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Derived Equivalence Classification of the Gentle Two-Cycle Algebras

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Abstract We complete a derived equivalence classification of the gentle two-cycle algebras initiated in earlier papers by Avella-Alaminos and Bobiński–Malicki.

Keywords Gentle two-cycle algebra · Derived category · Derived equivalence

Mathematics Subject Classification (2010) 16G20 · 18E30

1 Introduction and the Main Result

Throughout the paper k denotes a fixed algebraically closed field. For a (finite-dimensional basic connected) algebra Λ one considers its (bounded) derived category $\mathcal{D}^b(\Lambda)$, which has a structure of a triangulated category. Derived categories seem to be a proper setup to do homological algebra. Derived categories appearing in representation theory of algebras have connections with derived categories studied in algebraic geometry (see for example [11, 24, 31]). Moreover, these categories serve as a source for constructions of categorifications of cluster algebras (this line of research was initiated by a fundamental paper by Buan, Marsh, Reineke, Reiten and Todorov [20]) and have links to theoretical physics (including famous Orlov's theorem [36]).

Algebras Λ' and Λ'' are said to be derived equivalent if the categories $\mathcal{D}^b(\Lambda')$ and $\mathcal{D}^b(\Lambda'')$ are triangle equivalent. A study of derived categories (in particular derived equivalences) in the representation theory of algebras was initiated by papers of Happel [28,

Presented by Henning Krause.

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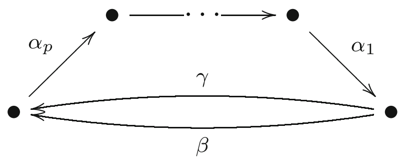
29] and motivated by tilting theory, and is now an important direction of research (see for example [3, 10, 13, 15, 16, 18, 19, 21, 25, 30, 32, 34, 35, 37, 38]).

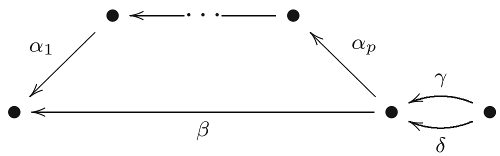
Gentle algebras were introduced by Assem and Skowroński [6] in their study of the algebras derived equivalent to the hereditary algebras of Euclidean type $\tilde{\mathbb{A}}$. Namely, they have proved that the algebras derived equivalent to the hereditary algebras of Euclidean type $\tilde{\mathbb{A}}$ are precisely the gentle one-cycle algebras which satisfy the clock condition. On the other hand, the algebras derived equivalent to the hereditary algebras of Dynkin type $\mathbb{A}$ are precisely the gentle tree algebras [4]. Moreover, the gentle one-cycle algebras which do not satisfy the clock condition are precisely the discrete derived algebras, which are not locally finite [42]. The above motivates study of a derived equivalence classification for the gentle algebras. One should note that the class of gentle algebras is closed with respect to the derived equivalence [40].

By the above results the derived equivalence classes of the gentle algebras with at most one-cycle are known and they are distinguished by the invariant of Avella-Alaminos and Geiss [8]. It is natural to study as the next step a derived equivalence classification of the gentle two-cycle algebras. Here a gentle algebra Λ is called two-cycle if the number of edges in the Gabriel quiver of Λ exceeds by one the number of vertices in this quiver. Before formulating the main result of the paper we define some families of gentle two-cycle algebras.

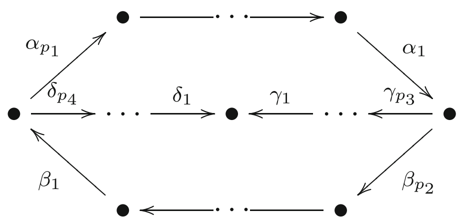
By $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_+$ we denote the sets of integers, nonnegative integers and positive integers, respectively. If i and j are integers, then $[i, j]$ denotes the set of integers l such that $i \leq l \leq j$. For $p \in \mathbb{N}_+$ and $r \in [0, p-1]$, $\Lambda_0(p, r)$ is the algebra of the quiver



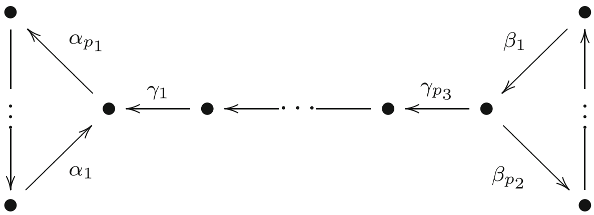
bound by $\alpha_p\beta$, $\alpha_i\alpha_{i+1}$ for $i \in [1, r]$, and $\gamma\alpha_1$. Moreover, for $p \in \mathbb{N}_+$, $\Lambda_0(p+1, -1)$ is the algebra of the quiver



bound by $\alpha_p\gamma$ and $\beta\delta$. Furthermore, for $p_1, p_2 \in \mathbb{N}_+$, $p_3, p_4 \in \mathbb{N}$, and $r_1 \in [0, p_1 - 1]$, such that $p_2 + p_3 \geq 2$ and $p_4 + r_1 \geq 1$, $\Lambda_1(p_1, p_2, p_3, p_4, r_1)$ is the algebra of the quiver



bound by $\alpha_i\alpha_{i+1}$ for $i \in [p_1 - r_1, p_1 - 1]$, $\alpha_{p_1}\beta_1$, $\beta_i\beta_{i+1}$ for $i \in [1, p_2 - 1]$, and $\beta_{p_2}\alpha_1$. Finally, for $p_1, p_2 \in \mathbb{N}_+$, $p_3 \in \mathbb{N}$, $r_1 \in [0, p_1 - 1]$, and $r_2 \in [0, p_2 - 1]$, such that $p_3 + r_1 + r_2 \geq 1$, $\Lambda_2(p_1, p_2, p_3, r_1, r_2)$ is the algebra of the quiver



bound by $\alpha_i\alpha_{i+1}$ for $i \in [p_1 - r_1, p_1 - 1]$, $\alpha_{p_1}\alpha_1$, $\beta_i\beta_{i+1}$ for $i \in [p_2 - r_2, p_2 - 1]$, and $\beta_{p_2}\beta_1$.

The main aim of this paper is to prove the following theorem.

Theorem A *The above defined algebras are representatives of the derived equivalence classes of the gentle two-cycle algebras. More precisely,*

(1) *if Λ is a gentle two-cycle algebra, then Λ is derived equivalent to one of the above defined algebras, and*
(2) *the above defined algebras are pairwise not derived equivalent.*

Parts of Theorem A have been already proved in [17] (see also [7]). More precisely, the following claims have been proved there:

(1) If Λ is a gentle two-cycle algebra, then Λ is derived equivalent to an algebra from one of the families Λ_0, Λ_1 and Λ_2.
(2) The algebras from different families are not derived equivalent.
(3) The algebras from family Λ_1 (Λ_2) are pairwise not derived equivalent.

Thus in order to prove Theorem A, we have to show the following.

Theorem B *If $p', p'' \in \mathbb{N}_+$, $r' \in [-1, p'-1]$, $r'' \in [-1, p''-1]$, and $(p', r') \neq (1, -1) \neq (p'', r'')$, then the algebras $\Lambda_0(p', r')$ and $\Lambda_0(p'', r'')$ are not derived equivalent.*

Partial versions of Theorem B have been obtained independently by Amiot [1] and Kalck [33]. In particular, Amiot has proved this result in the case when r's are "small" relative to p's (see Proposition 2.3 for a precise statement) by refining her earlier joint results with Grimeland on surface algebras [2]. The new ingredient of the paper is Corollary 3.2, which says that if $\Lambda(p, r')$ and $\Lambda(p, r'')$ are derived equivalent, then $\Lambda(p+1, r')$ and $\Lambda(p+1, r'')$ are derived equivalent. Using this and induction we reduce the situation to the setup of Amiot's result.

We note that one can replace derived equivalence by tilting-cotilting equivalence (see for example [6]) in Theorems A and B. Indeed, obviously if algebras are not derived equivalent, then they are not tilting-cotilting equivalent. On the other hand, every derived equivalence obtained in [17] is realized via a tilting-cotilting equivalence.

The paper consists of two sections. In Section 2 we recall necessary tools, including the invariant of Avella-Alaminos and Geiss, Auslander–Reiten quivers, (generalized APR) reflections, and behavior of derived equivalences under one-point coextensions. Next in

Section 3 we prove Theorem B. In the paper we use a formalism of bound quivers introduced by Gabriel [23]. For related background see for example [5].

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2 Preliminaries

2.1 Quivers and Their Representations

By a quiver Δ we mean a set Δ_0 of vertices and a set Δ_1 of arrows together with two maps $s = s_\Delta, t = t_\Delta \colon \Delta_1 \to \Delta_0$, which assign to $\alpha \in \Delta_1$ the starting vertex $s\alpha$ and the terminating vertex $t\alpha$, respectively. We assume that all considered quivers Δ are locally finite, i.e. for each $x \in \Delta_0$ there is only a finite number of $\alpha \in \Delta_1$ such that either $s\alpha = x$ or $t\alpha = x$. A quiver Δ is called finite if Δ_0 (and, consequently, also Δ_1) is a finite set. For technical reasons we assume that if Δ is a quiver, then $\Delta_0 \neq \emptyset$ and Δ has no isolated vertices, i.e. there is no $x \in \Delta_0$ such that $s\alpha \neq x \neq t\alpha$ for each $\alpha \in \Delta_1$. In particular, $\Delta_1 \neq \emptyset$.

Let Δ be a quiver. If $l \in \mathbb{N}_+$, then by a path in Δ of length l we mean every sequence $\sigma = \alpha_1 \cdots \alpha_l$ such that $\alpha_i \in \Delta_1$ for each $i \in [1, l]$ and $s\alpha_i = t\alpha_{i+1}$ for each $i \in [1, l-1]$. In the above situation we put $s\sigma := s\alpha_l$ and $t\sigma := t\alpha_1$. Moreover, we call α_1 and α_l the terminating and the starting arrows of σ, respectively. Observe that each $\alpha \in \Delta$ is a path in Δ of length 1. Moreover, for each $x \in \Delta_0$ we introduce the path $\mathbf{1}_x$ in Δ of length 0 such that $s\mathbf{1}_x := x =: t\mathbf{1}_x$. We denote the length of a path σ by $\ell(\sigma)$. If σ' and σ'' are two paths in Δ such that $s\sigma' = t\sigma''$, then we define the composition $\sigma'\sigma''$ of σ' and σ'', which is a path in Δ of length $\ell(\sigma') + \ell(\sigma'')$, in the obvious way (in particular, $\sigma\mathbf{1}_{s\sigma} = \sigma = \mathbf{1}_{t\sigma}\sigma$ for each path σ). A path σ_0 is called a subpath of a path σ, if there exist paths σ' and σ'' such that $\sigma = \sigma'\sigma_0\sigma''$.

By a (monomial) bound quiver we mean a pair $\Lambda = (\Delta, R)$ consisting of a finite quiver Δ and a set R of paths in Δ, such that:

(1) $\ell(\rho) > 1$ for each $\rho \in R$, and
(2) there exists $n \in \mathbb{N}_+$ such that every path σ in Δ with $\ell(\sigma) = n$ has a subpath which belongs to R.

If $\Lambda = (\Delta, R)$ is a bound quiver, then by a path in Λ we mean a path in Δ which does not have a subpath from R. A path σ in Λ is said to be maximal in Λ if σ is not a subpath of a longer path in Λ. The lack of isolated vertices in Δ implies that $\ell(\sigma) > 0$ for each maximal path σ in Λ.

By a representation V of a bound quiver $\Lambda = (\Delta, R)$ we mean a collection of finite-dimensional vector spaces V_x, $x \in \Delta_0$, and linear maps $V_\alpha \colon V_{s\alpha} \to V_{t\alpha}$, $\alpha \in \Delta_1$, such that the induced map $V_\rho \colon V_{s\rho} \to V_{t\rho}$ is zero for every $\rho \in R$. If V and W are representations, then a homomorphism $f \colon V \to W$ is a collection of linear maps $f_x \colon V_x \to W_x$, $x \in \Delta_0$, such that $f_{t\alpha} V_\alpha = W_\alpha f_{s\alpha}$ for every arrow α in Δ. The category $\operatorname{rep} \Lambda$ of representations of Λ is an abelian category. We call bound quivers Λ' and Λ'' derived equivalent (and write $\Lambda' \simeq_{\mathrm{der}} \Lambda''$), if the derived categories $\mathcal{D}^b(\operatorname{rep} \Lambda')$ and $\mathcal{D}^b(\operatorname{rep} \Lambda'')$ are triangle equivalent. We will usually write shortly $\mathcal{D}^b(\Lambda)$ instead of $\mathcal{D}^b(\operatorname{rep} \Lambda)$ if Λ is a bound quiver.

A connected bound quiver $\Lambda = (\Delta, R)$ is called gentle if the following conditions are satisfied:

(1) R consists of paths of length 2,
(2) for each $x \in \Delta_0$ there are at most two $\alpha \in \Delta_1$ such that $s\alpha = x$ and at most two $\alpha \in \Delta_1$ such that $t\alpha = x$,
(3) for each $\alpha \in \Delta_1$ there is at most one $\alpha' \in \Delta_1$ such that $s\alpha' = t\alpha$ and $\alpha'\alpha \notin R$, and at most one $\alpha' \in \Delta_1$ such that $t\alpha' = s\alpha$ and $\alpha\alpha' \notin R$,
(4) for each $\alpha \in \Delta_1$ there is at most one $\alpha' \in \Delta_1$ such that $s\alpha' = t\alpha$ and $\alpha'\alpha \in R$, and at most one $\alpha' \in \Delta_1$ such that $t\alpha' = s\alpha$ and $\alpha\alpha' \in R$.

Let $\Lambda = (\Delta, R)$ be a gentle bound quiver. Note that by condition (1) above a path $\alpha_1 \ldots \alpha_l$ in Δ is a path in Λ if and only if $\alpha_i\alpha_{i+1} \notin R$ for all $i \in [1, l-1]$. We call a path $\alpha_1 \ldots \alpha_l$ in Δ an antipath in Λ if $\alpha_i\alpha_{i+1} \in R$ for all $i \in [1, l-1]$. In particular, every path of length at most 1 is an antipath. Again we call an antipath ω maximal if ω is not a subpath of a longer antipath in Λ.

2.2 The Invariant of Avella-Alaminos and Geiss

Throughout this subsection $\Lambda = (\Delta, R)$ is a fixed gentle bound quiver.

By a permitted thread in Λ we mean either a maximal path in Λ or $\mathbf{1}_x$, for $x \in \Delta_0$, such that there is at most one arrow α with $s\alpha = x$, there is at most one arrow β with $t\beta = x$, and if such α and β exist, then $\alpha\beta \notin R$. Similarly, by a forbidden thread we mean either a maximal antipath in Λ or $\mathbf{1}_x$, for $x \in \Delta_0$, such that there is at most one arrow α with $s\alpha = x$, there is at most one arrow β with $t\beta = x$, and if such α and β exist, then $\alpha\beta \in R$.

Denote by $\mathcal{P}$ and $\mathcal{F}$ the sets of the permitted and forbidden threads in Λ, respectively. We define bijections $\Phi_1 \colon \mathcal{P} \to \mathcal{F}$ and $\Phi_2 \colon \mathcal{F} \to \mathcal{P}$. First, if σ is a maximal path in Λ, then we put $\Phi_1(\sigma) := \omega$, where ω is the unique forbidden thread such that $t\omega = t\sigma$ and either $\ell(\omega) = 0$ or $\ell(\omega) > 0$ and the terminating arrows of σ and ω differ. If $\mathbf{1}_x$, for $x \in \Delta_0$, is a permitted thread, there are two cases to consider. If there is an arrow β such that $t\beta = x$ (note that such β is uniquely determined), then $\Phi_1(\mathbf{1}_x)$ is the (unique) forbidden thread whose terminating arrow is β. Otherwise we put $\Phi_1(\mathbf{1}_x) := \mathbf{1}_x$. We define Φ_2 dually. Namely, if ω is a maximal antipath, then $\Phi_2(\omega) := \sigma$, where σ is the permitted thread such that $s\sigma = s\omega$ and either $\ell(\sigma) = 0$ or $\ell(\sigma) > 0$ and the starting arrows of ω and σ differ. Now, let $x \in \Delta_0$ and $\mathbf{1}_x$ be a forbidden thread. If there is $\alpha \in \Delta_1$ such that $s\alpha = x$, then $\Phi_2(\mathbf{1}_x)$ is the permitted thread whose starting arrow is α. Otherwise, $\Phi_2(\mathbf{1}_x) := \mathbf{1}_x$. Finally, we put $\Phi := \Phi_1\Phi_2 \colon \mathcal{F} \to \mathcal{F}$.

Let $\mathcal{F}'$ be the set of arrows in Δ which are not subpaths of any maximal antipath in Λ (i.e. every antipath containing α can be extended to a longer antipath). For every $\alpha \in \mathcal{F}'$ there exists uniquely determined $\alpha' \in \mathcal{F}'$ such that $\alpha\alpha' \in R$. We put $\Phi'(\alpha) := \alpha'$. In this way we get a bijection $\Phi' \colon \mathcal{F}' \to \mathcal{F}'$. In other words, $\mathcal{F}'$ is the set of arrows which lie on oriented cycles with full relations. Moreover, two arrows in $\mathcal{F}'$ belong to the same orbit with respect to the action of Φ' if and only if they lie on the same oriented cycle with full relations.

The following result seems to be well-known, however we could not find a reference for it, hence we include its proof for completeness.

Proposition 2.1 *Let* $\Lambda = (\Delta, R)$ *be a gentle bound quiver. Then* $\operatorname{gldim} \Lambda < \infty$ *if and only if* $\mathcal{F}' = \emptyset$.

Proof For a vertex x of Δ we denote by S_x and P_x the simple and the projective representations of Λ at x, respectively. For $\alpha \in \Delta_1$ we denote by P_α the corresponding map $P_{t\alpha} \to P_{s\alpha}$.

Assume first $\mathcal{F}' = \emptyset$ and fix $x \in \Delta_0$. Assume there are exactly two arrows α and β starting at x. Let $\alpha_n \cdots \alpha_1$ and $\beta_m \cdots \beta_1$ be the maximal antipaths, whose starting arrows are α and β, respectively (in particular, $\alpha_1 = \alpha$ and $\beta_1 = \beta$) – such antipaths exist, since $\mathcal{F}' = \emptyset$. Then

$$\cdots \to P_{t\alpha_2} \oplus P_{t\beta_2} \xrightarrow{\begin{bmatrix} P_{\alpha_2} & 0 \\ 0 & P_{\beta_2} \end{bmatrix}} P_{t\alpha_1} \oplus P_{t\beta_1} \xrightarrow{\begin{bmatrix} P_{\alpha_1} & P_{\beta_1} \end{bmatrix}} S_x \to 0$$

is a minimal projective presentation of S_x, so $\operatorname{pdim}_\Lambda S_x = \max\{n, m\} < \infty$. If there is only one arrow starting at x, then we have a degenerate version of the above. Finally, if there is no arrow starting at x, then $S_x = P_x$.

Now assume $\mathcal{F}' \neq \emptyset$, choose $\alpha \in \mathcal{F}'$, and put $\alpha_i := \Phi'^{-i}(\alpha)$, $i \in \mathbb{N}$. Then

$$\cdots \to P_{t\alpha_1} \xrightarrow{P_{\alpha_1}} P_{t\alpha_0} \xrightarrow{P_{\alpha_0}} P_{s\alpha} \to \operatorname{Coker} P_\alpha \to 0$$

is a minimal projective presentation of $\operatorname{Coker} P_\alpha$, so $\operatorname{pdim}_\Lambda \operatorname{Coker} P_\alpha = \infty$. □

Let $\mathcal{F}/\Phi$ be the set of orbits in $\mathcal{F}$ with respect to the action of Φ. For each $\mathcal{O} \in \mathcal{F}/\Phi$ we put $n_\mathcal{O} := |\mathcal{O}|$ and $m_\mathcal{O} := \sum_{\omega \in \mathcal{O}} \ell(\omega)$. Similarly, if $\mathcal{O} \in \mathcal{F}'/\Phi'$, then $n_\mathcal{O} := 0$ and $m_\mathcal{O} := |\mathcal{O}|$. We define $\phi_\Lambda : \mathbb{N}^2 \to \mathbb{N}$ by the formula:

$$\phi_\Lambda(n, m) := |\{\mathcal{O} \in \mathcal{F}/\Phi \cup \mathcal{F}'/\Phi' : (n_\mathcal{O}, m_\mathcal{O}) = (n, m)\}| \qquad (n, m \in \mathbb{N}).$$

Avella-Alaminos and Geiss have proved [8] that ϕ_Λ is a derived invariant, i.e. if Λ' and Λ'' are derived equivalent gentle bound quivers, then $\phi_{\Lambda'} = \phi_{\Lambda''}$.

For a function $\phi\colon \mathbb{N}^2 \to \mathbb{N}$ we put $\|\phi\| := \sum_{(n,m) \in \mathbb{N}^2} \phi(n, m)$. If Λ is a gentle bound quiver, then $\|\phi_\Lambda\|$ equals $|\mathcal{F}/\Phi| + |\mathcal{F}'/\Phi'|$. We will need the following observation.

Lemma 2.2 *Let $\Lambda = (\Delta, R)$ be a gentle bound quiver such that $\|\phi_\Lambda\| = 1$. Then $\mathcal{F}' = \emptyset$, hence* gldim $\Lambda < \infty$. *Moreover, if $\mathcal{O} \in \mathcal{F}/\Phi$, then $n_\mathcal{O} \neq m_\mathcal{O}$.*

Proof Let $\mathcal{O}$ be the unique element of $\mathcal{F}/\Phi \cup \mathcal{F}'/\Phi'$ (i.e. either $\mathcal{O} = \mathcal{F}$ or $\mathcal{O} = \mathcal{F}'$). It follows from [15, Lemma 3.2], that $n_\mathcal{O} = 2|\Delta_0| - |\Delta_1|$ and $m_\mathcal{O} = |\Delta_1|$. If $\mathcal{O} = \mathcal{F}'$, then $n_\mathcal{O} = 0$, hence $|\Delta_1| = 2|\Delta_0|$. By condition (2) of the definition of a gentle bound quiver this means that for each $x \in \Delta_0$ there are exactly two arrows starting at x. Consequently, condition (4) of the definition implies that for each $\alpha \in \Delta_1$ there exists $\alpha' \in \Delta_1$ such that $s\alpha' = t\alpha$ and $\alpha'\alpha \notin R$. Thus, there exist paths in Λ of arbitrary length, which contradicts condition (2) of the definition of a bound quiver. Consequently, $\mathcal{O} = \mathcal{F}$, hence $\mathcal{F}' = \emptyset$.

Now assume $n_\mathcal{O} = m_\mathcal{O}$. Then $2|\Delta_0| - |\Delta_1| = |\Delta_1|$, i.e. $|\Delta_0| = |\Delta_1|$, hence Λ is a one-cycle gentle bound quiver. However in this case $\|\phi_\Lambda\| = 2$ (see [8, Section 7]), hence the claim follows. □

2.3 Boundary Complexes

Let $\Lambda = (\Delta, R)$ be a gentle bound quiver. One defines the Auslander–Reiten quiver $\Gamma(\mathcal{D}^b(\Lambda))$ of $\mathcal{D}^b(\Lambda)$ in the following way: the vertices of $\Gamma(\mathcal{D}^b(\Lambda))$ are (representatives of) the isomorphism classes of the indecomposable complexes in $\mathcal{D}^b(\Lambda)$ and the number of arrows between vertices X and Y equals the dimension of the space of irreducible maps between X and Y.

Since the gentle bound quivers are Gorenstein (see [27]), the Auslander–Reiten translation τ (see [30]) is an autoequivalence on the subcategory of perfect complexes (i.e. complexes, which are quasi-isomorphic to bounded complexes of projective representations). In particular, if gldim $\Lambda < \infty$, then τ is an automorphism of $\mathcal{D}^b(\Lambda)$.

An indecomposable complex $X \in \mathcal{D}^b(\Lambda)$ is called *boundary* if X is perfect and there is only one arrow in $\Gamma(\mathcal{D}^b(\Lambda))$ terminating at X. Equivalently, X is perfect and in the Auslander–Reiten triangle (see [30]) terminating at X the middle term is indecomposable.

The invariant of Avella-Alaminos and Geiss describes the action of the shift Σ on the components of $\Gamma(\mathcal{D}^b(\Lambda))$ containing boundary complexes. We will use the following excerpt from their results in [8, Sections 5 and 6]. First, there exist homogeneous tubes in $\Gamma(\mathcal{D}^b(\Lambda))$ if and only if there exists an orbit $\mathcal{O} \in \mathcal{F}/\Phi$ such that $n_{\mathcal{O}} = 1 = m_{\mathcal{O}}$. Let $\mathcal{C}$ be the family of components of $\Gamma(\mathcal{D}^b(\Lambda))$, which contain boundary complexes, but are not homogeneous tubes. If $\mathcal{C}/\Sigma$ is the set of orbits in $\mathcal{C}$ with respect to the action of Σ and $\mathcal{X}$ is the set of orbits $\mathcal{O} \in \mathcal{F}/\Phi$ such that $(n_{\mathcal{O}}, m_{\mathcal{O}}) \neq (1, 1)$, then $|\mathcal{C}/\Sigma| = |\mathcal{X}|$. In particular, if $|\mathcal{X}| = 1$ and X and Y are boundary complexes, which do not lie in homogeneous tubes, then there exists $p \in \mathbb{Z}$ such that $\Sigma^p X$ and Y belong to the same component. If $\|\phi_\Lambda\| = 1$, we have even more.

Lemma 2.3 *Let Λ be a gentle bound quiver such that $\|\phi_\Lambda\| = 1$. If X and Y are boundary complexes in $\mathcal{D}^b(\Lambda)$, then there exists an autoequivalence F of $\mathcal{D}^b(\Lambda)$ such that $FX = Y$.*

Proof Assume first that Λ is derived equivalent to a hereditary algebra of Dynkin type $\mathbb{A}$, i.e. Λ is a gentle tree. In this case $\Gamma(\mathcal{D}^b(\Lambda))$ is $\mathbb{Z}\mathbb{A}_n$ for some $n \in \mathbb{N}_+$ (see [29, Section I.5]), hence the boundary complexes form two orbits with respect to the action of τ, which is an autoequivalence of $\mathcal{D}^b(\Lambda)$, since gldim $\Lambda < \infty$ by Lemma 2.2. Moreover, Σ interchanges these orbits, hence the claim follows in this case.

If Λ is one-cycle gentle bound quiver, then $\|\phi_\Lambda\| = 2 \neq 1$ by [8, Section 7], hence we may assume Λ is not of polynomial growth by [39, Theorem 1.1]. Let $\mathcal{O}$ be the unique element of $\mathcal{F}/\Phi \cup \mathcal{F}'/\Phi'$. Lemma 2.2 implies that $\mathcal{O} \in \mathcal{F}/\Phi$ and $(n_{\mathcal{O}}, m_{\mathcal{O}}) \neq (1, 1)$. In particular, there are no homogeneous tubes in $\Gamma(\mathcal{D}^b(\Lambda))$. Consequently, by the discussion above we know there exists $p \in \mathbb{Z}$ such that $\Sigma^p X$ and Y belong to the same component of $\Gamma(\mathcal{D}^b(\Lambda))$. Moreover, [26, Theorem 2.6] implies that $\Sigma^p X$ and Y belong to the same τ-orbit, i.e. there exists $q \in \mathbb{Z}$ such that $\tau^q \Sigma^p X = Y$. Finally, gldim $\Lambda < \infty$ by Lemma 2.2, hence τ is an autoequivalence of $\mathcal{D}^b(\Lambda)$, and the claim follows. □

If σ is a path in Λ, then we have the corresponding (string) representation $M(\sigma)$ (see for example [22]). We have the following observation.

Lemma 2.4 *Let Λ be a gentle bound quiver. If σ is a maximal path in Λ, then $M(\sigma)$ (viewed as a complex concentrated in degree 0) is a boundary complex in $\mathcal{D}^b(\Lambda)$.*

Proof In the terminology of [14] (see also [12]) a projective presentation of $M(\sigma)$ is given by the complex which corresponds to the antipath $\Phi_2^{-1}(\sigma)$. In particular, this implies that $M(\sigma)$ is a perfect complex in $\mathcal{D}^b(\Lambda)$. Moreover, if one uses results of [14] in order to calculate the Auslander–Reiten triangle terminating at $M(\sigma)$, then one gets that its middle term is indecomposable. Alternatively, one may use the Happel functor [28, 29] and well-known formulas (see for example [22, 41]) for calculating the Auslander–Reiten triangles in the stable category of the category of representations of the repetitive category $\hat{\Lambda}$ of Λ. We leave details to the reader. □

We formulate the following consequence.

Corollary 2.5 *Let Λ' and Λ'' be derived equivalent gentle bound quivers such that $\|\phi_{\Lambda'}\| = 1 = \|\phi_{\Lambda''}\|$. If σ' and σ'' are maximal paths in Λ' and Λ'', respectively, then there exists a derived equivalence $F: \mathcal{D}^b(\Lambda') \to \mathcal{D}^b(\Lambda'')$ such that $F(M(\sigma')) = M(\sigma'')$.*

Proof Let $G: \mathcal{D}^b(\Lambda') \to \mathcal{D}^b(\Lambda'')$ be a derived equivalence. We know from Lemma 2.4 that $M(\sigma')$ and $M(\sigma'')$ are boundary complexes in $\mathcal{D}^b(\Lambda')$ and $\mathcal{D}^b(\Lambda'')$, respectively. Consequently, $G(M(\sigma'))$ and $M(\sigma'')$ are boundary complexes in $\mathcal{D}^b(\Lambda'')$. Thus, by Lemma 2.3, there exists an autoequivalence H of $\mathcal{D}^b(\Lambda'')$ such that $H(G(M(\sigma'))) = M(\sigma'')$. We take $F = H \circ G$. □

2.4 One-point Coextensions

If Λ is a bound quiver and M is a representation of Λ, then one defines a bound quiver $[M]\Lambda$, called the one-point coextension of Λ by M (see for example [9]). However, usually $[M]\Lambda$ is not monomial, even if Λ is. Consequently, in the paper we only consider one-point coextensions of the form $[M(\sigma)]\Lambda$, where Λ is a gentle bound quiver and σ is a maximal path in Λ.

Let $\Lambda = (\Delta, R)$ be a gentle bound quiver and σ a maximal path in Λ. We define the one-point coextension $[M(\sigma)]\Lambda$ of Λ by $M(\sigma)$ as follows: $[M(\sigma)]\Lambda := (\Delta', R')$, where

(1) Δ' is obtained from Δ by adding a new arrow α starting at $t\sigma$ and terminating at a new vertex x;
(2) if there exists (necessarily unique) arrow α' in Δ, which terminates at $t\sigma$, but is not the terminating arrow of σ, then $R' := R \cup \{\alpha\alpha'\}$; otherwise, $R' := R$.

We write shortly $[\sigma]\Lambda$ instead of $[M(\sigma)]\Lambda$. One easily gets the following.

Lemma 2.6 *Let Λ be gentle bound quiver. If σ is a maximal path in Λ, then $[\sigma]\Lambda$ is a gentle bound quiver.*

Proof Exercise. □

The following is a special version of the dual of Barot and Lenzing's [9, Theorem 1].

Proposition 2.7 *Let σ' and σ'' be maximal paths in gentle bound quivers Λ' and Λ'', respectively. If there exists a triangle equivalence $F: \mathcal{D}^b(\Lambda') \to \mathcal{D}^b(\Lambda'')$ such that $F(M(\sigma')) = M(\sigma'')$, then $[\sigma']\Lambda'$ and $[\sigma'']\Lambda''$ are derived equivalent.*

Combining Proposition 2.7 with Corollary 2.5 we obtain.

Corollary 2.8 *Let Λ' and Λ'' be derived equivalent gentle bound quivers such that $\|\phi_{\Lambda'}\| = 1 = \|\phi_{\Lambda''}\|$. If σ' and σ'' are maximal paths in Λ' and Λ'', respectively, then $[\sigma']\Lambda'$ and $[\sigma'']\Lambda''$ are derived equivalent.*

2.5 Reflections

Let $\Lambda = (\Delta, R)$ be a gentle bound quiver. Let x be a vertex in Δ such that there is no $\alpha \in \Delta_1$ with $s\alpha = x = t\alpha$ and for each $\alpha \in \Delta_1$ with $s\alpha = x$ there exists $\beta_\alpha \in \Delta_1$ with

$t\beta_\alpha = x$ and $\alpha\beta_\alpha \notin R$. We define a bound quiver $\Lambda' = (\Delta', R')$ in the following way: $\Delta'_0 = \Delta_0$, $\Delta'_1 = \Delta_1$,

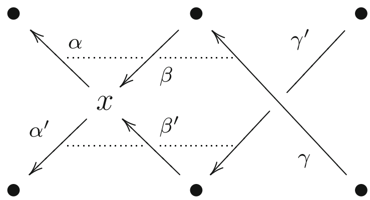
$$s_{\Delta'}\alpha = \begin{cases} x & \text{if } t_\Delta\alpha = x, \\ s_\Delta\beta_\alpha & \text{if } s_\Delta\alpha = x, \\ s_\Delta\alpha & \text{otherwise,} \end{cases}$$

$$t_{\Delta'}\alpha = \begin{cases} s_\Delta\alpha & \text{if } t_\Delta\alpha = x, \\ x & \text{if there exists } \beta \in \Delta_1 \text{ such that} \\ & \qquad t_\Delta\beta = x,\ s_\Delta\beta = t_\Delta\alpha \text{ and } \beta\alpha \in R, \\ t_\Delta\alpha & \text{otherwise,} \end{cases}$$

and R' consists of the following relations:

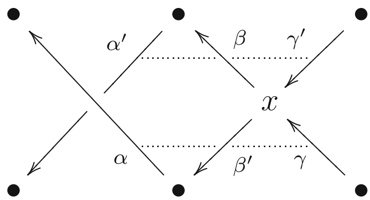
- $\alpha\beta$, where $\alpha\beta \in R$ and $t_\Delta\alpha \neq x \neq s_\Delta\alpha$,
- $\alpha\beta_\alpha$, where $\alpha \in \Delta_1$ and $s_\Delta\alpha = x$,
- $\alpha\beta$, where $\alpha, \beta \in \Delta_1$ are such that $t_\Delta\alpha = x$ and $\gamma\beta \in R$ for some $\gamma \in \Delta_1$, $\gamma \neq \alpha$, with $t_\Delta\gamma = x$.

The following pictures, where the relations are indicated by dots, illustrate the situation: if locally (in a neighbourhood of x) Δ has the form



,

then locally Δ' has the form

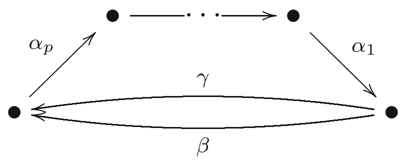


.

In the above situation we say that Λ' is obtained from Λ by applying the (generalized APR) reflection at x. The bound quiver Λ' is derived equivalent to Λ (see [17, Section 1]).

We will need the following application of this operation, which is a special version of [17, Lemma 1.1].

Lemma 2.9 *Let $\Lambda = (\Delta, R)$ be a gentle bound quiver such that Δ is of the form*



for $p \in \mathbb{N}_+$. Assume that $\alpha_{i-1}\alpha_i \notin R$ and $\alpha_i\alpha_{i+1} \in R$ for some $i \in [2, p-1]$. Then Λ is derived equivalent to the gentle bound quiver $\Lambda' := (\Delta, R')$, where

$$R' := (R \setminus \{\alpha_i\alpha_{i+1}\}) \cup \{\alpha_{i-1}\alpha_i\}.$$

Proof We obtain Λ' from Λ by applying the reflection at $t\alpha_i$, hence Λ and Λ' are derived equivalent by the discussion above. □

In the above situation we say that Λ' is obtained from Λ by a shift of the relation $\alpha_i\alpha_{i+1}$.

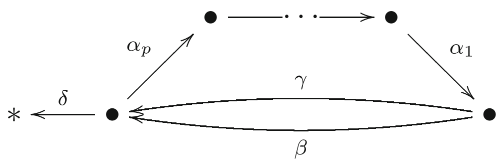
3 Proof of the Main Result

The aim of this section is to prove that the bound quivers $\Lambda_0(p, r)$, $p \in \mathbb{N}_+$, $r \in [-1, p+1]$, $(p, r) \neq (1, -1)$, are pairwise not derived equivalent. Observe (see also [17, Lemma 3.1]) that $\|\phi_{\Lambda_0(p,r)}\| = 1$. The following observation is crucial.

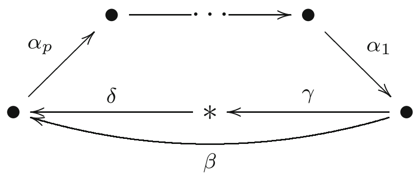
Lemma 3.1 *Let $p \in \mathbb{N}_+$ and $r \in [-1, p-1]$, $(p, r) \neq (1, -1)$. If σ is a maximal path in $\Lambda_0(p, r)$, then $[\sigma]\Lambda_0(p, r)$ is derived equivalent to $\Lambda_0(p + 1, r)$.*

Proof If σ' and σ'' are maximal paths in $\Lambda_0(p, r)$, then Corollary 2.8 implies that $[\sigma']\Lambda_0(p, r)$ and $[\sigma'']\Lambda_0(p, r)$ are derived equivalent. Thus it is enough to consider one particular σ.

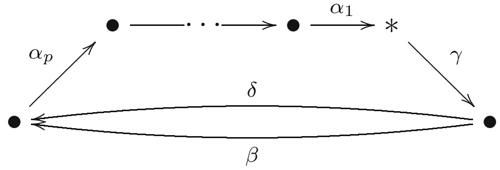
First assume that $r \geq 0$ and let σ be the maximal path whose terminating arrow is β, i.e. $\sigma := \beta\alpha_1$, if $r > 0$, and $\sigma := \beta\alpha_1 \cdots \alpha_p\gamma$, if $r = 0$. Then $[\sigma]\Lambda_0(p, r)$ is the quiver



bound by relations $\alpha_p\beta$, $\alpha_i\alpha_{i+1}$ for $i \in [1, r]$, $\gamma\alpha_1$ and $\delta\gamma$. If we apply the reflection at the vertex denoted by $*$, then we obtain the quiver

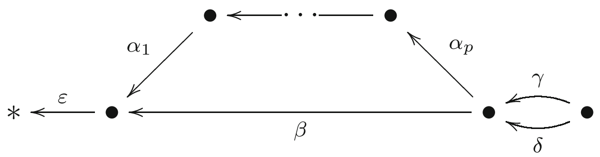


bound by relations $\alpha_p\beta$, $\alpha_i\alpha_{i+1}$ for $i \in [1, r]$, and $\gamma\alpha_1$. Now we apply again the reflection at the vertex denoted by $*$ and obtain the quiver



bound by relations $\alpha_p\beta, \alpha_i\alpha_{i+1}$ for $i \in [1, r]$, and $\delta\gamma$. Finally we shift relations (see Lemma 2.9) r times and obtain (a bound quiver isomorphic to) $\Lambda_0(p+1, r)$.

We proceed similarly if $r = -1$. If $\sigma := \beta\gamma$, then $[\sigma]\Lambda_0(p, -1)$ is the quiver



bound by relations $\alpha_p\gamma$, $\beta\delta$ and $\varepsilon\alpha_1$. By applying the reflection at the vertex denoted by $*$ we obtain $\Lambda_0(p+1, -1)$. □

We have the following consequence of Lemma 3.1.

Corollary 3.2 *Let $p \in \mathbb{N}_+$ and $r', r'' \in [-1, p-1]$, $(p, r') \neq (1, -1) \neq (p, r'')$. If $\Lambda_0(p, r')$ and $\Lambda_0(p, r'')$ are derived equivalent, then $\Lambda_0(q, r')$ and $\Lambda_0(q, r'')$ are derived equivalent for all $q \geq p$.*

Proof By induction it is enough to prove that $\Lambda_0(p+1, r')$ and $\Lambda_0(p+1, r'')$ are derived equivalent provided $\Lambda_0(p, r')$ and $\Lambda_0(p, r'')$ are derived equivalent. Let σ' and σ'' be maximal paths in $\Lambda_0(p, r')$ and $\Lambda_0(p, r'')$, respectively. Corollary 2.8 implies that $[\sigma']\Lambda_0(p, r')$ and $[\sigma'']\Lambda_0(p, r'')$ are derived equivalent. Since according to Lemma 3.1 $[\sigma']\Lambda_0(p, r') \simeq_{\text{der}} \Lambda_0(p+1, r')$ and $[\sigma'']\Lambda_0(p, r'') \simeq_{\text{der}} \Lambda_0(p+1, r'')$, the claim follows. □

An important role in our proof is played by the following result due to Amiot [1, Corollary 4.4].

Proposition 3.3 *Let $q \geq 3$ and $-1 \leq r', r'' \leq \frac{q}{2} - 1$. If $r' \neq r''$, then the algebras $\Lambda_0(q, r')$ and $\Lambda_0(q, r'')$ are not derived equivalent.*

Now we are ready to prove Theorem B.

Proof of Theorem B Let $p', p'' \in \mathbb{N}$, $r' \in [-1, p'-1]$ and $r'' \in [-1, p''-1]$ be such that $(p', r') \neq (1, -1) \neq (p'', r'')$. Obviously, $\Lambda_0(p', r')$ and $\Lambda_0(p'', r'')$ are not derived equivalent if $p' \neq p''$ (e.g. they have different numbers of vertices). Thus assume that $p' = p''$ and denote this common value by p. Choose $q \geq p$ such that $r', r'' \leq \frac{q}{2} - 1$. If $\Lambda_0(p, r')$ and $\Lambda_0(p, r'')$ are derived equivalent, then Corollary 3.2 implies that $\Lambda_0(q, r')$ and $\Lambda_0(q, r'')$ are derived equivalent as well. Consequently, $r' = r''$ according to Proposition 3.3 and the claim follows. □

References

1. Amiot, C.: The derived category of surface algebras: the case of torus with one boundary component. Algebr. Represent. Theory **19**, 1059–1080 (2016)
2. Amiot, C., Grimeland, Y.: Derived invariants for surface algebras. J. Pure Appl. Algebra **220**, 3133–3155 (2016)
3. Asashiba, H.: The derived equivalence classification of representation-finite selfinjective algebras. J. Algebra **214**, 182–221 (1999)
4. Assem, I., Happel, D.: Generalized tilted algebras of type A_n. Comm. Algebra **9**, 2101–2125 (1981)
5. Assem, I., Simson, D., Skowroński, A.: Elements of the representation theory of associative algebras. Vol. 1, London Mathematical Social Study Texts, p. x+458. Cambridge Univ. Press, Cambridge (2006)
6. Assem, I., Skowroński, A.: Iterated tilted algebras of type $\tilde{\mathbf{A}}_n$. Math. Z. **195**, 269–290 (1987)
7. Avella-Alaminos, D.: Derived classification of gentle algebras with two cycles. Bol. Soc. Mat. Mexicana (3) **14**, 177–216 (2008)
8. Avella-Alaminos, D., Geiss, Ch.: Combinatorial derived invariants for gentle algebras. J. Pure Appl. Algebra **212**, 228–243 (2008)
9. Barot, M., Lenzing, H.: One-point extensions and derived equivalence. J. Algebra **264**, 1–5 (2003)
10. Bastian, J.: Mutation classes of $\tilde{A}_n$-quivers and derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$. Algebra Number Theory **5**, 567–594 (2011)
11. Beĭlinson, A.A.: Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. Funktsional. Anal. i Prilozhen. **12**, 68–69 (1978) (Russian); English transl., Funct. Anal. Appl. **12**, 214–216 (1978)
12. Bekkert, V., Merklen, H.A.: Indecomposables in derived categories of gentle algebras. Algebr. Represent. Theory **6**, 285–302 (2003)
13. Białkowski, J., Holm, T., Skowroński, A.: Derived equivalences for tame weakly symmetric algebras having only periodic modules. J. Algebra **269**, 652–668 (2003)
14. Bobiński, G.: The almost split triangles for perfect complexes over gentle algebras. J. Pure Appl. Algebra **215**, 642–654 (2011)
15. Bobiński, G., Buan, A.B.: The algebras derived equivalent to gentle cluster tilted algebras. J. Algebra Appl. **11**, 1250012, 26 pp. (2012)
16. Bobiński, G., Geiß, Ch., Skowroński, A.: Classification of discrete derived categories. Cent. Eur. J. Math. **2**, 19–49 (2004)
17. Bobiński, G., Malicki, P.: On derived equivalence classification of gentle two-cycle algebras. Colloq. Math. **112**, 33–72 (2008)
18. Bocian, R., Holm, T., Skowroński, A.: Derived equivalence classification of one-parametric self-injective algebras. J. Pure Appl. Algebra **207**, 491–536 (2006)
19. Brüstle, T.: Derived-tame tree algebras. Compositio Math. **129**, 301–323 (2001)
20. Buan, A.B., Marsh, R., Reineke, M., Reiten, I., Todorov, G.: Tilting theory and cluster combinatorics. Adv. Math. **204**, 572–618 (2006)
21. Buan, A.B., Vatne, D.F.: Derived equivalence classification for cluster-tilted algebras of type A_n. J. Algebra **319**, 2723–2738 (2008)
22. Butler, M.C.R., Ringel, C.M.: Auslander-Reiten sequences with few middle terms and applications to string algebras. Comm. Algebra **15**, 145–179 (1987)
23. Gabriel, P.: Unzerlegbare Darstellungen. I. Manuscripta Math. **6**, 71–103; correction, ibid. **6** (1972), 309 (1972)
24. Geigle, W., Lenzing, H.: A class of weighted projective curves arising in representation theory of finite-dimensional algebras, Singularities, Representation of Algebras, and Vector Bundles, Lecture Notes in Math., vol. 1273, pp. 265–297. Springer, Berlin (1987)
25. Geiss, Ch.: Derived tame algebras and Euler-forms. Math. Z. **239**, 829–862 (2002). With an appendix by the author and B. Keller
26. Geiss, Ch., de la Peña, J.A.: Auslander-Reiten components for clans. Bol. Soc. Mat. Mexicana (3) **5**(2), 307–326 (1999)
27. Geiss, Ch., Reiten, I. In: Buchweitz, R.-O., Lenzing, H. (eds.): Gentle algebras are gorenstein, Representations of algebras and related topics, pp. 129–133. Fields Institute Communications, vol. 45, American Mathematical Society, Providence, RI (2005)
28. Happel, D.: On the derived category of a finite-dimensional algebra. Comment. Math. Helv. **62**, 339–389 (1987)
29. Happel, D.: Triangulated Categories in the Representation Theory of Finite-dimensional Algebras, London Mathematical Society Lecture Note Series, vol. 119. Cambridge University Press, Cambridge (1988)

30. Happel, D.: Auslander-Reiten triangles in derived categories of finite-dimensional algebras. Proc. Amer. Math. Soc. **112**, 641–648 (1991)
31. Hille, L., Perling, M.: Tilting bundles on rational surfaces and quasi-hereditary algebras. Ann. Inst. Fourier (Grenoble) **64**, 625–644 (2014)
32. Holm, T.: Derived equivalence classification of algebras of dihedral, semidihedral, and quaternion type. J. Algebra **211**, 159–205 (1999)
33. Kalck, M.: Gentle two-cycle algebras and stratifications of derived module categories (2015), preprint
34. Keller, B.: Deriving DG categories. Ann. Sci. École Norm. Sup. (4) **27**, 63–102 (1994)
35. Koenig, S., Zimmermann, A.: Derived Equivalences for Group Rings. Lecture Notes in Math., vol. 1685, p. x+246. Springer, Berlin (1998). With contributions by B. Keller, M. Linckelmann, J. Rickard and R. Rouquier
36. Orlov, D.: Derived categories of coherent sheaves and triangulated categories of singularities, Algebra, Arithmetic, and Geometry: in Honor of Yu. I. Manin. Vol. II (Y. Tschinkel and Y. Zarhin, eds.), Progress Mathematics, vol. 270, pp. 503–531. Birkhäuser Boston, Inc., Boston, MA (2009)
37. Rickard, J.: Morita theory for derived categories. J. London Math. Soc. (2) **39**, 436–456 (1989)
38. Rickard, J.: Derived categories and stable equivalence. J. Pure Appl. Algebra **61**, 303–317 (1989)
39. Ringel, C.M.: The repetitive algebra of a gentle algebra. Bol. Soc. Mat. Mexicana (3) **3**, 235–253 (1997)
40. Schröer, J., Zimmermann, A.: Stable endomorphism algebras of modules over special biserial algebras. Math. Z. **244**, 515–530 (2003)
41. Skowroński, A., Waschbüsch, J.: Representation-finite biserial algebras. J. Reine Angew. Math. **345**, 172–181 (1983)
42. Vossieck, D.: The algebras with discrete derived category. J. Algebra **243**, 168–176 (2001)