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RESEARCH PAPER

# Large deformation plasticity

## From basic relations to finite deformation

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**Abstract**
The theory of plasticity as a special field of continuum mechanics deals with the irreversible, i.e. permanent, deformation of solids. Under the action of given loads or deformations, the state of the stresses and strains or the strain rates in these bodies is described. In this way, it complements the theory of elasticity for the reversible behavior of solids. In practice, it has been observed that many materials behave elastically up to a certain load (yield point), beyond that load, however, increasingly plastic or liquid-like. The combination of these two material properties is known as elastoplasticity. The classical elastoplastic material behavior is assumed to be time-independent or rate-independent. In contrast, we call a time- or rate-dependent behavior visco-elastoplastic and visco-plastic—if the elastic part of the deformation is neglected. In plasticity theory, because of the given loads the states of the state variables stress, strain and temperature as well as their changes are described. For this purpose, the observed phenomena are introduced and put into mathematical relationships. The constitutive relations describing the specific material behavior are finally embedded in the fundamental relations of continuum theory and physics. Historically, the theory of plasticity was introduced in order to better estimate the strength of constructions. An analysis based purely on elastic codes is not in a position to do this, and can occasionally even lead to incorrect interpretations. On the other hand, the entire field of forming techniques requires a theory for the description of plastic behavior. Starting from the classical description of plastic behavior with small deformations, the present review is intended to provide an insight into the state of the art when taking into account finite deformations.

**Keywords** Finite deformation · Elasto-plasticity · Constitutive relations · Thermodynamics · History of plasticity

## 1 Small deformation theories

Since people have learned to melt and process metals, they know that these can be deformed under the influence of heat and large forces and that their properties can change as well. Whereas in ancient times, the knowledge about these processes was only accessible to a few selected people, who carefully guarded it as art, a rethinking has taken place with the New Age. At least since the beginning of industrialization in Europe, people were forced to investigate the governing processes and their interrelations more precisely. This was the beginning of the scientific treatment of plasticity, which, perhaps somewhat arbitrarily, can be identified with a series of experiments by the French engineer Tresca towards the end of the 19th century[1] Tresca [5] as well as later von Mises [6] introduced a limit in stress space, separating the range of elastic (or rigid) behavior from that of plastic deformations. The investigations then began with purely phenomenological considerations to describe the constitutive relations between the stresses and rates of strain and strains, respectively.

Until about 1940, for the case of small deformations, three different types of relations were proposed. These were (i) the Lévy-von Mises equations:

$$\dot{\boldsymbol{\varepsilon}}' = \Lambda \boldsymbol{\sigma}' , \quad (\bullet)' = (\bullet) - \frac{1}{3}\,\mathrm{tr}(\bullet)\boldsymbol{I} , \tag{1}$$

for a rigid plastic material[2], (ii) the Hencky equations:

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[1] A somewhat more detailed outline of the history of plasticity can be found in Refs. [1–4]. Parts of these works were included in the present review article.

[2] Here and in what follows, a prime will mark the deviator of a second order tensor.

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$$\boldsymbol{\varepsilon}' = \frac{1+\varphi}{2\mu}\,\boldsymbol{\sigma}'\,, \quad \mathrm{tr}(\boldsymbol{\varepsilon}) = \frac{1}{3K}\,\mathrm{tr}(\boldsymbol{\sigma})\,, \quad \dot{\boldsymbol{\varepsilon}}' = \frac{1}{2\mu}\,\dot{\boldsymbol{\sigma}}'\,. \tag{2}$$

under loading condition beyond the yield limit and with Eq. $(2)_3$ during unloading and purely elastic processes, and (iii) the Prandtl-Reuss equations:

$$\dot{\boldsymbol{\varepsilon}}' = \frac{1}{2\mu}\,\dot{\boldsymbol{\sigma}}' + \Lambda\boldsymbol{\sigma}'\,, \quad \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}) = \frac{1}{3K}\,\mathrm{tr}(\dot{\boldsymbol{\sigma}})\,, \tag{3}$$

for loading beyond the yield limit and with a plastic multiplier $\Lambda = 0$ for unloading or purely elastic behavior, where $\mu$ is the shear modulus and $K$ the bulk modulus. In addition, during the plastic part of the deformations, preferably von Mises type yield conditions like

$$F = \mathrm{tr}(\boldsymbol{\sigma}'^2) - 2\,k^2 = 0, \tag{4}$$

were prescribed, where $k$ is the shear yield limit.

It is obvious that Eq. (1) is valid only for plastic deformation. Below the yield limit, and hence $\Lambda = 0$, this model describes a rigid behavior. Thus, representing a special case of the more general Prandtl-Reuss model for vanishing elastic deformations or for those cases where the latter can be neglected compared with the plastic deformations. On the other hand, the $\varphi$ in Eq. $(2)_1$ is a yet unspecified proportionality factor, analogous to $\Lambda$.

Subsequently, the two models of Prandtl-Reuss and Hencky[3], were a matter of discussion whether they could adequately reproduce experimental observations. With this respect, Hill [7] finally stated that the Hencky equations were unsuitable to describe a complete plastic behavior of a metal, and further: *"None the less, in situations where the loading is continuous, the Hencky equations may lead to results in approximate agreement with observations. In some problems, too, total strain theories have certain advantages of mathematical convenience. It is apparently for this reason that the Hencky (...) relations have often been used in applications where the strains are small, particularly by Russian writers."* It thus turned out that applying the Hencky equations could be associated with great inaccuracies, especially in those cases where the loads are prescribed in the form of non-radial processes. Therefore, this model, which is also referred to as deformation theory, is no longer used today.

A more general description of the Prandtl-Reuss equations can be found in the form

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_\mathrm{e} + \dot{\boldsymbol{\varepsilon}}_\mathrm{p} = \mathbb{C}^{-1}\!:\dot{\boldsymbol{\sigma}} + \varrho\Lambda\,\frac{1}{2}\,\frac{\partial F}{\partial \boldsymbol{\sigma}'}\,, \tag{5}$$

where $\mathbb{C}$ is the elastic stiffness tensor for an isotropic material, and the colon marks a double contracting product. A so-called loading factor $\varrho$ has been introduced taking the values 1 and 0 during loading and unloading (elastic behavior), respectively:

$$\varrho = \begin{cases} 1, \text{ wherever } \; F = 0 \;\text{ and } \dot{F} = 0\,, \\ 0, \text{ wherever } \; F < 0, \text{ or where } F = 0 \text{ and } \dot{F} < 0\,. \end{cases} \tag{6}$$

Thus, for a von Mises material with yield condition (4), we determine

$$\dot{\boldsymbol{\varepsilon}} = \mathbb{C}^{-1}\!:\dot{\boldsymbol{\sigma}} + \varrho\Lambda\boldsymbol{\sigma}'\,. \tag{7}$$

For an isotropic, work-hardening hardening material, where function $k^2$ in Eq. (4) is described as function of the (accumulated) plastic work $w^\mathrm{p} = \int \boldsymbol{\sigma}\!:\dot{\boldsymbol{\varepsilon}}_\mathrm{p}\,\mathrm{d}t$, this relation is reformulated as

$$k^2 = k_0^2 + f(w^\mathrm{p})\,. \tag{8}$$

Now with the help of the consistency condition $\dot{F} = 0$, $\Lambda$ can be calculated as[4]

$$\Lambda = \frac{1}{H}\,\boldsymbol{\sigma}'\!:\dot{\boldsymbol{\sigma}}'\,, \quad H = 2k^2\,\frac{\mathrm{d}k^2}{\mathrm{d}w^\mathrm{p}}\,, \tag{9}$$

where $H = H(w^\mathrm{p})$ is a hardening function which has to be determined from appropriate experiments. With this result, we finally arrive at

$$\dot{\boldsymbol{\varepsilon}} = \mathbb{C}^{-1}\!:\dot{\boldsymbol{\sigma}} + \frac{\varrho}{H}\,(\boldsymbol{\sigma}'\!:\dot{\boldsymbol{\sigma}}')\boldsymbol{\sigma}'\,. \tag{10}$$

For application, especially for numerical calculations, an inversion of Eq. (10) is necessary. We therefore multiply it from the left by $\mathbb{C}$. Thus, $\mathbb{C}\!:\mathbb{C}^{-1} = \mathbb{I}$, where $\mathbb{I}$ is the fourth-order identity tensor, and

$$\dot{\boldsymbol{\sigma}} = \mathbb{C}\!:\left(\dot{\boldsymbol{\varepsilon}} - \frac{\varrho}{H}\,(\boldsymbol{\sigma}'\!:\dot{\boldsymbol{\sigma}}')\boldsymbol{\sigma}'\right). \tag{11}$$

Now again multiplying this result from the left by $\boldsymbol{\sigma}'$ yields

$$\boldsymbol{\sigma}'\!:\dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}'\!:\dot{\boldsymbol{\sigma}}' = \boldsymbol{\sigma}'\!:\mathbb{C}\!:\dot{\boldsymbol{\varepsilon}} - \frac{\varrho}{H}\,(\boldsymbol{\sigma}'\!:\dot{\boldsymbol{\sigma}}')\boldsymbol{\sigma}'\!:\mathbb{C}\!:\boldsymbol{\sigma}'\,.$$

From this, we finally can derive[5] the inverted form of Eq. (10)

$$\dot{\boldsymbol{\sigma}} = \mathbb{C}\!:\left(\dot{\boldsymbol{\varepsilon}} - \varrho\boldsymbol{\sigma}'\,\frac{\boldsymbol{\sigma}'\!:\mathbb{C}\!:\dot{\boldsymbol{\varepsilon}}}{H + \boldsymbol{\sigma}'\!:\mathbb{C}\!:\boldsymbol{\sigma}'}\right). \tag{12}$$

[3] In the English literature, these models are also referred to as *incremental* and *total* theories.

[4] We note that during loading $\varrho$ takes the value 1.

[5] This calculation was first carried out in Ref. [7] for the specific case of an isotropic material with hardening.

### 1.1 Some amendments

Numerous modifications of this classical flow rule have been introduced in the last centuries in various directions:

Firstly, with a modification of the yield condition (4), the concept of kinematic hardening has been adopted to account for the Bauschinger effect during cyclic loading. According to Reuss [8], Prager [9], and Melan [10], we find

$$F = (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) - 2k^2 \,, \quad \dot{\boldsymbol{\alpha}} = c\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} \,, \tag{13}$$

where $\boldsymbol{\alpha}$ is the so-called backstress, and $c$ is an additional hardening parameter. A slightly modified expression for the plastic part of the power $\dot{w}^{\mathrm{p}}$, namely

$$\boldsymbol{s} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} = \dot{w}^{\mathrm{p}} \,, \quad \boldsymbol{s} = \boldsymbol{\sigma}' - \boldsymbol{\alpha} \,, \tag{14}$$

leads to a new Prandtl-Reuss equation

$$\dot{\boldsymbol{\sigma}} = \mathbb{C} : \left( \dot{\boldsymbol{\varepsilon}} - \varrho \boldsymbol{s} \, \frac{\boldsymbol{s} : \mathbb{C} : \dot{\boldsymbol{\varepsilon}}}{\bar{H} + \boldsymbol{s} : \mathbb{C} : \boldsymbol{s}} \right) , \tag{15}$$

where, compared with Eq. (11), $\boldsymbol{\sigma}'$ has been replaced by the reduced deviatoric stresses $\boldsymbol{s}$ and the hardening function $H$ of Eq. $(9)_2$ by

$$\bar{H} = H + 2k^2 c \,, \tag{16}$$

combining the isotropic and the kinematic hardening.

Moreover, to describe in a more appropriate manner the behavior during cyclic loading and to account for the different phenomena observed during corresponding experiments, the linear evolution Eq. $(13)_2$ has been modified by introducing a second term, e.g.,

$$\dot{\boldsymbol{\alpha}} = c \left( \dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} - \frac{\boldsymbol{\alpha}}{2k^2} \, \dot{w}^{\mathrm{p}} \right) . \tag{17}$$

This generalization was first discussed in Ref. [11] and then propagated in several papers (see, e.g., Ref. [12])[6].

Applying this modification to the Prandtl-Reuss equation, we finally arrive at

$$\dot{\boldsymbol{\sigma}} = \mathbb{C} : \left( \dot{\boldsymbol{\varepsilon}} - \varrho \boldsymbol{s} \, \frac{\boldsymbol{s} : \mathbb{C} : \dot{\boldsymbol{\varepsilon}}}{\hat{H} + \boldsymbol{s} : \mathbb{C} : \boldsymbol{s}} \right) , \tag{18}$$

where now

$$\hat{H} = \bar{H} - c \, \mathrm{tr}(\boldsymbol{s}\boldsymbol{\alpha}) = H + c[2k^2 - \mathrm{tr}(\boldsymbol{s}\boldsymbol{\alpha})] \,, \tag{19}$$

allowing for different evolutions in the *tensile* and the *compressive* parts of the hysteresis loop.

In addition, several kinematic variables can be used, with the generalization

$$\boldsymbol{\alpha} = \sum_i \boldsymbol{\alpha}_i \,, \quad (i = 1, 2, \ldots, n). \tag{20}$$

Finally, numerous applications of plasticity, e.g., in metal forming processes, made it necessary to extend the relations to large deformations. Whereas Hencky's deformation theory from its setting was restricted to applications within infinitesimal small deformations, this was not the case for the Prandtl-Reuss theory. Having in mind its setting as a combination of fluid-like and solid-like materials, its description of the fluid-like part originally was introduced as a relation of the stress $\boldsymbol{\sigma}$ as function of the rate of deformation (stretching) tensor $\boldsymbol{D}$. Thus, it should be straightforward to replace the rates of the strains in Eq. (5) by the respective parts of the rate of deformation, viz.

$$\boldsymbol{D} = \boldsymbol{D}_{\mathrm{e}} + \boldsymbol{D}_{\mathrm{p}} \,. \tag{21}$$

Applying this to Eq. (3), yields

$$\boldsymbol{D}' = \frac{1}{2\mu} \, \dot{\boldsymbol{\sigma}}' + \Lambda \boldsymbol{\sigma}', \quad \mathrm{tr}(\boldsymbol{D}) = \mathrm{tr}(\boldsymbol{D}_{\mathrm{e}}) = \frac{1}{3K} \, \mathrm{tr}(\dot{\boldsymbol{\sigma}}) \,. \tag{22}$$

This, however, would cause at least two new problems related with the elastic part of the aforementioned decomposition. A first seminal discussion of the question how to define in a physically reasonable way the different parts of strains or strain rates that may contribute to the composite behavior of an elastic-plastic material was given in Ref. [13][7].

(1) If the rates of the strain in the foregoing relations are replaced by the stretching, a relation between strain and stretching would become necessary, e.g., to determine the strains in the plastically deformed structure. Provided such a relation exists. We will see that it took several decades to properly answer this first question.
(2) Then the question arises, what kind of rate should be used for the herein incorporated stresses and stress-like quantities?
(3) Finally, a fundamental question needs to be answered: In which way should the splitting of the different solid-like elastic and fluid-like plastic contributions to the total deformation be applied? Moreover, if strains are used, which strain measure should be chosen?

[6] In several of these works, the progress of the plastic processes is described by accumulated plastic strains $\boldsymbol{\varepsilon}_{\mathrm{p}}$ rather than by the accumulated plastic work $w^{\mathrm{p}}$. This, however, does not change the results significantly.

[7] The interested reader may in particular follow the discussion in Sects. 3 and 4 of this critical review.

### 1.2 Cyclic loading

Cyclically varying loads occur in many problems of engineering applications. Already with his work from 1924, H. Hencky [14] pointed out the necessity of an adequate treatment. At the time, he explained that, depending on the specific loading, a cyclic load could give rise to an elastic shakedown or to damage with rupture and subsequent failure of the structure. These thoughts were taken up again in the 1930s by many authors [8–10,15] when introducing kinematic hardening and describing the Bauschinger effect. Another very simple approach to describe the Bauschinger effect can be traced back to Ref. [16]. Since the second half of the last century, such challenging engineering projects as aircraft constructions, high-speed railways, aerospace structures and, in particular, the construction of nuclear reactors have provided a significant stimulus to improve the description of cyclic plasticity phenomena. These developments required a particularly high level of security for these structures. Since then, many attempts have been made to include hardening and recovery, ratchetting, amplitude dependence of hardening, the influence of non-radial loading, etc., in model formulations.

One of the first more realistic models that has been widely used in practice describes a back stress increasing along the plastic strain rate orientation and decreasing proportionally to the back stress itself, refer to Eq. (17). That is, the back stress evolves toward its limit value specified by a hardening saturation. This model was originally introduced by Armstrong and Frederick (A-F model) [11] and then generalized by many authors [17–25]. Comparable results can be achieved with a multi-surface model such as provided by Mróz [26], cf. Refs. [27,28]. Ohno and Wang [29] succeeded in significantly improving the A-F model by incorporating the dynamic recovery of the $\boldsymbol{\alpha}_i$. Amendments were discussed e.g. in Refs. [30,31]. Further typical examples of current models are given as Refs. [32–35]. An extension to model large deformations and to include temperature effects is described with Refs. [36–38].

### 1.3 Influence of temperature and rate dependence

Although from the very beginning plastic behavior of a material body should be closely related with thermomechanical energy transformations, e.g., due to the energy dissipation into heat during plastic work, this effect was for long time neglected (refer, e.g., to the textbook [7]).

This situation began to change, when in the 1950s and 60s, several researchers studied the thermodynamic foundations of elastoplasticity. In the case of small deformation, e.g., Prager [39], Ziegler [40], and Naghdi [41] started to incorporate thermodynamic principles into their considerations. In earlier works, e.g., Green [42] and Prager [43] already used specific elements of thermodynamics, when demanding a positive value of the rate of plastic work (plastic power) throughout the whole body. Later these thoughts were broadened in Refs. [44–47]. More recent contributions were made in Refs. [48–50], and many others. Experimental observations describing a temperature dependency of plastic behavior were reported, e.g., in Refs. [51–54].

In order to consider the temperature dependence of a material behavior, it is not sufficient to simply insert the temperature into the material description, e.g., as an additional parameter. The modeling in addition to the fundamental laws of mechanics has to incorporate also the natural laws of thermodynamics.

First, it is assumed that there exists a quantity $T$, called the absolute temperature, which is always positive:

$$T > 0\,. \tag{23}$$

This statement is sometimes referred to as the zeroth law of thermodynamics.

Second, from the observation of interconvertibility of heat and mechanical work, it is assumed that the mechanical work done by the external loading and the internal stresses and the non-mechanical work contributed by the heat flux and the heat supply are converted into kinetic energy and internal energy of the material body. Accordingly, a new physical quantity, the internal energy, is introduced. Let $\varphi$ represent the internal energy per unit reference volume. Then, from the energy balance, together with the Euler-Cauchy law of motion and the continuity equation, the following relation may be derived:

$$\dot{\varphi} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \nabla \cdot \boldsymbol{q} + r\,, \tag{24}$$

for all possible thermodynamic processes. Here, $\nabla$ is used to designate the differentiation with respect to the current position vector. Let $\boldsymbol{n}$ be the outward normal at a point on a closed material surface. Then $\boldsymbol{q} \cdot \boldsymbol{n} > 0$ means the efflux of the heat through the material surface and thus follows the minus sign in Eq. (24) for the term contributed by the heat flux $\boldsymbol{q}$. This is the first law of thermodynamics: the changing rate of internal energy is furnished by the sum of the (internal) stress power and the (internal) non-mechanical power due to heat flux and heat supply $r$.

Next, based on the observation of the irreversibility of macroscopic physical phenomena, it is assumed that there exists a scalar field quantity $\eta$ measured per unit reference volume, called specific entropy, so that the following inequality holds:

$$\dot{\eta} \geq -\nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) + \frac{r}{T}\,, \tag{25}$$

where $\dot{\eta}$ represents the total local dissipation. This inequality is the second law.

Entropy $\eta$ and the internal energy $\varphi$ are related to each other through the first and second laws. To render this relation direct, alternate forms are derivable by a Legendre transform introducing the Helmholtz free energy per unit reference volume:

$$\psi = \varphi - T\eta \,. \tag{26}$$

Then, the energy balance (24) may be reformulated as

$$\dot{\psi} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \nabla \cdot \boldsymbol{q} + r - \eta \dot{T} - T\dot{\eta} \,. \tag{27}$$

Using this relation, we may recast inequality (25)

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - (\dot{\psi} + \eta \dot{T}) - \frac{1}{T} \boldsymbol{q} \cdot \nabla T \geq 0 \,. \tag{28}$$

Inequality (25) or its alternate form Eq. (28), also known as Clausius-Duhem inequality, express the entropy principle for the irreversibility of physical phenomena in deformable bodies.

Furthermore, we introduce a quantity $\mathscr{D}$

$$\mathscr{D} = T\dot{\eta} - (r - \nabla \cdot \boldsymbol{q}) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - (\dot{\psi} + \eta \dot{T}) \,. \tag{29}$$

This quantity, known as internal dissipation, provides the remaining part after the deduction of the total local heating from the total local entropy rate times the temperature. The Planck inequality (cf., e.g., Ref. [55])

$$\mathscr{D} \geq 0 \tag{30}$$

requires that the internal dissipation should always be non-negative. It should be positive for a process of dissipative deformation such as in elastoplasticity.

For a thermomechanical behavior of a material body, there will be a strong coupling between the deformation field, the stress field, and the temperature field. The material behavior will be characterized by constitutive relations relating these field quantities and their histories. The thermodynamic laws as expressed in Eqs. (23–25), or their alternate forms Eqs. (27) and (28), place restrictions on various kinds of the material behavior.

We note here that the stress power $\dot{w} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$ per unit volume in Eq. (24) plays an essential role, especially when deriving constitutive relations. If thermodynamic consistency of a constitutive relation is analyzed—or in other words—if the bounds of such consistency with thermodynamics are examined, the internal dissipation $\mathscr{D}$ should fulfill Planck's inequality (30).

Since $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\varepsilon}}$ constitute the stress power, the stress and the strain may be regarded as pair of energetically conjugated quantities [56–58]. Thus, for an elastoplastic material, e.g., with state variables $\boldsymbol{\epsilon}_{\mathrm{e}}$ and $T$, process variables $\boldsymbol{\alpha}$ and $\kappa = k^2(w^{\mathrm{p}}, T)$, and the Helmholtz free energy $\psi = \psi(\boldsymbol{\varepsilon}_{\mathrm{e}}, T, \boldsymbol{\alpha}, \kappa)$, inequality (30) becomes

$$\begin{aligned} \mathscr{D} = &\left\{ \boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}_{\mathrm{e}}} \right\} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{e}} - \left\{ \eta + \frac{\partial \psi}{\partial T} \right\} \dot{T} \\ &+ \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} - \frac{\partial \psi}{\partial \boldsymbol{\alpha}} \dot{\boldsymbol{\alpha}} - \frac{\partial \psi}{\partial \kappa} \dot{\kappa} \geq 0 \,. \end{aligned} \tag{31}$$

With this notation, we follow Ref. [59], and only for simplicity, we replace the $k^2$ of prior relations by a process variable $\kappa$, which now may be a function of plastic work and the temperature.

Since this inequality is assumed to hold true for all admissible processes, two equations of state can be deduced as

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}_{\mathrm{e}}} \quad \text{and} \quad \eta = -\frac{\partial \psi}{\partial T} \,. \tag{32}$$

In addition, as the time rates of the state variables are independent of $\nabla T$, the Clausius-Duhem inequality (28) was recast into an internal dissipation term and a heat conduction inequality. The latter requires

$$-\frac{1}{T} \boldsymbol{q} \cdot \nabla T \geq 0 \,. \tag{33}$$

With these results, the production of free energy becomes

$$\dot{\psi} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{e}} - \eta \dot{T} - \mathscr{D}_i \,, \tag{34}$$

where

$$\mathscr{D}_i = -\frac{\partial \psi}{\partial \boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}} - \frac{\partial \psi}{\partial \kappa} \dot{\kappa} \tag{35}$$

is the dissipation due to the inelastic deformation of the internal structure. Thus, introducing the balance (27), the entropy production may be reformulated as

$$T\dot{\eta} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} + \mathscr{D}_i - \nabla \cdot \boldsymbol{q} + r \,. \tag{36}$$

This is guaranteed for arbitrary $\dot{\boldsymbol{\varepsilon}}_{\mathrm{e}}$ and $\dot{T}$, if the two equations of state (32) hold. The first term herein is the so-called inelastic stress power.[8]

With the help of the equations of state (32), the above result may be converted to an evolution equation for the temperature (refer, e.g., to Ref. [60]). We thus arrive at the equation of heat production:

$$c_T \dot{T} = T \frac{\partial \boldsymbol{\sigma}}{\partial T} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{e}} - \nabla \cdot \boldsymbol{q} + r + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} + \mathscr{D}_i - T \frac{\partial \mathscr{D}_i}{\partial T} \,, \tag{37}$$

[8] In passing, we note that for an adiabatic process, i.e. for $-\nabla \cdot \boldsymbol{q} + r \equiv 0$, due to the different sources of dissipation the remaining thermodynamic process is not isentropic.

where

$$c_T = -T\,\frac{\partial^2 \psi}{\partial T^2}$$

is the specific heat at constant strain.

To complete the equation of heat production, a constitutive relation for the heat flux $\boldsymbol{q}$ has to be inserted into Eq. (37). It is a common practice to introduce here Fourier's law of isotropic heat conduction

$$\boldsymbol{q} = -\lambda \nabla T\,, \tag{38}$$

where $\lambda \geq 0$ is here the so-called thermal conductivity and such that inequality (33) will take its maximum value.

Fourier's law means that the heat flux can only take place in the direction of decreasing temperature. It is obvious that for one-dimensional problems, this equation reduces to the parabolic diffusion equation, implying that thermal disturbances propagate with infinite velocity. A more appropriate generalized law has been discussed in Ref. [61]

$$c_\tau\,\dot{\boldsymbol{q}} + \boldsymbol{q} = -\lambda \nabla T\,, \tag{39}$$

the so-called Maxwell-Cattaneo relation, where now wave-like solutions can be obtained from this equation. Accordingly, $c_\tau$ is the characteristic relaxation time.

We note that both laws of heat conduction constitute rate-dependent relations and thus introduce a rate dependency into the description of thermoplasticity, although the setting of the elastic-plastic material description was introduced as rate independent.

It is well known that in many practical problems, the actual behavior of a material is governed by plastic as well as viscous effects, e.g., visible as a more or less pronounced strain rate dependence of respective uniaxial stress-strain curves. This, however, is contradicting the fundamental assumption of plasticity, namely, a time independence of the constitutive equations. It is therefore a common practice that for a material with less rate sensitivity, or for a low velocity of loading, this dependence is neglected. For many materials, however, it is observed that these strain-rate effects are more pronounced after the plastic state has been reached. In these cases, it may be assumed that the material displays viscous properties in the plastic range, too. Before yielding, the material remains elastic, and after yielding, it exhibits viscous as well as plastic properties. This composite behavior is called viscoplasticity.

We note that for most materials, the rate sensitivity increases with the temperature. Thus, it should be clear that at higher temperatures, these strain-rate effects could not be neglected.

In viscoplasticity, the development of a mathematical model heads back to the beginning of the twentieth century with Refs. [62–64,66] and some earlier considerations in Refs. [65,67]. Norton [68] developed a one-dimensional dashpot model, which linked the rate of (secondary) creep to the stress. In 1934, Odqvist [69] generalized Norton's law to the multiaxial case. Hohenemser and Prager proposed a first model for slow viscoplastic flow.This model provided a relation between the deviatoric stress and the strain rate for an incompressible Bingham solid [70]. Malvern [71] has taken up these ideas proposing a one-dimensional constitutive law of the form

$$E\dot{\varepsilon} = \dot{\sigma} + k[\sigma - f(\varepsilon)]\,. \tag{40}$$

Here, the plastic strain rate is proportional to $\sigma - f(\varepsilon)$, the excess of stress over the stress at the same strain in a static test, and $\sigma = f(\varepsilon)$ expressing the static plastic stress-strain relation. The basic idea of this so-called overstress description has been adopted later by numerous other authors.

The first IUTAM Symposium on "Creep in Structures" [72] provided a major development in viscoplasticity. Perzyna in Refs. [73,74] and later with his seminal work [75] introduced a viscosity coefficient that is temperature and time dependent, e.g.,

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2\mu}\dot{\boldsymbol{\sigma}}' + \gamma\langle\Phi(F)\rangle\frac{\partial f}{\partial \boldsymbol{\sigma}}\,, \quad \mathrm{tr}\dot{\boldsymbol{\varepsilon}} = \frac{1}{3K}\mathrm{tr}\dot{\boldsymbol{\sigma}}\,, \tag{41}$$

where $\mu$ and $K$ are parameters of elasticity and $\gamma$ is a viscosity constant of the material. The symbol $\langle\bullet\rangle$ herein is defined as:

$$\langle\bullet\rangle = \begin{cases} 0, & \text{for} \quad F \leq 0, \\ \bullet, & \text{for} \quad F > 0, \end{cases} \tag{42}$$

and $F$ is a static yield function:

$$F(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\mathrm{p}}) = \frac{f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\mathrm{p}})}{\kappa} - 1\,, \tag{43}$$

wherein $\kappa = \kappa(w^{\mathrm{p}})$ is a work-hardening parameter.

In cases where the influence of the temperature can no longer be neglected, these equations are reformulated as

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2\mu}\dot{\boldsymbol{\sigma}}' + \gamma(T)\langle\Phi(F)\rangle\frac{\partial f}{\partial \boldsymbol{\sigma}}\,, \quad \mathrm{tr}\dot{\boldsymbol{\varepsilon}} = \frac{1}{3K}\mathrm{tr}\dot{\boldsymbol{\sigma}} + \bar{\alpha}\dot{T}\,, \tag{44}$$

with $\bar{\alpha}$ the coefficient of thermal expansion. Here $\gamma$ and $\kappa$, and thus $F$ and $\Phi$ are temperature-dependent quantities.

## 2 Step towards finite deformations

Hencky [76] was one of the first to use tensor analysis in continuum mechanics. He also in Ref. [77] introduced a logarithmic strain measure

$$\boldsymbol{h} = \frac{1}{2}\ln \boldsymbol{B}\,, \quad \boldsymbol{B} = \boldsymbol{V}^2 = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}, \tag{45}$$

in today's notation, and

$$\boldsymbol{h} = \boldsymbol{e}^{(0)} = \frac{1}{2}\ln \boldsymbol{B} = \frac{1}{2}\sum_{\sigma=1}^{n}(\ln \chi_\sigma)\boldsymbol{B}_\sigma\,, \tag{46}$$

where $\boldsymbol{F}$ is the deformation gradient, and $\chi_\sigma$ and $\boldsymbol{B}_\sigma$ are $n$ distinct eigenvalues and corresponding eigenprojections, respectively, of the left Cauchy-Green tensor $\boldsymbol{B}$. The symbol $(\bullet)^{\mathrm{T}}$ herein is used to represent the transpose of the second-order tensor. Moreover, $\boldsymbol{e}^{(0)}$ marks an Eulerian strain out of the family of the Seth-Hill-Doyle-Ericksen strains [78]:

$$\boldsymbol{e}^{(m)} = \frac{1}{2m}\left(\boldsymbol{B}^m - \boldsymbol{I}\right), \tag{47}$$

for $m = 0$.[9]

The corresponding Lagrangian counterparts are

$$\boldsymbol{E}^{(m)} = \frac{1}{2m}\left(\boldsymbol{C}^m - \boldsymbol{I}\right), \tag{48}$$

and in particular for $m = 0$,

$$\boldsymbol{H} = \frac{1}{2}\ln \boldsymbol{C}\,, \quad \boldsymbol{C} = \boldsymbol{U}^2 = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} \tag{49}$$

are the Lagrangian Hencky strains with the property

$$\boldsymbol{H} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{h}\boldsymbol{R}\,. \tag{50}$$

Herein $\boldsymbol{C}$ is the right Cauchy-Green tensor, and according to the polar decomposition, the deformation gradient $\boldsymbol{F}$ has the unique left and right multiplicative decompositions

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{U}, \quad \boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{-1}, \quad \det \boldsymbol{R} = 1\,, \tag{51}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are right and left stretch tensors, and $\boldsymbol{R}$ is the rotation tensor. Thus, due to the above correspondence (50), the Lagrangian Hencky strain $\boldsymbol{H}$ may be interpreted as the back-rotated Eulerian Hencky strain $\boldsymbol{h}$.

Hencky introduced Lagrangian and Eulerian descriptions and discussed in this context the importance of time derivatives occurring in the relevant constitutive laws. For a Lagrangian analysis, these were the material time derivatives. In an Eulerian description, which he preferred for physical reasons, he noted that the time derivative must be independent of the respective rigid body rotation.

[9] The proof of this limit is given with Ref. [4].

In Ref. [77], he therefore replaced the time derivative of the stress tensor by[10]

$$\dot{\boldsymbol{\sigma}} \quad \Rightarrow \quad \mathring{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{\sigma}\,. \tag{52}$$

Herein, the spin tensor (vorticity) $\boldsymbol{W}$ has been introduced as skew-symmetrical part of the velocity gradient $\boldsymbol{L}$:

$$\boldsymbol{W} = \frac{1}{2}\left(\boldsymbol{L} - \boldsymbol{L}^{\mathrm{T}}\right), \quad \boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}\right), \quad \boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}. \tag{53}$$

Today, we recognize the Jaumann derivative, previously discussed in Refs. [79,80]. At that time, however, this finding was ignored. Until much later the idea was taken up again, e.g. in Refs. [81–87], and others. Prager [88] also mentioned that *"Jaumann's work does not seem to be well known: the definition ... is frequently used in the recent literature without reference to Jaumann."*

## 2.1 Different objective rates and hypoelasticity

The first who rediscovered the above mentioned problems of describing finite deformations was Oldroyd [81]. Like Hencky he introduced convected coordinates and a *convected differentiation with respect to time* which must replace the material derivative *when equations of state are transformed from a convected to a fixed system of reference.*

As convected coordinates were used, four different relations may be given in a spatial description depending on the different representations of the (second rank) tensor components as covariant, contravariant, or mixed quantities. It can be shown that these different forms are particular cases of the Lie derivative (refer to Refs. [89,90]). For the stress tensor, a contravariant description was preferred, and thus the *upper-convected* derivative or Oldroyd rate $(54)_1$ was obtained

$$\begin{aligned}
(\delta^{cc}/\delta t)\boldsymbol{\sigma} &= \mathring{\boldsymbol{\sigma}}_u = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{L}^{\mathrm{T}} - \boldsymbol{L}\boldsymbol{\sigma}\,,\\
(\delta^{\cdot c}_{c}/\delta t)\boldsymbol{\sigma} &= \mathring{\boldsymbol{\sigma}}_m = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{L}^{\mathrm{T}} + \boldsymbol{L}^{\mathrm{T}}\boldsymbol{\sigma}\,,\\
(\delta^{c}_{\cdot c}/\delta t)\boldsymbol{\sigma} &= \mathring{\boldsymbol{\sigma}}_m = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{L} - \boldsymbol{L}\boldsymbol{\sigma}\,,\\
(\delta_{cc}/\delta t)\boldsymbol{\sigma} &= \mathring{\boldsymbol{\sigma}}_l = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}\boldsymbol{\sigma}\,,
\end{aligned} \tag{54}$$

for the Cauchy stresses. Here the superscript or subscript index $c$ is used as a mnemonic notation to distinguish contravariant or covariant, respectively (refer to Ref. [57]).[11] In

[10] Unfortunately, the original work of Hencky contained a small error. Instead of the above spin tensor, he used an alternative definition, which differs by a minus sign. Due to this deviation, however, his derivative (52) loses its objectivity.

[11] We note that in the original literature instead of the small circle a wavy line was used to designate an objective time derivative. In some more recent literature, the raising and lowering of the indices $c$ is in analogy to the accidentals of music notation designated by a sharp (♯) or a flat (♭).

the above, the remaining three forms are with Eq. $(54)_4$ the Cotter-Rivlin or *lower-convected* rate according to Ref. [91]. Moreover, due to the symmetry of the Cauchy stress tensor, the two mixed descriptions coincide.

In a comprehensive work on the foundations of elasticity and fluid dynamics, Truesdell [92] discussed the general form of a material where the rate of stress is related to the rate of deformation. Applying the principle of invariance against a rigid rotation, he arrived at the following expression for the rate of stress:

$$\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{L}^{\mathrm{T}} - \boldsymbol{L}\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}(\boldsymbol{D}) \tag{55}$$

in a spatial description, where $\overset{\circ}{\boldsymbol{\sigma}}$ is named a relative time flux. It is evident that this is the same result as Oldroyd's upper-convected rate, where the differential operation is applied to the weighted Cauchy stress $J\boldsymbol{\sigma}$, where $J$ is the Jacobian of deformation and $\dot{J} = J\mathrm{tr}(\boldsymbol{D})$.

Thus, the defining relation for the simplest law satisfying this principle is

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbb{A} : \boldsymbol{D}\,, \tag{56}$$

where the material tensor $\mathbb{A}$ may be a function of $\boldsymbol{\sigma}$. This class of materials is named hypoelastic bodies with the properties that in general they neither have a preferred state nor a preferred stress (refer to Ref. [93]). We note that Ref. [85] emphasized that *"this new theory does not employ any concept of strain."*

As a special case, an isotropic hypoelastic body of grade zero is considered, where the right-hand side of Eq. (56) appears to be independent of $\boldsymbol{\sigma}$ [93]. In the simplest case, this expression will depend upon the rate of deformation in the same way as the stress depends upon the strain in the classical elasticity theory, viz.

$$\overset{\circ}{\boldsymbol{\sigma}} = 2\mu\boldsymbol{D} + \lambda\mathrm{tr}(\boldsymbol{D})\boldsymbol{I}\,, \tag{57}$$

where here $\mu$ and $\lambda$ are the Lamé constants. It was shown that these equations reduce to those of classical linear elasticity under the usual assumptions of infinitesimal deformations.

Under the assumption of invariance against a rigid body motion, Refs. [83,84] arrived at a description for the rate of stress in the form of the Jaumann rate.[12] Hill [94] finally combined the convected derivatives of Eq. (54) and the Jaumann rate by introducing a relation

$$\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{\sigma} - m(\boldsymbol{\sigma}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{\sigma})\,, \tag{58}$$

which turns over to the Jaumann rate (52), the Oldroyd rate $(54)_1$, and the Cotter-Rivlin rate $(54)_4$, respectively, for $m = 0, 1, -1$. Hill indeed preferred the Kirchhoff stress $\boldsymbol{\tau}$, which is the weighted Cauchy stress $J\boldsymbol{\sigma}$, rather than the Cauchy stress $\boldsymbol{\sigma}$ itself.

During ensuing years, possible relations between hypoelasticity and elasticity were discussed extensively. From this discussion, it turned out that most hypoelastic materials fail to have the properties of an elastic let alone a hyperelastic (Green elastic) material. It was Bernstein [95] who showed that hypoelastic rate constitutive relations have to meet specific integrability conditions to gain the properties of an elastic material. Nevertheless these rates—preferably the Jaumann rate—were widely accepted and used, even in commercial computer codes. This preference of the Jaumann rate may be due to Prager [88], who in an elementary discussion of different at the time existing stress rates and with the objective to avoid non-uniqueness in the definition has introduced an additional restriction which has to be satisfied during plastic processes. Then the stress is at yield limit and the yield function is zero. The yield function should be stationary when the stress rate vanishes. This implies that the invariants of the stress tensor should be stationary, too. Applying this *stationary condition* to the above introduced stress rates, Prager argued that the convected (non-corotational) rates presented with the Oldroyd rates and the Truesdell rate could not be recommended, as these rates do not imply stationarity of the stress invariants.

Thus, with the Jaumann rate the above Prandtl-Reuss theory, in particular, the $J_2$-flow theory with a von Mises-type yield function (4), was developed by many researchers. However, the foundation of this classical theory was shaken by an unexpected discovery of spurious phenomena like the shear oscillations. It seems that Lehmann [96] was the first to reveal that a rigid-plastic $J_2$-flow theory with kinematic hardening would predict an oscillating shear stress response to monotonically progressing simple shearing[13].

Ten years later, this phenomenon was rediscovered in Ref. [97]. On the other hand, Dienes [98] demonstrated that a similar phenomenon would emerge for the hypoelastic rate Eq. (57) which was assumed to describe purely elastic behavior. The question on how to avoid these unexpected results, however, is then often reduced to a search for a proper definition of the objective rate. Dienes used the Green-Naghdi rate or *polar rate* (refer to Refs. [99,100]) where the material time derivative is applied to a rotated stress tensor, thus replacing the vorticity $\boldsymbol{W}$ of the Jaumann rate by a skew-symmetric rate of rotation $\boldsymbol{\Omega}^{\mathrm{R}}$ with

[12] Noll called this a general invariance requirement principle of isotropy of space, whereas Thomas used the term absolute time derivative.

[13] Unfortunately, this paper was written in German and submitted for publication to a Romanian journal in 1968. Due to severe production problems in those days, the article appeared not until 1972. These circumstances may explain why this paper was widely ignored.

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{\mathrm{R}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{\Omega}^{\mathrm{R}} - \boldsymbol{\Omega}^{\mathrm{R}}\boldsymbol{\sigma}\,, \qquad \boldsymbol{\Omega}^{\mathrm{R}} = \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}. \tag{59}$$

In an entirely comparable way, e.g., the spin $\boldsymbol{\Omega}^{\mathrm{E}}$ and the rotation $\boldsymbol{R}^{\mathrm{E}}$ of the Eulerian triad may be used to define another objective rate:

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{\mathrm{E}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{\Omega}^{\mathrm{E}} - \boldsymbol{\Omega}^{\mathrm{E}}\boldsymbol{\sigma}\,, \qquad \boldsymbol{\Omega}^{\mathrm{E}} = \dot{\boldsymbol{R}}^{\mathrm{E}}\boldsymbol{R}^{\mathrm{E}\mathrm{T}}. \tag{60}$$

This rate is sometimes called Sowerby-Chu rate (refer to Ref. [101]). It may be clear that numerous additional objective rates could be defined, e.g. simply by combining the above mentioned rules.

The hypoelastic rate Eq. (57) is intended for finite elastic deformation behavior. Although it may be clear that the small deformation case yields the conventional elastic relation, i.e., Hooke's law, essential difference emerges for the direct extension of Eq. (22) to finite deformations. Typical examples were presented in Refs. [102,103]. Kleiber [102] also discussed non-radial cyclic processes of a hypoelastic material with a Jaumann derivative and observed a significant linearly increasing accuracy loss with the number of cycles. It was stated that within several cycles (with not uncommon rotations) the solution could lead to an energy error comparable with the maximum energy attained during the cycle, *"thus rendering the results totally useless"*, as the accumulated energy should vanish at the end of a closed cycle for an elastic material. In view of this result, he concluded that it might be doubtful that the use of an objective rate on the left-hand side of Eq. (56) (with a constant material tensor $\mathbb{A}$) would yield more accurate results than those corresponding to a simple (non-objective) material time derivative.

Another unexpected finding was made Simo and Pister [89], who demonstrated that for each of the, at that time, well-known objective rates, except for certain unrealistic particular cases for elastic constants, the widely used hypoelastic rate Eq. (57) fails to be exactly integrable. Since this finding, a general tendency was to believe that this non-integrability property would likely be true for all possible rates (see, e.g., Refs. [104,105]). This would imply that the classical Eulerian elastoplasticity and the Prandtl-Reuss theory might be self-inconsistent in the sense of formulating elastic behavior via the hypoelastic equation.

Only recently it was demonstrated that both problems, namely, the shear oscillation and the non-integrability of the hypoelastic relation (57), were closely related. To prove this, a new objective time derivative had to be defined which, applied to a yet undetermined Eulerian strain measure, could give the stretching $\boldsymbol{D}$. It was shown that this strain was the Hencky strain $\boldsymbol{h}$ and the new time derivative, in turn, was the logarithmic rate. Only with this logarithmic rate applied to $\boldsymbol{\sigma}$, Eq. (57) could be integrated to give an elastic relation.

## 2.2 The logarithmic rate and related properties

The development of the logarithmic rate as a remedy out of the above-mentioned dilemma has started with the wish to resolve the following questions: Can the stretching $\boldsymbol{D}$ be represented as a direct flux of an Eulerian strain measure, say $\boldsymbol{e}$? And could this eventually be the logarithmic strain? Although the stretching is frequently referred to as the rate of deformation tensor or the Eulerian strain rate, it was believed for a long time that the former would not be the case (see, e.g., Refs. [57] and [106]).

With respect to the latter, there were different attempts to relate the logarithmic Hencky strain $\boldsymbol{h}$ to the stretching $\boldsymbol{D}$. Hill [56] showed that the material time derivative of the Lagrangian counterpart $\dot{\boldsymbol{H}}$ equals $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{R}$ within a second-order term (see also Refs. [57,107]). With respect to the Eulerian logarithmic strain, Hoger [108], like Refs. [58,106] earlier, stated that $\boldsymbol{h}$ does not have a conjugate stress. The proof, however, was somewhat flawed as the material time derivative of $\ln \boldsymbol{V}$ introduced in Ref. [108] failed to be objective (see also Ref. [109]). Instead, she should have introduced an objective time derivative and thus broaden the definition of conjugacy introduced in Ref. [56].

Gurtin and Spear [110] and Hoger [111] derived conditions when under very specific circumstances the Jaumann rate of the logarithmic strain $(\ln \boldsymbol{V})^{\circ}$ equals the stretching. It was shown, e.g., that a condition $\boldsymbol{D}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{D}$ would be necessary and sufficient for this case. Moreover, it was shown that $(\ln \boldsymbol{V})^{\circ}$ would be a very good approximation to $\boldsymbol{D}$ for sufficiently small deformations, a result which already has been demonstrated in Ref. [80]. Inspired by these works and using an explicit formula for the gradient of the general strain measure $\boldsymbol{e}$ with respect to the stretch tensor $\boldsymbol{V}$ (see in Refs. [112,113]), Xiao et al. [114] could prove that an objective corotational rate of the logarithmic strain measure $\ln \boldsymbol{V}$ could be identical with the stretching tensor $\boldsymbol{D}$ and furthermore that in all possible strain measures, only $\ln \boldsymbol{V}$ would enjoy this property, i.e. any corotational rate of any other strain measure could not be identical with $\boldsymbol{D}$.

This result was gained by broadening the work-conjugacy relation introduced in Ref. [56]:

$$\dot{w} = \mathrm{tr}(\boldsymbol{S}\dot{\boldsymbol{E}}) = \mathrm{tr}(\boldsymbol{\tau}\boldsymbol{D})\,. \tag{61}$$

A stress $\boldsymbol{S}$ and a strain $\boldsymbol{E}$ are said to be a conjugate pair, if $\mathrm{tr}(\boldsymbol{S}\dot{\boldsymbol{E}})$ represents the stress power $\dot{w}$. A classical solution of this problem is given with $\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-\mathrm{T}}$ as second Piola-Kirchhoff stress and the Lagrangian Green strain $\boldsymbol{E}$. Relation (61) now has been generalized by introducing a pair of objective Eulerian stress and strain measures, say $(\boldsymbol{s}, \boldsymbol{e})$, both symmetric, and an Eulerian spin:

$$\boldsymbol{\Omega}^{*} = -\boldsymbol{Q}^{*\mathrm{T}}\dot{\boldsymbol{Q}}^{*} = \dot{\boldsymbol{Q}}^{*\mathrm{T}}\boldsymbol{Q}^{*}\,, \tag{62}$$

where $\boldsymbol{Q}^*$ is a proper orthogonal tensor.

In an $\boldsymbol{\Omega}^*$-frame relative to a fixed background frame, this pair becomes $(\boldsymbol{Q}^*\boldsymbol{s}\,\boldsymbol{Q}^{*\mathrm{T}},\ \boldsymbol{Q}^*\boldsymbol{e}\,\boldsymbol{Q}^{*\mathrm{T}})$. Then the inner product $(\boldsymbol{Q}^*\boldsymbol{s}\,\boldsymbol{Q}^{*\mathrm{T}}) : \overline{(\boldsymbol{Q}^*\boldsymbol{e}\,\boldsymbol{Q}^{*\mathrm{T}})}$ is formed by the observer in the $\boldsymbol{\Omega}^*$-frame just as an observer in the fixed background frame does for a Lagrangian stress and strain pair.

Following the idea of Hill, which is concerned with a fixed background frame, the observer in the $\boldsymbol{\Omega}^*$-frame feels that the pair $(\boldsymbol{s}, \boldsymbol{e})$ is an $\boldsymbol{\Omega}^*$-work-conjugate pair if the above inner product furnishes the stress power, i.e.

$$(\boldsymbol{Q}^*\boldsymbol{s}\,\boldsymbol{Q}^{*\mathrm{T}}) : \overline{(\boldsymbol{Q}^*\boldsymbol{e}\,\boldsymbol{Q}^{*\mathrm{T}})}^{\cdot} = \boldsymbol{s} : \overset{\circ}{\boldsymbol{e}}{}^* = \dot{w}, \tag{63}$$

where $\overset{\circ}{\boldsymbol{e}}{}^*$ is the corotational rate of the strain defined by the spin $\boldsymbol{\Omega}^*$,

$$\overset{\circ}{\boldsymbol{e}}{}^* = \dot{\boldsymbol{e}} + \boldsymbol{e}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{e}\,. \tag{64}$$

A corotational rate of an Eulerian tensor need not be objective. The spin tensors defining objective corotational rates are called material spins [78]. We now want to know whether or not a strain measure $\boldsymbol{e}$ and a material spin $\boldsymbol{\Omega}^*$ can be found such that the objective corotational rate of $\boldsymbol{e}$ defined by $\boldsymbol{\Omega}^*$ is identical with the stretching $\boldsymbol{D}$, i.e.,

$$\overset{\circ}{\boldsymbol{e}}{}^* = \dot{\boldsymbol{e}} + \boldsymbol{e}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{e} = \boldsymbol{D}\,, \quad \boldsymbol{e} = \boldsymbol{h} \quad \text{and} \quad \boldsymbol{\Omega}^* = \boldsymbol{\Omega}^{\log}. \tag{65}$$

It turned out that this expression, where both the strain and the spin can be chosen arbitrarily, holds if and only if $\boldsymbol{e}$ is the logarithmic strain $\boldsymbol{h} = \ln \boldsymbol{V}$.[14] The main idea in finding Eq. (65) was inspired by the wish to express in an Eulerian description of the energy balance the stress power as a function of Cauchy stress and an objective rate of a conjugate strain.

It turned out that Eq. (65) has a unique continuous solution for the logarithmic spin:

$$\boldsymbol{\Omega}^{\log} = \boldsymbol{W} + \sum_{\sigma\neq\tau}^{n} \left[ \frac{1 + (\chi_\sigma/\chi_\tau)}{1 - (\chi_\sigma/\chi_\tau)} + \frac{2}{\ln(\chi_\sigma/\chi_\tau)} \right] \boldsymbol{B}_\sigma \boldsymbol{D} \boldsymbol{B}_\tau \,. \tag{66}$$

We note that explicit basis-free and unified expressions for $\boldsymbol{\Omega}^{\log}$ have been presented with Refs. [114,116].

The spin $\boldsymbol{\Omega}^{\log}$ given by Eq. (66) is referred to as the logarithmic spin because of its unique relationship with the logarithmic strain. For an objective Eulerian tensor $\boldsymbol{A}$, the corotational rate defined by the logarithmic spin, i.e.,

$$\overset{\circ}{\boldsymbol{A}}{}^{\log} \equiv \dot{\boldsymbol{A}} + \boldsymbol{A}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\boldsymbol{A} \tag{67}$$

is called the logarithmic rate of $\boldsymbol{A}$. Then, Eq. (67) together with the relationship (65) yields the following exact kinematical relationship between the Hencky strain $\boldsymbol{h}$ and the stretching $\boldsymbol{D}$:

$$\boldsymbol{D} = \overset{\circ}{\boldsymbol{h}}{}^{\log} = \dot{\boldsymbol{h}} + \boldsymbol{h}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\boldsymbol{h}\,. \tag{68}$$

It seems that Lehmann et al. [117] were the first to establish relationship (68) in discussing the work-conjugacy between the Cauchy stress $\boldsymbol{\sigma}$ and Hencky strain $\boldsymbol{h}$.[15]

With reference to the idea of Kleiber [102], e.g., a hypoelastic model of grade zero with different known objective rates has been subjected to different non-radial cyclic processes in strain space. It turned out from the calculations of the corresponding stresses that only a hypoelastic body with a logarithmic rate exhibits the properties of an elastic material, where no dissipation should be observed at the end of a cycle, i.e., where the stresses should return to their initial values (refer to Lin [123], Lin et al. [124] and Meyers et al. [125,126]).

If several evolution equations for tensor-valued variables are used, e.g., for a description with kinematic hardening, several objective rates might be discussed. To avoid contradictions, Xiao et al. [127] showed that any other additional objective rate included in the set of constitutive relations has to be of the same kind. This is especially true for the rates of internal variables included in the evolution equations of these variables.

## 3 Large deformations

In the late 1970s Green and Naghdi [99] established a rigorous theory of plasticity in the framework of modern continuum mechanics, where the restrictions on the general form of the constitutive relations were derived from thermodynamic principles. The kinematical basis of this work was the assumption that the total strain could be decomposed into the sum of an elastic-like and a plastic strain tensor, respectively. Although in their paper the plastic strain has been introduced as a primitive variable and the elastic-like strain is merely defined by the difference of the total strain and this

[14] It seems that a direct precursor of Eq. (65) is relation (2.13) with Eqs. (2.11) and (2.15) in Ref. [115], which says that the Jaumann rate of $\boldsymbol{h}$ should exactly give $\boldsymbol{D}$ in some cases; see also Refs. [110,111].

[15] Almost at the same time several groups were seeking a solution for this problem, namely to express the stretching $\boldsymbol{D}$ as an objective rate of an Eulerian strain. It is reported that P.A. Zhilin 1995 starting from a quite different idea came to a conclusion comparable to Eq. (68). Unfortunately, however, his result remained unpublished then (refer to Ref. [118]). Later, Eq. (68) was also discovered by Reinhardt and Dubey [119,120]. This relationship was derived in the general sense of studying Eq. (65) independently, and its intrinsic uniqueness property was thus revealed for the first time in Refs. [78,114,121,122] from different contexts. Regrettably, Profs. Lehmann, Guo, and Zhilin were not able to realize the publication of their works and their seminal ideas.

plastic variable, it is a common practice to refer to this sum as an additive decomposition of the strain tensor into its elastic and plastic parts.

Besides the additive decomposition of the stretching discussed so far, an alternative approach to a finite deformation theory was presented in Refs. [128,129]. This approach was shortly later modified Refs. [130,131] by adding so-called director triads and the notion of an isoclinic intermediate configuration. In contrast to Green and Naghdi's approach in these papers a multiplicative decomposition of the deformation gradient into elastic and plastic parts is assumed.

### 3.1 Basic facts for a deforming continuous body

We now switch to some relevant facts in finite deformation kinematics of continua. For a deformable body in the pure mechanical sense, as basic field variables, the deformation gradient $\boldsymbol{F}$ and the Cauchy or *true* stress $\boldsymbol{\sigma}$ at each particle characterize the local deformation state relative to a reference configuration and the local stressed state, respectively. Let $\boldsymbol{X}$ and $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ be the reference and the current position vector of a material particle, respectively. Then, the deformation gradient is given by (see also Sect. 2 of this article)

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}} \,. \tag{69}$$

We consider the local deformation state occurring at the infinitesimal neighborhood of each particle with $\mathrm{d}\boldsymbol{X}$ and $\mathrm{d}\boldsymbol{x}$, the reference and the current line element, respectively. At the infinitesimal neighborhood of each particle, we have the transformation formula between the line elements:

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\,\mathrm{d}\boldsymbol{X}, \quad \det \boldsymbol{F} > 0 \,. \tag{70}$$

The particle velocity and the velocity gradient are $\boldsymbol{v} = \dot{\boldsymbol{x}}$ and $\boldsymbol{L}$, respectively.

A general class of strain measures is defined through a smooth increasing function $g(\chi)$ with $g(1) = g'(1) - \frac{1}{2} = 0$, where the $\chi_\sigma$ are the distinct eigenvalues of either $\boldsymbol{B}$ or $\boldsymbol{C}$. Their forms can be given by (see Eqs. (47) and (48))

$$\begin{aligned} \boldsymbol{e}^{(m)} &= \boldsymbol{g}(\boldsymbol{B}) = \sum_{\sigma=1}^{n} g(\chi_\sigma)\boldsymbol{B}_\sigma = \frac{1}{2m}(\boldsymbol{B}^m - \boldsymbol{I}) \,, \\ \boldsymbol{E}^{(m)} &= \boldsymbol{g}(\boldsymbol{C}) = \sum_{\sigma=1}^{n} g(\chi_\sigma)\boldsymbol{C}_\sigma = \frac{1}{2m}(\boldsymbol{C}^m - \boldsymbol{I}), \end{aligned} \tag{71}$$

defining Lagrangian and Eulerian strain tensors. Two important examples are the Green strain of Lagrangian type for $m = 1$

$$\boldsymbol{E}^{(1)} = \boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}) \tag{72}$$

and the Hencky strains of Lagrangian and Eulerian type with Eqs. (46) and (49) for $m = 0$.

This class would be further generalized if we introduce as additional requirements

$$\lim_{\chi\to\infty} g(\chi) = \infty \,, \quad \lim_{\chi\to 0} g(\chi) = -\infty \,, \tag{73}$$

and thus take into account in a more realistic way the physical properties of a deforming body. This, however, would exclude several classical measures, e.g., the Green strain Eq. (72), and underline the importance of the Hencky strain.

Recently, in Ref. [132] a further generalization was introduced in which a combination of two functions was proposed:

$$g(\chi) = \frac{1}{m_1 + m_2}(\chi^{m_1} - \chi^{-m_2}) \,, \quad m_1 m_2 \geq 0 \,. \tag{74}$$

With various strain measures, various stress measures may be introduced via the unified concept of work conjugacy. This idea was exemplified in Ref. [58] and fully developed in Refs. [56,57,94]. Hill [57] also introduced the notion of *work conjugacy*, although work rate or stress power are discussed (see also the Appendix of Ref. [56]).

### 3.2 Eulerian formulations with the logarithmic rate

This subsection is primarily devoted to the development of the Prandtl-Reuss relations—or in other words—the additive decomposition of the stretching during the 1980s when with the rapid development of fast and powerful computers in conjunction with efficient numerical methods (e.g. the FEM) new trends in plasticity were initiated.

The basic Prandtl-Reuss relations (refer to Eqs. (21) and (22)) were formulated in Sect. 1. Several steps towards a more general theory were undertaken shortly later. To this end scalar-valued and second-order tensor-valued internal variables have been introduced to model these phenomena by means of the evolution of these variables. Thus, the history of a process is represented by these internal variables and their history. Here, we also should mention Refs. [133,134] where starting out from the additive decomposition (21) a rate-type description of elastic-plastic behavior was introduced. Scalar and tensorial internal variables were used to account for the isotropic and kinematic hardening during plastic flow. The elastic part is modeled as hypoelastic material with $\boldsymbol{D}_\mathrm{e}$ linearly related to an objective corotational rate of the stress $\overset{\circ}{\boldsymbol{\sigma}}$, i.e.,

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbb{C} : \boldsymbol{D}_\mathrm{e} \,. \tag{75}$$

Herein, Tokuoka has taken a Jaumann rate—without mentioning this source.

Thus, the extension of Eqs. (21) and (22) seems to be canonical. Likewise, however, it should be borne in mind that neither the material time derivatives incorporated in the solid-like elastic part of Eq. (22) meet the principle of objectivity nor does a hypoelastic material in general really behave like an elastic material.

We will reformulate the Prandtl-Reuss relations for large deformation within the context of an Eulerian description. Therefore, in Eqs. (22) and (5) we replace the material time derivative by a logarithmic rate and moreover, for physical reasons, the Cauchy stress $\boldsymbol{\sigma}$ by the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, namely

$$\boldsymbol{D} = \boldsymbol{D}_{\mathrm{e}} + \boldsymbol{D}_{\mathrm{p}} = \mathbb{C}^{-1} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + \varrho \Lambda \frac{1}{2} \frac{\partial F}{\partial \boldsymbol{\tau}} . \tag{76}$$

This leads to a constitutive relation whose hypoelastic part $\boldsymbol{D}_{\mathrm{e}}$ is exactly integrable to deliver an elastic relation:

$$\boldsymbol{h}_{\mathrm{e}} = \mathbb{C}^{-1} : \boldsymbol{\tau} . \tag{77}$$

In the same way, if, e.g., according to Eq. $(13)_2$ an additional tensorial variable $\boldsymbol{\alpha}$ is introduced, with reference to Prager's stationarity condition, this should read

$$\overset{\circ}{\boldsymbol{\alpha}}{}^{\log} = c \boldsymbol{D}_{\mathrm{p}} , \tag{78}$$

since Ref. [135] could show that in an Eulerian description, if different rates are used, these rates must be the same, namely, logarithmic rates.

Finally the evolution equation for the scalar-valued internal variable $\kappa$ may be taken as

$$\dot{\kappa} = \dot{w}^{\mathrm{p}} = \boldsymbol{\tau} : \boldsymbol{D}_{\mathrm{p}} , \tag{79}$$

corresponding to Eq. (14).

As yield condition the v. Mises condition $(13)_1$ will be used with

$$F = (\boldsymbol{\tau}' - \boldsymbol{\alpha}) : (\boldsymbol{\tau}' - \boldsymbol{\alpha}) - \kappa^2 = 0 . \tag{80}$$

From the condition of consistency $\dot{F} = 0$, we thus get

$$\frac{\partial F}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + \frac{\partial F}{\partial \boldsymbol{\alpha}} : \overset{\circ}{\boldsymbol{\alpha}}{}^{\log} - 2 \frac{\mathrm{d}\kappa}{\mathrm{d}w^{\mathrm{p}}} \dot{w}^{\mathrm{p}} = 0 . \tag{81}$$

Introducing here Eqs. (78) and (79) allows to eliminate expression $\Lambda$ from Eq. (76):

$$\boldsymbol{D} = \boldsymbol{D}_{\mathrm{e}} + \boldsymbol{D}_{\mathrm{p}} = \mathbb{C}^{-1} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + \frac{\varrho (\boldsymbol{\tau}' - \boldsymbol{\alpha}) : \overset{\circ}{\boldsymbol{\tau}}{}^{\log}}{2c\kappa + \frac{\mathrm{d}\kappa}{\mathrm{d}w^{\mathrm{p}}} \boldsymbol{\tau} : (\boldsymbol{\tau}' - \boldsymbol{\alpha})} (\boldsymbol{\tau}' - \boldsymbol{\alpha}) . \tag{82}$$

These are the Prandtl-Reuss equations for a material with linear kinematic and non-linear isotropic work hardening.

For a temperature-dependent material, we will finally adopt the hyperelasticity of the above derived relation (77). Thus, following the procedure outlined before for small deformations, a complementary thermo-hyperelastic potential $\bar{W}(\boldsymbol{\tau}, T)$ may be introduced, such that the reversible part of the logarithmic strain $\boldsymbol{h}_{\mathrm{e}}$ is derivable from this potential with respect to the work-conjugated stress $\boldsymbol{\tau}$, i.e.,

$$\boldsymbol{h}_{\mathrm{e}} = \ln \boldsymbol{V}_{\mathrm{e}} = \frac{\partial \bar{W}(\boldsymbol{\tau}, T)}{\partial \boldsymbol{\tau}} , \tag{83}$$

and its time derivative

$$\boldsymbol{D}_{\mathrm{e}} = \frac{\partial^2 \bar{W}(\boldsymbol{\tau}, T)}{\partial \boldsymbol{\tau}^2} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + \frac{\partial^2 \bar{W}(\boldsymbol{\tau}, T)}{\partial \boldsymbol{\tau} \partial T} \dot{T} . \tag{84}$$

This thermoelastic potential $\bar{W}(\boldsymbol{\tau}, T)$ is part of a complementary free enthalpy function $g = g(\boldsymbol{\tau}, T, \boldsymbol{\alpha}, \kappa)$ per unit reference volume (refer to Ref. [59]):

$$g = \varphi - T\eta - \boldsymbol{\tau} : \boldsymbol{h}_{\mathrm{e}} , \tag{85}$$

containing a formal elastic Hencky strain $\boldsymbol{h}_{\mathrm{e}}$. Then, the energy balance relation may be formulated as

$$\dot{g} = \boldsymbol{\tau} : \boldsymbol{D} - J \nabla \cdot \boldsymbol{q} + r - \eta \dot{T} - T\dot{\eta} - \overset{\cdot}{\boldsymbol{\tau} : \boldsymbol{h}_{\mathrm{e}}} . \tag{86}$$

Hence, similar as for small deformations, we may recast the Planck inequality (30) in the form

$$\mathscr{D} = \boldsymbol{\tau} : \boldsymbol{D}_{\mathrm{p}} - (\dot{g} + \eta \dot{T}) - \overset{\circ}{\boldsymbol{\tau}}{}^{\log} : \boldsymbol{h}_{\mathrm{e}} \geq 0 . \tag{87}$$

Upon introducing here

$$\dot{g} = \frac{\partial g}{\partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} + \frac{\partial g}{\partial T} \dot{T} + \frac{\partial g}{\partial \boldsymbol{\alpha}} : \overset{\circ}{\boldsymbol{\alpha}}{}^{\log} + \frac{\partial g}{\partial \kappa} \dot{\kappa} , \tag{88}$$

the above inequality takes the form

$$\mathscr{D} = - \left\{ \boldsymbol{h}_{\mathrm{e}} + \frac{\partial g}{\partial \boldsymbol{\tau}} \right\} : \overset{\circ}{\boldsymbol{\tau}}{}^{\log} - \left\{ \eta + \frac{\partial g}{\partial T} \right\} \dot{T} + \boldsymbol{\tau} : \boldsymbol{D}_{\mathrm{p}} - \frac{\partial g}{\partial \boldsymbol{\alpha}} : \overset{\circ}{\boldsymbol{\alpha}}{}^{\log} - \frac{\partial g}{\partial \kappa} \dot{\kappa} \geq 0 , \tag{89}$$

and the corresponding equations of state are

$$\boldsymbol{h}_{\mathrm{e}} = - \frac{\partial g}{\partial \boldsymbol{\tau}} \quad \text{and} \quad \eta = - \frac{\partial g}{\partial T} . \tag{90}$$

The complementary thermoelastic potential $\bar{W}(\boldsymbol{\tau}, T)$ is part of this enthalpy function $g = g(\boldsymbol{\tau}, T, \boldsymbol{\alpha}, \kappa)$ (see Ref. [136]). We now introduce the above state equations into Eq. (88)

$$\dot{g} = -\boldsymbol{h}_{\mathrm{e}} \colon \overset{\circ}{\boldsymbol{\tau}}{}^{\log} - \eta \dot{T} - \mathscr{D}_i \,, \tag{91}$$

to describe the enthalpy production, where

$$\mathscr{D}_i = -\frac{\partial g}{\partial \boldsymbol{\alpha}} \colon \overset{\circ}{\boldsymbol{\alpha}}{}^{\log} - \frac{\partial g}{\partial \kappa} \dot{\kappa}, \tag{92}$$

is the dissipation due to the inelastic deformation of the internal structure. Introducing here the balance into Eq. (86), the entropy production may be reformulated:

$$T\dot{\eta} = \boldsymbol{\tau} \colon \boldsymbol{D}_{\mathrm{p}} - J \nabla \cdot \boldsymbol{q} + r + \mathscr{D}_i \,. \tag{93}$$

With the help of the above state equations, this result may be converted to an evolution equation for the temperature:

$$\begin{aligned} c_T \dot{T} = &-T \frac{\partial \boldsymbol{h}_{\mathrm{e}}}{\partial T} \colon \overset{\circ}{\boldsymbol{\tau}}{}^{\log} - J \nabla \cdot \boldsymbol{q} + r + \boldsymbol{\tau} \colon \boldsymbol{D}_{\mathrm{p}} + \mathscr{D}_i \\ &-T \frac{\partial \mathscr{D}_i}{\partial T} \,, \end{aligned} \tag{94}$$

where

$$c_T = -T \frac{\partial^2 g}{\partial T^2},$$

is the specific heat.

Towards modeling different elastoplastic features of a material, suitable forms of the elastic potential $\bar{W}$ in the elastic rate Eq. (84) and of the yield function (80) in the flow rule should be presented in a sense consistent with experimental facts. It may be essential to note that a given form of $\bar{W}$ is merely applicable for a certain range of deformation and would have no relevance to any realistic material behavior in an extreme case far beyond this range, irrespective of the fact that from a formal mathematical standpoint, it may be well defined over the whole range of deformation.

A simple isothermal form of the complementary potential $\bar{W}$ is quadratic and referred to as Hencky elastic potential, namely,

$$\boldsymbol{h}_{\mathrm{e}} = \frac{\partial \bar{W}(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}} = \mathbb{C}^{-1} \colon \boldsymbol{\tau} \,. \tag{95}$$

Prior to the initial yielding, the integration produces a finite strain hyperelastic equation of Hookean type:

$$\boldsymbol{h} = \frac{\partial \bar{W}(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}} = \mathbb{C}^{-1} \colon \boldsymbol{\tau} \,. \tag{96}$$

Equation (96) is known as Hencky's elastic equation [137]. It is known that this equation can well represent moderate elastic deformations with each principal stretch falling within the range [0.7,1.3]. Beyond this range, however, Eq. (96) would be no longer applicable [138] and merely of formal sense.

The logarithmic rate in its present form (67) is not undisputed, especially when applied to large elastoplastic deformations, e.g., in metal forming. Very recently, Jiao and Fish [139] have presented a generalization of this rate, introducing their so-called kinetic logarithmic spin:

$$\boldsymbol{\Omega}^{\mathrm{klog}} = \boldsymbol{W} + \boldsymbol{A}^{\log}(\boldsymbol{B}^k(\boldsymbol{\tau}), \boldsymbol{D}) \,, \tag{97}$$

replacing in the sum on the right-hand side of Eq. (66) the left Cauchy green tensor $\boldsymbol{B}$ by a new variable $\boldsymbol{B}^k$, as function of the stress $\boldsymbol{\tau}$. To this end, $\boldsymbol{V}^k = \sqrt{\boldsymbol{B}^k}$ is defined as exponential function of the Hencky elastic potential (95)

$$\boldsymbol{V}^k = \exp\left[\boldsymbol{h}_{\mathrm{e}}(\boldsymbol{\tau})\right] . \tag{98}$$

This kinetic extension is intended to avoid errors that may occur when calculating a cyclic process with extremely high loading and unloading using the logarithmic rate. We note, however, that at least one requirement for applying the logarithmic rate was violated in this comparison: The reference frame should be kept fixed.

### 3.3 Lagrangian formulations with plastic strain

In contrast to the aforementioned Eulerian formulation, a Lagrangian-type formulation of finite deformations was proposed in Ref. [99].[16] This theory represents the first effort towards a rigorous treatment of finite elastoplasticity within the framework of continuum thermodynamics.

As extension of the classical small deformation theory to finite deformations, a perhaps more direct idea is to use a finite strain measure and its conjugate stress. Let $\boldsymbol{A}$ be any given Lagrangian strain. Then we may define

$$\boldsymbol{A} = \boldsymbol{A}_{\mathrm{e}} + \boldsymbol{A}_{\mathrm{p}} \,, \tag{99}$$

where $\boldsymbol{A}_{\mathrm{e}}$ and $\boldsymbol{A}_{\mathrm{p}}$ are labeled elastic and plastic parts of $\boldsymbol{A}$, respectively, with the intention that in conjunction with the conjugate stress of $\boldsymbol{A}$, they will be used as basic variables to formulate an elastic relation and a flow rule.

It is known that the additive separation of a total strain into elastic and plastic parts is restricted to very particular cases. In general, such separation might be of formal sense only. On account of the difficulties involved in defining a plastic strain, Green and Naghdi [99] introduced a strain-like variable of Lagrangian type, denoted $\boldsymbol{E}_{\mathrm{p}}$, and regarded it as a primitive variable, *"stating certain of its properties but not defining it explicitly, and thus relegated its explicit identification to special assumptions or situations."*

[16] For details, we refer to Ref. [13].

This variable is associated with the total Green strain $\boldsymbol{E}$ as given in Eq. (72). Well understanding the limited applicability of the additive separation of $\boldsymbol{E}$, they did not interpret the difference $\boldsymbol{E} - \boldsymbol{E}_\mathrm{p}$ as an elastic strain or part but as an alternative convenient variable used for well-motivated purposes.

With the plastic strain $\boldsymbol{E}_\mathrm{p}$ as additional variable, the set of basic variables entering the constitutive formulations will be given by $(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)$. Note that herein $\boldsymbol{\alpha}$ and $\kappa$ are internal variables representing the progress of the plastic deformations by suitably defined evolution equations. In accord with the general setting of their theory, the tensor-valued internal variable $\boldsymbol{\alpha}$ is introduced as back stress of Lagrangian type to model the Bauschinger effect, whereas the isotropic hardening is modeled through the evolution of $\kappa$. Alternatively, the sets $(\boldsymbol{E} - \boldsymbol{E}_\mathrm{p}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)$ as well as $(\boldsymbol{S}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)$ may be useful for certain specific purposes. For instance, the latter is used in a stress-space formulation.

The set $(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)$ of basic variables is representing a strain-space formulation (see, e.g., Refs. [140–142]).[17]

In general, it is assumed that the conjugate stress of the Green strain $\boldsymbol{E}$, i.e., the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$, is determined by the foregoing set of basic variables. This leads to the total stress-strain relation

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)\,, \tag{100}$$

where $\hat{\boldsymbol{S}}$ is a tensor-valued constitutive function relying on all four variables. It is assumed that this tensor function is twice differentiable and establishes a one-one relationship between the stress tensor $\boldsymbol{S}$ and the total strain $\boldsymbol{E}$. Then, the inverted form of Eq. (100) yields

$$\boldsymbol{E} = \hat{\boldsymbol{E}}(\boldsymbol{S}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)\,. \tag{101}$$

In addition, $\mathrm{g}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)$ is a yield function in a strain-space formulation. It is assumed that the time derivative of each of the three variables $\boldsymbol{E}_\mathrm{p}$, $\boldsymbol{\alpha}$, and $\kappa$ is linear in $\dot{\boldsymbol{E}}$ with coefficients that are functions of the whole set of variables. Thus, the flow rule is given by

$$\dot{\boldsymbol{E}}_\mathrm{p} = \zeta \left( \frac{\partial \mathrm{g}}{\partial \boldsymbol{E}} \colon \dot{\boldsymbol{E}} \right) \boldsymbol{\varrho}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)\,, \tag{102}$$

where $\boldsymbol{\varrho}$ is a tensor-valued constitutive function, and $\zeta$ is the plastic multiplier differentiating elastic behavior (including unloading) from elastic-plastic one by taking the values 0 and 1, respectively.

The loading-unloading criterion in strain space is shown to possess a simple, unified form for perfect elastic-plastic as well as hardening and softening behavior (see, e.g., Ref. [13]). The evolution equations for the hardening variable $\kappa$ and for the back stress $\boldsymbol{\alpha}$ are given in the forms:

$$\dot{\kappa} = \zeta \left( \frac{\partial \mathrm{g}}{\partial \boldsymbol{E}} \colon \dot{\boldsymbol{E}} \right) \lambda(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)\,, \tag{103}$$

$$\dot{\boldsymbol{\alpha}} = \zeta \left( \frac{\partial \mathrm{g}}{\partial \boldsymbol{E}} \colon \dot{\boldsymbol{E}} \right) \boldsymbol{\beta}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \boldsymbol{\alpha}, \kappa)\,. \tag{104}$$

Here, $\lambda$ and $\boldsymbol{\beta}$ are additional scalar- and tensor-valued constitutive functions.

Within the general context of the above Lagrangian theory, Naghdi and coworkers made a rigorous, systematic study of the consequences implied by the work postulate and broadened the scope of Ilyushin's postulate, introducing

$$\int_{t_0}^{t_f} \boldsymbol{S} \colon \dot{\boldsymbol{E}}\ \mathrm{d}t \geq 0\,, \tag{105}$$

for every homogeneous finite strain cycle. From this postulate, the essential structure of the Green-Naghdi theory was derived. For simplicity, these results here are presented in the absence of a back stress $\boldsymbol{\alpha}$.

It is demonstrated that there is a stress potential $\hat{\psi}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$ such that the general relation (100) for the stress response is reduced to

$$\boldsymbol{S} = \frac{\partial \hat{\psi}}{\partial \boldsymbol{E}}\,. \tag{106}$$

In a general context, Refs. [143,144] have demonstrated that such a relation holds true with $\hat{\psi}$ being a fully general elastic potential relying on the *"prior history of inelastic deformation"* and with $(\boldsymbol{S}, \boldsymbol{E})$ any given work-conjugate pair.

Moreover, it is shown that the constitutive function $\boldsymbol{\varrho}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$ characterizing the flow rule is related to the yield function $\mathrm{g}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$ and the stress potential $\hat{\psi}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$ as well as the hardening function $\lambda(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$ in the following manner:

$$\hat{\boldsymbol{\sigma}} \equiv \frac{\partial^2 \hat{\psi}}{\partial \boldsymbol{E}_\mathrm{p} \partial \boldsymbol{E}} \colon \boldsymbol{\varrho} + \frac{\partial^2 \hat{\psi}}{\partial \kappa\, \partial \boldsymbol{E}}\, \lambda = -\gamma\, \frac{\partial \mathrm{g}}{\partial \boldsymbol{E}}\,, \tag{107}$$

where $\gamma$ is an undetermined scalar function relying on $(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$. From the above, it follows that the tensor-valued constitutive function $\boldsymbol{\varrho}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$ is obtainable from three scalar constitutive functions, i.e., the yield function $\mathrm{g}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$, the stress potential $\hat{\psi}(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$, and the hardening function $\lambda(\boldsymbol{E}, \boldsymbol{E}_\mathrm{p}, \kappa)$, if the second gradient $\partial^2 \hat{\psi} / \partial \boldsymbol{E}_\mathrm{p} \partial \boldsymbol{E}$ is invertible.

A direct relation between the stress rate and the total strain rate is derivable

$$\dot{\boldsymbol{S}} = \mathbb{K} \colon \mathbb{L} \colon \dot{\boldsymbol{E}}\,, \tag{108}$$

[17] For a discussion of the pros and cons of the strain-space formulation, we refer to Ref. [13].

and its inverse is (see, e.g., Ref. [13])

$$\dot{\boldsymbol{E}} = (\mathbb{K}\!:\!\mathbb{L})^{-1}\!:\dot{\boldsymbol{S}}\,, \tag{109}$$

where $\mathbb{L}$ is the tensor of elasticity and $\mathbb{K}$ is a second fourth-order material tensor

$$\mathbb{L}=\frac{\partial^2\hat{\psi}}{\partial\boldsymbol{E}^2}\,,\quad \mathbb{K}=\mathbb{I}+\zeta\hat{\boldsymbol{\sigma}}\otimes\frac{\partial \mathrm{f}}{\partial\boldsymbol{S}}=\mathbb{I}+\zeta\left(\hat{\boldsymbol{\sigma}}\otimes\frac{\partial \mathrm{g}}{\partial\boldsymbol{E}}\right)\!:\mathbb{L}^{-1}\,. \tag{110}$$

Here, $\mathrm{f}=\mathrm{f}(\boldsymbol{S},\boldsymbol{E}_\mathrm{p},\kappa)$ is the yield function in stress space, which is obtained by substituting the inverted form (101) into the yield function $\mathrm{g}=\mathrm{g}(\boldsymbol{E},\boldsymbol{E}_\mathrm{p},\kappa)$ in strain space, and the fourth-order tensor $\mathbb{I}$ is the identity transformation over the symmetric second-order tensor space.

The critical point within this theory is mainly hidden in its setting, namely, in the additive splitting of the total strain $\boldsymbol{E}$ to determine the elastic-like quantity $\boldsymbol{E}_\mathrm{e}$, if the plastic strain is introduced as a primitive variable $\boldsymbol{E}_\mathrm{p}$. This point has been addressed first in Ref. [128] and has led to a long discussion about the admissibility or even the significance of the additivity of strains in a finite deformation theory, in order to achieve an effective uncoupling of elastic and plastic properties. In this respect, we refer, e.g., to Refs. [145,146].

A second issue is related with the specific free Helmholtz energy function $\hat{\psi}(\boldsymbol{E},\boldsymbol{E}_\mathrm{p},\kappa)$ or $\hat{\psi}(\boldsymbol{E}-\boldsymbol{E}_\mathrm{p},\boldsymbol{E}_\mathrm{p},\kappa)$ in an alternative form which is incorporated as an essential quantity of this theory. It might be not clear how this function apart from some very special cases will be determined. While forms of this energy function for elastic and even thermoelastic solids are well established, the introduction of plastic, i.e. irreversible, processes into this function will produce new if not intractable problems.

### 3.4 Formulations with unstressed configurations

In recent years the multiplicative separation of elastic-plastic deformations has become popular and found increasing applications in the phenomenological study of finite elastoplasticity. This was initiated by Refs. [128,129], and use was made by Refs. [147–149], and [150] et al., albeit it may be traced back to earlier works by Refs. [151–153], and [154], et al.; see Ref. [155] and the references therein.

It is not derived merely from the direct extension of the small deformation case but motivated by physical considerations. The central idea is the notion of a local intermediate unstressed configuration at each particle defined by an imaginary destressing process. If such configuration could be defined, then at each particle elastic and plastic deformations could be separated from the total elastic-plastic deformation in a definite and accurate manner.

Extending our previous considerations, we consider a continuous material body with initial configuration $\mathscr{B}_0$ experiencing finite elastic-plastic deformations in the current configuration $\mathscr{B}$. According to Ref. [145], we may introduce a straining-destressing experiment: *"Following elastic-plastic deformation from the undisturbed configuration $\boldsymbol{X}$ to $\boldsymbol{x}$, destressing to zero stress occurs from $\boldsymbol{x}$ to $\boldsymbol{p}$ $\cdots$. Since the configuration $\boldsymbol{p}$ is unstressed, the elastic strain there is zero and the strain in $\boldsymbol{p}$ is therefore totally plastic."* Here, $\boldsymbol{X}$ and $\boldsymbol{x}$ are the position vectors of a generic material particle in the initial and current configurations $\mathscr{B}_0$ and $\mathscr{B}$, respectively, and $\boldsymbol{p}$ is the position vector of the same particle in the unstressed configuration.

In the above cited statement, the total elastic-plastic deformation $\boldsymbol{F}$ from $\boldsymbol{X}$ to $\boldsymbol{x}$ is actually undergone by the material body, while the plastic deformation from $\boldsymbol{X}$ to $\boldsymbol{p}$ and the elastic deformation from $\boldsymbol{p}$ to $\boldsymbol{x}$, denoted $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{F}_\mathrm{p}$, are introduced by an imaginary destressing procedure and hence not actually undergone by the material body. $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{F}_\mathrm{p}$ will serve as additional deformation-like variables. The question as to how the destressing procedure is achieved is at the moment left open and will be discussed later.

The deformation gradient $\boldsymbol{F}$ is related to a local affine configuration and based upon the notion of line elements at the infinitesimal neighborhood of a particle. Following Ref. [145], the deformation of a material line element $\mathrm{d}\boldsymbol{X}$ in the aforementioned straining-destressing experiment is given by

$$\mathrm{d}\boldsymbol{p}=\boldsymbol{F}_\mathrm{p}\,\mathrm{d}\boldsymbol{X}\,,\quad \mathrm{d}\boldsymbol{x}=\boldsymbol{F}_\mathrm{e}\,\mathrm{d}\boldsymbol{p}\,,\quad\rightarrow\quad \mathrm{d}\boldsymbol{x}=\boldsymbol{F}\,\mathrm{d}\boldsymbol{X}\,, \tag{111}$$

In the above, $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{p}$ are the (actual) spatial line element in the current configuration $\mathscr{B}$ and the line element in the fictitious unstressed configuration, which are the counterparts of the material line element $\mathrm{d}\boldsymbol{X}$ in $\mathscr{B}_0$ after experiencing the actual elastic-plastic deformation and the plastic deformation induced by the destressing procedure, respectively. The transformation relations yield the widely used multiplicative separation:

$$\boldsymbol{F}=\boldsymbol{F}_\mathrm{e}\boldsymbol{F}_\mathrm{p}\,. \tag{112}$$

Once the above separation could unambiguously be established, the elastic and plastic deformations $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{F}_\mathrm{p}$ would be separated exactly from the total elastic-plastic deformation $\boldsymbol{F}$ and hence endowed with the desired physical features. Therefore, $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{F}_\mathrm{p}$ could be employed as additional variables to realize physically pertinent formulations of elastic and plastic behavior. However, a central issue with the separation is the non-uniqueness in the following sense: If $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{F}_\mathrm{p}$ obey the separation (112), then the same may be true for $\boldsymbol{F}_\mathrm{e}\boldsymbol{Q}^\mathrm{T}$ and $\boldsymbol{Q}\boldsymbol{F}_\mathrm{p}$ with an arbitrary rotation $\boldsymbol{Q}$. This means that the rotational parts of $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{F}_\mathrm{p}$, i.e., $\boldsymbol{R}_\mathrm{e}$ and $\boldsymbol{R}_\mathrm{p}$, would

be rendered indeterminate. In other words, a non-unique separation (112) would fail to separate the just-mentioned two rotations, refer to Ref. [156].

Sometimes the rotational parts incorporated in $\boldsymbol{F}_{\mathrm{e}}$ and $\boldsymbol{F}_{\mathrm{p}}$ and even in $\boldsymbol{F}$ are loosely said to be *superimposed rigid body rotations*. This expression may produce an impression as if these rotations might be not so essential. However, essential difference exists between each such rotation and any truly rigid body rotation. The latter is constant at all points in a body and should have no effect on both basic equations of motion and constitutive formulations of material behavior, whereas the former varies from point to point and exhibits essential effects on both. In fact, the rotational parts of $\boldsymbol{F}$ and $\boldsymbol{F}_{\mathrm{e}}$ and $\boldsymbol{F}_{\mathrm{p}}$ are inseparable parts of deformations and deformation rates and incorporated in constitutive formulations for both elastic and plastic behaviors.

The decoupling represented by Eq. (112) enables us to accomplish a direct formulation of elastic behavior. Now, the Kirchhoff stress $\boldsymbol{\tau}$ may be specified by a single variable, i.e., the elastic deformation $\boldsymbol{F}_{\mathrm{e}}$. Emphasizing the substantial invariance property of the elastic moduli in the process of elastic-plastic deformations, Lee [129] assumed the following invariable elastic relation:

$$\boldsymbol{\tau} = 2\boldsymbol{F}_{\mathrm{e}}\,\frac{\partial \psi}{\partial \boldsymbol{C}_{\mathrm{e}}}\,\boldsymbol{F}_{\mathrm{e}}{}^{\mathrm{T}}\,. \tag{113}$$

Note that all elastic domains correspond to the same elastic potential $\psi = \psi(\boldsymbol{C}_{\mathrm{e}})$ with $\boldsymbol{C}_{\mathrm{e}} = \boldsymbol{F}_{\mathrm{e}}{}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{e}}$. This means that the elastic behavior will be described by that for the initial elastic domain prior to the occurrence of yielding.

The foregoing non-uniqueness renders Eq. (113) incomplete. To eliminate this, an extra condition was introduced (see, e.g., Refs. [128,129] and [156])[18]:

$$\boldsymbol{F}_{\mathrm{e}}{}^{\mathrm{T}} = \boldsymbol{F}_{\mathrm{e}}\,. \tag{114}$$

and then the elastic relation (113) in the case of isotropy would become

$$\boldsymbol{\tau} = 2\boldsymbol{B}_{\mathrm{e}}\,\frac{\partial \psi}{\partial \boldsymbol{B}_{\mathrm{e}}}\,. \tag{115}$$

We note that in this specific case, we also have $\boldsymbol{C}_{\mathrm{e}} = \boldsymbol{V}_{\mathrm{e}}^2 = \boldsymbol{F}_{\mathrm{e}}\boldsymbol{F}_{\mathrm{e}}{}^{\mathrm{T}} = \boldsymbol{B}_{\mathrm{e}}$, i.e., the left and right Cauchy-Green tensors are equal. Moreover, for an isotropic material, $\boldsymbol{V}_{\mathrm{e}}$, $\boldsymbol{B}_{\mathrm{e}}$, and $\partial\psi/\partial\boldsymbol{B}_{\mathrm{e}}$ have the same principal axes so that the products are commutative.

The relation (113), in particular Eq. (115), assumes the standard form for the classical hyperelastic relation, which is usually regarded to describe the elastic behavior included in but separated from the elastic-plastic behavior as a single entity.

For a physically pertinent formulation of plastic flow, it is desirable to have a proper separation of the total deformation rate $\boldsymbol{D}$ into elastic and plastic parts, as shown in Eq. (21). Although the separation (112) could realize the decoupling of elastic and plastic deformation except for an arbitrary rotation, a definite deformation rate separation based on it has proved to be not so clear and simple. In fact, we have

$$\begin{aligned}\boldsymbol{D} &= \mathrm{sym}(\dot{\boldsymbol{F}}_{\mathrm{e}}\boldsymbol{F}_{\mathrm{e}}{}^{-1}) + \mathrm{sym}(\boldsymbol{F}_{\mathrm{e}}\dot{\boldsymbol{F}}_{\mathrm{p}}\boldsymbol{F}_{\mathrm{p}}{}^{-1}\boldsymbol{F}_{\mathrm{e}}{}^{-1})\\ &\neq \mathrm{sym}(\dot{\boldsymbol{F}}_{\mathrm{e}}\boldsymbol{F}_{\mathrm{e}}{}^{-1}) + \mathrm{sym}(\dot{\boldsymbol{F}}_{\mathrm{p}}\boldsymbol{F}_{\mathrm{p}}{}^{-1})\,,\end{aligned} \tag{116}$$

where the last two terms may be called the elastic and plastic deformation rates and will be denoted $\bar{\boldsymbol{D}}_{\mathrm{e}}$ and $\bar{\boldsymbol{D}}_{\mathrm{p}}$.

The above inequality shows that the total deformation rate $\boldsymbol{D}$ cannot be split into the sum of the two rates $\bar{\boldsymbol{D}}_{\mathrm{e}}$ and $\bar{\boldsymbol{D}}_{\mathrm{p}}$. We also refer to the discussions of Refs. [158–163]. Moreover, the non-uniqueness property of Eq. (112) leaves each rate term in Eq. (116) unspecified. With $\boldsymbol{F}_{\mathrm{e}} = \boldsymbol{V}_{\mathrm{e}}\boldsymbol{R}_{\mathrm{e}}$ and $\boldsymbol{F}_{\mathrm{p}} = \boldsymbol{R}_{\mathrm{p}}\boldsymbol{U}_{\mathrm{p}}$, the following relations make this clear:

$$\begin{aligned}\dot{\boldsymbol{F}}_{\mathrm{e}}\boldsymbol{F}_{\mathrm{e}}^{-1} &= \dot{\boldsymbol{V}}_{\mathrm{e}}\boldsymbol{V}_{\mathrm{e}}^{-1} + \boldsymbol{V}_{\mathrm{e}}\dot{\boldsymbol{R}}_{\mathrm{e}}\boldsymbol{R}_{e}{}^{\mathrm{T}}\boldsymbol{V}_{\mathrm{e}}^{-1},\\ \dot{\boldsymbol{F}}_{\mathrm{p}}\boldsymbol{F}_{\mathrm{p}}^{-1} &= \dot{\boldsymbol{R}}_{\mathrm{p}}\boldsymbol{R}_{p}{}^{\mathrm{T}} + \boldsymbol{R}_{\mathrm{p}}\dot{\boldsymbol{U}}_{\mathrm{p}}\boldsymbol{U}_{\mathrm{p}}^{-1}\boldsymbol{R}_{p}{}^{\mathrm{T}}.\end{aligned} \tag{117}$$

They show that each rate term in Eq. (116) is essentially dependent on either the elastic rotation $\boldsymbol{R}_{\mathrm{e}}$ or the plastic rotation $\boldsymbol{R}_{\mathrm{p}}$ or on both. To resolve this difficulty, additional assumptions and procedures would be needed. It has been shown (see Refs. [145,162]) that the deformation rate separation (21) may be re-established by assuming the extra condition (114) and defining the elastic and plastic deformation rates by

$$\begin{aligned}\boldsymbol{D}_{\mathrm{e}} &= \hat{\mathbb{K}}^{-1}\!:(\dot{\boldsymbol{B}}_{\mathrm{e}} + \boldsymbol{B}_{\mathrm{e}}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{B}_{\mathrm{e}}),\\ \boldsymbol{D}_{\mathrm{p}} &= \hat{\mathbb{K}}^{-1}\!:(2\boldsymbol{V}_{\mathrm{e}}\bar{\boldsymbol{D}}_{\mathrm{p}}\boldsymbol{V}_{\mathrm{e}})\,,\end{aligned} \tag{118}$$

with $\hat{\mathbb{K}}$ a fourth-order tensor given by

$$\hat{\mathbb{K}}_{ijkl} = 2(\boldsymbol{B}_{\mathrm{e}})_{ik}\delta_{jl} = 2(\boldsymbol{B}_{\mathrm{e}})_{jl}\delta_{ik}\,. \tag{119}$$

With the deformation rate separation described above, the isotropic elastic relation (115) may be reformulated in an equivalent Eulerian rate form as given by

$$\boldsymbol{D}_{\mathrm{e}} = (\hat{\mathbb{L}}\!:\hat{\mathbb{K}})^{-1}\!:\ \mathring{\boldsymbol{\tau}}^{J}\,, \tag{120}$$

where $\mathring{\boldsymbol{\tau}}^{J}$ is the Jaumann rate of $\boldsymbol{\tau}$ and

$$\hat{\mathbb{L}}_{ijkl} = 2\delta_{ik}\left(\frac{\partial \psi}{\partial \boldsymbol{B}_{\mathrm{e}}}\right)_{jl} + 2(\boldsymbol{B}_{\mathrm{e}})_{im}\left(\frac{\partial^2 \psi}{\partial \boldsymbol{B}_{\mathrm{e}}^2}\right)_{mjkl}\,. \tag{121}$$

[18] Here, $\boldsymbol{R}_{\mathrm{e}} = \boldsymbol{I}$ was used. An alternative approximation has been introduced in Ref. [157] with $\boldsymbol{W}_{\mathrm{p}} = \boldsymbol{0}$.

We note that the material tensor $\mathbb{N} = \hat{\mathbb{L}} : \hat{\mathbb{K}}$ herein has the same major and minor symmetry properties as the elasticity tensor $\mathbb{L}$, namely, $\mathbb{N}_{ijkl} = \mathbb{N}_{klij} = \mathbb{N}_{ijlk} = \mathbb{N}_{jikl}$.

It appears that within the context of the general constitutive formulation sketched above, the consequences implied by the work postulate have not yet been derived. Results have been given, e.g., in Refs. [164,165], assuming maximal plastic dissipation principles. In this case, the normality rule is accepted as a plausible assumption, but the convexity property of the yield surface may or may not be assumed. With the normality rule, the total stress rate-strain rate relationship is established as follows:

$$\boldsymbol{D} = \left(\mathbb{N} + \zeta\, \frac{1}{h}\frac{\partial \mathrm{f}}{\partial \boldsymbol{\tau}} \otimes \frac{\partial \mathrm{f}}{\partial \boldsymbol{\tau}}\right) : \overset{\circ}{\boldsymbol{\tau}}{}^{J}, \tag{122}$$

where $\mathrm{f} = \mathrm{f}(\boldsymbol{\tau}, \boldsymbol{\alpha}, \kappa)$ is the yield surface in stress space and $h$ is a hardening function, which can be derived from the normality rule for the plastic deformation rate $\boldsymbol{D}_\mathrm{p}$, the consistency condition of plasticity, and the evolution equations of the internal variables $\boldsymbol{\alpha}$ and $\kappa$. For details we refer e.g. to Ref. [1]. As before $\zeta$ is a plastic multiplier, differentiating the cases of loading and unloading.

Similar to the situation with the Green-Naghdi theory, a long discussion evolved about different issues related with this multiplicative decomposition or the Lee theory.[19]

### 3.5 Director triads and isoclinic configurations

With the particular assumption (114), a complete elastic-plastic formulation may be established, but it is confined to the case of an isotropic elastic potential $\psi$. Towards a more general treatment, one approach is to use director triads and isoclinic configurations, which originated from Refs. [170,171] and the following works [130,131,163,172,173]. The main idea is quoted as follows: *"... to determine in some way the orientation of the present (stressed or released) configuration, so that an orientation variable must be added to the state variables. We shall use the following mode of orientation. We consider a material plane of unit normal* $\boldsymbol{n}$*, and in this plane a material direction* $\boldsymbol{m}$ *(a unit vector). The element is oriented ... by the orthonormal triad formed from* $\boldsymbol{m}$ *and* $\boldsymbol{n}$*, and which will be called a director triad"* [173] and then *"we assume that at time t the material element is very rapidly unloaded ... This unloading process is elastic ... We thus obtain a present released configuration* ($\kappa$) *which is only defined up to an arbitrary rotation."* This idea may be realized by selecting a triad formed by three orthonormal vectors, say, $\boldsymbol{\xi} = (\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$, for each material element. Embedded in the present released configuration, i.e., the unstressed configuration, such a triad $\boldsymbol{\xi}$ is rotating as the former is changing, so that it determines the orientation of the former by specifying the related rotation. Such a triad is said to be a director triad. Further, if a particular director triad $\boldsymbol{\xi}_0 = (\boldsymbol{d}_1^0, \boldsymbol{d}_2^0, \boldsymbol{d}_3^0)$ is chosen in such a manner that it always keeps the same orientation with respect to the fixed axes, then it may be called an *isoclinic triad.* Accordingly, the unstressed configuration with the orientation specified by an isoclinic triad $\boldsymbol{\xi}_0$ is referred to as an isoclinic configuration.

Evidently, the isoclinic configuration with a well defined isoclinic triad $\boldsymbol{\xi}_0$ results in a unique separation (112), and as such the elastic and plastic deformations $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{F}_\mathrm{p}$ are accordingly specified. Now the constitutive relations are formulated in a somewhat different way. It is assumed that the potential $\psi$ relies on both the elastic Green strain $\boldsymbol{E}_\mathrm{e} = \frac{1}{2}(\boldsymbol{F}_\mathrm{e}{}^\mathrm{T}\boldsymbol{F}_\mathrm{e} - \boldsymbol{I})$ and the internal variables, whereas the yield function f depends on the stress

$$\boldsymbol{S}_\mathrm{e} = \boldsymbol{F}_\mathrm{e}^{-1}\boldsymbol{\tau}\boldsymbol{F}_\mathrm{e}^{-\mathrm{T}}, \tag{123}$$

and the internal variables. Here $\boldsymbol{S}_\mathrm{e}$ is the so-called Mandel stress, acting on the unstressed intermediate configuration. Then, the elastic relation (113) is converted to

$$\boldsymbol{S}_\mathrm{e} = \frac{\partial \psi}{\partial \boldsymbol{E}_\mathrm{e}}. \tag{124}$$

Besides, the flow rule is non-symmetric and formulated for $\dot{\boldsymbol{F}}_\mathrm{p}\boldsymbol{F}_\mathrm{p}^{-1}$ in a 9-dimensional space.[20]

The formulation with an isoclinic triad $\boldsymbol{\xi}_0$ may, finally, be changed to a more general type with an arbitrary director triad $\boldsymbol{\xi} = \boldsymbol{Q}\boldsymbol{\xi}_0$ and a rotation variable $\boldsymbol{Q}$. Such general results involve the corotational rate relative to the triad $\boldsymbol{\xi}$, and the rotation $\boldsymbol{Q}$ will enter as an additional variable into each constitutive function.

An observation on the notion of director triad is as follows. Let $\boldsymbol{\xi}$ be an arbitrary director triad embedded in the unstressed configuration. At the initial instant $t_0$, it is given by the fixed triad $\boldsymbol{\xi}_0$ in the undeformed configuration $\mathscr{B}_0$. Since a director triad is always orthogonal at any instant $t$, i.e., $\boldsymbol{d}_i \cdot \boldsymbol{d}_j = 0$ for $i \neq j$, and Eq. $(111)_1$ yields $\boldsymbol{d}_i = \boldsymbol{F}_\mathrm{p}\boldsymbol{d}_i^0$, we deduce $\boldsymbol{d}_i^0 \cdot (\boldsymbol{F}_\mathrm{p}{}^\mathrm{T}\boldsymbol{F}_\mathrm{p})\boldsymbol{d}_j^0 = 0$ for $i \neq j$. This implies that the three orthonormal vectors $\boldsymbol{d}_i^0$ are just the eigenvectors of the plastic stretch $\boldsymbol{U}_\mathrm{p} = \sqrt{\boldsymbol{F}_\mathrm{p}^\mathrm{T}\boldsymbol{F}_\mathrm{p}}$. From this and the right polar decomposition of $\boldsymbol{F}_\mathrm{p}$, we may infer

[19] This discussion was primarily between the two schools and their followers and lasted several years. We therefore refer to Refs. [166–169] and the instructive discussion therein.

[20] There might, however, be some doubt about the physical pertinence of such non-symmetric flow rule in a 9-dimensional space. It would imply that nine, instead of six, rate equations governing plastic flow should be needed even in the case of infinitesimal deformation, except for some particular cases. We also refer to Ref. [172] and the discussion therein.

$$\boldsymbol{F}_{\mathrm{p}} = \boldsymbol{R}_{\mathrm{p}}\boldsymbol{U}_{\mathrm{p}}, \qquad \boldsymbol{U}_{\mathrm{p}} = \sum_{i=1}^{3} \lambda_i^p \boldsymbol{d}_i^0 \otimes \boldsymbol{d}_i^0.$$

In the above, the plastic rotation $\boldsymbol{R}_{\mathrm{p}}$ is arbitrary for a general director triad, whereas it is constant for an isoclinic triad.

This result would mean that the principal axes of the plastic stretch $\boldsymbol{U}_{\mathrm{p}}$ always keep unchanged, and this might be a too strong restriction. It could be eliminated by assuming that the rotation of the director triad at each particle to be independent of the deformation of the material element at this particle. But this would not only offer no assistance in clarifying the issues concerning the separation (112) but also go beyond the scope of a classical continuum. In fact, no director triad, let alone an isoclinic triad, could be defined in a classical continuum. Generally, any three mutually orthogonal line elements at each particle in the initial configuration could not always maintain the orthogonality property, since any line element in the unstressed configuration, no matter whether real or imaginary, should obey the basic kinematic relation (111).

However, the above remark would in no way invalidate the constitutive formulation in the foregoing. Actually, the latter may be independent of the notion of director triad, albeit it was thought to be the starting point. Here, the essential point might be that three additional conditions or relations would be needed to eliminate the non-uniqueness of the separation (112). A flow rule for $\dot{\boldsymbol{F}}_{\mathrm{p}}\boldsymbol{F}_{\mathrm{p}}^{-1}$ in a 9-dimensional space would furnish adequate constitutive relations specifying $\boldsymbol{F}_{\mathrm{p}}$. As such, however, three additional constitutive relations incorporated in the non-symmetric flow rule for $\dot{\boldsymbol{F}}_{\mathrm{p}}\boldsymbol{F}_{\mathrm{p}}^{-1}$ would have to be introduced, as compared to a symmetric flow rule for $\boldsymbol{D}_{\mathrm{p}}$.

Numerous modifications of the above concepts were reported in the literature of the last almost 50 years. Therein, also their implementation into robust and efficient numerical codes and the improvement of the latter became increasingly important.

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