Representation Theory of Polyadic Groups

W. A. Dudek · M. Shahryari

Abstract In this article, we introduce the notion of representations of polyadic groups and we investigate the connection between these representations and those of retract groups and covering groups.

1 Introduction

A non-empty set G together with an n-ary operation $f : G^n \to G$ is called an *n-ary groupoid* and is denoted by (G, f). We will assume that $n > 2$.

According to the general convention used in the theory of n-ary systems, the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. In the case $j < i$ it is the

Presented by Yuri Drozd.

W. A. Dudek (✉)
Institute of Mathematics and Computer Science, Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
e-mail: dudek@im.pwr.wroc.pl

M. Shahryari
Department of Pure Mathematics, Faculty of Mathematical Sciences,
University of Tabriz, Tabriz, Iran
e-mail: mshahryari@tabrizu.ac.ir

empty symbol. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we write $\overset{(t)}{x}$. In this convention $f(x_1, \ldots, x_n) = f(x_1^n)$ and

$$f\left(x_1, \ldots, x_i, \underbrace{x, \ldots, x}_{t}, x_{i+t+1}, \ldots, x_n\right) = f\left(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n\right).$$

An n-ary groupoid (G, f) is called (i, j)-*associative*, if

$$f\left(x_1^{i-1}, f\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right) \tag{1.1}$$

holds for all $x_1, \ldots, x_{2n-1} \in G$. If this identity holds for all $1 \leqslant i < j \leqslant n$, then we say that the operation f is *associative* and (G, f) is called an n-*ary semigroup*.

If, for all $x_0, x_1, \ldots, x_n \in G$ and fixed $i \in \{1, \ldots, n\}$, there exists an element $z \in G$ such that

$$f\left(x_1^{i-1}, z, x_{i+1}^n\right) = x_0, \tag{1.2}$$

then we say that this equation is *i-solvable* or *solvable at the place i*. If this solution is unique, then we say that Eq. 1.2 is *uniquely i-solvable*.

An n-ary groupoid (G, f) uniquely solvable for all $i = 1, \ldots, n$, is called an *n-ary quasigroup*. An associative n-ary quasigroup is called an *n-ary group* or a *polyadic group*. In the binary case (i.e., for $n = 2$) it is a usual group.

Now, such and similar n-ary systems have many applications in different branches. For example, in the theory of automata, (cf. [11]), n-ary semigroups and n-ary groups are used, some n-ary groupoids are applied in the theory of quantum groups (cf. [15]). Different applications of ternary structures in physics are described by R. Kerner (cf. [13]). In physics there are used also such structures as n-ary Filippov algebras (cf. [16]) and n-Lie algebras (cf. [18]).

The idea of investigations of such groups seems to be going back to E. Kasner's lecture [12] at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first paper concerning the theory of n-ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (see [2]). In this paper Dörnte observed that any n-ary groupoid (G, f) of the form $f(x_1^n) = x_1 \circ x_2 \circ \ldots x_n \circ b$, where $(G, \circ)$ is a group and b is its fixed element belonging to the center of $(G, \circ)$, is an n-ary group. Such n-ary groups, called b-*derived* from the group $(G, \circ)$, are denoted by $der_b(G, \circ)$. In the case when b is the identity of $(G, \circ)$ we say that such n-ary group is *reducible* to the group $(G, \circ)$ or *derived* from $(G, \circ)$. But for every $n > 2$ there are n-ary groups which are not derived from any group. An n-ary group (G, f) is derived from some group iff it contains an element e (called an *n-ary identity*) such that

$$f\left(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}\right) = x \tag{1.3}$$

holds for all $x \in G$ and $i = 1, \ldots, n$.

It is worthwhile to note that in the definition of an n-ary group, under the assumption of the associativity of the operation f, it suffices only to postulate the existence of a solution of Eq. 1.2 at the places $i = 1$ and $i = n$ or at one place i other than 1 and n (cf. [17], p. 213). Other useful characterizations of n-ary groups one can find in [3] and [6].

From the definition of an n-ary group (G, f), we can directly see that for every $x \in G$, there exists only one $z \in G$ satisfying the equation

$$f\left(\overset{(n-1)}{x}, z\right) = x. \tag{1.4}$$

This element is called *skew* to x and is denoted by $\bar{x}$. In a ternary group $(n = 3)$ derived from the binary group $(G, \circ)$ the skew element coincides with the inverse element in $(G, \circ)$. Thus, in some sense, the skew element is a generalization of the inverse element in binary groups. Dörnte proved (see [2]) that in ternary groups we have $\overline{f(x, y, z)} = f(\bar{z}, \bar{y}, \bar{x})$ and $\bar{\bar{x}} = x$, but for $n > 3$ this is not true. For $n > 3$ there are n-ary groups in which one fixed element is skew to all elements (cf. [4]) and n-ary groups in which any element is skew to itself.

Nevertheless, the concept of skew elements plays a crucial role in the theory of n-ary groups. Namely, as Dörnte proved (see also [6]), the following theorem is true.

Theorem 1.1 *In any n-ary group* (G, f) *the following identities*

$$f\left(\overset{(i-2)}{x}, \bar{x}, \overset{(n-i)}{x}, y\right) = f\left(y, \overset{(n-j)}{x}, \bar{x}, \overset{(j-2)}{x}\right) = y, \tag{1.5}$$

$$f\left(\overset{(k-1)}{x}, \bar{x}, \overset{(n-k)}{x}\right) = x \tag{1.6}$$

hold for all $x, y \in G$, $2 \leqslant i, j \leqslant n$ *and* $1 \leqslant k \leqslant n$.

One can prove (cf. [3]) that for $n > 2$ an n-ary group can be defined as an algebra $(G, f, ^-)$ with one associative n-ary operation f and one unary operation $^- : x \to \bar{x}$ satisfying for some $2 \leqslant i, j \leqslant n$ the identities (1.5). This means that a non-empty subset H of an n-ary group (G, f) is its subgroup iff it is closed with respect to the operation f and $\bar{x} \in H$ for every $x \in H$.

Fixing in an n-ary operation f all inner elements $a_2, \ldots, a_{n-1}$ we obtain a new binary operation

$$x * y = f\left(x, a_2^{n-1}, y\right).$$

Such obtained groupoid $(G, *)$ is called a *retract* of (G, f). Choosing different elements $a_1, \ldots, a_{n-1}$ we obtain different retracts. Retracts of n-ary groups are groups. Retracts of a fixed n-ary group are isomorphic (cf. [8]). So, we can consider only retracts of the form

$$x * y = f\left(x, \overset{(n-2)}{a}, y\right).$$

Such retracts will be denoted by $Ret_a(G, f)$, or simply by $Ret_a(G)$. The identity of the group $Ret_a(G)$ is $\bar{a}$. One can verify that the inverse element to x has the form

$$x^{-1} = f\left(\bar{a}, \overset{(n-3)}{x}, \bar{x}, \bar{a}\right). \tag{1.7}$$

Binary retracts of an n-ary group (G, f) are commutative only in the case when there exists an element $a \in G$ such that

$$f\left(x, \overset{(n-2)}{a}, y\right) = f\left(y, \overset{(n-2)}{a}, x\right)$$

holds for all $x, y \in G$. An n-ary group with this property is called *semiabelian*. It satisfies the identity

$$f\left(x_1^n\right) = f\left(x_n, x_2^{n-1}, x_1\right) \tag{1.8}$$

(cf. [3]).

One can prove (cf. [9]) that a semiabelian n-ary group is *medial*, i.e., it satisfies the identity

$$f\left(f\left(x_{11}^{1n}\right), f\left(x_{21}^{2n}\right), \ldots, f\left(x_{n1}^{nn}\right)\right) = f\left(f\left(x_{11}^{n1}\right), f\left(x_{12}^{n2}\right) \ldots, f\left(x_{1n}^{nn}\right)\right). \tag{1.9}$$

In such n-ary groups

$$\overline{f\left(x_1, x_2, x_3, \ldots, x_n\right)} = f\left(\overline{x}_1, \overline{x}_2, \overline{x}_3, \ldots, \overline{x}_n\right) \tag{1.10}$$

for all $x_1, \ldots, x_n \in G$.

Any n-ary group can be uniquely described by its retract and some automorphism of this retract. Namely, the following Hosszú-Gluskin Theorem (cf. [5] or [7]) is valid.

Theorem 1.2 *An n-ary groupoid* (G, f) *is an n-ary group iff*

(1)   *on G one can define an operation* $\cdot$ *such that* $(G, \cdot)$ *is a group,*
(2)   *there exist an automorphism* φ *of* $(G, \cdot)$ *and* $b \in G$ *such that* $\varphi(b) = b$,
(3) $\varphi^{n-1}(x) = b \cdot x \cdot b^{-1}$ *for every* $x \in G$,
(4) $f(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \cdots \cdot \varphi^{n-1}(x_n) \cdot b$ *for all* $x_1, \ldots, x_n \in G$.

One can prove that $(G, \cdot) = Ret_a(G, f)$ for some $a \in G$. In connection with this we say that an n-ary group (G, f) is (φ, b)-derived from the group $(G, \cdot)$.

The main aim of this article is to introduce *representations* of n-ary groups and to investigate their main properties, with a special focus on ternary groups. Note that, this is not the first attempt to study representations of n-ary groups, because there are some other articles, with different point of views concerning representations on n-ary groups, (cf. [1, 10, 17] and [19]). However, our method seems to be the most natural generalization of the notion of representation from binary to n-ary groups.

2 Action of an *n*-ary Group on a Set

Suppose that (G, f) is an n-ary group and A is a non-empty set. We say that (G, f) *acts on* A if for all $x \in G$ and $a \in A$ corresponds a unique element $x.a \in A$ such that

(i) $f(x_1^n).a = x_1.(x_2.(x_3. \ldots .(x_n.a)) \ldots)$ for all $x_1, \ldots, x_n \in G$,
(ii)   for all $a \in A$, there exists $x \in G$ such that $x.a = a$,
(iii)   the map $a \mapsto x.a$ is a bijection for all $x \in G$.

For $a \in A$, we define the *stabilizer* G_a of a as follows

$$G_a = \{x \in G : x.a = a\}.$$

Proposition 2.1 G_a *is an n-ary subgroup of* (G, f).

Proof By condition (ii) of the above definition G_a is non-empty. Since for $x_1, x_2, \ldots, x_n \in G_a$ we have

$$f\left(x_1^n\right).a = x_1.(x_2.(x_3. \ldots . (x_n.a))\ldots) = a,$$

$f(x_1^n) \in G_a$. Hence G_a is closed with respect to the operation f.

Now if $x \in G_a$, then by Eq. 1.6 we obtain

$$a = x.a = f\left(\overline{x}, \overset{(n-1)}{x}\right).a = \overline{x}.(x. \ldots .x.(x.a))\ldots) = \overline{x}.a,$$

which implies $\overline{x} \in G_a$. This completes the proof. □

Proposition 2.2 *If an n-ary group* (G, f) *acts on a set* A, *then the relation* $\sim$ *defined on* A *by*

$$a \sim b \iff \exists x \in G : x.a = b$$

is an equivalence relation.

Proof For each $a \in A$ there is $x \in G$ such that $x.a = a$, so $a \sim a$. If $a \sim b$ for $a, b \in A$, then $z.a = b$ for some $z \in G$. Let y be the unique solution of the equation

$$f\left(y, z, \overset{(n-2)}{x}\right) = x,$$

where $x \in G$ is such that $x.a = a$. For this y we have $y.b = a$ since

$$a = x.a = f\left(y, z, \overset{(n-2)}{x}\right).a = y.z.a = y.b.$$

Thus $b \sim a$. Finally, let $a \sim b$ and $b \sim c$. Then there are $x, y, z \in G$ such that $x.a = b$, $y.b = c$ and $z.b = b$. In this case for $u = f(y, \overset{(n-2)}{z}, x)$ we have

$$u.a = f\left(y, \overset{(n-2)}{z}, x\right).a = y.b = c,$$

which proves $a \sim c$. □

Theorem 2.3 *The formula* $x.a = f(x, a, \overset{(n-3)}{x}, \overline{x})$ *defines an action of an n-ary group* G *on itself.*

Proof The last condition of Theorem 1.2 can be written in the form

$$f\left(x_1^n\right) = x_1 \cdot \varphi\left(x_2\right) \cdot \varphi^2(x_3) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n.$$

Thus $\overline{x} = (\varphi(x) \cdot \varphi^2(x) \cdot \ldots \cdot \varphi^{n-2}(x) \cdot b)^{-1}$. Consequently

$$x.a = x \cdot \varphi(a) \cdot \varphi\left(x^{-1}\right). \tag{2.1}$$

Hence

$$y.(x.a) = y \cdot \varphi(x) \cdot \varphi^2(a) \cdot \varphi^2\left(x^{-1}\right) \cdot \varphi(y)^{-1}$$
$$= y \cdot \varphi(x) \cdot \varphi^2(a) \cdot \varphi\left((y \cdot \varphi(x))^{-1}\right).$$

Iterating this procedure we obtain

$$x_1.(x_2.(x_3 \ \ldots \ .(x_n.a))\ldots)$$
$$= x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot \varphi^n(a) \cdot \varphi\left(\left(x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \ldots \cdot \varphi^{n-1}(x_n)\right)^{-1}\right).$$

Since $\varphi^n(a) = b \cdot \varphi(a) \cdot b^{-1}$ from the above we obtain

$$x_1.(x_2.(x_3 \ \ldots \ .(x_n.a))\ldots) = f\left(x_1^n\right) \cdot \varphi(a) \cdot \varphi\left(f\left(x_1^n\right)^{-1}\right).$$

This by Eq. 2.1 gives $f(x_1^n).a = x_1.(x_2.(x_3 \ \ldots \ .(x_n.a))\ldots)$. □

Proposition 2.4 *In semiabelian n-ary groups the relation*

$$a \sim b \iff \exists x \in G: \ f\left(x, a, \overset{(n-3)}{x}, \overline{x}\right) = b$$

is a congruence.

Proof Indeed, by Proposition 2.2 it is an equivalence relation. To prove that it is a congruence let $a_i \sim b_i$, i.e., $f(x_i, a_i, \overset{(n-3)}{x_i}, \overline{x}_i) = b_i$ for some $x_i \in G$ and all $i = 1, \ldots, n$. Then

$$f\left(b_1^n\right) = f\left(f\left(x_1, a_1, \overset{(n-3)}{x_1}, \overline{x}_1\right), f\left(x_2, a_2, \overset{(n-3)}{x_2}, \overline{x}_2\right), \ldots, f\left(x_n, a_n, \overset{(n-3)}{x_n}, \overline{x}_n\right)\right),$$

which by the mediality and Eq. 1.10 gives

$$f\left(b_1^n\right) = f\left(f\left(x_1^n\right), f\left(a_1^n\right), \underbrace{f\left(x_1^n\right), \ldots, f\left(x_1^n\right)}_{n-3}, \overline{f\left(x_1^n\right)}\right).$$

Thus $f(a_1^n) \sim f(b_1^n)$. □

Remark 2.5 The formula (2.1) says that in *n*-ary groups b-derived from a group $(G, \cdot)$ the above relation coincides with the conjugation in $(G, \cdot)$. Thus in non-semiabelian *n*-ary groups it may not be a congruence.

Elements belonging to the same equivalence class are called *conjugate*. The equivalence classes are called *conjugate classes* of an *n*-ary group G and have the form

$$Cl_G(a) = \left\{f\left(x, a, \overset{(n-3)}{x}, \overline{x}\right) : x \in G\right\}.$$

As a simple consequence of Eqs. 1.9 and 1.10 we obtain

Proposition 2.6 *In semiabelian n-ary group the set containing all elements of G conjugated with elements of a given n-ary subgroup also is an n-ary subgroup.*

For $a \in G$, we define the *centralizer* of a, as follows

$$C_G(a) = \left\{ x \in G : f\left(x, a, \overset{(n-3)}{x}, \overline{x} \right) = a \right\}.$$

From Theorem 1.1 it follows that in n-ary groups b-derived from a group $(G, \cdot)$ the centralizer of any $a \in G$ coincides with the centralizer of a in $(G, \cdot)$.

Proposition 2.7 *For every $x \in C_G(a)$ and every $0 \leqslant i, j, k \leqslant n - 2$ such that $i + j + k = n - 2$ we have*

$$f\left(\overset{(i)}{x}, a, \overset{(j)}{x}, \overline{x}, \overset{(k)}{x} \right) = f\left(\overset{(i)}{x}, \overline{x}, \overset{(j)}{x}, a, \overset{(k)}{x} \right) = a.$$

Proof For every $x \in C_G(a)$, we have $f(x, a, \overset{(n-3)}{x}, \overline{x}) = a$. Multiplying this equation on the left by x and on the right by $x, \ldots, x, \overline{x}$ ($n - 2$ elements), we obtain

$$f\left(x, f\left(x, a, \overset{(n-3)}{x}, \overline{x} \right), \overset{(n-3)}{x}, \overline{x} \right) = f\left(x, a, \overset{(n-3)}{x}, \overline{x} \right) = a,$$

which in view of the associativity of the operation f and Eq. 1.6 gives

$$f\left(x, x, a, \overset{(n-4)}{x}, \overline{x} \right) = a.$$

Repeating this procedure we obtain

$$f\left(\overset{(i)}{x}, a, \overset{(n-i-2)}{x}, \overline{x} \right) = a$$

for every $1 \leqslant i \leqslant n - 2$. Theorem 1.1 completes the proof. □

3 G-modules and Representations

All vector spaces in this section are defined over the field of complex numbers and have finite dimension.

Definition 3.1 Suppose that an n-ary group G acts on a vector space V and we have

(1) $x.(\lambda v + u) = \lambda x.v + x.u,$
(2) $\exists p \in G \, \forall v \in V : p.v = v.$

Then we call (V, p), or simply V, a *G-module*.

Notions, such as G-submodule, G-homomorphism, irreducibility and so on, are defined by the ordinary way.

Definition 3.2 A map $\Lambda : G \to GL(V)$ with the property

$$\Lambda\left(f\left(x_1^n \right) \right) = \Lambda(x_1)\Lambda(x_2)\ldots\Lambda(x_n)$$

is a *representation* of G, provided that ker Λ is non-empty. The function

$$\chi(x) = Tr \, \Lambda(x)$$

is called the corresponding *character* of Λ.

Remark 3.3 If V is a G-module, then Λ defined by

$$\Lambda(x)(v) = x.v$$

is a representation of G. The converse is also true.

Example 3.4 Let A be an arbitrary binary group with a normal subgroup H. Let $a \in A \setminus H$ be an involution. Then $G = aH$ with the operation

$$f(x, y, z) = xyz$$

is a ternary group. If Λ is an ordinary representation of A with the property $a \in$ ker Λ, then, clearly Λ is also a representation of G. For example, suppose $A = GL_n(\mathbb{C})$ and $H = SL_n(\mathbb{C})$. Let $a = diag(-1, 1, \ldots, 1)$ and define $G = aH$. Then, every representation of A in which $a \in$ ker Λ is also a representation of a ternary group G.

Example 3.5 For any subgroup H of an ordinary group A and any element $a \in Z(A) \setminus H$ with the order n we define on $G = aH$ an n-ary operation by

$$f(x_1, x_2, \ldots, x_n) = ax_1x_2 \ldots x_n.$$

This operation is associative, because $a \in Z(A)$. Also, G is closed under this operation, since $o(a) = n$. So, G is an n-ary group. Any A-representation Λ with $a \in$ ker Λ is also a G-representation.

Example 3.6 The set $G = \mathbb{Z}_n$ with the ternary operation

$$f(x, y, z) = x - y + z \ (\text{mod } n)$$

is, by Theorem 1.2, a ternary group. We want to classify all representations of G.

Let $\Lambda : G \rightarrow GL_m(\mathbb{C})$ be any representation. Then we have

$$\Lambda(f(x, y, z)) = \Lambda(x)\Lambda(y)\Lambda(z),$$

equivalently,

$$\Lambda(x - y + z) = \Lambda(x)\Lambda(y)\Lambda(z).$$

We have

$$\Lambda(x + y) = \Lambda(x)\Lambda(0)\Lambda(y), \quad \Lambda(x - y) = \Lambda(x)\Lambda(y)\Lambda(0).$$

Suppose $A = \Lambda(0)$. We have

$$\Lambda(x + y) = \Lambda(x)A\Lambda(y).$$

It is easy to see that $A^2 = I$. Now, define $\Lambda'(x) = A\Lambda(x)$. Then

$$\Lambda'(x + y) = \Lambda'(x)\Lambda'(y),$$

and so, Λ' is an ordinary representation of $(\mathbb{Z}_n, +)$. Hence, every representation of the ternary group G is of the form $\Lambda(x) = A\Lambda'(x)$, where A is an involution and Λ' is an ordinary representation of $(\mathbb{Z}_n, +)$.

Similarly, we can classify all representations of ternary groups of the form $G = (A, f)$, where A is an ordinary abelian group and

$$f(x, y, z) = x - y + z.$$

Theorem 3.7 (Maschke) *Let G be a finite n-ary group. Then every G-module is completely reducible.*

Proof Let (V, p) be a G-module and $W \leq_G V$. Suppose $V = W \oplus X$, where X is just a subspace. Let $\varphi : V \to W$ be the corresponding projection. Define a new map $\theta : V \to V$ as

$$\theta(v) = \frac{1}{|G|} \sum_{x \in G} \overline{x}.\varphi(x.v).$$

It is easy to see that

$$\theta(x.v) = x.p. \ldots .p.\theta(v) = x.\theta(v).$$

So θ is a G-homomorphism and hence its kernel is a G-submodule. For all $w \in W$, we have $\theta(w) = w$ and so $\theta^2 = \theta$. Now, we have $V = W \oplus \ker\theta$. □

Remark 3.8 Any G-module (V, p) is also an ordinary $Ret_p(G)$-module, because

$$(x * y).v = f\left(x, \overset{(n-2)}{p}, y\right).v = x.p. \ldots .p.y.v = x.y.v.$$

From now on, we will assume that $e \in G$ is an arbitrary fixed element. For all $p \in G$, we have $Ret_e(G) \cong Ret_p(G)$ and further the isomorphism is given by the following rule

$$h(x) = f\left(\overset{(n-2)}{e}, x, \overline{p}\right).$$

By $\hat{G}$ we denote the binary group $Ret_e(G)$. If (V, p) is a G-module, then we can define a $\hat{G}$-module structure on V by $x \circ v = h(x).v$. So, we have

$$x \circ v = f\left(\overset{(n-2)}{e}, x, \overline{p}\right).v = e. \ldots .e.x.\overline{p}.v.$$

But, we have $\overline{p}.v = \overline{p}.p. \ldots .p.v = f(\overline{p}, \overset{(n-1)}{p}).v = p.v = v$. Hence

$$x \circ v = \underbrace{e. \ldots .e}_{n-2}.x.v.$$

Now, every G-module is also a $\hat{G}$-module, but the converse is not true in general. During this article, we will give some necessary and sufficient conditions for a $\hat{G}$-module to be also a G-module. The next proposition is the first condition of this type.

Proposition 3.9 *Let V be a $\hat{G}$-module. Then V is a G-module iff*

$$\forall x_2, \ldots, x_{n-1} \in G \,\, \forall v : \,\, f\left(\overline{e}, x_2^{n-1}, \overline{e}\right).v = x_2.x_3. \,\, \ldots \,\, .x_{n-1}.v.$$

Proof We have

$$f\left(x_1^n\right) = f\left(f\left(x_1, \overset{(n-2)}{e}, \overline{e}\right), x_2^n\right)$$

$$= f\left(x_1, \overset{(n-2)}{e}, f\left(\overline{e}, x_2^n\right)\right)$$

$$= x_1 * f\left(\overline{e}, x_2^n\right)$$

$$= x_1 * f\left(\overline{e}, x_2^{n-1}, f\left(\overline{e}, \overset{(n-2)}{e}, x_n\right)\right)$$

$$= x_1 * f\left(\overline{e}, x_2^{n-1}, \overline{e}\right) * x_n.$$

So, the equality

$$f\left(x_1^n\right).v = x_1.x_2. \,\, \ldots \,\, .x_{n-1}.x_n.v$$

holds, iff

$$f\left(\overline{e}, x_2^{n-1}, \overline{e}\right).v = x_2.x_3. \,\, \ldots \,\, .x_{n-1}.v$$

for all $x_2, \ldots, x_{n-1}$ and v. $\qquad\square$

Remark 3.10 Suppose that V is a G-module in which the corresponding representation is Λ. We know that V is also a $\hat{G}$-module. The corresponding representation of this last module is

$$\hat{\Lambda}(x) = \underbrace{\Lambda(e) \ldots \Lambda(e)}_{n-2} \Lambda(x).$$

Because in $\hat{G}$, the identity element is $\overline{e}$, we have

$$\hat{\Lambda}(\overline{e}) = id.$$

So $\Lambda(e)^{n-2}\Lambda(\overline{e}) = id$ and hence

$$\Lambda(\overline{e}) = \Lambda(e)^{2-n}.$$

In the sequel, the corresponding character of $\hat{\Lambda}$, will be denoted by $\hat{\chi}$.

Proposition 3.11 *Suppose that Λ is a representation of G with the character χ. Then χ is constant on the conjugate classes of G.*

Proof Indeed, for every $b \in Cl_G(a)$ we have

$$\Lambda(b) = \Lambda \left(f \left(x, a, \overset{(n-3)}{x}, \overline{x} \right) \right) = \Lambda(x)\Lambda(a)\Lambda(x)^{n-3}\Lambda(\overline{x}),$$

so

$$
\begin{aligned}
\chi(b) &= Tr \left(\Lambda(x)\Lambda(a)\Lambda(x)^{n-3}\Lambda(\overline{x}) \right) \\
&= Tr \left(\Lambda(x)\Lambda(a)\Lambda(e)^{n-2}\Lambda(\overline{e})\Lambda(x)^{n-3}\Lambda(\overline{x}) \right) \\
&= Tr \left(\Lambda(a)\Lambda(e)^{n-2}\Lambda(\overline{e})\Lambda(x)^{n-3}\Lambda(\overline{x})\Lambda(x) \right) \\
&= Tr \left(\Lambda(a)\Lambda(e)^{n-2}\Lambda \left(f \left(\overline{e}, \overset{(n-3)}{x}, \overline{x}, x \right) \right) \right) \\
&= Tr \left(\Lambda(a)\Lambda(e)^{n-2}\Lambda(\overline{e}) \right) \\
&= Tr\, \Lambda(a) \\
&= \chi(a).
\end{aligned}
$$

This completes the proof. $\square$

Proposition 3.12 *Suppose that* $\Lambda : (G, f) \to GL(V)$ *is a representation of the finite n-ary group* (G, f) *with the corresponding character* χ. *Let*

$$\ker \chi = \{x \in G : \chi(x) = \dim V\}.$$

Then $\ker \chi = \ker \Lambda$.

Proof Let $\dim V = m$. It is clear that $\ker \Lambda \subseteq \ker \chi$. Moreover, for each $x \in G$ of order k we have

$$\Lambda(x)^{m^k} = \Lambda(x).$$

Hence $\Lambda(x)$ is a root of the polynomial $T^{m^k-1} - 1$. But, this polynomial has distinct roots in $\mathbb{C}$, so $\Lambda(x)$ can be diagonalized, i.e.,

$$\Lambda(x) \sim \text{diag}(\varepsilon_1, \ldots, \varepsilon_m),$$

where all ε_i are roots of unity. Now, we have

$$\chi(x) = \varepsilon_1 + \cdots + \varepsilon_m.$$

If $\chi(x) = m$, then $\varepsilon_i = 1$ for all i. Hence $\Lambda(x) = id$ and so $x \in \ker \Lambda$. This completes the proof. $\square$

In the next proposition, we obtain the explicit form of the character $\hat{\chi}$.

Proposition 3.13 *Let* χ *be a character of an n-ary group* (G, f). *Then for any* $p \in \ker \chi$ *we have*

$$\hat{\chi}(x) = \chi \left(f \left(\overset{(n-2)}{e}, x, \overline{p} \right) \right).$$

Proof We know that χ is a character of $Ret_p(G)$. On the other hand there is an isomorphism

$$h : Ret_e(G) \to Ret_p(G),$$

where $h(x) = f(\overset{(n-2)}{e}, x, \overline{p})$. So, the composite map $\chi \circ h$ is a character of $Ret_e(G)$. Let Λ be the corresponding representation of χ. Now, we have

$$\chi(h(x)) = Tr\ \left(\Lambda(e)^{n-2}\Lambda(x)\Lambda(\overline{p})\right)$$

$$= Tr\ \left(\Lambda(e)^{n-2}\Lambda(x)\right)$$

$$= Tr\ \hat{\Lambda}(x).$$

Hence $\hat{\chi}(x) = \chi(f(\overset{(n-2)}{e}, x, \overline{p}))$.   □

Remark 3.14 Now, for any irreducible character χ of an n-ary group (G, f), we have an ordinary irreducible character $\hat{\chi}$ of the binary group $\hat{G} = Ret_e(G)$. So, we obtain the following orthogonality relation for the irreducible characters of G:

$$\frac{1}{|G|} \sum_{x \in G} \chi_1 \left(f \left(\overset{(n-2)}{e}, x, \overline{p}_1\right) \right) \overline{\chi_2 \left(f \left(\overset{(n-2)}{e}, x, \overline{p}_2\right) \right)} = \delta_{\hat{\chi}_1, \hat{\chi}_2},$$

where $p_1 \in \ker \chi_1$ and $p_2 \in \ker \chi_2$ are arbitrary elements.

Proposition 3.15 *If a representation $\Gamma : Ret_e(G, f) \to GL(V)$ is also a representation of the n-ary group (G, f), then*

$$\Gamma(\overline{x}) = \Gamma(x)^{2-n}$$

for every $x \in G$.

Proof Indeed, $f(\overset{(n-1)}{x}, \overline{x}) = x$ implies $\Gamma(x)^{n-1}\Gamma(\overline{x}) = \Gamma(x)$, which gives $\Gamma(\overline{x}) = \Gamma(x)^{2-n}$.   □

Corollary 3.16 *Let (G, f) be a ternary group. Then a representation $\Gamma : Ret_e(G, f) \to GL(V)$ is also a representation of (G, f) iff*

$$\Gamma(\overline{x}) = \Gamma(x)^{-1}$$

for every $x \in G$.

Proof From Proposition 3.9 it follows that $\Gamma : Ret_e(G, f) \to GL(V)$ is a representation of a ternary group (G, f) iff it satisfies the identity

$$\Gamma \left(f \left(\overline{e}, x, \overline{e}\right) \right) = \Gamma(x).$$

If $\Gamma(\overline{x}) = \Gamma(x)^{-1}$ holds for all $x \in G$, then, in view of Eq. 1.7, for all $x \in G$ we have

$$\Gamma(f(\overline{e}, x, \overline{e})) = \Gamma\left(f\left(\overline{e}, \overline{\overline{x}}, \overline{e}\right)\right) = \Gamma\left(\overline{x}^{-1}\right) = \Gamma(\overline{x})^{-1} = \Gamma(x).$$

Hence Γ is a representation of (G, f).

The converse statement is a consequence of Proposition 3.15. $\square$

Remark 3.17 We can use the above proposition to obtain some deeper results in the case when G has a central element. Note that, according to [8], an n-ary group (G, f) has a central element iff it is b-derived from a binary group $(G, \cdot)$ and $b \in Z(G, \cdot)$. Obviously, in this case $Z(G, f) = Z(G, \cdot)$.

Proposition 3.18 *Let e be a central element of an n-ary group $(G, f) = der_b(G, \cdot)$. Then a representation $\Gamma : Ret_e(G) \to GL(V)$ is a representation of (G, f) iff*

$$\Gamma\left(x_2 x_3 \ldots x_n e^{2-n}\right) = \Gamma(x_2)\Gamma(x_3) \ldots \Gamma(x_n)$$

for all $x_2, \ldots, x_n \in G$.

Proof Since $(G, f) = der_b(G, \cdot)$ the binary operation in $Ret_e(G, f)$ has the form

$$x * y = f\left(x, \overset{(n-2)}{e}, y\right) = xye^{n-2}b.$$

For a representation Γ of $Ret_e(G, f)$, we have

$$\Gamma(x * y) = \Gamma(x)\Gamma(y). \tag{3.1}$$

Now, for Γ to be a representation of (G, f), it is necessary and sufficient that

$$\Gamma\left(f\left(x_1^n\right)\right) = \Gamma(x_1 x_2 \ldots x_n b) = \Gamma(x_1)\Gamma(x_2) \ldots \Gamma(x_n).$$

If we replace in Eq. 3.1, y by $x_2 \ldots x_n e^{2-n}$, we obtain

$$\Gamma(x_1 x_2 \ldots x_n b) = \Gamma(x_1)\Gamma\left(x_2 \ldots x_n e^{2-n}\right).$$

So Γ is a representation of (G, f), iff

$$\Gamma\left(x_2 x_3 \ldots x_n e^{2-n}\right) = \Gamma(x_2)\Gamma(x_3) \ldots \Gamma(x_n)$$

for all $x_2, \ldots, x_n \in G$. $\square$

In an n-ary group $(G, f) = der_b(G, \cdot)$ we have $\overline{x} = x^{2-n}b^{-1}$. Hence, comparing the above result with Proposition 3.15 we obtain

Corollary 3.19 *Let e be a central element of an n-ary group $(G, f) = der_b(G, \cdot)$. If a representation $\Gamma : Ret_e(G) \to GL(V)$ is a representation of (G, f), then $\Gamma(x^{2-n}b^{-1}) = \Gamma(x)^{2-n}$ for every $x \in G$.*

In the case of ternary groups, by Corollary 3.16, we obtain stronger result.

Corollary 3.20 *Let* $(G, f) = der_b (G, \cdot)$ *be a ternary group. Then a representation* $\Gamma :$ $Ret_e(G, f) \to GL(V)$ *is also a representation of* (G, f), *iff* $\Gamma((bx)^{-1}) = \Gamma(x)^{-1}$ *for every* $x \in G$.

Proposition 3.21 *Let e be a central element of an ternary group* $(G, f) = der_b (G, \cdot)$. *Then a character* χ *of* $Ret_e(G, f)$ *is a character of* (G, f) *iff for all* $x \in G$ *we have* $\chi(\bar{x}) = \overline{\chi(x)}$.

Proof Let $\Gamma : Ret_e(G, f) \to GL(V)$ be a representation corresponding to χ. If χ is a character of (G, f), then Γ is also a representation of (G, f) and so $\Gamma(\bar{x}) = \Gamma(x)^{-1}$. Hence we have $\chi(\bar{x}) = \overline{\chi(x)}$.

Conversely, if $\chi(\bar{x}) = \overline{\chi(x)}$ holds for all $x \in G$, then in particular $\overline{\chi(e)} = \chi(\bar{e})$. Thus $\chi(e) = \chi(\bar{e})$ because $\chi(\bar{e})$ is real. Now, for all $x \in G$, we have $x * \bar{x} = f(x, e, \bar{x}) = f(e, x, \bar{x}) = e$, so $\chi(x * \bar{x}) = \chi(e) = \chi(\bar{e})$. Hence,

$$x * \bar{x} \in \ker \chi = \ker \Gamma.$$

This shows that $\Gamma(x^{-1}) = \Gamma(\bar{x})$ and so Γ is a representation of G. Hence χ is also a character of G. $\qquad\qquad\square$

Proposition 3.22 *Let e be a central element of a ternary group* $(G, f) = der_b (G, \cdot)$. *If* χ *is a common character of* (G, f) *and* $Ret_e(G, f)$, *then* $\hat{\chi} = \chi$.

Proof We have $\chi(\bar{e}) = \overline{\chi(e)}$, so $\chi(e)$ is real, and hence $\chi(e) = \chi(\bar{e})$. So $e \in \ker \chi$. Now, suppose $p = e$. Then

$$\hat{\chi}(x) = \chi(f(e, x, \bar{p})) = \chi(f(e, x, \bar{e})) = \chi(f(x, e, \bar{e})) = \chi(x),$$

which completes the proof. $\qquad\qquad\square$

In the remaining part of this section, we try to answer the problem: when $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$? We give an answer to this question for n-ary groups with some central elements.

Proposition 3.23 *For an n-ary group* (G, f) *with a central element e the following assertions are true:*

(1) *Let* (V, p) *be a G-module and* $h : V \to V$ *be a* $\hat{G}$-*homomorphism. Then h is also a G-homomorphism.*

(2) *Let* (V_1, p_1) *and* (V_2, p_2) *be two G-modules and* $h : V_1 \to V_2$ *be a* $\hat{G}$-*homomorphism. Then h is a G-homomorphism, iff* $h(e.v) = e.h(v)$.

(3) *Let (V_1, p_1) and (V_2, p_2) be two G-modules and $h : V_1 \to V_2$ be a $\hat{G}$-homomorphism. Then h is a G-homomorphism, iff $p_1.h(v) = h(v)$ for every $v \in V_1$.*

(4) *Let (V_1, p_1) and (V_2, p_2) be two G-modules and*

$$V_1 \cong_{\hat{G}} V_2.$$

Then $V_1 \cong_G V_2$, iff for all $u \in V_2$, $p_1.u = u$.

Proof

(1) In view of $x * y = f(x, \overset{(n-2)}{e}, y)$, for a G-module (V, p), we have

$$h(e.v) = h\left(f\left(\overset{(n-1)}{e}, \bar{e} \right).v \right)$$

$$= h\left(f\left(f\left(\overset{(n-1)}{e}, \bar{e} \right), \overset{(n-1)}{p} \right).v \right)$$

$$= h\left(f\left(f\left(e, \overset{(n-2)}{p}, \bar{e} \right), \overset{(n-2)}{e}, p \right).v \right)$$

$$= h\left(f\left(e, \overset{(n-2)}{p}, \bar{e} \right) \circ v \right)$$

$$= f\left(e, \overset{(n-2)}{p}, \bar{e} \right) \circ h(v)$$

$$= f\left(f\left(e, \overset{(n-2)}{p}, \bar{e} \right), \overset{(n-2)}{e}, p \right).h(v)$$

$$= f\left(e, \overset{(n-2)}{p}, f\left(\bar{e}, \overset{(n-2)}{e}, p \right) \right).h(v)$$

$$= f\left(e, \overset{(n-1)}{p} \right).h(v)$$

$$= e.p. \ldots .p.h(v)$$

$$= e.h(v).$$

Now for all $x \in G$, we have $h(x \circ v) = x \circ h(v)$, so

$$h(\underbrace{e. \ldots .e}_{n-2}.x.v)) = e. \ldots .e.x.h(v).$$

Hence

$$\underbrace{e. \ldots .e}_{n-2}.h(x.v) = e. \ldots .e.x.h(v).$$

Since the map $u \mapsto e.u$ is bijection, we have $h(x.v) = x.h(v)$.

(2) The proof of this part is just as the above.

(3) Suppose h is a G-homomorphism. Then $p_1.h(v) = h(p_1.v) = h(v)$ for every $v \in V_1$.

Conversely, assume that for all $v \in V_1$ holds $p_1.h(v) = h(v)$. Then

$$
h(e.v) = h\left(f\left(\overset{(n-1)}{e}, \bar{e} \right) \cdot \underbrace{p_1. \ \ldots \ .p_1}_{n-2} .v \right)
$$

$$
= h\left(\underbrace{e. \ \ldots \ .e}_{n-1} .\bar{e}. \underbrace{p_1. \ \ldots \ .p_1}_{n-2} .v \right)
$$

$$
= h\left(f\left(e, \overset{(n-2)}{p_1}, \bar{e} \right) \cdot \underbrace{e. \ \ldots \ .e}_{n-2} .v \right)
$$

$$
= h\left(f\left(e, \overset{(n-2)}{p_1}, \bar{e} \right) \circ v \right)
$$

$$
= f\left(e, \overset{(n-2)}{p_1}, \bar{e} \right) \circ h(v)
$$

$$
= f\left(e, \overset{(n-2)}{p_1}, \bar{e} \right) \cdot \underbrace{e. \ \ldots \ .e}_{n-2} .h(v)
$$

$$
= f\left(e, \overset{(n-2)}{p_1}, \bar{e} \right) \cdot \underbrace{e. \ \ldots \ .e}_{n-2} .p_1.h(v)
$$

$$
= f\left(f\left(e, \overset{(n-2)}{p_1}, \bar{e} \right), \overset{(n-2)}{e}, p_1 \right) .h(v)
$$

$$
= f\left(e, \overset{(n-2)}{p_1}, f\left(\bar{e}, \overset{(n-2)}{e}, p_1 \right) \right) .h(v)
$$

$$
= f\left(e, \overset{(n-1)}{p_1} \right) .h(v)
$$

$$
= e.h(v).
$$

(4) Let $h : V_1 \to V_2$ be a G-isomorphism. Then h is also a $\hat{G}$-homomorphism, and hence $p_1.h = h$. Because h is onto, we obtain $p_1.u = u$, for all $u \in V_2$.
Conversely, suppose $p_1.u = u$, for all $u \in V_2$. Let $h : V_1 \to V_2$ be a $\hat{G}$-isomorphism. Then $p_1.h = h$, and so h is a G-isomorphism. $\qquad \square$

Proposition 3.24 *Let (G, f) be an n-ary group with a central element and let $\Lambda_1, \Lambda_2 : G \to GL(V)$ be two representations of (G, f), such that $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$. Then $\Lambda_1 \sim \Lambda_2$, iff $\ker \Lambda_1 = \ker \Lambda_2$.*

Proof Let $p \in \ker \Lambda_1 = \ker \Lambda_2$. We define two G-modules V_1 and V_2, as follows: V_1 is the vector space V with the action $x.v = \Lambda_1(x)(v)$, V_2 is the vector space V with the action $x.v = \Lambda_2(x)(v)$. Then $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$ implies

$$V_1 \cong_{\hat{G}} V_2,$$

and $p.u = u$, for all $u \in V_2$. So, $V_1 \cong_G V_2$. This proves $\Lambda_1 \sim \Lambda_2$.

Conversely, let $\Lambda_1 \sim \Lambda_2$. Hence, we have $V_1 \cong_G V_2$. By the previous proposition, for $p \in \ker \Lambda_1$ and $u \in V_2$, we have $p.u = u$. Thus $\Lambda_2(p) = id$. Therefore, $\ker \Lambda_1 = \ker \Lambda_2$. $\qquad\square$

Corollary 3.25 *Let Λ_1 and Λ_2 be two representations of an n-ary group (G, f) with a central element e. If $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$, then $\Lambda_1 \sim \Lambda_2$ iff $\Lambda_1(\overline{e}) \sim \Lambda_2(\overline{e})$.*

Proof By the above proposition, $\Lambda_1 \sim \Lambda_2$, iff $\ker \Lambda_1 = \ker \Lambda_2$. But, we have

$$\ker \Lambda_1 = \left\{ x \in G : \hat{\Lambda}_1(x) = \Lambda_1(e)^{n-2} \right\},$$

$$\ker \Lambda_2 = \left\{ x \in G : \hat{\Lambda}_2(x) = \Lambda_2(e)^{n-2} \right\}.$$

Hence $\Lambda_1 \sim \Lambda_2$, iff $\Lambda_1(e)^{n-2} \sim \Lambda_2(e)^{n-2}$. But we have $\Lambda_1(e)^{n-2} = \Lambda_1(\overline{e})^{-1}$ and similarly for Λ_2. So $\Lambda_1 \sim \Lambda_2$, iff $\Lambda_1(\overline{e})^{-1} \sim \Lambda_2(\overline{e})^{-1}$. $\qquad\square$

Remark 3.26 In the last two propositions and Corollary 3.25 the assumption that e is a central element can be replaced by the assumption that the n-ary group (G, f) is semiabelian.

4 Connection with the Representations of the Covering Group

According to Post's Coset Theorem (cf. [17] or [14]) for any n-ary group (G, f) there exists a binary group $(G^*, \cdot)$ and its normal subgroup H such that $G^*/H \simeq \mathbb{Z}_{n-1}$ and $G \subseteq G^*$ and

$$f\left(x_1^n\right) = x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n$$

for all $x_1, \ldots, x_n \in G$.

The group $(G^*, \cdot)$ is called the *covering group* for (G, f). We know several methods of a construction of such group. The smallest covering group has the form $G_a^* = G \times \mathbb{Z}_{n-1}$, where

$$\langle x, r \rangle \cdot \langle y, s \rangle = \left\langle f_* \left(x, \overset{(r)}{a}, y, \overset{(s)}{a}, \overline{a}, \overset{(n-2-r\diamond s)}{a} \right), r \diamond s \right\rangle,$$

$r \diamond s = (r + s + 1)(\mathrm{mod}\,(n-1))$ and $a \in G$ an arbitrary but fixed element. The symbol f_* means that the operation f is used one or two times (depending on the value s and t). Clearly fixing various element a of G, we obtain various groups but all these groups are isomorphic (cf. [14]).

The element $\langle \bar{a}, n-2 \rangle$ is the identity of the group $(G_a^*, \cdot)$. The inverse element has the form

$$\langle x, t \rangle^{-1} = \left\langle f_* \left(\bar{a}, \overset{(n-2-t)}{a}, \bar{x}, \overset{(n-3)}{x}, \bar{a}, \overset{(t+1)}{a} \right), k \right\rangle,$$

where $k = (n - 3 - t) (\mathrm{mod}\,(n-1))$.

The set G is identified with the subset $\{\langle x, 0 \rangle : x \in G\}$. Every retract of (G, f) is isomorphic to the normal subgroup

$$H = \{\langle x, n-2 \rangle : x \in G\}.$$

Suppose that V is a G_a^*-module. Then for $x_1, \ldots, x_n \in G$ we have

$$x_1.x_2.x_3.\ \ldots\ .x_n.v = \langle x_1, 0 \rangle . \langle x_2, 0 \rangle . \langle x_3, 0 \rangle . \ldots . \langle x_n, 0 \rangle .v$$

$$= \left\langle f \left(x_1, x_2, \bar{a}, \overset{(n-3)}{a} \right), 1 \right\rangle . \langle x_3, 0 \rangle . \ldots . \langle x_n, 0 \rangle .v$$

$$= \left\langle f \left(f \left(x_1^2, \bar{a}, \overset{(n-3)}{a} \right), a, x_3, \bar{a}, \overset{(n-4)}{a} \right), 2 \right\rangle . \ldots . \langle x_n, 0 \rangle .v$$

$$= \left\langle f \left(x_1^2, f \left(\bar{a}, \overset{(n-2)}{a}, x_3 \right), \bar{a}, \overset{(n-4)}{a} \right), 2 \right\rangle . \ldots . \langle x_n, 0 \rangle .v$$

$$= \left\langle f \left(x_1^3, \bar{a}, \overset{(n-4)}{a} \right), 2 \right\rangle . \ldots . \langle x_n, 0 \rangle .v$$

$$\vdots$$

$$= \left\langle f \left(x_1^n \right), 0 \right\rangle .v$$

$$= f \left(x_1^n \right) .v$$

So, we obtain

Proposition 4.1 *Let $(G_a^*, \cdot)$ be the covering group for an n-ary group (G, f). Then for a G_a^*-module V to be a G-module it is necessary and sufficient that*

$$\exists p \in G \; \forall v \in V : \; p.v = v.$$

Hence, we proved

Proposition 4.2 *Let $(G_a^*, \cdot)$ be the covering group for an n-ary group (G, f). A representation Γ of G_a^* is a representation of G, iff $\ker \Gamma \cap G \neq \varnothing$. If Γ is irreducible G^*-representation, then it is also irreducible as a representation of G.*

Now, suppose (V, p) is a G-module. For the covering group $(G_p^*, \cdot)$ of (G, f) we can define an action of G_p^* on V as

$$\langle x, k \rangle .v = x.v.$$

Then, it can be easily verified that V is a G_p^*-module. But, we know that $G_a^* \cong G_p^*$, so let $h : G_a^* \to G_p^*$ be any isomorphism. For any $x \in G_a^*$, define $x.v = h(x).v$. Hence V becomes a G_a^*-module. Further, if W is a G-submodule of G, then it is also a G_p^*-submodule and so a G_a^*-submodule. Hence, we proved

Theorem 4.3 *There is a bijection between the set of all irreducible representations of* (G, f) *and the set of all irreducible representations of* G_a^* *with kernels not disjoint from G.*

5 Normal Subgroups in Polyadic Groups

In this section, we show that the representation theory of n-ary groups reduces to the representation theory of binary groups. For this we introduce the concept of normal n-ary subgroup.

Definition 5.1 An n-ary subgroup H of an n-ary group (G, f) is called *normal* if

$$f\left(\overset{(n-3)}{a}, \bar{a}, h, a\right) \in H$$

for all $h \in H$ and $a \in G$. A normal subgroup $H \neq G$ containing at least two elements is called *proper*. If G has no any proper normal subgroup, then we say that it is *simple*. If $H = G$ is the only simple subgroup of G, then we say it is *strongly simple*.

Definition 5.2 For any n-ary subgroup H of an n-ary group (G, f) we define the relation $\sim_H$ on G, by

$$a \sim_H b \iff \exists x, y \in H : b = f\left(a, \overset{(n-2)}{x}, y\right).$$

Lemma 5.3 $a \sim_H b \iff \exists x_2, \ldots, x_n \in H : b = f\left(a, x_2^n\right).$

Proof Indeed, if $b = f(a, x_2^n)$ for some $x_2, \ldots, x_n \in H$, then, in view of Theorem 1.1, for every $x \in H$ we have

$$b = f\left(a, x_2^n\right) = f\left(a, f\left(\overset{(n-2)}{x}, \bar{x}, x_2\right), x_3^n\right) = f\left(a, \overset{(n-2)}{x}, y\right),$$

where $y = f(\bar{x}, x_2^n) \in H$, so $a \sim_H b$. The converse is obvious. $\qquad\square$

Now it is easy to see that such defined relation is an equivalence on G. The equivalence class of G, containing a is denoted by aH and is called the *left coset* of H with the representative a. By Lemma 5.3 it has the form

$$aH = \left\{f\left(a, \overset{(n-2)}{x}, y\right) : x, y \in H\right\} = \left\{f\left(a, h_2^n\right) : h_2, \ldots, h_n \in H\right\}.$$

The n-ary group (G, f) is partitioned by cosets of H.

Proposition 5.4 *If H is a finite n-ary subgroup of (G, f), then for all $a \in G$, we have* $|aH| = |H|$.

Proof By Theorem 1.2, for an *n*-ary group (G, f) there is a binary group $(G, \cdot)$, $\varphi \in Aut(G, \cdot)$ and an element $b \in G$ such that

$$f\left(x_1^n\right) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \ldots \cdot \varphi^{n-1}(x_n) \cdot b,$$

for all $x_1, \ldots, x_n \in G$. So, we have

$$aH = \left\{a \cdot \varphi(x_2) \cdot \varphi^2(x_3) \ldots \cdot \varphi^{n-1}(x_n) \cdot b : x_2, \ldots, x_n \in H\right\}.$$

But, clearly this set is in one-one correspondence with the set

$$\left\{\varphi(x_2) \cdot \varphi^2(x_3) \ldots \cdot \varphi^{n-1}(x_n) \cdot b : x_2, \ldots, x_n \in H\right\},$$

which does not depend on a. So, we have $|aH| = |H|$. $\qquad\qquad\qquad\square$

On the set $G/H = \{aH : a \in G\}$ we introduce the operation

$$f_H(a_1H, a_2H, \ldots, a_nH) = f\left(a_1^n\right) H.$$

Proposition 5.5 *If H is a normal n-ary subgroup of* (G, f)*, then* $(G/H, f_H)$ *is an n-ary group derived from the group* $Ret_H(G/H)$.

Proof First we show that the operation f_H is well-defined. For this let $a_iH = b_iH$ for some $a_i, b_i \in G, i = 1, 2, \ldots, n$. Then

$$b_1 = f\left(a_1, x_2^n\right), \quad b_2 = f\left(a_2, y_2^n\right), \quad \ldots, \quad b_n = f\left(a_n, z_2^n\right)$$

for some $x_i, y_i, \ldots, z_i \in H$
Now, using Theorem 1.2 we obtain

$$
\begin{aligned}
f\left(b_1^n\right) &= f\left(f\left(a_1, x_2^n\right), f\left(a_2, y_2^n\right), \ldots, f\left(a_n, z_2^n\right)\right) \\
&= f\left(f\left(a_1, x_2^{n-1}, f\left(\overset{(n-2)}{a_2}, \bar{a}_2, x_n\right)\right), f\left(a_2, y_2^n\right), \ldots, f\left(a_n, z_1^n\right)\right) \\
&= f\left(f\left(a_1, x_2^{n-1}, a_2\right), f\left(f\left(\overset{(n-3)}{a_2}, \bar{a}_2, x_n, a_2\right), y_2^n\right), \ldots, f\left(a_n, z_n^n\right)\right) \\
&= f\left(f\left(a_1, x_2^{n-1}, a_2\right), f\left(w_n, y_2^n\right), \ldots, f\left(a_n, z_1^n\right)\right) \\
&= f\left(f\left(a_1, x_2^{n-2}, f\left(\overset{(n-2)}{a_2}, \bar{a}_2, x_{n-1}\right), a_2\right), f\left(w_n, y_2^n\right), \ldots, f\left(a_n, z_2^n\right)\right) \\
&= f\left(f\left(a_1, x_2^{n-2}, a_2, f\left(\overset{(n-3)}{a_2}, \bar{a}_2, x_{n-1}, a_2\right)\right), f\left(w_n, y_2^n\right), \ldots, f\left(a_n, z_2^n\right)\right) \\
&= f\left(f\left(a_1, x_2^{n-2}, a_2, w_{n-1}\right), f\left(w_n, y_2^n\right), \ldots, f\left(a_n, z_2^n\right)\right) \\
&\;\;\vdots \\
&= f\left(f\left(a_1, a_2, w_3^{n-1}\right), f\left(w_n, y_2^n\right), \ldots, f\left(a_n, z_2^n\right)\right),
\end{aligned}
$$

where $w_i = f(\overset{(n-3)}{a_2}, \bar{a}_2, x_i, a_2) \in H$.

Repeating this procedure for a_3, a_4 and so on, we obtain

$$f\left(b_1^n\right) = f\left(f\left(a_1^n\right), h_2^n\right).$$

This means that the operation f_H is well-defined.

It is easy to verify that $(G/H, f_H)$ is an n-ary group. Using the above procedure it is not difficult to see that H is the identity of G/H. Hence the n-ary group G/H is derived from the group $Ret_H(G/H)$. □

Now, we return to the representations, again. Consider a representation $\Lambda : (G, f) \to GL(V)$. It is easy to see that $\ker \Lambda$ is a normal subgroup of G. Let H be a normal n-ary subgroup of (G, f) such that $H \subseteq \ker \Lambda$. Then, there is a representation $\bar{\Lambda} : G/H \to GL(V)$ such that

$$\bar{\Lambda}(aH) = \Lambda(a).$$

Conversely, from every representation of G/H, we obtain a representation of G. On the other hand, G/H is of reduced type, and hence its representations are the same as the ordinary representations of $Ret_H(G/H)$. So, we proved,

Proposition 5.6 *There is a bijection between ordinary representations of $Ret_H(G/H)$ and the set of representations of G with the property $H \subseteq \ker \Lambda$.*

Proposition 5.7 *A simple n-ary group which is not strongly simple is b-derived from an abelian group or it is reducible to a non-abelian group.*

Proof Suppose $H = \{p\}$ is a normal n-ary subgroup of (G, f). Then we have

$$f(p, p, \ldots, p) = p, \quad \overline{p} = p, \quad \forall x \in G : f\left(\overset{(n-3)}{x}, \overline{x}, p, x\right) = p.$$

Hence

$$f\left(p, x_2^n\right) = f\left(f\left(\overset{(n-2)}{x_2}, \overline{x}_2, p\right), x_2^n\right)$$

$$= f\left(x_2, f\left(\overset{(n-3)}{x_2}, \overline{x}_2, p, x_2\right), x_3^n\right)$$

$$= f\left(x_2, p, x_3^n\right).$$

This shows that p is a central element and, according to [8], the n-ary group (G, f) is b-derived from a binary group $(G, \cdot)$. Hence, $Z(G, f) = Z(G, \cdot)$ is a normal n-ary subgroup of (G, f). But G has no proper normal subgroups, so there are two cases:

(1) $Z(G, \cdot) = G$ and so (G, f) is b-derived from an abelian group,
(2) $Z(G, \cdot)$ is singleton and hence $b = 1$. In this case (G, f) is reducible to a non-abelian group $(G, \cdot)$.

□

Remark 5.8 To find representations of an *n*-ary group (G, f), we have four cases, as follow,

(1) only $H = G$ is a normal subgroup of (G, f), (in this case (G, f) has only trivial representation),
(2) (G, f) is *b*-derived from an abelian group,
(3) $(G, f) = der(G, \cdot)$, (in this case representations of (G, f) are the same as the representations of $(G, \cdot)$),
(4) (G, f) has proper normal *n*-ary subgroups, (in this case, if we know the set of normal *n*-ary subgroups of (G, f), then we obtain all its representations from representations of the groups $Ret_H(G/H)$).

Finally, summarizing results of this section, we have the following theorem:

Theorem 5.9 *Representation theory of n-ary groups, reduces to the following three problems,*

a) *representations of b-derived n-ary groups from abelian groups,*
b) *determining all normal n-ary subgroup,*
c) *representation theory of ordinary groups.*

References

1. Borowiec, A., Dudek, W.A., Duplij, S.: Bi-element representations of ternary groups. Commun. Algebra **34**, 1651–1670 (2006)
2. Dörnte, W.: Unterschungen über einen verallgemeinerten Gruppenbegriff. Math. Z. **29**, 1–19 (1929)
3. Dudek, W.A.: Remarks on *n*-groups. Demonstr. Math. **13**, 165–181 (1980)
4. Dudek, W.A.: On *n*-ary group with only one skew element. Rad. Mat. (Sarajevo) **6**, 171–175 (1990)
5. Dudek, W.A., Glazek, K.: Around the Hosszú–Gluskin Theorem for *n*-ary groups. Discrete Math. **308**, 4861–4876 (2008)
6. Dudek, W.A., Glazek, K., Gleichgewicht, B.: A note on the axioms of *n*-groups. Colloq. Math. Soc. János Bolyai **29**, 195–202 (1977)
7. Dudek, W.A., Michalski, J.: On a generalization of Hosszú theorem. Demonstr. Math. **15**, 437–441 (1982)
8. Dudek, W.A., Michalski, J.: On retract of polyadic groups. Demonstr. Math. **17**, 281–301 (1984)
9. Głazek, K., Gleichgewicht, B.: Abelian *n*-groups. Colloq. Math. Soc. János Bolyai **29**, 321–329 (1977)
10. Gleichgewicht, B., Wanke-Jakubowska, M.B., Wanke-Jerie, M.E.: On representations of cyclic *n*-groups. Demonstr. Math. **16**, 357–365 (1983)
11. Grzymala-Busse, J.W.: Automorphisms of polyadic automata. J. Assoc. Comput. Mach. **16**, 208–219 (1969)
12. Kasner, E.: An extension of the group concept. Bull. Am. Math. Soc. **10**, 290–291 (1904)
13. Kerner, R.: Ternary and non-associative algebraic structures and their applications in physics. Univ. P. and M. Curie, Paris (2000)
14. Michalski, J.: Covering *k*-groups of *n*-groups. Arch. Math. (Brno) **17**, 207–226 (1981)

15. Nikshych, D., Vainerman, L.: Finite quantum groupoids and their applications. Univ. California, Los Angeles (2000)
16. Pojidaev, A.P.: Enveloping algebras of Fillipov algebras. Commun. Algebra **31**, 883–900 (2003)
17. Post, E.L.: Polyadic groups. Trans. Am. Math. Soc. **48**, 208–350 (1940)
18. Vainerman, L., Kerner, R.: On special classes of n-algebras. J. Math. Phys. **37**, 2553–2565 (1996)
19. Wanke-Jakubowska, M.B., Wanke-Jerie, M.E.: On representations of n-groups. Ann. Sci. Math. Polonae. Comment. Math. **24**, 335–341 (1984)