega\sigma\eta\gamma\beta\varepsilon)^m - \varepsilon(\alpha\delta\mu\omega\sigma\eta\gamma\beta\varepsilon)^m = 0. \end{aligned}$$

□

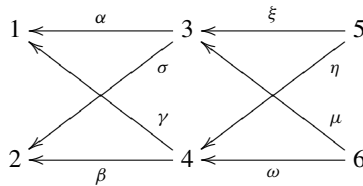
Corollary 5.5 *The algebra $\Omega(m)$ degenerates to the special biserial algebra $\Sigma(m, 0)$. In particular, $\Omega(m)$ is a tame algebra.*

Proof It follows from Lemmas 5.3 and 5.4 that $\Sigma(m, 1)$, $t \in K$, is an algebraic family in the variety $\text{alg}_K(d)$ with $d = 81m + 3$ such that $\Sigma(m, 1) \cong \Sigma(m, 1) \cong \Omega(m)$ for any

$t \in K \setminus \{0\}$ and $\Sigma(m, 0)$ is a special biserial algebra. Then it follows from Propositions 2.1 and 2.2 that $\Omega(m)$ is a tame algebra. □

Proposition 5.6 *For each $\lambda \in K$, $\Lambda(m, \lambda)$ is a tame algebra of non-polynomial growth.*

Proof It follows from Lemma 5.2 (ii), Corollary 5.5 and [6, Theorem] that $\Lambda(m, 0)$ is a tame algebra. Then, applying Propositions 2.2 and 5.1, we conclude that $\Lambda(m, \lambda)$ is a tame algebra for any $\lambda \in K \setminus \{0\}$. $\Lambda = \Lambda(m, \lambda)$ for an arbitrary $\lambda \in K$. Consider now the quotient algebra Γ of Λ by the ideal generated by the arrows $\delta, \nu, \varepsilon, \varrho$. Then Γ is the algebra given by the quiver



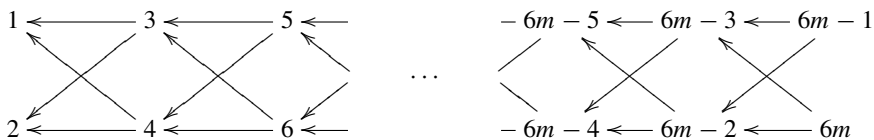
and the relations

$$\omega\beta = \mu\sigma, \quad \eta\gamma = \xi\alpha, \quad \mu\alpha = \omega\gamma, \quad \xi\sigma = \eta\beta.$$

Then Γ is the tame minimal non-polynomial growth algebra (30) from [18]. Therefore, Λ is of non-polynomial growth. □

We end this section with a Galois covering interpretation of the singular higher tetrahedral algebras.

Let $m \geq 2$ be a natural number. We denote by $B(m)$ the fully commutative algebra of the following quiver



Consider the repetitive category $\widehat{B(m)}$ of $B(m)$. Then the Nakayama automorphism $v_{\widehat{B(m)}}$ of $\widehat{B(m)}$ admits an m -th root φ_m such that $(\varphi_m)^m = v_{\widehat{B(m)}}$. Let $\Gamma(m)$ be the orbit algebra $\widehat{B(m)}/(\varphi_m)$ of $\widehat{B(m)}$ with respect to the infinite cyclic group (φ_m) generated by φ_m (see [24] for relevant definitions).

Then we obtain the following proposition.

Proposition 5.7 *The algebras $\Lambda(m, 0)$ and $\Gamma(m)$ are isomorphic.*

We would like to stress that, for any $\lambda \in K \setminus \{0\}$, the non-singular higher tetrahedral algebra $\Lambda(m, \lambda)$ is not the orbit algebra of the repetitive category of an algebra.

6 Proof of Theorem 3

Throughout this section, $\Lambda = \Lambda(m, \lambda)$, for some $m \geq 2$ and $\lambda \in K^*$.

We show first that every simple Λ -module is periodic of period four. This will then tell us what the terms of a minimal projective bimodule resolution of Λ must be (see Proposition 3.1). As for notation, we write Ω for syzygies of right Λ -modules, and we write Ω_{Λ^e} for syzygies of right Λ^e -modules (Λ - Λ -bimodules).

Proposition 6.1 *Each simple Λ -module is periodic of period four. There is an exact sequence*

$$0 \rightarrow S_i \rightarrow P_i \rightarrow P_x \oplus P_y \rightarrow P_j \oplus P_k \rightarrow P_i \rightarrow S_i \rightarrow 0$$

where the arrows adjacent to i end at j, k and start at x, y .

Proof The automorphism φ of Λ induces an equivalence of the module category $\text{mod } \Lambda$, with two orbits on simple modules. We only need to prove periodicity for one simple from each orbit. We will consider S_1 and S_4 .

(1) We compute $\Omega^2(S_1)$ which we identify with the kernel of the map $d_1 : P_6 \oplus P_5 \rightarrow P_1$ defined by

$$d_1(a, b) := \nu a + \delta b,$$

for $a \in P_6$ and $b \in P_5$. Since $\nu\omega = \delta\eta$ and $\nu\mu = \delta\xi$, the kernel contains the submodule generated by ϕ and ψ , where

$$\phi = (-\omega, \eta) \quad \text{and} \quad \psi = (\mu, -\xi).$$

We will show that $\text{Ker } d_1 = \phi\Lambda + \psi\Lambda$. Since we have one inclusion, it suffices to show that both spaces have the same dimension, that is, we must show that $\phi\Lambda + \psi\Lambda$ has dimension $6m + 1$. We observe that $\phi\Lambda$ is isomorphic to $\Omega^{-1}(S_4)$ since ω, η are the arrows ending at vertex 4. Similarly, $\psi\Lambda \cong \Omega^{-1}(S_3)$. In particular, $\dim_K \phi\Lambda = 6m - 1 = \dim_K \psi\Lambda$. It follows that we must show that $\dim_K \phi\Lambda \cap \psi\Lambda = 6m - 3$, that is,

$$\dim_K \phi\Lambda / (\phi\Lambda \cap \psi\Lambda) = 2.$$

(1a) We identify the intersections of $\phi\Lambda$ and $\psi\Lambda$ with $0 \oplus P_5$. We claim that each of $\phi\Lambda \cap (0 \oplus P_5)$ and $\psi\Lambda \cap (0 \oplus P_5)$ is 1-dimensional, spanned by $(0, X_5^m)$. Indeed, suppose $\phi p = (0, z)$ for some $p \in \Lambda$ and $0 \neq z$. We may assume that p is a monomial in the arrows. To have $\omega p = 0$ the monomial p must have length $\geq 3m - 1$. To have $\eta p = 0$ and $\eta p = z \neq 0$, we must have that p has length $3m - 1$ and ends at vertex 4, and then $\eta p = X_5^m$. For the converse, take $p = X_4^{m-1}\gamma\delta$. Similarly one proves the second statement.

(1b) We claim that $\phi J^2 + \phi\gamma K$ is contained in the intersection $\phi\Lambda \cap \psi\Lambda$. Namely, we have $\phi\gamma = -\psi\alpha$, by the relations. Next, we have

$$\phi\beta = (-\omega\beta, \eta\beta) = -\psi\sigma - (0, \lambda X_5^{m-1}\eta\beta). \tag{*}$$

Hence $\phi\beta\varepsilon = \psi\sigma\varepsilon - (0, \lambda X_5^m)$ (using Lemmas 4.3, 4.4, 4.5). By (1a) above, this belongs to the intersection and it follows from these that $\phi J^2 \subseteq \phi\Lambda \cap \psi\Lambda$. We note that if $\lambda = 0$, then $\Omega^2(S_i)$ has more than two minimal generators, and hence S_i is not periodic of period 4.

(1c) Note that $\phi J^2 + \phi\gamma K$ has dimension $6m - 3$. We have the chain of submodules

$$\phi J^2 + \phi\gamma K \subseteq \phi\Lambda \cap \psi\Lambda \subseteq \phi\Lambda,$$

and the quotient $\phi\Lambda / (\phi J^2 + \phi\gamma K)$ is spanned by the cosets of ϕ and $\phi\beta$.

Assume for a contradiction that $\phi\beta \in \phi\Lambda \cap \psi\Lambda$. Then, by (*), we have $(0, X_5^{m-1}\eta\beta) \in \psi\Lambda$, but this contradicts (1a). So $\phi\beta$ is not in the intersection, and therefore the dimension of $\phi\Lambda / (\phi\Lambda \cap \psi\Lambda)$ is 2, as required.

(1d) Now it is easy to see that S_1 has period four. Namely, define $d_2 : P_4 \oplus P_3 \rightarrow \Omega^2(S_1)$ by

$$d_2(u, v) := \phi u + \psi v,$$

for $u \in P_4$ and $v \in P_3$. The kernel of d_2 , that is, $\Omega^3(S_1)$ has dimension $2(6m) - (6m + 1) = 6m - 1$. We have seen that $\phi\gamma = -\psi\alpha$, and therefore $(\gamma, \alpha)\Lambda \subseteq \text{Ker}(d_2)$. This submodule is isomorphic to $\Omega^{-1}(S_1)$ and has dimension $6m - 1$. We deduce that

$$\Omega^{-1}(S_1) \cong (\gamma, \alpha)\Lambda \cong \Omega^3(S_1).$$

So S_1 is periodic of period dividing 4, and then equal to 4.

(2) We compute $\Omega^2(S_4)$, which we identify with the kernel of $d_1 : P_1 \oplus P_2 \rightarrow P_4$ defined as

$$d_1(w, z) = \gamma w + \beta z,$$

for $w \in P_1$ and $z \in P_2$. This is analogous to (1), there is only a small difference in the formulae. Using the relations, the kernel of d_1 contains ϕ and ψ , where

$$\phi = (\delta, -\varepsilon - \lambda(\rho\omega\beta)^{m-1}\varepsilon) \quad \text{and} \quad \psi = (-\nu, \rho).$$

By the same arguments as in (1), to prove that $\text{Ker}(d_1) = \phi\Lambda + \psi\Lambda$, we must show that $\dim_K \phi\Lambda / (\phi\Lambda \cap \psi\Lambda) = 2$. We have $\phi\eta = -\psi\omega$, which is in the intersection, and we have

$$\phi\xi = -\psi\mu - (0, \lambda X_2^{m-1}\varepsilon\xi).$$

As before one shows that $\phi J^2 = \psi J^2$ and hence is in the intersection. Suppose $\phi\xi$ is in the intersection. Then it follows that $(0, -\lambda X_2^{m-1}\varepsilon\xi)$ is in $\psi\Lambda$, which is a contradiction to the analog of (1a). It follows that $\phi\Lambda / (\phi\Lambda \cap \psi\Lambda)$ is 2-dimensional. Then as in (1d) one concludes that S_4 has Ω -period four. □

We use the notation as in Section 3, in particular the description of \mathbb{P}_0 and \mathbb{P}_1 . For the higher tetrahedral algebra, we need to specify \mathbb{P}_2 , which has generators corresponding to the minimal relations involving paths of length two. Each of these minimal relations has a term $\theta f(\theta)$ for θ an arrow, and this gives a bijection between arrows and minimal relations involving paths of length two. So we take

$$\mathbb{P}_2 := \bigoplus_{\theta \in Q_1} \Lambda(e_{s(\theta)} \otimes e_{t(f(\theta))})\Lambda.$$

We may denote the minimal relation with term $\theta f(\theta)$ by μ_θ . Then the definition of R in Section 3 specializes to

$$R : \mathbb{P}_2 \rightarrow \mathbb{P}_1, \quad R(e_{s(\theta)} \otimes e_{t(f(\theta))}) := \pi(\mu_\theta).$$

Lemma 6.2 *The homomorphism $R : \mathbb{P}_2 \rightarrow \mathbb{P}_1$ induces a projective cover of $\Omega_{\Lambda^e}^2(\Lambda)$ in mod Λ^e . In particular, $\Omega_{\Lambda^e}^3(\Lambda) = \text{Ker } R$.*

Proof This is similar as that of Lemma 7.2 of [11], and uses Lemma 3.4. □

By Propositions 3.1 and 6.1, we can take $\mathbb{P}_3 = \bigoplus_{i \in Q_0} \Lambda(e_i \otimes e_i)\Lambda$. For each vertex i of Q , we define an element ψ_i as follows. Let $\tau, \bar{\tau}$ be the arrows starting at i , and let $\theta, \bar{\theta}$ be the arrows ending at i . Set

$$\psi_i := (e_i \otimes e_{t(\theta)})\theta + (e_i \otimes e_{t(\bar{\theta})})\bar{\theta} - \tau(e_{t(\tau)} \otimes e_i) - \bar{\tau}(e_{t(\bar{\tau})} \otimes e_i).$$

Then we define a Λ^e -module homomorphism $S : \mathbb{P}_3 \rightarrow \mathbb{P}_2$ by

$$S(e_i \otimes e_i) := \psi_i, \quad \text{for } i \in Q_0.$$

Lemma 6.3 *The homomorphism $S : \mathbb{P}_3 \rightarrow \mathbb{P}_2$ induces a projective cover of $\Omega_{\Lambda^e}^3(\Lambda)$ in mod Λ^e . In particular, we have $\Omega_{\Lambda^e}^4(\Lambda) = \text{Ker}(S)$.*

Proof We know that the kernel of R is $\Omega_{\Lambda^e}^3(\Lambda)$, and we know that it has minimal generators corresponding to the vertices of Q . As well, from the definition, the element ψ_i does not lie in $(\text{rad } \mathbb{P}_2)^2$. Therefore, it is enough to show that $R(\psi_i) = 0$ for all i .

The algebra automorphism φ of Λ defined in Section 4, extends to an automorphism of Λ^e . One checks that it commutes with the map R and that it takes ψ_i to $\psi_{\varphi(i)}$. So it is enough to take $i = 1$ and $i = 4$.

(1) We compute $R(\psi_1)$. This is equal to

$$\begin{aligned} &R((e_1 \otimes e_4)\gamma + (e_1 \otimes e_3)\alpha - v(e_6 \otimes e_1) - \delta(e_5 \otimes e_1)) \\ &= \pi(\delta\eta - v\omega)\gamma + \pi(v\mu - \delta\xi)\alpha - v(\pi(\mu\alpha - \omega\gamma)) - \delta(\pi(\eta\gamma - \xi\alpha)) \\ &= (e_1 \otimes \eta\gamma + \delta \otimes \gamma - e_1 \otimes \omega\gamma - v \otimes \gamma) \\ &\quad + (e_1 \otimes \mu\alpha + v \otimes \alpha - e_1 \otimes \xi\alpha - \delta \otimes \alpha) \\ &\quad - (v \otimes \alpha + v\mu \otimes e_1 - v \otimes \gamma - v\omega \otimes e_1) \\ &\quad - (\delta \otimes \gamma + \delta\eta \otimes e_1 - \delta \otimes \alpha - \delta\xi \otimes e_1). \end{aligned}$$

The terms of the form $\alpha_1 \otimes \alpha_2$ for α_i arrows, cancel. The terms in $(e_1 \otimes e_5)\Lambda$ are

$$e_1 \otimes \eta\gamma - e_1 \otimes \xi\alpha = e_1 \otimes (\eta\gamma - \xi\alpha) = 0.$$

Similarly, there are two terms in $(e_1 \otimes e_6)\Lambda$ and two terms in $\Lambda(e_4 \otimes e_1)$ and two terms in $\Lambda(e_3 \otimes e_1)$, and they all cancel. Hence $R(\psi_1) = 0$.

(2) We compute $R(\psi_4)$. This is equal to

$$\begin{aligned} &R((e_4 \otimes e_5)\eta + (e_4 \otimes e_6)\omega - \gamma(e_1 \otimes e_4) - \beta(e_2 \otimes e_4)) \\ &= \pi(\gamma\delta - \beta\varepsilon - \lambda X_4^{m-1}\beta\varepsilon)\eta + \pi(\beta\rho - \gamma v)\omega \\ &\quad - \gamma(\pi(\delta\eta - v\omega)) - \beta(\pi(\rho\omega - \varepsilon\eta - \lambda X_2^{m-1}\varepsilon\eta)). \end{aligned} \tag{*}$$

We must choose a version of X_4 and of X_2 . It is natural to take $X_4 = \beta\rho\omega$ and $X_2 = \rho\omega\beta$. We continue the calculation. With this, (*) is equal to

$$\begin{aligned} &(e_4 \otimes \delta\eta + \gamma \otimes \eta - e_4 \otimes \varepsilon\eta - \beta \otimes \eta) - \lambda\pi((\beta\rho\omega)^{m-1}\beta\varepsilon)\eta \\ &\quad + (e_4 \otimes \rho\omega + \beta \otimes \omega - e_4 \otimes v\omega - \gamma \otimes \omega) \\ &\quad - (\gamma \otimes \eta + \gamma\delta \otimes e_4 - \gamma \otimes \omega - \gamma v \otimes e_4) \\ &\quad - (\beta \otimes \omega + \beta\rho \otimes e_4 - \beta \otimes \eta - \beta\varepsilon \otimes e_4) + \lambda\beta(\pi((\rho\omega\beta)^{m-1}\varepsilon\eta)). \end{aligned}$$

The terms of the form $\alpha_1 \otimes \alpha_2$ with α_i arrows all cancel. Using the relations $\delta\eta = v\omega$ and $\beta\rho = \gamma v$, four of the other terms cancel. This leaves

$$-e_4 \otimes \varepsilon\eta + e_4 \otimes \rho\omega - \lambda\pi((\beta\rho\omega)^{m-1}\beta\varepsilon)\eta - \gamma\delta \otimes e_4 + \beta\varepsilon \otimes e_4 + \lambda\beta\pi((\rho\omega\beta)^{m-1}\varepsilon\eta).$$

The first two terms combine, and the fourth and fifth term combine, and we can rewrite the expression as

$$\lambda(e_4 \otimes X_2^{m-1}\varepsilon\eta) - \lambda\pi((\beta\rho\omega)^{m-1}\beta\varepsilon)\eta - \lambda(X_4^{m-1}\beta\varepsilon \otimes e_4) + \lambda\beta\pi((\rho\omega\beta)^{m-1}\varepsilon\eta). \tag{**}$$

Now we combine the second and fourth term of (**), and we expand both. All terms except the ones $- \otimes e_4$ and $e_4 \otimes -$ cancel, and we are left with

$$\lambda((\beta\rho\omega)^{m-1}\beta\varepsilon \otimes e_4 - e_4 \otimes (\rho\omega\beta)^{m-1}\varepsilon\eta). \tag{***}$$

The first term of (***) is the negative of the third term in (**) since $\beta\rho\omega = X_4$. The second term of (***) is the negative of the first term of (**) since $\rho\omega\beta = X_2$. Hence, everything cancels and $R(\psi_4) = 0$, as required. \square

Theorem 6.4 *There is an isomorphism $\Omega_{\Lambda^e}^4(\Lambda) \cong \Lambda$ in mod Λ^e .*

Proof This is similar as in the proof of Theorem 7.4 in [11]. We have defined a symmetrizing bilinear form of Λ in the proof of Theorem 4.7. We define elements $\xi_i \in \mathbb{P}_3$ by

$$\xi_i = \sum_{b \in \mathcal{B}_i} b \otimes b^*,$$

where $\{b^* : b \in \mathcal{B}\}$ is the dual basis corresponding to \mathcal{B} , defined by $(-, -)$. As in [11], it follows that the map

$$\theta : \Lambda \rightarrow \mathbb{P}_3, \text{ with } \theta(e_i) = \xi_i \text{ for all } i \in Q_0,$$

is a monomorphism of Λ - Λ -bimodules. Moreover, one shows that $S(\xi_i) = 0$, exactly as in [11], and hence the image of θ is contained in $\Omega_{\Lambda^e}^4(\Lambda) = \text{Ker } S$ (Lemma 6.3). This only uses general properties of the dual basis and no details on a specific algebra. Furthermore, $\Omega_{\Lambda^e}^4(\Lambda)$ is free of rank 1 as a left or right Λ -module. Namely, by Lemmas 3.3, 6.2, and 6.3, we have the exact sequence of bimodules

$$0 \rightarrow \Omega_{\Lambda^e}^4(\Lambda) \rightarrow \mathbb{P}_3 \xrightarrow{S} \mathbb{P}_2 \xrightarrow{R} \mathbb{P}_1 \xrightarrow{d} \mathbb{P}_0 \rightarrow \Lambda \rightarrow 0.$$

We have $\mathbb{P}_0 \cong \mathbb{P}_3$, and moreover \mathbb{P}_1 and \mathbb{P}_2 have obviously the same rank as free Λ -modules on each side. By the exactness, it follows that Λ and $\Omega_{\Lambda^e}^4(\Lambda)$ have the same rank. Therefore, the map θ gives an isomorphism of Λ with $\Omega_{\Lambda^e}^4(\Lambda)$.

Alternatively, for the last step one may apply [14] to show that $\Omega_{\Lambda^e}^4(\Lambda)$ must be isomorphic to ${}_1\Lambda_\sigma$ for some algebra automorphism σ , and therefore has rank 1 on each side. \square

Theorem 3 follows from Proposition 6.1, Theorem 6.4, and the following proposition.

Proposition 6.5 *Let $A = \Lambda(m, 0)$. Then mod A does not admit a periodic simple module.*

Proof Take $i \in \{1, 3, 5\}$. Observe that, for the indecomposable projective A -modules $P_i = e_i A$ and $P_{i+1} = e_{i+1} A$, we have $\text{rad } P_i / \text{soc } P_i \cong \text{rad } P_{i+1} / \text{soc } P_{i+1}$ in mod A . Then, by general theory, $P_i / \text{soc } P_i$ and $P_{i+1} / \text{soc } P_{i+1}$ are not in stable tubes of the stable Auslander-Reiten quiver Γ_A^s of A . Since A is a symmetric algebra, we conclude that S_i and S_{i+1} are not periodic modules. \square

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