Annali di Matematica Pura ed Applicata (1923 -) (2021) 200:2313–2331
https://doi.org/10.1007/s10231-021-01082-w

Symmetry results for *p*-Laplacian systems involving a first-order term

Francesco Esposito[1] · Susana Merchán[2] · Luigi Montoro[1]

Received: 26 February 2020 / Accepted: 8 February 2021 / Published online: 2 March 2021

Abstract
In this paper we obtain symmetry and monotonicity results for positive solutions to some p-Laplacian cooperative systems in bounded domains involving first-order terms and under zero Dirichlet boundary condition.

Keywords p-Laplace systems · First order terms · Non-variational systems · Axial symmetry · Radial symmetry · Qualitative properties

Mathematics Subject Classification 35J47 · 35J62 · 35J92

1 Introduction

The aim of this work is to get some symmetry and monotonicity results for nontrivial solutions $(u_1, u_2, \ldots, u_m) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \ldots \times C^1(\overline{\Omega})$ to the following quasilinear elliptic system

F. Esposito and L. Montoro were partially supported by PRIN project 2017JPCAPN (Italy): *Qualitative and quantitative aspects of nonlinear PDEs* and also by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). S. Merchán and L. Montoro were partially supported by project MTM2016-80474-P, MINECO (Spain): *Problemas elipticos y parabolicos basados en potencias del Laplaciano.*

✉ Luigi Montoro
montoro@mat.unical.it

Francesco Esposito
esposito@mat.unical.it

Susana Merchán
susana.merchan@upm.es

[1] Dipartimento di Matematica e Informatica, Università della Calabria, Ponte Pietro Bucci 31B, I-87036 Arcavacata di Rende, Cosenza, Italy

[2] Departamento de Matemáticas, Escuela de Caminos, Canales y Puertos, Universidad Politécnica de Madrid, Profesor Aranguren, 3, 28040 Madrid, Spain

$$\begin{cases} -\Delta_{p_i} u_i + a_i(u_i)|\nabla u_i|^{q_i} = f_i(u_1, u_2, \dots, u_i, \dots, u_m) & \text{in } \Omega \\ u_i > 0 & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{S})$$

where $i = 1, \dots, m$, $p_i > 1$, $q_i = \max\{1, p_i - 1\}$, Ω is a smooth bounded domain (connected open set) of $\mathbb{R}^N$, $N \geq 2$, $\Delta_{p_i} u_i := \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p-Laplace operator and a_i,f_i are problem data that obey to the set of assumptions (hp^*). The solution $(u_1, u_2, \dots, u_m)$ has to be understood in the weak distributional meaning. Our result will be obtained by means of the moving plane method, which goes back to the papers of Alexandrov [1] and Serrin [27]. In this work we use a nice variant of this technique: in particular the one of the celebrated papers of Berestycki-Nirenberg [3] and Gidas-Ni-Nirenberg [16], where the authors used, as essential ingredient, the maximum principle by comparing the values of the solution of the equation at two different points after a suitable reflection. Such a technique can be performed in general convex domains providing partial monotonicity results near the boundary and symmetry properties when the domain is convex and symmetric. For simplicity of exposition and without loss of generality, since the system ($\mathcal{S}$) is invariant with respect to translations and rotations, we assume directly in all the paper that Ω is a convex domain in the x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. When $m = 1$ the system ($\mathcal{S}$) is reduced to a scalar equation, that was already studied in [15] in the case of $\Omega = \mathbb{R}^N_+$ and $1 < p < 2$.

The moving plane procedure was applied to investigate symmetry properties of solutions of cooperative semilinear elliptic systems in bounded domains, firstly by Troy [28] (see also [11, 12, 26]): in this paper, the author considers the case $p_i = 2$ and $a_i = 0$ of ($\mathcal{S}$). This technique is very powerful and was adapted also in the case of cooperative semilinear systems in the half-space $\mathbb{R}^N_+$ by Dancer [10] and in the entire space $\mathbb{R}^N$ by Busca and Sirakov [4]. For other results regarding semilinear elliptic systems in bounded or unbounded domains, involving also critical nonlinearities, we refer to [13].

The moving plane method for quasilinear elliptic equations in bounded domains was developed in several papers by Damascelli, Pacella and Sciunzi [7–9] and in [14, 18] for quasilinear elliptic equations involving the Hardy–Leray potential and other more general singular nonlinearities. For the case of quasilinear elliptic systems in bounded domains we refer to [23, 24], where the authors considered the case $m = 2$ and $a_1 = a_2 = 0$ of ($\mathcal{S}$). Moreover, for other questions regarding existence, non-existence and Liouville type results, in the case of (pure, i.e., $a_i = 0$ in ($\mathcal{S}$)) p-Laplace systems, we refer the readers to the papers (and references therein) [2, 5, 6, 20, 21].

In this work we consider the general case of m p-Laplace equations with first-order terms.

To deal with the study of the qualitative properties of solutions to ($\mathcal{S}$), first we point out some regularity properties of the solutions to ($\mathcal{S}$), see Sect. 2. Indeed the fact that solutions to p-Laplace equations are not in general $C^2(\Omega)$, leads to the study of the summability properties of the second derivatives of the solutions. Thanks to these regularity results, we are able to prove a weak comparison principle in small domains, i.e., Proposition 2.5, that is a first crucial step in the proof of the main result of the paper, namely Theorem 1.1. Moreover, we also get some comparison and maximum principles that we will exploit in the Proof of Theorem 1.1.

Through all the paper, we assume that the following hypotheses (denoted by (hp^*) in the sequel) hold:

(hp^*) $\quad$ (i) For any $1 \leq i \leq m$, $a_i : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions.

(ii) $\quad$ For any $1 \leq i \leq m$, $f_i : \overline{\mathbb{R}}_+^m \to \mathbb{R}$ are locally $\mathcal{C}^1$ functions, i.e., $f_i \in C^1_{loc}(\overline{\mathbb{R}}_+^m)$, and assume that

$$f_i(t_1, t_2, \ldots, t_m) > 0,$$

for all $t_i > 0$. Moreover, the functions f_i satisfy

$$\frac{\partial f_i}{\partial t_k}(t_1, t_2, \ldots, t_m) \geq 0 \quad \text{for} \quad k \neq i, \;\; 1 \leq i, k \leq m. \tag{1.1}$$

The monotonicity conditions (1.1) are also known as *cooperativity conditions*, see [10, 24, 26, 28].

Finally we have the following

Theorem 1.1 *Assume that hypotheses* (hp^*) *hold. If* Ω *is convex in the* x_1*-direction and symmetric with respect to the hyperplane* $T_0 = \{x \in \mathbb{R}^N : x_1 = 0\}$, *then any solution* $(u_1, u_2, \ldots, u_m) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \ldots \times C^1(\overline{\Omega})$ *to* $(\mathcal{S})$ *is symmetric with respect to the hyperplane* T_0 *and nondecreasing in the* x_1*-direction in the set* $\Omega_0 = \{x_1 < 0\}$, *namely*

$$u_i(x_1, x_2, \ldots, x_N) = u_i(-x_1, x_2, \ldots, x_N) \quad \text{in } \Omega$$

and

$$\frac{\partial u_i}{\partial x_1}(x) \geq 0 \quad \text{in } \Omega_0, \tag{1.2}$$

for every $i \in \{1, \ldots, m\}$. *In particular, if* Ω *is a ball, then* u_i *are radially symmetric and radially decreasing, i.e.,*

$$\frac{\partial u_i}{\partial r}(r) < 0 \quad \text{for } r \neq 0.$$

Moreover, if $p_i > (2N+2)/(N+2)$ *for every* $i \in \{1, \ldots, m\}$, *then we have*

$$\frac{\partial u_i}{\partial x_1}(x) > 0 \quad \text{in } \Omega_0, \tag{1.3}$$

for every $i \in \{1, \cdots, m\}$.

The paper is organized as follows: In Section 2 we recall some preliminary results and we prove Proposition 2.5. The Proof of the Theorem 1.1 is contained in Sect. 3.

2 Preliminaries

In this section, we are going to give some results for p-Laplace equations involving a first-order term. Through all the paper, generic fixed and numerical constants will be denoted by C (with subscript or superscript in some case) and it will be allowed to vary within a single line or formula. Moreover, by $\mathcal{L}(\Omega)$ we will denote the Lebesgue measure of a measurable set Ω.

Firstly, we recall the following inequalities (see, for example, [7]) that we are going to use along the paper:

For all $\mu, \mu' \in \mathbb{R}^N$ with $|\mu| + |\mu'| > 0$ there exist two positive constants $C, \bar{C}$ depending on p such that

$$\begin{aligned} [|\mu|^{p-2}\mu - |\mu'|^{p-2}\mu'][\mu - \mu'] \geq C(|\mu| + |\mu'|)^{p-2}|\mu - \mu'|^2, \\ ||\mu|^{p-2}\mu - |\mu'|^{p-2}\mu'| \leq \bar{C}(|\mu| + |\mu'|)^{p-2}|\mu - \mu'|. \end{aligned} \tag{2.1}$$

In the following two theorems we give some regularity results and comparison/maximum principles for the solutions to $(\mathcal{S})$.

Theorem 2.1 (See [19, 22]). *Let Ω a bounded smooth domain of $\mathbb{R}^N$, $N \geq 2$, $1 < p < \infty$, $q \geq \max\{p-1, 1\}$ and consider $u \in C^1(\Omega)$ a positive weak solution to*

$$-\Delta_p u + a(u)|\nabla u|^q = f(x, u) \quad in \quad \Omega,$$

with

(*i*) $a : \mathbb{R} \to \mathbb{R}$ *a locally Lipschitz continuous function*;
(*ii*) $f \in C^1(\overline{\Omega} \times [0, +\infty))$.

Denoting $u_{x_i} = \partial u/\partial x_i$ and setting $\nabla u_{x_i} = 0$ on Z_u, for any $\Omega' \subset \Omega'' \subset\subset \Omega$, we have

$$\int_{\Omega'} \frac{|\nabla u|^{p-2-\beta}|\nabla u_{x_i}|^2}{|x-y|^\gamma}\, dx \leqslant \mathcal{C} \qquad \forall\, i = 1, \dots, N, \tag{2.2}$$

uniformly for any $y \in \Omega'$, with

$$\mathcal{C} := \mathcal{C}\Big(a, f, p, q, \beta, \gamma, \|u\|_{L^\infty(\Omega'')}, \|\nabla u\|_{L^\infty(\Omega'')}\Big),$$

for any $0 \leqslant \beta < 1$ and $\gamma < N-2$ if $N \geq 3$, or $\gamma = 0$ if $N = 2$.

Moreover, if $f(x, \cdot)$ is positive in Ω'', then it follows that

$$\int_{\Omega'} \frac{1}{|\nabla u|^{r(p-1)}} \frac{1}{|x-y|^\gamma}\, dx \leqslant \mathcal{C}^*, \tag{2.3}$$

uniformly for any $y \in \Omega'$, with

$$\mathcal{C}^* := \mathcal{C}^*\Big(a, f, p, q, r, \gamma, \|u\|_{L^\infty(\Omega'')}, \|\nabla u\|_{L^\infty(\Omega'')}\Big),$$

for any $r < 1$ and $\gamma < N-2$ if $N \geq 3$, or $\gamma = 0$ if $N = 2$.

In particular, these regularity results apply to the solutions u_i to $(\mathcal{S})$ with

$$f(x, u_i) = f_i(u_1, u_2, \dots, u_i, \dots, u_m). \tag{2.4}$$

Proof The proof follows exploiting and adapting some arguments contained in [19, 22] to (2.4)-type nonlinearities. This would imply some technicalities which we rather avoid here. □

For $\rho \in L^1(\Omega)$ and $1 \le s < \infty$, the weighted space $H^{1,s}_{\rho}(\Omega)$ (with respect to ρ) is defined as the completion of $C^1(\overline{\Omega})$ (or $C^\infty(\overline{\Omega})$) with the following norm

$$\|v\|_{H^{1,s}_\rho} = \|v\|_{L^s(\Omega)} + \|\nabla v\|_{L^s(\Omega,\rho)}, \tag{2.5}$$

where

$$\|\nabla v\|^s_{L^s(\Omega,\rho)} := \int_\Omega \rho(x)|\nabla v(x)|^s dx.$$

The space $H^{1,s}_{0,\rho}(\Omega)$ is, consequently, defined as the closure of $C^1_c(\Omega)$ (or $C^\infty_c(\Omega)$), with respect to the norm (2.5). We refer to [9] for more details about weighted Sobolev spaces and also to [17, Chapter 1] and the references therein. Theorem 2.1 provides also the right summability of the weight $|\nabla u(x)|^{p-2}$ in order to obtain a weighted Poincaré–Sobolev type inequality that will be useful in the sequel. For the proof we refer to [9, Section 3].

Theorem 2.2 (Weighted Poincaré-Sobolev type inequality). *Assume that hypotheses* (hp^*) *hold and let* $(u_1, u_2, \dots, u_m) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \dots \times C^1(\overline{\Omega})$ *be a solution to* $(\mathcal{S})$. *Assume that* $p_i \ge 2$ *for some* $i \in \{1, \dots, m\}$ *and set* $\rho_i = |\nabla u_i|^{p_i-2}$. *Then, for every* $w \in H^{1,2}_0(\Omega, \rho_i)$, *we have*

$$\|w\|_{L^2(\Omega)} \leqslant C_P\|\nabla w\|_{L^2(\Omega,\rho_i)} = C_P\left(\int_\Omega \rho_i\,|\nabla w|^2\right)^{\frac{1}{2}}, \tag{2.6}$$

with $C_P = C_P(\Omega) \to 0$ *if* $\mathcal{L}(\Omega) \to 0$.

The following theorem collects some comparison and maximum principles for solutions to the system $(\mathcal{S})$. We have

Theorem 2.3 (See [19, 22]). *Let* Ω *a bounded smooth domain of* $\mathbb{R}^N$, $N \ge 2$,

$$p_i > \frac{(2N+2)}{(N+2)} \tag{2.7}$$

and $q_i \ge \max\{p_i - 1, 1\}$ *for* $i = 1, \dots, m$. *Let* $(u_1, u_2, \dots, u_m), (v_1, v_2, \dots, v_m) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times \dots \times C^1(\overline{\Omega})$, *with* $(u_1, u_2, \dots, u_m)$ *a solution to* $(\mathcal{S})$ *and let us assume that assumptions* (hp^*) *hold.*

(1) *Then, for* $i = 1, 2, \dots, m$, *any connected domain* $\Omega' \subseteq \Omega$ *and for some constant* $\Lambda > 0$, *such that*

$$-\Delta_{p_i} u_i + a_i(u_i)|\nabla u_i|^{q_i} + \Lambda u_i \le -\Delta_{p_i} v_i + a_i(v_i)|\nabla v_i|^{q_i} + \Lambda v_i, \quad u_i \le v_i \quad in \quad \Omega'$$

in the weak distributional meaning, it follows that

$$u_i < v_i \quad in \quad \Omega',$$

unless $u_i \equiv v_i$ in Ω'.

(2) *For any* $i = 1, 2, \ldots, m$, *for any* $j = 1, 2, \ldots, N$, *and for any connected domain* $\Omega' \subseteq \Omega$ *such that*

$$\frac{\partial u_i}{\partial x_j} \geq 0 \quad in \quad \Omega',$$

it follows that

$$\frac{\partial u_i}{\partial x_j} > 0 \quad in \quad \Omega', \qquad \text{unless} \qquad \frac{\partial u_i}{\partial x_j} = 0 \quad \text{in} \quad \Omega'.$$

Proof The part (1) of the statement, follows using the regularity results contained in Theorem 2.1 and then exploiting [19, Theorem 1.2].

To prove the part (2) we need to define the linearized equations to the system $(\mathcal{S})$. In order to do this, since $(u_1, u_2, \ldots, u_m) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times \cdots \times C^1(\overline{\Omega})$ is a weak solution of $(\mathcal{S})$, then we set

$$\begin{aligned}
&L_{(u_1,\ldots,u_m)}\big((\partial_{x_j} u_1, \ldots, \partial_{x_j} u_i, \ldots, \partial_{x_j} u_m), (\varphi_1, \ldots, \varphi_m)\big) \\
&\quad = \Big(L^1_{(u_1,\ldots,u_m)}\big((\partial_{x_j} u_1, \ldots, \partial_{x_j} u_i, \ldots, \partial_{x_j} u_m), \varphi_1\big), \ldots, \\
&L^i_{(u_1,\ldots,u_m)}\big((\partial_{x_j} u_1, \ldots, \partial_{x_j} u_i, \ldots, \partial_{x_j} u_m), \varphi_i\big), \ldots, \\
&L^m_{(u_1,\ldots,u_m)}\big((\partial_{x_j} u_1, \ldots, \partial_{x_j} u_i, \ldots, \partial_{x_j} u_m), \varphi_m\big)\Big),
\end{aligned}$$

where for $p_i > 1$,

$$\begin{aligned}
&L^i_{(u_1,\ldots,u_m)}\big((\partial_{x_j} u_1, \ldots, \partial_{x_j} u_i, \ldots, \partial_{x_j} u_m), \varphi_i\big) \\
&\quad = \int_\Omega |\nabla u_i|^{p_i-2}(\nabla \partial_{x_j} u_i, \nabla \varphi_i) + (p_i - 2)\int_\Omega |\nabla u_i|^{p_i-4}(\nabla u_i, \nabla \partial_{x_j} u_i)(\nabla u_i, \nabla \varphi_i) \\
&\qquad + \int_\Omega a_i'(u_i)|\nabla u_i|^{q_i}\partial_{x_j} u_i \, \varphi_i + q_i \int_\Omega a_i(u_i)|\nabla u_i|^{q_i-2}(\nabla u_i, \nabla \partial_{x_j} u_i)\varphi_i \\
&\qquad - \int_\Omega \sum_{k=1}^m \frac{\partial f_i}{\partial u_k}(u_1, \ldots, u_i, \ldots, u_m)\partial_{x_j} u_k \, \varphi_i,
\end{aligned}$$

for any $\varphi_1, \ldots, \varphi_m \in C^1_0(\Omega)$. Moreover, using the regularity results contained in Theorem 2.1 (see [22]), the following equation holds

$$L_{(u_1,\ldots,u_m)}\big((\partial_{x_j} u_1, \ldots, \partial_{x_j} u_i, \ldots, \partial_{x_j} u_m), (\varphi_1, \ldots, \varphi_m)\big) = 0, \tag{2.8}$$

for all $(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_m)$ in $H^{1,2}_{0,\rho_{u_1}}(\Omega) \times \cdots \times H^{1,2}_{0,\rho_{u_i}}(\Omega) \times \cdots \times H^{1,2}_{0,\rho_{u_m}}(\Omega)$ where

$$\rho_{u_i}(x) := |\nabla u_i(x)|^{p_i-2}, \qquad i = 1, \ldots, m.$$

Since f_i are locally C^1 functions and $\|u_i\|_{L^\infty(\Omega)} \leq C$ for any $i \in \{1, \ldots, m\}$, there exists a positive constant Θ such that

$$\frac{\partial f_i}{\partial u_i} + \Theta \geq 0 \text{ for all } u_1, u_2, \dots, u_m > 0. \tag{2.9}$$

Moreover, in light of (1.1) we have

$$\frac{\partial f_i}{\partial u_k}(u_1, \dots, u_i, \dots, u_m) \geq 0 \tag{2.10}$$

for $i \neq k$. Therefore, using (2.9) and (2.10) and taking into account (2.8), it follows, for all $j = 1, \dots, N$ and for all $i = 1, \dots, m$, that $\partial_{x_j} u_i$ are nonnegative functions solving the inequalities

$$\begin{aligned}
&\int_\Omega |\nabla u_i|^{p_i-2}(\nabla \partial_{x_j} u_i, \nabla \varphi_i) + (p_i - 2)\int_\Omega |\nabla u_i|^{p_i-4}(\nabla u_i, \nabla \partial_{x_j} u_i)(\nabla u_i, \nabla \varphi_i) \\
&\quad + \int_\Omega a_i'(u_i)|\nabla u_i|^{q_i}\partial_{x_j} u_i\, \varphi_i + q_i \int_\Omega a_i(u_i)|\nabla u_i|^{q_i-2}(\nabla u_i, \nabla \partial_{x_j} u_i)\varphi_i \\
&\quad + \Theta \int_\Omega \partial_{x_j} u_i\, \varphi_i \geq 0
\end{aligned}$$

for all nonnegative test functions $\varphi_i \geq 0$.

Therefore, we can apply [22, Theorem 3.1] to each $\partial_{x_j} u_i$ separately obtaining that, for every $s > 1$ sufficiently close to 1 and some positive δ sufficiently small, there exists a positive constant C such that

$$\|\partial_{x_j} u_i\|_{L^s(B(x,2\delta))} \leq C_1 \inf_{B(x,\delta)} \partial_{x_j} u_i. \tag{2.11}$$

Then, the sets $\{x \in \Omega' : \partial_{x_j} u_i = 0\}$ are both closed (by continuity) and open (via inequality (2.11)) in the domain Ω'. This yields the assertion. □

Remark 2.4 We point out that Theorem 2.3 holds without any a priori assumption on the critical set of the solution $(u_1, u_2, \dots, u_m)$, that is, the set where the gradients ∇u_i vanish. On the other hand, though, condition (2.7) can be removed when we work in connected domain Ω' such that $\nabla u_i \neq 0$ for all $x \in \Omega'$ and for all $i \in \{1, \dots, m\}$. Indeed, the statements (1) and (2) of Theorem 2.3 hold in the whole range $p_i > 1$.

Note that the positivity of $f(x, \cdot)$, is actually needed to obtain (2.3). Furthermore, by (2.3) it follows that the critical set $\{x \in \Omega : \nabla u(x) = 0\}$ has zero Lebesgue measure.

An essential tool in the Proof of Theorem 1.1 is Proposition 2.5, i.e., a weak comparison principle in small domains. To prove it, we start giving the following assumptions:

(∗) We suppose that $(u_1, u_2, \dots, u_m) \in C^1(\overline{\Omega}_1) \times C^1(\overline{\Omega}_1) \times \dots \times C^1(\overline{\Omega}_1)$ is a solution to $(\mathcal{S})$ in the smooth bounded domain $\Omega_1 \subset \mathbb{R}^N$ and $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m) \in C^1(\overline{\Omega}_2) \times C^1(\overline{\Omega}_2) \times \dots \times C^1(\overline{\Omega}_2)$ is a solution to $(\mathcal{S})$ in the smooth bounded domain $\Omega_2 \subset \mathbb{R}^N$, with

$$\Omega_1 \cap \Omega_2 \neq \emptyset.$$

Proposition 2.5 *Assume that (∗) holds, $p_i > 1$, $q_i = \max\{1, p_i - 1\}$ for every $i \in \{1, 2, \dots, m\}$ and let $\Omega \subset \Omega_1 \cap \Omega_2$ be a connected set. Then, there exists a positive* *number* δ, *depending upon* $m, p_i, q_i, a_i, f_i, \|u_i\|_{L^\infty(\Omega)}, \|\nabla u_i\|_{L^\infty(\Omega)}, \|\nabla \tilde{u}_i\|_{L^\infty(\Omega)}, i = 1, 2, \dots, m$, *such that if* $\Omega_0 \subset \Omega$ *with*

$$\mathcal{L}(\Omega_0) \leq \delta \quad \text{and} \quad u_i \leq \tilde{u}_i \text{ on } \partial\Omega_0 \text{ for every } i \in \{1, \dots, m\},$$

then

$$u_i \leq \tilde{u}_i \text{ in } \Omega_0,$$

for every $i \in \{1, \dots, m\}$.

Proof Let us set

$$U_i = (u_i - \tilde{u}_i)^+.$$

We will prove the result by showing that

$$(u_i - \tilde{u}_i)^+ \equiv 0,$$

for every $i \in \{1, 2, \dots, m\}$. Since $u_i \leq \tilde{u}_i$ on $\partial\Omega_0$, then the functions $(u_i - \tilde{u}_i)^+$ belong to $W_0^{1,p_i}(\Omega_0)$. Therefore, since $u_i, \tilde{u}_i$ are both weak solutions to $(\mathcal{S})$ in Ω, for all $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega |\nabla u_i|^{p_i-2}(\nabla u_i, \nabla\varphi)dx + \int_\Omega a_i(u_i)|\nabla u_i|^{q_i}\varphi dx = \int_\Omega f_i(u_1, u_2, \dots, u_m)\varphi dx \tag{2.12}$$

and

$$\int_\Omega |\nabla \tilde{u}_i|^{p_i-2}(\nabla \tilde{u}_i, \nabla\varphi)dx + \int_\Omega a_i(\tilde{u}_i)|\nabla \tilde{u}_i|^{q_i}\varphi dx = \int_\Omega f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)\varphi dx, \tag{2.13}$$

for $i = 1, 2, \dots, m$. By a density argument, we can put, respectively, $\varphi = (u_i - \tilde{u}_i)^+$ in Eqs. (2.12) and (2.13). Subtracting, we get for any i

$$\begin{aligned}
&\int_{\Omega_0} \left(|\nabla u_i|^{p_i-2}\nabla u_i - |\nabla \tilde{u}_i|^{p_i-2}\nabla \tilde{u}_i, \nabla(u_i - \tilde{u}_i)^+\right) dx \\
&\quad + \int_{\Omega_0} \left(a_i(u_i)|\nabla u_i|^{q_i} - a_i(\tilde{u}_i)|\nabla \tilde{u}_i|^{q_i}\right)(u_i - \tilde{u}_i)^+ dx \\
&= \int_{\Omega_0} [f_i(u_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)](u_i - \tilde{u}_i)^+ dx.
\end{aligned} \tag{2.14}$$

The second term on the left-hand side of (2.14) can be estimated as follows

$$\begin{aligned}
&\left|\int_{\Omega_0} \big(a_i(u_i)|\nabla u_i|^{q_i} - a_i(\tilde{u}_i)|\nabla \tilde{u}_i|^{q_i}\big)(u_i - \tilde{u}_i)^+ \, dx\right| \\
&\quad = \left|\int_{\Omega_0} \big(a_i(u_i)|\nabla u_i|^{q_i} - a_i(u_i)|\nabla \tilde{u}_i|^{q_i} + a_i(u_i)|\nabla \tilde{u}_i|^{q_i} - a_i(\tilde{u}_i)|\nabla \tilde{u}_i|^{q_i}\big)(u_i - \tilde{u}_i)^+ \, dx\right| \\
&\quad \le \int_{\Omega_0} |a_i(u_i)|\big||\nabla u_i|^{q_i} - |\nabla \tilde{u}_i|^{q_i}\big|(u_i - \tilde{u}_i)^+ \, dx \\
&\qquad + \int_{\Omega_0} |\nabla \tilde{u}_i|^{q_i}(a_i(u_i) - a_i(\tilde{u}_i))(u_i - \tilde{u}_i)^+ \, dx.
\end{aligned}$$

Since a_i is a locally Lipschitz continuous function (see (hp^*)), it follows that there exists a positive constant $K_{a_i} = K_{a_i}(\|u_i\|_{L^\infty(\Omega)})$ such that for every $u_i \in [0, \|u_i\|_{L^\infty(\Omega)}]$

$$|a_i(u_i)| \le K_{a_i}.$$

Moreover, denoting by $L_{a_i} = L_{a_i}(\|u_i\|_{L^\infty(\Omega)})$ the Lipschitz constant of a_i, we obtain

$$\begin{aligned}
&\left|\int_{\Omega_0} \big(a_i(u_i)|\nabla u_i|^{q_i} - a_i(\tilde{u}_i)|\nabla \tilde{u}_i|^{q_i}\big)(u_i - \tilde{u}_i)^+ \, dx\right| \\
&\quad \le K_{a_i} \int_{\Omega_0} \big||\nabla u_i|^{q_i} - |\nabla \tilde{u}_i|^{q_i}\big|(u_i - \tilde{u}_i)^+ \, dx \\
&\qquad + C(q_i, L_{a_i}, \|\nabla \tilde{u}_i\|_{L^\infty(\Omega)}) \int_{\Omega_0} [(u_i - \tilde{u}_i)^+]^2 \, dx.
\end{aligned} \tag{2.15}$$

By the mean value's theorem and taking into account that $q_i \ge 1$, it follows that

$$\begin{aligned}
&K_{a_i} \int_{\Omega_0} \big||\nabla u_i|^{q_i} - |\nabla \tilde{u}_i|^{q_i}\big|(u_i - \tilde{u}_i)^+ \, dx \\
&\quad \le C(q_i, K_{a_i}) \int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{q_i - 1}|\nabla (u_i - \tilde{u}_i)^+|(u_i - \tilde{u}_i)^+ \, dx.
\end{aligned}$$

The last term (recall that $q_i \ge \max\{1, p_i - 1\}$) can be written as follows,

$$\begin{aligned}
&C \int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{q_i - 1}|\nabla (u_i - \tilde{u}_i)^+|(u_i - \tilde{u}_i)^+ \, dx \\
&\quad = C \int_{\Omega_0} \frac{(|\nabla u_i| + |\nabla \tilde{u}_i|)^{q_i - 1}}{(|\nabla u_i| + |\nabla \tilde{u}_i|)^{\frac{p_i - 2}{2}}} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{\frac{p_i - 2}{2}} |\nabla (u_i - \tilde{u}_i)^+|(u_i - \tilde{u}_i)^+ \, dx \\
&\quad \le C \int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{\frac{p_i - 2}{2}} |\nabla (u_i - \tilde{u}_i)^+|(u_i - \tilde{u}_i)^+ \, dx,
\end{aligned} \tag{2.16}$$

with $C = C(p_i, q_i, K_{a_i}, \|\nabla u_i\|_{L^\infty(\Omega)}, \|\nabla \tilde{u}_i\|_{L^\infty(\Omega)})$ is a positive constant. Exploiting Young's inequality in the right-hand side of (2.16) we finally obtain

$$\begin{aligned} C & \int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{q_i - 1} |\nabla (u_i - \tilde{u}_i)^+| |(u_i - \tilde{u}_i)^+| \, dx \\ & \leqslant \varepsilon C \int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{p_i - 2} |\nabla (u_i - \tilde{u}_i)^+|^2 \, dx \\ & \quad + \frac{C}{\varepsilon} \int_{\Omega_0} [(u_i - \tilde{u}_i)^+]^2 \, dx. \end{aligned}$$

Therefore, collecting the previous estimates, from (2.15), we obtain

$$\begin{aligned} & \left| \int_{\Omega_0} \left(a_i(u_i) |\nabla u_i|^{q_i} - a_i(\tilde{u}_i) |\nabla \tilde{u}_i|^{q_i} \right) (u_i - \tilde{u}_i)^+ \, dx \right| \\ & \quad \leqslant \varepsilon C \int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{p_i - 2} |\nabla (u_i - \tilde{u}_i)^+|^2 \, dx \\ & \quad \quad + \frac{C}{\varepsilon} \int_{\Omega_0} [(u_i - \tilde{u}_i)^+]^2 \, dx. \end{aligned}$$

Finally, using (2.1) and fixing ε sufficiently small, from (2.14) we get

$$\begin{aligned} & \int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{p_i - 2} |\nabla (u_i - \tilde{u}_i)^+|^2 \, dx \\ & \quad \le \int_{\Omega_0} \left(|\nabla u_i|^{p_i - 2} \nabla u_i - |\nabla \tilde{u}_i|^{p_i - 2} \nabla \tilde{u}_i, \nabla (u_i - \tilde{u}_i)^+ \right) dx \\ & \quad \le C \int_{\Omega_0} [f_i(u_1, u_2, \ldots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_m)] (u_i - \tilde{u}_i)^+ \, dx \\ & \quad \quad + C \int_{\Omega_0} [(u_i - \tilde{u}_i)^+]^2 \, dx, \end{aligned} \tag{2.17}$$

where $C = C(p_i, q_i, K_{a_i}, L_{a_i}, \|\nabla u_i\|_{L^\infty(\Omega)}, \|\nabla \tilde{u}_i\|_{L^\infty(\Omega)})$ is a positive constant.

The first term on the right-hand side of (2.17) can be arranged as follows

$$
\begin{aligned}
&\int_{\Omega_0} [f_i(u_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)](u_i - \tilde{u}_i)^+ \, dx \\
&\quad = \int_{\Omega_0} [f_i(u_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, u_2, \dots, u_m) + f_i(\tilde{u}_1, u_2, \dots, u_m) \\
&\qquad - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)](u_i - \tilde{u}_i)^+ \, dx \\
&\quad = \int_{\Omega_0} [f_i(u_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, u_2, \dots, u_m) \\
&\qquad + f_i(\tilde{u}_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, u_m) \\
&\qquad + \dots + f_i(\tilde{u}_1, \tilde{u}_2, \dots, u_i, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_i, \dots, u_m) \\
&\qquad + f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_i, \dots, u_m) \\
&\qquad \vdots \\
&\qquad \dots + f_i(\tilde{u}_1, \tilde{u}_2, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)](u_i - \tilde{u}_i)^+ \, dx.
\end{aligned}
\tag{2.18}
$$

Using the fact that f_i are C^1_{loc} functions satisfying (1.1), see (hp^*), by (2.18) we have

$$
\begin{aligned}
&\int_{\Omega_0} [f_i(u_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)](u_i - \tilde{u}_i)^+ \, dx \\
&\quad \le \int_{\Omega_0} \frac{f_i(u_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, u_2, \dots, u_m)}{(u_1 - \tilde{u}_1)^+} (u_1 - \tilde{u}_1)^+ (u_i - \tilde{u}_i)^+ \, dx \\
&\qquad + \int_{\Omega_0} \frac{f_i(\tilde{u}_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, u_m)}{(u_2 - \tilde{u}_2)^+} (u_2 - \tilde{u}_2)^+ (u_i - \tilde{u}_i)^+ \, dx \\
&\qquad \vdots \\
&\qquad + \int_{\Omega_0} \frac{f_i(\tilde{u}_1, \tilde{u}_2, \dots, u_i, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_i, \dots, u_m)}{(u_i - \tilde{u}_i)} [(u_i - \tilde{u}_i)^+]^2 \, dx \\
&\qquad \vdots \\
&\qquad + \int_{\Omega_0} \frac{f_i(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3 \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)}{(u_m - \tilde{u}_m)^+}](u_m - \tilde{u}_m)^+ (u_i - \tilde{u}_i)^+ \, dx \\
&\quad \le L_{f_i} \sum_{j=1}^{m} \int_{\Omega_0} (u_j - \tilde{u}_j)^+ (u_i - \tilde{u}_i)^+ \, dx,
\end{aligned}
\tag{2.19}
$$

where L_{f_i} is the Lipschitz constant of f_i that depends on the $\max\limits_{1 \le j \le m} \{\|u_j\|_{L^\infty(\Omega)}\}$. Exploiting Young's inequality on the right-hand side of (2.19), we get

$$
\begin{aligned}
&\int_{\Omega_0} [f_i(u_1, u_2, \dots, u_m) - f_i(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)](u_i - \tilde{u}_i)^+ \, dx \\
&\quad \le C \sum_{j=1}^{m} \int_{\Omega_0} [(u_j - \tilde{u}_j)^+]^2 \, dx,
\end{aligned}
\tag{2.20}
$$

where $C = C\big(m, L_{f_i}\big)$ is a positive constant. Finally, from (2.17) and (2.20) we infer for $i = 1, \dots, m$

$$\int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{p_i-2} |\nabla (u_i - \tilde{u}_i)^+|^2 dx \le C_i \sum_{j=1}^{m} \int_{\Omega_0} [(u_j - \tilde{u}_j)^+]^2 \, dx, \tag{2.21}$$

where $C_i = C_i(m, p_i, q_i, K_{a_i}, L_{a_i}, L_{f_i}, \|\nabla u_i\|_{L^\infty(\Omega)}, \|\nabla \tilde{u}_i\|_{L^\infty(\Omega)})$ is a positive constant.

In the case $p_j \ge 2$, a weighted Poincaré inequality holds true on the right-hand side of (2.21), see Theorem 2.2. Indeed, Eq. (2.6) yields

$$\int_{\Omega_0} [(u_j - \tilde{u}_j)^+]^2 dx \le C_{P,j}(\Omega_0) \int_{\Omega_0} (|\nabla u_j| + |\nabla \tilde{u}_j|)^{p_j-2} |\nabla (u_j - \tilde{u}_j)^+|^2 dx, \quad \text{if } p_j \ge 2, \tag{2.22}$$

where the Poincaré constant $C_{P,j}(\Omega_0) \to 0$, when the Lebesgue measure $\mathcal{L}(\Omega_0) \to 0$. Actually, we used the fact that, since $p_j \ge 2$,

$$|\nabla u_j|^{p_j-2} \le (|\nabla u_j| + |\nabla \tilde{u}_j|)^{p_j-2}.$$

In the case $p_j < 2$, we use the standard Poincaré inequality on the right-hand side of (2.21), namely

$$\int_{\Omega_0} [(u_j - \tilde{u}_j)^+]^2 \, dx \le C_{P,j}(\Omega_0) \int_{\Omega_0} |\nabla (u_j - \tilde{u}_j)^+|^2 \, dx, \quad \text{if } p_j < 2,$$

and $C_{P,j}(\Omega_0) \to 0$ if $\mathcal{L}(\Omega_0) \to 0$. Moreover, in the case $p_j < 2$ since $u_j, \tilde{u}_j \in C^1(\overline{\Omega})$, we deduce also

$$\begin{aligned} &\int_{\Omega_0} |\nabla (u_j - \tilde{u}_j)^+|^2 dx \\ &\quad \le C(p_j, \|\nabla u_j\|_{L^\infty(\Omega)}, \|\nabla \tilde{u}_j\|_{L^\infty(\Omega)}) \int_{\Omega_0} (|\nabla u_j| + |\nabla \tilde{u}_j|)^{p_j-2} |\nabla (u_j - \tilde{u}_j)^+|^2 dx. \end{aligned} \tag{2.23}$$

Using (2.23), up to redefine the Poincaré constant in this case, we obtain

$$\int_{\Omega_0} [(u_j - \tilde{u}_j)^+]^2 \, dx \le C_{P,j}(\Omega_0) \int_{\Omega_0} (|\nabla u_j| + |\nabla \tilde{u}_j|)^{p_j-2} |\nabla (u_j - \tilde{u}_j)^+|^2 \, dx, \quad \text{if } p_j < 2, \tag{2.24}$$

and $C_{P,j}(\Omega_0) \to 0$ if $\mathcal{L}(\Omega_0) \to 0$. Let us set now

$$C_P(\Omega_0) = \max_{1 \le j \le m} \{C_{P,j}(\Omega_0)\}. \tag{2.25}$$

Furthermore, by combining (2.21) with (2.22), (2.24) and (2.25), we obtain for $i = 1, \dots, m$

$$\begin{aligned} &\int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{p_i-2} |\nabla (u_i - \tilde{u}_i)^+|^2 dx \\ &\quad \le C_i C_P(\Omega_0) \sum_{j=1}^{m} \int_{\Omega_0} (|\nabla u_j| + |\nabla \tilde{u}_j|)^{p_j-2} |\nabla (u_j - \tilde{u}_j)^+|^2 dx. \end{aligned} \tag{2.26}$$

Let us define $\hat{C} = m \cdot \max_{1 \le i \le m} \{C_i\}$. By adding Eq. (2.26) and setting

$$I(\Omega_0) = \sum_{i=1}^{m} \int_{\Omega_0} (|\nabla u_i| + |\nabla \tilde{u}_i|)^{p_i-2} |\nabla (u_i - \tilde{u}_i)^+|^2 dx,$$

we obtain

$$I(\Omega_0) \leq \hat{C} C_P(\Omega_0) I(\Omega_0). \tag{2.27}$$

Now, we choose $\delta > 0$ sufficiently small such that the condition $\mathcal{L}(\Omega_0) \leq \delta$ implies

$$\hat{C} C_P(\Omega_0) < 1.$$

Therefore, from (2.27) we get the desired contradiction, namely

$$U_i = (u_i - \tilde{u}_i)^+ \equiv 0,$$

for all $i = 1, \ldots, m$. □

3 Symmetry results for solutions to ($\mathcal{S}$): Proof of Theorem 1.1

In this section, we prove our main result. As we said in the introduction, without loss of generality and for the sake of simplicity, since the problem is invariant with respect to translations, reflections and rotations, we suppose that Ω is a bounded smooth domain which is convex in the x_1-direction and symmetric with respect to $\{x_1 = 0\}$. Let us now recall the main ingredients of the moving plane method. We set

$$T_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}.$$

Given $x \in \mathbb{R}^N$ and $\lambda < 0$, we define

$$x_\lambda = R_\lambda(x) := (2\lambda - x_1, x_2, \ldots, x_N)$$

and the reflected functions

$$u_{i,\lambda}(x) := u_i(x_\lambda), \qquad i = 1, 2, \ldots, m.$$

We also set

$$\begin{aligned} &\Omega_\lambda := \{x \in \Omega : x_1 < \lambda\}, \\ &a := \inf_{x \in \Omega} x_1, \end{aligned} \tag{3.1}$$

$$\Lambda := \left\{ a < \lambda < 0 \;:\; u_i \leq u_{i,t} \text{ in } \Omega_t, \text{ for all } t \in (a, \lambda] \text{ and for all } i = 1, 2, \ldots, m \right\} \tag{3.2}$$

and (if $\Lambda \neq \emptyset$)

$$\bar{\lambda} = \sup \Lambda.$$

Finally, for $i = 1, \ldots, m$, we define the *critical sets*

$$Z_{u_i} := \{x \in \Omega : \nabla u_i(x) = 0\}.$$

Proof of Theorem 1.1 For $a < \lambda < 0$ [see (3.1)] and λ sufficiently close to a, we assume that $\mathcal{L}(\Omega_\lambda)$ is as small as we need. In particular, we may assume that Proposition 2.5 works with $\Omega_1 = \Omega$, $\Omega_2 = R_\lambda(\Omega)$, $\Omega_0 = \Omega_\lambda$ and $\tilde{u}_i = u_{i,\lambda}$. Therefore, we set

$$W_{i,\lambda} := u_i - u_{i,\lambda}, \quad i = 1, 2, \dots, m$$

and we observe that, by construction, we have

$$W_{i,\lambda} \le 0 \text{ on } \partial\Omega_\lambda, \quad i = 1, 2, \dots, m.$$

By Proposition 2.5, it follows that

$$W_{i,\lambda} \le 0 \text{ in } \Omega_\lambda, \quad i = 1, 2, \dots, m.$$

Hence, the set Λ [see (3.2)] is not empty and $\bar{\lambda} \in (a, 0]$. Note that, by continuity, it follows $u_i \le u_{i,\bar{\lambda}}$. We have to show that, actually $\bar{\lambda} = 0$. Hence, we assume by contradiction that $\bar{\lambda} < 0$ and we argue as follows.

First of all, we point out that $\mathcal{L}(Z_{u_i}) = 0$ for all i. Indeed, if we apply Theorem 2.1, for u_i with $f(x, u_i) = f_i(u_1, u_2, \dots, u_i, \dots, u_m)$, from (2.3) the conclusion follows. Hence, let A be an open set such that for $i = 1, \dots, m$

$$Z_{u_i} \cap \Omega_{\bar{\lambda}} \subset A \subset \Omega_{\bar{\lambda}},$$

with the Lebesgue measure $\mathcal{L}(A)$ small as we like. Notice now that, since f_i are locally C^1 functions and $\|u_i\|_{L^\infty(\Omega)} \le C$ for any $i \in \{1, \dots, m\}$, there exists a positive constant Θ such that

$$\frac{\partial f_i}{\partial u_i} + \Theta \ge 0 \text{ for all } u_1, u_2, \dots, u_m > 0. \tag{3.3}$$

Furthermore, using (1.1) we obtain

$$\begin{aligned} &-\Delta_{p_i} u_i + a_i(u_i)|\nabla u_i|^{q_i} + \Theta u_i = f_i(u_1, u_2, \dots, u_m) + \Theta u_i \\ &\le f_i(u_{1,\lambda}, u_{2,\lambda}, \dots, u_{m,\lambda}) + \Theta u_{i,\lambda} = -\Delta_{p_i} u_{i,\lambda} + a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i} + \Theta u_{i,\lambda} \end{aligned} \tag{3.4}$$

for any $a < \lambda \le \bar{\lambda}$. In light of (3.4) we have

$$\begin{cases} -\Delta_{p_i} u_i + a_i(u_i)|\nabla u_i|^{q_i} + \Theta u_i \le -\Delta_{p_i} u_{i,\lambda} + a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i} + \Theta u_{i,\lambda} & \text{in } \Omega_\lambda, \\ u_i \le u_{i,\lambda} & \text{in } \Omega_\lambda. \end{cases} \tag{3.5}$$

Then, by (3.5) and the strong comparison principle, see statement (1) of Theorem 2.3, for any $i = 1, 2, \dots, m$ such that $p_i \ge 2$, we have

$$u_i < u_{i,\bar{\lambda}} \qquad \text{or} \qquad u_i \equiv u_{i,\bar{\lambda}},$$

in $\Omega_{\bar{\lambda}}$.

In the case $1 < p_i < 2$, we prove first the following

Claim: The case $u_i \equiv u_{i,\bar{\lambda}}$ in some connected component $\mathcal{C}$ of $\Omega_{\bar{\lambda}} \setminus Z_{u_i}$, such that $\overline{\mathcal{C}} \subset \Omega$, is not possible.

We proceed by contradiction. Let us assume that such component exists, namely

$$\mathcal{C} \subset \Omega \quad \text{such that} \quad \partial\mathcal{C} \subset Z_{u_i}.$$

For all $\varepsilon > 0$, let us define $G_\varepsilon : \mathbb{R}_0^+ \to \mathbb{R}$ by setting

$$G_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \le t \le \varepsilon \\ 2t - 2\varepsilon & \text{if } \varepsilon \le t \le 2\varepsilon \\ t & \text{if } t \ge 2\varepsilon. \end{cases} \tag{3.6}$$

Let $\chi_{\mathcal{A}}$ be the characteristic function of a set $\mathcal{A}$. We define

$$\Psi_\varepsilon := e^{-s_i(u_i)} \frac{G_\varepsilon(|\nabla u_i|)}{|\nabla u_i|} \chi_{(\mathcal{C} \cup \mathcal{C}^\lambda)}, \tag{3.7}$$

where $\mathcal{C}^\lambda$ is the reflected set of $\mathcal{C}$ with respect to the hyperplane $T_{\bar{\lambda}}$ and

$$s_i(t) = \hat{C}_i \cdot \int_0^t a_i^+(t')dt', \tag{3.8}$$

where $a_i^+ := \max\{0, a_i\}$ ($a_i^- := -\min\{0, a_i\}$) and $\hat{C}_i$ denotes some positive constant to be chosen later.

We point out that $\operatorname{supp}\Psi_\varepsilon \subset \mathcal{C} \cup \mathcal{C}^\lambda$, which implies $\Psi_\varepsilon \in W_0^{1,p}(\mathcal{C} \cup \mathcal{C}^\lambda)$. Indeed by definition of $\mathcal{C}$ we have that $\nabla u_i = 0$ on $\partial(\mathcal{C} \cup \mathcal{C}^\lambda)$. Moreover, using the test function Ψ_ε defined in (3.7), we are able to integrate on the boundary $\partial(\mathcal{C} \cup \mathcal{C}^\lambda)$ which could be not regular.

Hence, we obtain

$$\begin{aligned} &\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2}(\nabla u_i, \nabla \Psi_\varepsilon)dx + \int_{\mathcal{C}\cup\mathcal{C}^\lambda} a_i^+(u_i)|\nabla u_i|^{q_i}\Psi_\varepsilon dx \\ &\quad = \int_{\mathcal{C}\cup\mathcal{C}^\lambda} a_i^-(u_i)|\nabla u_i|^{q_i}\Psi_\varepsilon dx + \int_{\mathcal{C}\cup\mathcal{C}^\lambda} f_i(u_1, u_2, \dots, u_m)\Psi_\varepsilon dx. \end{aligned} \tag{3.9}$$

It is easy to see that for every $x \in [0, M]$ and for every $l, q \ge 1$ and $\sigma > 0$, there exists a positive constant $C = C(l, q, \sigma, M)$ such that

$$x^q \le C \cdot x^l + \sigma, \quad x \in [0, M]. \tag{3.10}$$

Therefore, (3.9) and (3.10) imply:

$$\begin{aligned} &\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2}(\nabla u_i, \nabla \Psi_\varepsilon)dx \\ &\quad + C_i(\sigma_i, p_i, q_i, \|\nabla u_i\|_{L^\infty(\Omega)}) \int_{\mathcal{C}\cup\mathcal{C}^\lambda} a_i^+(u_i)|\nabla u_i|^{p_i}\Psi_\varepsilon dx \\ &\quad + \sigma_i \int_{\mathcal{C}\cup\mathcal{C}^\lambda} a_i^+(u_i)\Psi_\varepsilon dx \\ &\ge \int_{\mathcal{C}\cup\mathcal{C}^\lambda} a_i^-(u_i)|\nabla u_i|^{q_i}\Psi_\varepsilon dx + \int_{\mathcal{C}\cup\mathcal{C}^\lambda} f_i(u_1, u_2, \dots, u_m)\Psi_\varepsilon dx \\ &\ge \int_{\mathcal{C}\cup\mathcal{C}^\lambda} f_i(u_1, u_2, \dots, u_m)\Psi_\varepsilon dx. \end{aligned} \tag{3.11}$$

By $(hp^*) - (ii)$, since $\overline{\mathcal{C} \cup \mathcal{C}^\lambda} \subset \Omega$ we have that there exists $\gamma_i > 0$ such that

$$f_i(u_1, u_2, \dots, u_m) \ge \gamma_i.$$

Hence, we can choose σ_i in (3.10), say $\bar{\sigma}_i$, small enough such that

$$\gamma_i - \bar{\sigma}_i \, \|a_i^+(u_i)\|_\infty = \tilde{C}_i > 0\,,$$

so that

$$\begin{aligned}&\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2}(\nabla u_i, \nabla \Psi_\varepsilon) dx \\ &+ C_i(\bar{\sigma}_i, p_i, q_i, \|\nabla u_i\|_{L^\infty(\Omega)}) \int_{\mathcal{C}\cup\mathcal{C}^\lambda} a_i^+(u_i)|\nabla u_i|^{p_i}\Psi_\varepsilon dx \\ &\geq \tilde{C}_i \int_{\mathcal{C}\cup\mathcal{C}^\lambda} \Psi_\varepsilon dx.\end{aligned} \tag{3.12}$$

Choosing $\hat{C}_i$ in (3.8) equal to $C_i(\bar{\sigma}_i, p_i, q_i, \|\nabla u_i\|_{L^\infty(\Omega)})$ in (3.12) we obtain

$$\begin{aligned}&\int_{\mathcal{C}\cup\mathcal{C}^\lambda} e^{-s_i(u_i)}|\nabla u_i|^{p_i-2}\left(\nabla u_i, \nabla \frac{G_\varepsilon(|\nabla u_i|)}{|\nabla u_i|}\right) dx \\ &\geq \tilde{C}_i \int_{\mathcal{C}\cup\mathcal{C}^\lambda} e^{-s_i(u_i)} \frac{G_\varepsilon(|\nabla u_i|)}{|\nabla u_i|} dx.\end{aligned} \tag{3.13}$$

We set $h_\varepsilon(t) = \dfrac{G_\varepsilon(t)}{t}$, meaning that $h_\varepsilon(t) = 0$ for $0 \leq t \leq \varepsilon$. We have:

$$\begin{aligned}&\left|\int_{\mathcal{C}\cup\mathcal{C}^\lambda} e^{-s_i(u_i)}|\nabla u_i|^{p_i-2}\left(\nabla u_i, \nabla \frac{G_\varepsilon(|\nabla u_i|)}{|\nabla u_i|}\right) dx\right| \\ &\leq \int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-1}|h_\varepsilon'(|\nabla u_i|)||\nabla(|\nabla u_i|)| dx \\ &\leq C_i \int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2}\Big(|\nabla u_i| h_\varepsilon'(|\nabla u_i|)\Big)\|D^2 u_i\| dx,\end{aligned} \tag{3.14}$$

where $\|D^2 u_i\|$ denotes the Hessian norm and C_i a positive constant.

We let $\varepsilon \to 0$. To this aim, let us first show that

- (i) $|\nabla u_i|^{p_i-2}\|D^2 u_i\| \in L^1(\mathcal{C}\cup\mathcal{C}^\lambda)$;
- (ii) $|\nabla u_i| h_\varepsilon'(|\nabla u_i|) \to 0$ a.e. in $\mathcal{C}\cup\mathcal{C}^\lambda$ as $\varepsilon \to 0$ and $|\nabla u_i| h_\varepsilon'(|\nabla u_i|) \leq C$ with C not depending on ε.

Let us prove (i). By Hölder's inequality it follows

$$\begin{aligned}&\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2}\|D^2 u_i\| dx \leq \sqrt{\mathcal{L}(\mathcal{C}\cup\mathcal{C}^\lambda)}\left(\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{2(p_i-2)}\|D^2 u_i\|^2 dx\right)^{\frac{1}{2}} \\ &\leq C_i\left(\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2-\beta_i}\|D^2 u_i\|^2 |\nabla u_i|^{p_i-2+\beta_i} dx\right)^{\frac{1}{2}} \\ &\leq C_i \|\nabla u_i\|_{L^\infty(\Omega)}^{(p_i-2+\beta_i)/2}\left(\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2-\beta_i}\|D^2 u_i\|^2 dx\right)^{\frac{1}{2}},\end{aligned} \tag{3.15}$$

with $0 \leq \beta_i < 1$ and C_i a positive constant.

Using (2.2) of Theorem 2.1, we infer that

$$\left(\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2-\beta_i}\|D^2u_i\|^2dx\right)^{\frac{1}{2}} \leq C.$$

Then, by (3.15) we obtain

$$\int_{\mathcal{C}\cup\mathcal{C}^\lambda} |\nabla u_i|^{p_i-2}\|D^2u_i\|dx \leq C.$$

Let us prove (ii). Recalling (3.6), we obtain

$$h'_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 < t \leq \varepsilon \\ \frac{2\varepsilon}{t^2} & \text{if } \varepsilon < t < 2\varepsilon \\ 0 & \text{if } t \geq 2\varepsilon, \end{cases}$$

and, then, $|\nabla u_i|h'_\varepsilon(|\nabla u_i|)$ tends to 0 almost everywhere in $\mathcal{C}\cup\mathcal{C}^\lambda$ as ε goes to 0 and $|\nabla u_i|h'_\varepsilon(|\nabla u_i|) \leq 2$.

Finally, by the Lebesgue's dominate convergence theorem, passing to the limit for $\varepsilon \to 0$ in (3.13) we obtain

$$0 \geq \tilde{C}_i \int_{\mathcal{C}\cup\mathcal{C}^\lambda} e^{-s_i(u_i)}dx > 0.$$

This gives a contradiction, hence the Claim holds.

Then, using also Hopf's boundary lemma (see [25, Theorem 5.5.1]) for

$$-\Delta_{p_i}u_i + a_i(u_i)|\nabla u_i|^{q_i} = f_i(u_1, u_2, \dots, u_i, \dots, u_m) \geq 0,$$

$u_i > 0$ in Ω and $u_i = 0$ on $\partial\Omega$, we deduce that the set $\Omega_{\bar\lambda}\backslash Z_{u_i}$ is connected. Indeed, thanks to Hopf's lemma, Z_{u_i} lies far from the boundary $\partial\Omega$. Moreover, we also remark that since Ω is convex in the x_1-direction, we have that the boundary $\partial\Omega$ is connected. Consequently, for any $i = 1, 2, \dots, m$ we get

$$u_i < u_{i,\bar\lambda} \tag{3.16}$$

in $\Omega_{\bar\lambda}\backslash Z_{u_i}$.

Consider now a compact set K in $\Omega_{\bar\lambda}$ such that $\mathcal{L}(\Omega_{\bar\lambda}\backslash K)$ is sufficiently small so that Proposition 2.5 can be applied. By what we proved before, for any $i \in \{1, \dots, m\}$, it holds that $u_i < u_{i,\bar\lambda}$ in $K\backslash A$, which is compact. Then, by (uniform) continuity, we find $\epsilon > 0$ such that, $\bar\lambda + \epsilon < 0$ and for $\bar\lambda < \lambda < \bar\lambda + \epsilon$ we have that $\mathcal{L}(\Omega_\lambda\backslash(K\backslash A))$ is small enough as before, and $u_{i,\lambda} - u_i > 0$ in $K\backslash A$ for any i. In particular, $u_{i,\lambda} - u_i > 0$ on $\partial(K\backslash A)$. Consequently, $u_i \leq u_{i,\lambda}$ on $\partial(\Omega_\lambda\backslash(K\backslash A))$. By Proposition 2.5 it follows $u_i \leq u_{i,\lambda}$ in $\Omega_\lambda\backslash(K\backslash A)$ and, consequently in Ω_λ, which contradicts the assumption $\bar\lambda < 0$. Therefore $\bar\lambda = 0$ and the thesis is proved. Finally, (1.2) follows by the monotonicity of the solution that is implicit in the moving plane method.

Finally, if Ω is a ball, repeating this argument along any direction, it follows that u_i, $i = 1, \dots, m$, are radially symmetric. The fact that $\dfrac{\partial u_i}{\partial r}(r) < 0$ for $r \neq 0$, follows by the Hopf's boundary lemma which works in this case since the level sets are balls and, therefore, fulfill the interior sphere condition.

Finally (1.3) follows by (1.2) using Theorem 2.3 (see the statement (2)) and the Dirichlet boundary condition of ($\mathcal{S}$). $\square$

Funding Open access funding provided by Università della Calabria within the CRUI-CARE Agreement.

References

1. Alexandrov, A.D.: A characteristic property of the spheres. Ann. Mat. Pura Appl. **58**, 303–354 (1962)
2. Azizieh, C., Clément, P., Mitidieri, E.: Existence and a priori estimates for positive solutions of p -Laplace systems. J. Differ. Equ. **184**(2), 422–442 (2002)
3. Berestycki, H., Nirenberg, L.: On the method of moving planes and the sliding method. Bull. Soc. Brasil. de Mat. Nova Ser. **22**(1), 1–37 (1991)
4. Busca, J., Sirakov, B.: Symmetry results for semilinear elliptic systems in the whole space. J. Differ. Equ. **163**(1), 41–56 (2000)
5. Clément, P., Fleckinger, J., Mitidieri, E., de Thélin, F.: Existence of positive solutions for a nonvariational quasilinear elliptic system. J. Differ. Equ. **166**(2), 455–477 (2000)
6. Clément, P., Manásevich, R., Mitidieri, E.: Positive solutions for a quasilinear system via blow up. Comm. Partial Differ. Equ. **18**(12), 2071–2106 (1993)
7. Damascelli, L.: Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. Ann. Inst. H. Poincaré Anal. Non Linéaire **15**(4), 493–516 (1998)
8. Damascelli, L., Pacella, F.: Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p < 2$, via the moving plane method. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26**(4), 689–707 (1998)
9. Damascelli, L., Sciunzi, B.: Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations. J. Differ. Equ. **206**(2), 483–515 (2004)
10. Dancer, E.N.: Moving plane methods for systems on half spaces. Math. Ann. **342**(2), 245–254 (2008)
11. De Figueiredo, D.G.: Monotonicity and symmetry of solutions of elliptic systems in general domains. NoDEA Nonlinear Differ. Equ. Appl. **1**(2), 119–123 (1994)
12. De Figueiredo, D.G., Yang, J.: Decay, symmetry and existence of solutions of semilinear elliptic systems. Nonlinear Anal. **33**(3), 211–234 (1998)
13. Esposito, F.: Symmetry and monotonicity properties of singular solutions to some cooperative semilinear elliptic systems involving critical nonlinearity. Discrete Contin. Dyn. Syst. **40**(1), 549–577 (2020)
14. Esposito, F., Montoro, L., Sciunzi, B.: Monotonicity and symmetry of singular solutions to quasilinear problems. J. Math. Pure Appl. **126**(9), 214–231 (2019)
15. Farina, A., Montoro, L., Riey, G., Sciunzi, B.: Monotonicity of solutions to quasilinear problems with a first-order term in half-spaces. Ann. Inst. H. Poincaré Anal. Non Linéaire **32**(1), 1–22 (2015)
16. Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry and related properties via the maximum principle. Comm. Math. Phys. **68**, 209–243 (1979)
17. Heinonen, J., Kilpeläinen, T., Martio, O.: Nonlinear Potential Theory of Degenerate Elliptic Equations. Oxford Mathematical Monographs. Clarendon Press, Oxford (1993)
18. Merchán, S., Montoro, L., Peral, I., Sciunzi, B.: Existence and qualitative properties of solutions to a quasilinear elliptic equation involving the Hardy–Leray potential. Ann. Inst. H. Poincaré Anal. Non Linéaire **31**(1), 1–22 (2014)
19. Merchán, S., Montoro, L., Sciunzi, B.: On the Harnack inequality for quasilinear elliptic equations with a first order term. Proc. Roy. Soc. Edinburgh Sect. A **148**(5), 1075–1095 (2018)

20. Mitidieri, E., Pokhozhaev, S.I.: Absence of global positive solutions of quasilinear elliptic inequalities. Dokl. Akad. Nauk **359**(4), 456–460 (1998)
21. Mitidieri, E., Pokhozhaev, S.I.: Absence of positive solutions for quasilinear elliptic problems in $\mathbb{R}^N$. Proc. Steklov Inst. Math. **227**(4), 186–216 (1999)
22. Montoro, L.: Harnack inequalities and qualitative properties for some quasilinear elliptic equations. NoDEA Nonlinear Differ. Equ. Appl. **26**(6), 33 (2019). (**Paper No. 45,**)
23. Montoro, L., Riey, G., Sciunzi, B.: Qualitative properties of positive solutions to systems of quasilinear elliptic equations. Adv. Differ. Equ. **20**(7–8), 717–740 (2015)
24. Montoro, L., Sciunzi, B., Squassina, M.: Symmetry results for nonvariational quasi-linear elliptic systems. Adv. Nonlinear Stud. **10**(4), 939–955 (2010)
25. Pucci, P., Serrin, J.: The Maximum Principle. Birkhauser, Boston (2007)
26. Reichel, W., Zou, H.: Non-existence results for semilinear cooperative elliptic systems via moving spheres. J. Differ. Equ. **161**(1), 219–243 (2000)
27. Serrin, J.: A symmetry problem in potential theory. Arch. Rational Mech. Anal. **43**, 304–318 (1971)
28. Troy, W.C.: Symmetry Properties in Systems of Semilinear Elliptic Equations. J. Differ. Equ. **42**, 400–413 (1981)