Annali di Matematica Pura ed Applicata (1923 -) (2019) 198:2195–2205
https://doi.org/10.1007/s10231-019-00862-9

Einstein and η-Einstein Sasakian submanifolds in spheres

Beniamino Cappelletti-Montano[1] · **Andrea Loi**[1]

Received: 24 March 2019 / Accepted: 23 April 2019 / Published online: 7 May 2019

Abstract

The aim of this paper is to study Sasakian immersions of compact Sasakian manifolds into the odd-dimensional sphere equipped with the standard Sasakian structure. We obtain a complete classification of such manifolds in the Einstein and η-Einstein cases when the codimension of the immersion is 4. Moreover, we exhibit infinite families of compact Sasakian η-Einstein manifolds which cannot admit a Sasakian immersion into any odd-dimensional sphere. Finally, we show that, after possibly performing a $\mathcal{D}$-homothetic deformation, a homogeneous Sasakian manifold can be Sasakian immersed into some odd-dimensional sphere if and only if S is regular and either S is simply connected or its fundamental group is finite cyclic.

Keywords Sasakian · Sasaki–Einstein · η-Einstein · Sasakian immersion · Kähler manifolds · Kähler immersions

Mathematics Subject Classification 53C25 · 53C55

1 Introduction

Sasakian manifolds were introduced by the foundational work of Sasaki [31] in 1960. A *contact metric manifold* is a contact connected manifold (S, η) admitting a Riemannian metric g compatible with the contact structure, in the sense that, defined the (1, 1)-tensor ϕ by $d\eta(X, Y) = 2g(X, \phi Y)$, the following conditions are fulfilled

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1}$$

The authors were supported by Prin 2015—Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis—Italy and by Fondazione di Sardegna and Regione Autonoma della Sardegna (Project GESTA). All the authors are members of INdAM-GNSAGA—Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni.

✉ Andrea Loi
loi@unica.it

Beniamino Cappelletti-Montano
b.cappellettimontano@unica.it

[1] Dipartimento di Matematica e Informatica, Università di Cagliari, Cagliari, Italy

where ξ denotes the *Reeb vector field* of the contact structure, that is, the unique vector field on S such that

$$i_\xi \eta = 1, \quad i_\xi d\eta = 0.$$

Moreover, a contact metric manifold is said to be *Sasakian* if the following integrability condition is satisfied

$$[\phi X, \phi Y] = -d\eta(X, Y)\xi, \tag{2}$$

for any vector fields X and Y on S. It follows from the definition that S must be of odd dimension, say $2n + 1$. Two Sasakian manifolds (S_1, η_1, g_1) and (S_2, η_2, g_2) are said to be *equivalent* if there exists a contactomorphism $F : S_1 \longrightarrow S_2$ between them which is also an isometry, i.e.,

$$F^*\eta_2 = \eta_1, \quad F^*g_2 = g_1. \tag{3}$$

One can prove that if (3) holds, then F satisfies also

$$F_{*_x} \circ \phi_1 = \phi_2 \circ F_{*_x}, \quad F_{*_x}\xi_1 = \xi_2$$

for any $x \in S_1$. An isometric contactomorphism $F : S \longrightarrow S$ from a Sasakian manifold (S, η, g) to itself will be called a *Sasakian transformation* of (S, η, g). A Sasakian manifold is *homogeneous* if it is acted upon transitively by its group of Sasakian transformations.

Sasakian geometry can be considered as the odd-dimensional counterpart of Kähler geometry. In fact in any contact manifold (S, η), one can consider the 1-dimensional foliation defined by the Reeb vector field. Actually one can prove that this foliation is transversely Kähler if and only if S is Sasakian. On the other hand, a Sasakian manifold can be also characterized as a Riemannian manifold (S, g) whose metric cone $(S \times \mathbb{R}^+, r^2 g + dr^2)$ is Kähler. In particular, one can prove that (S, g) is Sasaki–Einstein if and only if the corresponding Riemannian cone is Calabi–Yau. The classical example of Sasaki–Einstein manifold is given by the odd-dimensional sphere $\mathbb{S}^{2n+1}$ endowed with the usual Riemannian metric g_0 and the contact form induced by the form $x_1 dy_1 - y_1 dx_1 + \cdots + x_{n+1} dy_{n+1} - y_{n+1} dx_{n+1}$ on $\mathbb{R}^{2n+2}$. This is called the *standard Sasakian structure* of $\mathbb{S}^{2n+1}$. In all the paper, unless otherwise stated, whenever we speak of the $\mathbb{S}^{2n+1}$ as a Sasakian manifold, we are assuming that it is equipped with the standard Sasakian structure (η_0, g_0).

Sasaki–Einstein manifolds attracted the attention of several authors since it was pointed out their relation with string theory and the so-called Maldacena conjecture (see [25]). In this framework, Gauntlett, Martelli, Sparks and Waldram discovered the first known examples of irregular (see section below for the definition) Sasaki–Einstein metrics on $\mathbb{S}^2 \times \mathbb{S}^3$ [13]. We mention also the work of Boyer, Galicki and Kollár [8] on the existence of non-trivial Sasaki–Einstein metrics on the spheres and to the study by Martelli, Sparks and Yau on the relations between the critical points of Einstein–Hilbert action and Sasaki–Einstein manifolds [26].

Let us consider now the foliation defined by the Reeb vector field of a Sasakian manifold S. Using the theory of Riemannian submersions, one can show that the transverse geometry is Kähler–Einstein if and only if the Ricci tensor of S satisfies the following equality:

$$\mathrm{Ric} = \lambda g + \nu \eta \otimes \eta \tag{4}$$

for some constants λ and ν. Any Sasakian manifold satisfying (4) is said to be η-*Einstein*. Notice that in any η-Einstein Sasakian manifold the Einstein constants are related by

$$\lambda + \nu = 2n \tag{5}$$

(see, e.g., [7]). Another useful property of η-Einstein Sasakian manifolds is that, contrary to Sasaki–Einstein ones, they are preserved by $\mathcal{D}_a$-homothetic deformations, that is, the change of structure tensors of the form

$$\phi_a := \phi, \quad \xi_a := \frac{1}{a}\xi, \quad \eta_a := a\eta, \quad g_a := ag + a(a-1)\eta \otimes \eta \tag{6}$$

where $a > 0$. These transformations were first considered by Tanno in [33] and then used in several contexts. One proves (see [3] and [7]) that if (ϕ, ξ, η, g) is a Sasakian η-Einstein structure on S with Einstein constants (λ, ν), then, for any $a > 0$, the deformed structure $(\phi_a, \xi_a, \eta_a, g_a)$ is still a Sasakian η-Einstein structure with Einstein constants given by

$$\lambda_a = \frac{\lambda + 2 - 2a}{a}, \quad \nu_a = 2n - \frac{\lambda + 2 - 2a}{a}. \tag{7}$$

Combining (4) and (7) one sees that the $\mathcal{D}_a$-homothetic deformation with $a = \frac{\lambda+2}{2(1+n)}$ takes an η-Einstein Sasakian structure with $\lambda > -2$ into a Sasaki–Einstein one.

Examples of η-Einstein Sasakian manifolds with $\lambda > -2$ are provided by the tangent sphere bundle $T_1\mathbb{S}^m$ of any sphere $\mathbb{S}^m$ (see [36]). Thus, a suitable $\mathcal{D}_a$-homothetic deformation gives $T_1\mathbb{S}^m$ the structure of a homogeneous Sasaki–Einstein manifold. In particular, the standard homogeneous Sasaki–Einstein structure on $\mathbb{S}^2 \times \mathbb{S}^3 \simeq T_1\mathbb{S}^2$ can be obtained in this way.

In this paper, we study the Sasakian immersions of Sasakian manifolds into the odd-dimensional sphere. By a Sasakian immersion of a Sasakian manifold (S_1, η_1, g_1) into a Sasakian manifold (S_2, η_2, g_2), we mean an isometric immersion $\varphi : (S_1, g_1) \longrightarrow (S_2, g_2)$ that preserves the Sasakian structures, i.e., such that

$$\varphi^* g_2 = g_1, \quad \varphi^* \eta_2 = \eta_1, \tag{8}$$

$$\varphi_* \xi_1 = \xi_2, \quad \varphi_* \circ \phi_1 = \phi_2 \circ \varphi_*. \tag{9}$$

This definition was first considered in the early seventies, under different names, by Okumura [30], Harada [15–17], Kon [23,24], who mainly studied some geometric conditions ensuring the immersed manifold to be totally geodesic. However, despite the theory of Kähler immersions, which has widely developed in the last decades due to the fundamental work of Calabi (see [19] for an updated review of this topic), there are very few results about Sasakian immersions. Relapsing some conditions in (8)–(9), we can mention a recent, remarkable result of Ornea and Verbitsky [29]. Namely, they proved that a compact Sasakian manifold admits a CR-embedding (i.e., an embedding, non-necessarily isometric, satisfying (9)) into a Sasakian manifold diffeomorphic to a sphere. On the other hand, Takahashi [32] and Tanno [34] studied codimension one isometric immersions of a Sasakian manifold S in Riemannian manifolds of constant curvature, proving that, under some assumptions, S is of constant curvature 1.

As far as the knowledge of the authors, no general results concerning Sasakian immersions into the sphere are known. One of the aims of this paper is to start filling this gap. We start by the following two classification results (Theorem 1 and Theorem 2 and the corresponding corollaries), dealing with Sasaki–Einstein manifolds and Sasakian η-Einstein manifolds in small codimension, respectively.

Theorem 1 *Let S be a $(2n+1)$-dimensional compact Sasaki–Einstein manifold. Assume that there exists a Sasakian immersion of S into $\mathbb{S}^{2N+1}$ for some nonnegative integer N. Then S is Sasaki equivalent to $\mathbb{S}^{2n+1}$.*

A contact metric manifold is said to be *K-contact* if the Reeb vector field is Killing. In dimension greater than 3, this condition is weaker than the Sasakian condition. However, as proved by Boyer and Galicki [5] and in alternative way by Apostolov, Draghici and Moroianu [1], if the manifold is compact and Einstein, these two notions coincide. Using this fact and Theorem 1, we then obtain the following:

Corollary 1 *Let K be a $(2n+1)$-dimensional compact Einstein K-contact manifold. Assume that there exists a contact metric immersion of K into $\mathbb{S}^{2N+1}$ for some nonnegative integer N. Then K is Sasaki equivalent to $\mathbb{S}^{2n+1}$.*

In order to state Theorem 2, we recall the Boothby–Wang construction (see [4] and Sect. 2). To any regular and compact Sasakian manifold (S, η, g), we can associate a compact Hodge manifold M, namely a compact Kähler manifold with integral Kähler form ω (so M is projective algebraic by Kodaira's theorem) and a principal $\mathbb{S}^1$-bundle $\pi : S \to M$ with connection η such that $\pi^*\omega = ad\eta$, for a constant $a \neq 0$. The manifold M will be called *the Kähler manifold corresponding to S through the Boothby–Wang construction*. Notice that if (S, η_a, g_a), $a > 0$, is obtained by a regular Sasakian manifold (S, η, g) through a $\mathcal{D}_a$-homothetic deformation, then (S, η_a, g_a) is still regular and its corresponding Kähler manifold through the Boothby–Wang construction is the same as that of (S, η, g). Conversely, to any compact Hodge manifold M one can associate a regular compact Sasakian manifold (S, η, g) which is the total space of a principal $\mathbb{S}^1$-bundle over M and such that $\pi^*\omega = d\eta$. Also in this case the manifold S will be called *the Sasakian manifold corresponding to M through the Boothby–Wang construction*. If M is assumed to be simply connected, then S is unique up to Sasakian transformations and will be denoted by $S = \mathrm{BW}(M)$ and called the *Boothby–Wang manifold* corresponding to M (see Proposition 2 for a proof).

Theorem 2 *Let S be a $(2n+1)$-dimensional compact η-Einstein Sasakian manifold. Assume that there exists a Sasakian immersion of S into $\mathbb{S}^{2N+1}$. If $N = n + 2$, then S is Sasaki equivalent to $\mathbb{S}^{2n+1}$ or to $\mathrm{BW}(Q_n)$, where $Q_n \subset \mathbb{C}\mathrm{P}^{n+1}$ is the complex quadric equipped with the restriction of the Fubini-Study form of $\mathbb{C}\mathrm{P}^{n+1}$.*

Theorem 2 should be compared with part i) of the main theorem by Kenmotsu in [21], where the same conclusion is proved for $N = n + 1$ and when S is assumed to be complete and not necessarily compact. For general codimension, due to the corresponding conjecture in the Kähler case (see [19, Ch. 4]), we believe the validity of the following:

Conjecture *If a compact η-Einstein Sasakian manifold can be Sasakian immersed into a sphere, then S is Sasakian equivalent to $\mathrm{BW}(M)$ where M is a simply connected compact homogeneous Hodge manifold.*

The paper contains two further results (Theorem 3 and Theorem 4). In Theorem 3 (and its Corollary 2) we exhibit infinite families of examples of η-Einstein Sasakian structures on compact manifolds which cannot be induced by the Sasakian structure of the sphere. In Theorem 4 we prove that the sphere $\mathbb{S}^{2N+1}$ is, for a suitable N, the Sasakian manifold where all regular compact homogeneous Sasakian manifolds of the form $\mathrm{BW}(M)$ can be Sasakian immersed.

Theorem 3 *Let S be a compact regular Sasakian η-Einstein manifold of dimension $2n + 1$ with Einstein constant $\lambda < 2n$, according to the notation in (4). Then S cannot be Sasakian immersed into any sphere.*

Remark 1 It is worth pointing out that when $\lambda \leq -2$, by (7), a $\mathcal{D}_a$-homothetic deformation gives rise to an η-Einstein structure on S with Einstein constant $\lambda_a \leq -2$. Thus, by Theorem 3, any $\mathcal{D}_a$-homothetic deformation of an η-Einstein Sasakian manifold with $\lambda \leq -2$ cannot admit a Sasakian immersion into any sphere. On the other hand, a suitable $\mathcal{D}_a$-homothetic deformation of an η-Einstein Sasakian manifold with $-2 < \lambda < 2n$ gives rise to an η-Einstein Sasakian manifold with $\lambda_a \geq 2n$ (and vice versa). Notice also that the case $\lambda = 2n$ corresponds to the Einstein case, treated in Theorem 1.

Corollary 2 *Let M be either a K3 surface with the Calabi–Yau Kähler form, or the flat complex torus or a compact Riemann surface with the hyperbolic form*[1]*, and let S be a regular Sasakian manifold corresponding to M through the Boothby–Wang construction. Then S which cannot be Sasakian immersed into any sphere.*

Theorem 4 *Let S be a compact homogeneous Sasakian manifold. Then after possibly performing a $\mathcal{D}_a$-homothetic deformation, S admits a Sasakian immersion into $\mathbb{S}^{2N+1}$ if and only S is regular and either S is simply connected or its fundamental group is finite cyclic.*

The proof of Theorem 1 follows essentially by considering the induced Kähler immersion from the Calabi–Yau Kähler cone $C(S)$ of the Sasakian manifold S and the Kähler cone of $\mathbb{S}^{2N+1}$, namely $\mathbb{C}^{N+1} \setminus \{0\}$ and using a result of Umehara [38] which forces $C(S)$ to be flat. The proofs of Theorem 2 and Theorem 3, for $\lambda \leq -2$, are obtained by considering the induced Kähler immersions into the complex projective space obtained through the Boothby–Wang construction (see Proposition 1) and using some known results on Kähler immersions due to D. Hulin [18] and K. Tsukada [37], respectively. The case $-2 < \lambda < 2n$ in Theorem 3 is treated by the Gauss–Codazzi equations once one considers the induced map between the corresponding Kähler cones.

Finally, Theorem 4 is based on a lifting property (Proposition 3) and on the classification of Kähler immersions of compact homogeneous Kähler spaces due to the second author, Di Scala and Hishi [10].

The paper consists in two more sections. In Sect. 2 we prove Proposition 1, Proposition 2 and Proposition 3, while Sect. 3 is dedicated to the proofs of the main results, Theorems 1-4.

The authors would like to thank Gianluca Bande e Giovanni Placini for a careful reading of the paper.

2 Boothby–Wang fibrations and Sasakian immersions

The following result, due to Harada [17], will be one of the key ingredients in the proof of our main results.

Proposition 1 *Let $\varphi : S \rightarrow S'$ be a Sasakian immersion between two Sasakian manifolds S and S'. Assume that S and S' are compact, and S' is regular. Then, S is regular and there exists a Kähler immersion $i(\varphi) : M \rightarrow M'$ such that $i(\varphi) \circ \pi = \pi' \circ \varphi$, where $\pi : S \rightarrow M$ and $\pi' : S' \rightarrow M'$ are the Riemannian submersions given by Boothby–Wang construction.*

We also need to see if we can reverse the construction in Proposition 1 by lifting a Kähler immersion to a Sasakian one (see Proposition 3). So assume that M is a compact Hodge

---

[1] Let Σ_g be a compact Riemann surface of genus $g \geq 2$. One can realize Σ_g as the quotient D/Γ of the unit disk $D \subset \mathbb{C}$ where Γ is a Fuchsian subgroup $\Gamma \subset SU(1,1)$. The Kähler form $\omega_{hyp} = \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}$ is invariant by Γ so it defines an integral Kähler form on Σ_g, denoted by the same symbol ω_{hyp} and called the *hyperbolic form*.

manifold. As we have already pointed out in the introduction, there exists a compact regular Sasakian manifold (S, η, g) which has over it a $\mathbb{S}^1$-bundle $\pi : S \to M$ with connection η such that $d\eta = \pi^*\omega$. On the other hand, the integrality of ω implies the existence of a holomorphic (ample) line bundle $p : L \to M$ whose first Chern class is represented by the De Rham cohomology class of the integral Kähler form ω, namely $c_1(L) = [\omega]_{DR}$. Now, by a result of Ornea and Verbitsky [28] there exists a Hermitian metric h_* on the dual line bundle $p_* : L^* \to M$ such that the bundle $\pi : S \to M$ is the restriction of $p_* : L^* \to M$ to the subbundle consisting of unitary vectors of L^*, i.e., $S = \{v \in L^* \mid h_*(v, v) = 1\}$. The key point of their proof is that the cone $C(S)$ of S, viewed as a complex manifold, is equal to the set of nonzero vectors of L^*. Hence, the CR-structure on (S, η, g) and its $(1, 1)$-tensor ϕ is uniquely determined by the complex structure of L. Actually one can prove (see, e.g., [40, Section 2]) that h_* is the dual of the Hermitian metric h on L satisfying $\mathrm{Ric}(h) = \omega$, where $\mathrm{Ric}(h)$ is the 2-form on M whose local expression is given by

$$\mathrm{Ric}(h) = -\frac{i}{2\pi}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)) = \omega$$

for a trivializing holomorphic section $\sigma : U \to L\setminus\{0\}$ (here ∂ and $\bar{\partial}$ are the standard complex operator associated with the holomorphic structure of L). Moreover, the contact form η can be written in terms of h_* as follows (see, e.g., the first line of formula (8) p. 322 in [40]):

$$\eta = -i\partial h_{*|S} \tag{10}$$

Using these facts, we can prove the following uniqueness result.

Proposition 2 *Let M be a simply connected compact Hodge manifold with integral Kähler form ω. Let (S_j, η_j, g_j), $j = 1, 2$, be two Sasakian manifolds and assume there exist two principal $\mathbb{S}^1$-bundles $\pi_j : S_i \to M$ with connection η_j such that $d\eta_j = \pi_j^*\omega$, $j = 1, 2$. Then, (S_1, η_1, g_1) and (S_2, η_2, g_2) are Sasakian equivalent.*

Proof Let $p_j : L_j \to M$, $j = 1, 2$, be two holomorphic line bundles such that $c_1(L_j) = [\omega]_{DR}$. Since M is simply connected, these line bundles are holomorphically equivalent and so there exists a holomorphic diffeomorphism $\hat{F} : L_1^* \to L_2^*$ such that $p_2 \circ \hat{F} = p_1$. Let h_{j*}, $j = 1, 2$, be the Hermitian metric on L_j^* such that $S_j = \{v \in L_j^* \mid h_{j*}(v, v) = 1\}$. Since the dual Hermitian metric h_j on L_j, $j = 1, 2$, satisfies $\mathrm{Ric}(h_j) = \omega$ and M is compact, one easily gets $\hat{F}^*(h_{2*}) = \lambda h_{1*}$ for a positive constant λ. By denoting by F the restriction of $\hat{F}$ to S_1, one then gets a diffeomorphism $F : S_1 \to S_2$ such that $\pi_2 \circ F = \pi_1$ and, by the above-mentioned result of Ornea–Verbitsky, it preserves the tensors ϕ_j of S_j, namely

$$F_{*_x} \circ \phi_1 = \phi_2 \circ F_{*_x},$$

for all $x \in S_1$. Moreover, by (10), one gets

$$F^*\eta_2 = F^*(-i\partial h_{2*|S}) = -i\partial\hat{F}^*h_{2*|S} = -i\partial\lambda h_{1*|S} = -i\partial h_{1*|S} = \eta_1,$$

where we are denoting by the same symbol the ∂-operator of L_{j*}, $j = 1, 2$. The last two equations imply $F^*g_2 = g_1$ and we are done. □

When M is a simply connected compact Hodge manifold, we will denote by $\mathrm{BW}(M)$ the Sasakian manifold, which we call the Boothby–Wang manifold (unique up to Sasakian transformations by Proposition 2) such that there exists a principal $\mathbb{S}^1$-bundle $\pi : \mathrm{BW}(M) \to M$ whose connection form η satisfies $\pi^*\omega = d\eta$.

Example 5 When $M = \mathbb{C}P^n$ is the n-dimensional complex projective space and $\omega = \omega_{FS}$ is the Fubini-Study Kähler form[2], then $\mathrm{BW}(\mathbb{C}P^n) = \mathbb{S}^{2n+1}$ and the Boothby–Wang fibration $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$ is the Hopf fibration. Notice that in this case the line bundle L^* is the tautological line bundle over $\mathbb{C}\mathrm{P}^n$.

Remark 2 When (M, ω) is a compact but non-simply connected Kähler manifold, one could find an infinite family of non-equivalent regular Sasakian manifolds $(S, \eta) \to M$ which are the total space of a $\mathbb{S}^1$-bundle over M and satisfying $\pi^*\omega = d\eta$. This happens, for example, by taking $M = \Sigma_g$ a compact Riemann surface of genus $g \geq 2$ with the hyperbolic form ω_{hyp}. Indeed, there exists an infinite family of non-equivalent holomorphic line bundles over M whose first Chern class can be represented by ω_{hyp} (see, e.g., [14]), and thus, by Ornea–Verbitsky one gets an infinite family of non-equivalent regular Sasakian manifolds S which correspond to M through the Boothby–Wang construction. Notice that, by Corollary 2, none of these Sasakian manifolds can be Sasakian immersed into some sphere.

The following lifting result is the key ingredient in the proof of Theorem 4.

Proposition 3 *Let M and M' be simply connected compact Hodge manifolds, and let $(\mathrm{BW}(M), \eta, g)$ (resp. $(\mathrm{BW}(M'), \eta', g')$ be the corresponding Boothby–Wang manifolds. Given a Kähler immersion $i : M \to M'$, then there exists a Sasakian immersion $\varphi : \mathrm{BW}(M) \to \mathrm{BW}(M')$ such that $i \circ \pi = \pi' \circ \varphi$.*

Proof Consider the pull-back $\mathbb{S}^1$-bundle $\hat{B} \xrightarrow{\hat{\pi}} M$ induced by i, and let $\psi : \hat{B} \to \mathrm{BW}(M')$ be the bundle map (such that $\pi' \circ \psi = i \circ \hat{\pi}$). Since i is a Kähler immersion, it follows that $(\psi^*\eta', \psi^*g')$ is a Sasakian structure on $\hat{B}$ such that $\hat{\pi}^*\omega = d(\psi^*\eta')$. As M is simply connected, it follows by Proposition 2 that there exists a diffeomorphism $F : \mathrm{BW}(M) \to \hat{B}$ such that $F^*\psi^*\eta' = \eta$ and $F^*\psi^*g' = g$. Hence, $\varphi := \psi \circ F$ is the desired lifting. □

Example 6 It is interesting to construct explicit Sasakian immersions obtained as a lift of Kähler immersions. For example, if one considers the Segre embedding (which is a Kähler embedding)

$$i : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^3 : ([z_0, z_1], [w_0.w_1]) \mapsto [z_0w_0, z_0w_1, z_1w_0, z_1w_1],$$

then the map

$$\varphi : T_1\mathbb{S}^3 \cong \mathbb{S}^2 \times \mathbb{S}^3 \to \mathbb{S}^7 : ([z_0, z_1], \xi_0, \xi_1) \mapsto \frac{(\xi_0 z_0, \xi_0 z_1, \xi_1 z_0, \xi_1 z_1)}{\sqrt{|z_0|^2 + |z_1|^2}},$$

($\mathbb{S}^3 = \{(\xi_0, \xi_1) \in \mathbb{C}^2 \mid \xi_0^2 + \xi_1^2 = 1\}$ and $\mathbb{S}^2 = \mathbb{C}\mathrm{P}^1$), is a Sasakian immersion, where $T_1\mathbb{S}^3 \cong \mathbb{S}^2 \times \mathbb{S}^3$ is equipped with an η-Einstein Sasakian structure which can be also obtained as a $\mathcal{D}_a$-deformation of the standard homogeneous Sasaki–Einstein structure on $T_1\mathbb{S}^3 \cong \mathbb{S}^2 \times \mathbb{S}^3$ described in introduction (cf. [27] and Remark 3).

3 Proof of the main results

Proof of Theorem 1 Let $\varphi : S \longrightarrow \mathbb{S}^{2N+1}$ be a Sasakian immersion. Then, φ induces a Kähler immersion $\Phi = \varphi \times \mathrm{Id}_{\mathbb{R}^+} : C(S) \longrightarrow \mathbb{C}^{N+1} \setminus \{0\}$ between the corresponding Kähler cones.

---

[2] In homogeneous coordinates, the Fubini-Study form reads as $\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(|Z_0|^2 + \cdots + |Z_n|^2)$.

As already pointed out in introduction, it is well known (cf. [6]) that the Kähler cone $C(S)$ of a Sasaki–Einstein manifold is Calabi–Yau, i.e., the Kähler metric on $C(S)$ is Ricci flat. By a result of Umehara [38], a Ricci flat metric on a Kähler manifold which admits a Kähler immersion into $\mathbb{C}^N$ (equipped with the flat metric) is forced to be flat. Notice that the curvature tensors R and $\bar{R}$ of the Riemannian manifolds S and $C(S)$, respectively, are related by

$$\bar{R}\,(X, Y)\,Z = R\,(X, Y)\,Z + g(X, Z)Y - g(Y, Z)X$$

for any $X, Y \in \Gamma(TS)$ (see, for instance, [1]). Thus, being $C(S)$ flat, S becomes a manifold of constant curvature 1. By a result of Tanno [35], locally the Sasakian structure of S is isomorphic to the standard Sasakian structure of the $(2n+1)$-sphere. More precisely, being a complete Riemannian manifold of constant curvature 1, S is isometric to a quotient $\mathbb{S}^{2n+1}/\,\Gamma$ of a Euclidean sphere under a finite group of isometries [39]. We claim that Γ is the identity group, and so S is Sasakian equivalent to $\mathbb{S}^{2n+1}$. Indeed, let $\pi : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}/\,\Gamma$ be the universal covering map. Consider the Sasakian immersion $f = \varphi \circ \pi : \mathbb{S}^{2n+1} \to \mathbb{S}^{2N+1}$, and let $i : \mathbb{S}^{2n+1} \hookrightarrow \mathbb{S}^{2N+1}$ be the standard totally geodesic embedding. Then, $F = f \times \mathrm{Id}_{\mathbb{R}^+}$ and $I = i \times \mathrm{Id}_{\mathbb{R}^+}$ are two Kähler immersions from $\mathbb{C}^{n+1} \setminus \{0\}$ into $\mathbb{C}^{N+1} \setminus \{0\}$ (the latter is the natural inclusion). By the celebrated Calabi's rigidity theorem (see [9, Theorem 2]), there exists a unitary transformation U of $\mathbb{C}^{N+1}$ such that $U \circ F = I$. Therefore, F, and hence f, is forced to be injective. Thus, π is injective and Γ reduces to the identity group, proving our claim. □

Proof of Theorem 2 It follows by Proposition 1 and Example 5 that S is regular and if M denotes the complex n-dimensional compact Kähler manifold given by the Boothby–Wang construction, it admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$, with $N = n+2$. Since S is compact and η-Einstein, its base M is a compact Kähler–Einstein manifold (cf. [6]). By a result due to Tsukada [37], the codimension restriction forces M to be either the complex quadric $Q_n \subset \mathbb{C}\mathrm{P}^{n+1}$ or $\mathbb{C}\mathrm{P}^n$ which are both simply connected. Hence, the conclusion follows by Proposition 2.

Proof of Theorem 3 Let S be an η-Einstein Sasakian manifolds with Einstein constants (λ, ν) and assume by a contradiction that there exists a Sasakian immersion $\varphi : S \longrightarrow \mathbb{S}^{2N+1}$. We distinguish two cases: $-2 < \lambda < 2n$ and $\lambda \leq -2$. Let us first suppose $-2 < \lambda < 2n$. A straightforward computation shows that the Ricci tensor of the Riemannian cone $C(S)$ of S is given by

$$\mathrm{Ric}_{C(S)}\left(\frac{d}{dr}, \cdot\right) = 0, \quad \mathrm{Ric}_{C(S)}(X, Y) = -\nu\,(g(X, Y) - \eta(X)\eta(Y)) \tag{11}$$

for any $X, Y \in \Gamma(TS)$. Using (11), one can easily get a local basis on $C(S)$ with respect to which the Ricci tensor of $C(S)$ is represented by the following matrix

$$\mathrm{diag}(-\nu, \ldots, -\nu, 0, 0) \tag{12}$$

where the entry $-\nu$ is repeated $2n$ times. Now, our assumption that $-2 < \lambda < 2n$ together with (5) yields that $0 < \nu < 2 + 2n$. In particular, in view of (12), this implies that the Ricci tensor of the Kähler cone $C(S)$ is not negative semidefinite. On the other hand, as in the proof of Theorem 1, φ induces a Kähler immersion $\Phi : C(S) \longrightarrow \mathbb{C}^{N+1} \setminus \{0\}$ between the corresponding Kähler cones. Hence, by the Gauss–Codazzi equations (see, e.g., [22, Prop. 9.5, Ch. IX]) one deduces that the Ricci tensor of $C(S)$ is negative semidefinite, yielding the desired contradiction. Assume now that $\lambda \leq -2$, and let M be the Kähler manifold which corresponds to S through the Boothby–Wang construction. Using the O'Neill tensors of the theory of Riemannian submersions, one can easily prove that M is a compact Kähler–Einstein manifold with scalar curvature $2n(2+\lambda) \leq 0$. On the other hand, by Proposition 1 and Example 5, the existence of the Sasakian immersion $\varphi : S \longrightarrow \mathbb{S}^{2N+1}$ would give rise to a Kähler immersion from M into $\mathbb{C}\mathrm{P}^N$. But a result of Hulin [18] asserts that the scalar curvature of a projectively induced Kähler–Einstein metric must be strictly positive, in contrast with the inequality just proved. □

Proof of Theorem 4 In order to prove the theorem notice first that if S is a compact homogeneous Sasakian manifold then the compact Hodge manifold M corresponding to S through the Boothby–Wang construction is a compact homogeneous Kähler manifold. By a well-known result (see, e.g., [2, Theorem 8.97]), M is then the Kähler product of a flat complex torus and a simply connected compact homogeneous Kähler manifold, and hence, in particular, its fundamental group is either infinite or trivial.

Assume now that S admits a Sasakian immersion into a sphere $\mathbb{S}^{2N+1}$, for some N. Then, by Proposition 1 and Example 5, S is regular and M admits a Kähler immersion into the complex projective space $\mathbb{C}\mathrm{P}^N$. Thus, M is forced to be simply connected since the flat complex torus cannot admit a Kähler immersion into $\mathbb{C}\mathrm{P}^N$ (see, e.g., [10, Theorem 3]). Consider now the long exact sequence of homothopy groups associated with the Boothby–Wang fibration $\pi : S \to M$:

$$\cdots \to \pi_1(\mathbb{S}^1) \cong \mathbf{Z} \overset{\alpha}{\to} \pi_1(S) \overset{\beta}{\to} \pi_1(M) \to \pi_0(\mathbb{S}^1) = \{0\} \to \cdots \qquad (13)$$

The condition $\pi_1(M) = \{0\}$ implies that the map $\alpha : \mathbf{Z} \to \pi_1(S)$ is surjective. Thus, $\pi_1(S)$ is isomorphic to either $\{0\}$, $\mathbf{Z}$ or $\mathbf{Z}_m$ for some integer $m > 0$. The possibility $\pi_1(S) = \mathbf{Z}$ is excluded by the fact that the first Betti number of a compact Sasakian manifold must be even [12]. Then one implication of theorem follows.

Conversely, assume that S is a regular compact homogeneous Sasakian manifold whose fundamental group is either trivial or finite cyclic. Let M be the compact homogeneous Hodge manifold corresponding to S through the Boothby–Wang construction. By the long exact sequence (13) and the surjectivity of the map $\beta : \pi_1(S) \to \pi_1(M)$, we deduce that $\pi_1(M)$ is either trivial or finite cyclic. Therefore, M is forced to be simply connected since the fundamental group of a torus is not finite. Now, any simply connected homogeneous compact Hodge manifold admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$, for some N (see Theorem 1 in [20]). Thus, by Proposition 3, we can lift this Kähler immersion to a Sasakian immersion from $\mathrm{BW}(M)$ into $\mathbb{S}^{2N+1}$. Moreover, since M is simply connected, then, up to a $\mathcal{D}_a$-homothetic deformation, $S = \mathrm{BW}(M)$, and we are done.

Remark 3 To understand the necessity of a $\mathcal{D}_a$-homothetic deformation in Theorem 4, consider any compact simply connected homogeneous Hodge manifold M with an integral Kähler–Einstein form, which has necessarily strictly positive scalar curvature. By Theorem 1 in [20] M admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$, for some N, and then, by Proposition 3, its Boothby–Wang Sasakian manifold $\mathrm{BW}(M)$ admits a Sasakian immersion into $\mathbb{S}^{2N+1}$. Theorem 3 forces $\mathrm{BW}(M)$ to be η-Einstein with $\lambda > 2n$. Then, by a suitable $\mathcal{D}_a$-homothetic deformation of the Sasakian structure of $\mathrm{BW}(M)$ (cf. Remark 1) we get a compact and homogeneous η-Einstein Sasakian manifold S with $-2 < \lambda_a < 2n$ which, by Theorem 3, does not admit a Sasakian immersion into any sphere. (Example 6 is a particular case of this construction when $M = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$.)

We end this paper with an explicit example of compact homogeneous Sasakian manifold with finite cyclic group admitting a Sasakian immersion into the sphere.

Example 7 If m is a positive integer, then $\mathrm{BW}(\mathbb{C}P^n, m\omega_{FS})$ is the Sasakian manifold given by the lens space $\mathbb{S}^{2n+1}\ /\mathbf{Z}_m$ (for $m = 1$ one gets Example 5 while, for $m = 2$, one gets $SO(3)$ with the standard Sasakian structure). Indeed, one can show (see, e.g., [11, p. 908]) that the boundary of the disk bundle of the mth power $p : L^{*m} \to \mathbb{C}\mathrm{P}^n$ of the tautological bundle over $\mathbb{C}\mathrm{P}^n$ (cf. Example 5) is diffeomorphic to $\mathbb{S}^{2n+1}\ /\mathbf{Z}_m$ and by Ornea–Verbitsky [28] one gets that the restriction of p to $\mathbb{S}^{2n+1}\ /\mathbf{Z}_m$ is indeed the Boothby–Wang fibration. Now, since the fundamental group of $\mathbb{S}^{2n+1}\ /\mathbf{Z}_m$ is $\mathbf{Z}_m$, Theorem 4 yields a Sasakian immersion of $\mathbb{S}^{2n+1}\ /\mathbf{Z}_m$ into $\mathbb{S}^{2N+1}$, for some N. More precisely, this immersion is the lift of the Kähler immersion, $V_m : (\mathbb{C}P^n, m\omega_{FS}) \to (\mathbb{C}P^{\binom{n+m}{n}}, \omega_{FS})$ obtained by a suitable rescaling of the Veronese embedding (see [9, Theorem 13]) (hence, in this case, $N = 2\binom{n+m}{n} + 1$).

References

1. Apostolov, V., Draghici, T., Moroianu, A.: The odd-dimensional Goldberg conjecture. Math. Nachr. **279**, 948–952 (2006)
2. Besse, A.L.: Einstein manifolds, Reprint of the 1987 edition. Classics in Mathematics. Springer-Verlag, Berlin (2008)
3. Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds. Birkhäuser, Basel (2010)
4. Boothby, W.M., Wang, H.C.: On contact manifolds. Ann. Math. **68**, 721–734 (1958)
5. Boyer, C.P., Galicki, K.: Einstein manifolds and contact geometry. Proc. Am. Math. Soc. **129**, 2419–2430 (2001)
6. Boyer, C.P., Galicki, K.: Sasakian Geometry. Oxford Mathematical Monographs, Oxford University Press, Oxford (2008)
7. Boyer, C., Galicki, K., Matzeu, P.: η-Einstein Sasakian geometry. Commun. Math. Phys. **262**, 177–208 (2006)
8. Boyer, C.P., Galicki, K., Kollár, J.: Einstein metrics on spheres. Ann. Math. **162**, 557–580 (2005)
9. Calabi, E.: Isometric Imbedding of complex manifolds. Ann. Math. **58**, 1–23 (1953)
10. Di Scala, A.J., Ishi, H., Loi, A.: Kähler immersions of homogeneous Kähler manifolds into complex space forms. Asian J. Math. (3) **16**, 479–488 (2012)
11. Engliš, M., Zhang, G.: Ramadanov conjecture and line bundles over compact Hermitian symmetric spaces. Math. Z. **264**(4), 901–912 (2010)
12. Fujitani, T.: Complex-valued differential forms on normal contact Riemannian manifolds. Tôhoku Math. J. (2) **18**, 349–361 (1966)
13. Gauntlett, J.P., Martelli, D., Sparks, J., Waldram, D.: Sasaki–Einstein metrics on $S^2 \times S^3$. Adv. Theor. Math. Phys. **8**(4), 711–734 (2004)
14. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley, Hoboken (1978)
15. Harada, M.: Sasakian space forms immersed in Sasakian space forms. Bull. Tokyo Gakugey Univ. Ser. IV **24**, 7–11 (1972)
16. Harada, M.: On Sasakian submanifolds. Bull. Tokyo Gakugey Univ. Ser. IV **25**, 19–23 (1973)
17. Harada, M.: On Sasakian submanifolds. Tôhoku Math. J. **25**, 103–109 (1973)
18. Hulin, D.: Kähler–Einstein metrics and projective embeddings. J. Geom. Anal. **10**(3), 525–528 (2000)
19. Loi, A., Zedda, M.: Kähler Immersions of Kähler Manifolds into Complex Space Forms. Lecture Notes of the Unione Matematica Italiana, vol. 23. Springer (2018)
20. Loi, A., Zuddas, F.: On the Gromov width of Homogeneous Kähler manifolds. Differ. Geom. Appl. **47**, 130–132 (2016)
21. Kenmotsu, K.: Invariant submanifolds in a Sasakian manifold. Tôhoku Math. J. **21**, 495–500 (1969)
22. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. II. Interscience Pubishers, Geneva (1969)
23. Kon, M.: Invariant submanifolds of normal contact metric manifolds. Kōdai Math. Sem. Rep. **25**, 330–336 (1973)
24. Kon, M.: Invariant submanifolds in Sasakian manifolds. Math. Ann. **219**, 277–290 (1976)
25. Maldacena, J.: The large N limit of superconformal field theories and supergravity. Intt. J. Theor. Phys. **38**, 1113–1133 (1999)

26. Martelli, D., Sparks, J., Yau, S.-T.: Sasaki–Einstein manifolds and volume minimisation. Commun. Math. Phys. **280**, 611–673 (2008)
27. Ornea, L., Piccinni, P.: Induced Hopf Bundles and Einstein Metrics, New Developments in Differential Geometry, Budapest 1996, 295–305. Kluwer Acad. Publ, Dordrecht (1999)
28. Ornea, L., Verbitsky, M.: Sasakian structures on CR-manifolds. Geom. Dedicata **125**(1), 155–173 (2007)
29. Ornea, L., Verbitsky, M.: Embeddings of compact Sasakian manifolds. Math. Res. Lett. **14**(4), 703–710 (2007)
30. Okumura, M.: On contact metric immersion. Kōdai Math. Sem. Rep. **20**, 389–409 (1968)
31. Sasaki, S.: On differentiable manifolds with certain structures which are closely related to almost contact structure. Tôhoku Math. J. **2**, 459–476 (1960)
32. Takahashi, T.: Sasakian hypersurfaces in a space of constant curvature. Tôhoku Math. J. **21**, 644–653 (1969)
33. Tanno, S.: The topology of contact Riemannian manifolds. Ill. J. Math. **12**, 700–717 (1968)
34. Tanno, S.: Isometric immersions of Sasakian manifolds in spheres. Kōdai Math. Sem. Rep. **21**, 448–458 (1969)
35. Tanno, S.: Sasakian manifolds with constant ϕ-holomorphic sectional curvature. Tôhoku Math. J. **21**, 501–507 (1969)
36. Tanno, S.: Geodesic flows on C_L-manifolds and Einstein metrics on $S^2 \times S^3$. In: Minimal Submanifolds and Geodesics, pp. 283–292 (1979)
37. Tsukada, K.: Einstein–Kähler submanifolds with codimension two in a complex space form. Math. Ann. **274**, 503–516 (1986)
38. Umehara, M.: Einstein–Kähler submanifolds of complex linear or hyperbolic space. Tôhoku Math. J. **39**, 385–389 (1987)
39. Wolf, J.A.: Spaces of Constant Curvature, 6th edn. AMS Chelsea Publishing, Providence (2011)
40. Zelditch, S.: Szegö Kernels and a Theorem of Tian. Int. Math. Res. Not. **6**, 317–331 (1998)