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Inversion and extension of the finite Hilbert transform on (−1, 1)

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Abstract
The principle of optimizing inequalities, or their equivalent operator theoretic formulation, is well established in analysis. For an operator, this corresponds to extending its action to larger domains, hopefully to the largest possible such domain (i.e., its *optimal domain*). Some classical operators are already optimally defined (e.g., the Hilbert transform in $L^p(\mathbb{R})$, $1 < p < \infty$), and others are not (e.g., the Hausdorff–Young inequality in $L^p(\mathbb{T})$, $1 < p < 2$, or the Sobolev inequality in various spaces). In this paper, a detailed investigation is undertaken of the finite Hilbert transform T acting on rearrangement invariant spaces X on $(-1, 1)$, an operator whose singular kernel is neither positive nor does it possess any monotonicity properties. For a large class of such spaces X, it is shown that T is already optimally defined on X (this is known for $L^p(-1, 1)$ for all $1 < p < \infty$, except $p = 2$). The case $p = 2$ is significantly different because the range of T is a proper dense subspace of $L^2(-1, 1)$. Nevertheless, by a completely different approach, it is established that T is also optimally defined on $L^2(-1, 1)$. Our methods are also used to show that the solution of the airfoil equation, which is well known for the spaces $L^p(-1, 1)$ whenever $p \neq 2$ (due to certain properties of T), can also be extended to the class of r.i. spaces X considered in this paper.

Keywords Finite Hilbert transform · Rearrangement invariant space · Airfoil equation · Fredholm operator

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1 Introduction

For $1 \le p \le 2$, the Fourier transform F maps $L^p(\mathbb{T})$ into $\ell^{p'}(\mathbb{Z})$, with $\frac{1}{p} + \frac{1}{p'} = 1$. The Hausdorff–Young inequality $\|F(f)\|_{p'} \le \|f\|_p$ for $f \in L^p(\mathbb{T})$ ensures that F is continuous. The following question was raised by Edwards [14, p. 206], 50 years ago: *Given* $1 \le p \le 2$, *what can be said about the space* $\mathbf{F}^p(\mathbb{T})$ *consisting of those functions* $f \in L^1(\mathbb{T})$ *having the property that* $F(f\chi_A) \in \ell^{p'}(\mathbb{Z})$ *for all sets* A *in the Borel* σ*-algebra* $\mathcal{B}_{\mathbb{T}}$ *on* $\mathbb{T}$? A consideration of the functional

$$f \mapsto \sup_{A \in \mathcal{B}_{\mathbb{T}}} \|F(\chi_A f)\|_{p'}, \tag{1.1}$$

would be expected to be relevant in this regard. For $p = 2$, the operator $F\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is a Banach space *isomorphism*, which implies that $\mathbf{F}^2(\mathbb{T}) = L^2(\mathbb{T})$. What about the case $1 < p < 2$? It turns out that the functional (1.1) is a norm, that $\mathbf{F}^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$ is a Banach function space (briefly, B.f.s.) *properly* containing $L^p(\mathbb{T})$ and that $F\colon \mathbf{F}^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ is continuous. Moreover, $\mathbf{F}^p(\mathbb{T})$ is the *largest* such space in a certain sense. For the above facts we refer to [25]. The point is that the Hausdorff–Young inequality for functions in $L^p(\mathbb{T})$, $1 < p < 2$, *can* be extended to its *genuinely larger* optimal domain space $\mathbf{F}^p(\mathbb{T})$.

For many classical inequalities in analysis, or their equivalent operator theoretic formulation, an investigation along the lines of the Hausdorff–Young inequality alluded to above can be quite fruitful. One has a linear operator S defined on some B.f.s. $Z \subseteq L^0(\mu)$, with (Ω, Σ, μ) a measure space, taking values in a Banach space Y and a B.f.s. $X \subseteq Z$ such that $S\colon X \to Y$ is bounded. The above question posed by Edwards is also meaningful in this setting: What can be said about the space X_S consisting of those functions $f \in Z$ satisfying $S(f\chi_A) \in Y$ for all $A \in \Sigma$? In particular, is X_S genuinely larger than X? If so, can X_S be equipped with a function norm such that $X \subseteq X_S$ continuously and S has a Y-valued, continuous linear extension to X_S? And, of course, X_S should be the *largest* space with these properties. A few examples will illuminate this discussion.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $|\Omega| = 1$. The validity of the generalized Sobolev inequality $\|u^*\|_Y \le C\||\nabla u|^*\|_X$ for $u \in C_0^1(\Omega)$, where v^* is the decreasing rearrangement of a function v and X, Y are rearrangement invariant (briefly, r.i.) spaces on $[0, 1]$, is equivalent to the boundedness of the inclusion operator $j\colon W_0^1X(\Omega) \to Y(\Omega)$ for a suitable Sobolev space $W_0^1X(\Omega)$. By using a generalized Poincaré inequality, Cwikel and Pustylnik [10] and Edmunds et al. [13] showed that the boundedness of j is equivalent to the boundedness, from X into Y, of the one-dimensional operator S associated with Sobolev inequality, namely

$$(S(f))(t) := \int_t^1 f(s)s^{(1/n)-1}\mathrm{d}s, \quad t \in [0, 1],$$

which is generated by the kernel $K(t, s) := s^{(1/n)-1}\chi_{[t,1]}$ on $[0, 1] \times [0, 1]$. Accordingly, being able to extend the operator S is equivalent to extending the imbedding j and hence, to refining the generalized Sobolev inequality. The optimal extension of this kernel operator S is treated in [6,7]; whether or not the initial space becomes genuinely larger depends on properties of X and Y. A knowledge of the optimal domain of S has implications for the compactness of the Sobolev imbedding j [8,9].

For $0 < \alpha < 1$, the classical fractional integral operator in the spaces $L^p(0, 1)$, $1 \le p \le \infty$, has kernel (up to a constant) given by $K(t, s) = |s - t|^{\alpha-1}$. Its optimal extension has been investigated in [5]. For convolution (and more general Fourier multipliers) operators in $L^p(G)$, $1 \leq p < \infty$, with G a compact abelian group, see [24], [27, Ch.7] and the references therein. The optimal extension of the classical Hardy operator in $L^p(\mathbb{R})$, $1 < p < \infty$, with kernel $K(t, s) := (1/t)\chi_{[0,t]}(s)$ has been investigated in [11].

In this paper, we consider another classical singular integral operator. The Hilbert transform $H: L^p(\mathbb{R}) \to L^p(\mathbb{R})$, for $1 < p < \infty$ (whose boundedness is due to M. Riesz), is defined via convolution as a principal value integral; see, for example, [15, §6.7]. Since $H^2 = -I$, the operator H is a Banach space *isomorphism* on $L^p(\mathbb{R})$ for every $1 < p < \infty$ and so there is *no* larger B.f.s. which contains $L^p(\mathbb{R})$ and such that H has an $L^p(\mathbb{R})$-valued extension to this space. A related operator is the Hilbert transform $H_{2\pi}$ of 2π-periodic functions defined via the principal value integrals

$$(H_{2\pi}(f))(x) = p.v.\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\cot(u/2)\,\mathrm{d}u$$

for every measurable 2π-periodic function f and for every point $x \in [-\pi, \pi]$ for which the p.v.-integral exists. For each $1 < p < \infty$, the operator $H_{2\pi}$ is linear and continuous from $L^p(-\pi, \pi)$ into itself; denote this operator by $H^p_{2\pi}$. It is known that $H^p_{2\pi}$ has proper closed range, [3, Sect. 9.1]. Hence, $H^p_{2\pi}$ is surely not an isomorphism on $L^p(-\pi, \pi)$. Nevertheless, as for H, it turns out that there is *no* genuinely larger B.f.s. containing $L^p(-\pi, \pi)$ such that $H^p_{2\pi}$ has an $L^p(-\pi, \pi)$-valued extension to this space [27, Example 4.20].

The finite Hilbert transform $T(f)$ of $f \in L^1(-1, 1)$ is the principal value integral

$$(T(f))(t) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right)\frac{f(x)}{x-t}\,\mathrm{d}x,$$

which exists for a.e. $t \in (-1, 1)$ and is a measurable function. It is known to have important applications to aerodynamics, via the resolution of the so-called *airfoil equation,* [4], [18, Ch.11], [28,32,33]. More recently, the finite Hilbert transform has also found applications to problems arising in image reconstruction; see, for example, [17,29]. For each $1 < p < \infty$, the linear operator $f \mapsto T(f)$ maps $L^p(-1, 1)$ continuously into itself (denote this operator by T_p). Except when $p = 2$, the operator T_p behaves similarly, in some sense, to $H^p_{2\pi}$. Consequently, there is no larger B.f.s. containing $L^p(-1, 1)$ such that T_p has an $L^p(-1, 1)$-valued extension to this space, [27, Example 4.21]. However, for $p = 2$ the situation is significantly different, as already pointed out long ago in [30, p. 44]. One of the reasons is that the range of T_2 is a proper dense subspace of $L^2(-1, 1)$. The arguments used for T_p in the cases $1 < p < 2$ and $2 < p < \infty$ do not apply to T_2. Moreover, they fail to indicate whether or not T_2 has an $L^2(-1, 1)$-valued extension to a B.f.s. genuinely larger than $L^2(-1, 1)$. The atypical behavior of T when $p = 2$ has also been observed in [1], where T is considered to be acting in weighted L^p-spaces. Accordingly, the case $p = 2$ requires different arguments.

In this paper, we consider the inversion and the extension of the finite Hilbert transform T on function spaces on $(-1, 1)$. In Sect. 3, we extend known properties of T when it acts on the spaces $L^p(-1, 1)$, for $p \neq 2$, to a larger class of r.i. spaces X on $(-1, 1)$ satisfying certain restrictions on their Boyd indices, more precisely, that $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1/2$ or $1/2 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$; see Theorems 3.2 and 3.3. In particular, it is established that T is a Fredholm operator in such r.i. spaces. This allows a refinement of the solution of the airfoil equation by extending it to such r.i. spaces; see Corollary 3.5. In Sect. 4, we apply the results of the previous section to prove (cf. Theorem 4.7) the impossibility of extending the finite Hilbert transform when it acts on r.i. spaces X satisfying $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1/2$ or $1/2 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$. The proof relies on a deep result of Talagrand concerning L^0-valued measures. In the course of that investigation, we establish a rather unexpected characterization of when a function

$f \in L^1(-1,1)$ belongs to X in terms of the set of T-transforms $\{T(f\chi_A) : A \text{ measurable}\}$; see Proposition 4.2. In Sect. 5, we address the case $p = 2$. It is established (cf. Theorem 5.3), via a completely different approach, that $T\colon L^2(-1,1) \to L^2(-1,1)$ does not have a continuous $L^2(-1,1)$-valued extension to any larger B.f.s. The argument relies on showing that the norm

$$f \mapsto \sup_{|\theta|=1} \|T(\theta f)\|_2$$

(equivalent to (1.1) in the appropriate setting) is equivalent to the usual norm in $L^2(-1,1)$. We conclude Sect. 5 by extending the above mentioned characterization to show that $f \in L^2(-1,1)$ if and only if $T(f\chi_A) \in L^2(-1,1)$ for every measurable set $A \subseteq (-1,1)$; see Corollary 5.5.

Not all r.i. spaces X which T maps into itself (i.e., satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$) are covered. Except when $X = L^2(-1,1)$, for those r.i. spaces X not satisfying the conditions $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$ or $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ (e.g., the Lorentz spaces $L^{2,q}$ for $1 \le q \le \infty$ with $q \ne 2$) the techniques used here do not apply; see Remark 5.7.

2 Preliminaries

In this paper, the relevant measure space is $(-1,1)$ equipped with its Borel σ-algebra $\mathcal{B}$ and Lebesgue measure $|\cdot|$ (restricted to $\mathcal{B}$). We denote by $\operatorname{sim}\mathcal{B}$ the vector space of all $\mathbb{C}$-valued, $\mathcal{B}$-simple functions and by $L^0(-1,1) = L^0$ the space (of equivalence classes) of all $\mathbb{C}$-valued measurable functions, endowed with the topology of convergence in measure. The space $L^p(-1,1)$ is denoted simply by L^p, for $1 \le p \le \infty$.

A *Banach function space* (B.f.s.) X on $(-1,1)$ is a Banach space $X \subseteq L^0$ satisfying the ideal property, that is, $g \in X$ and $\|g\|_X \le \|f\|_X$ whenever $f \in X$ and $|g| \le |f|$ a.e. The *associate space* X' of X consists of all functions g satisfying $\int_{-1}^1 |fg| < \infty$, for every $f \in X$, equipped with the norm $\|g\|_{X'} := \sup\{|\int_{-1}^1 fg| : \|f\|_X \le 1\}$. The space X' is a closed subspace of the Banach space dual X^* of X. The second associate space X'' of X is defined as $X'' = (X')'$. The norm in X is absolutely continuous if, for every $f \in X$, we have $\|f\chi_A\|_X \to 0$ whenever $|A| \to 0$. The space X satisfies the Fatou property if, whenever $\{f_n\}_{n=1}^\infty \subseteq X$ satisfies $0 \le f_n \le f_{n+1} \uparrow f$ a.e. with $\sup_n \|f_n\|_X < \infty$, then $f \in X$ and $\|f_n\|_X \to \|f\|_X$.

A *rearrangement invariant* (r.i.) space X on $(-1,1)$ is a B.f.s. such that if $g^* \le f^*$ with $f \in X$, then $g \in X$ and $\|g\|_X \le \|f\|_X$. Here, $f^*\colon [0,2] \to [0,\infty]$ is the decreasing rearrangement of f, that is, the right continuous inverse of its distribution function: $\lambda \mapsto |\{t \in (-1,1) : |f(t)| > \lambda\}|$. The associate space X' of a r.i. space X is again a r.i. space. Every r.i. space on $(-1,1)$ satisfies $L^\infty \subseteq X \subseteq L^1$, [2, Corollary II.6.7]. Moreover, if $f \in X$ and $g \in X'$, then $fg \in L^1$ and $\|fg\|_{L^1} \le \|f\|_X \|g\|_{X'}$, i.e., Hölder's inequality is available. The fundamental function of X is defined by $\varphi_X(t) := \|\chi_A\|_X$ for $A \in \mathcal{B}$ with $|A| = t$, for $t \in [0,2]$.

In this paper, *all* B.f.s.' X (hence, all r.i. spaces) are on $(-1,1)$ relative to Lebesgue measure and, as in [2], satisfy the Fatou property. In this case, $X'' = X$ and hence, $f \in X$ if and only if $\int_{-1}^1 |fg| < \infty$, for every $g \in X'$. Moreover, X' is a norm-fundamental subspace of X^*, that is, $\|f\|_X = \sup_{\|g\|_{X'} \le 1} |\int_{-1}^1 fg|$ for $f \in X$, [2, pp.12-13]. If X is separable, then $X' = X^*$.

The family of r.i. spaces includes many classical spaces appearing in analysis, such as the Lorentz $L^{p,q}$ spaces, [2, Definition IV.4.1], Orlicz L^{φ} spaces [2, §4.8], Marcinkiewicz M_{φ} spaces, [2, Definition II.5.7], Lorentz Λ_{φ} spaces, [2, Definition II.5.12], and the Zygmund $L^p(\log \mathrm{L})^{\alpha}$ spaces, [2, Definition IV.6.11]. In particular, $L^p = L^{p,p}$, for $1 \le p \le \infty$. The space weak-L^1, denoted by $L^{1,\infty}(-1, 1) = L^{1,\infty}$, will play an important role; it is not a Banach space, [2, Definition IV.4.1]. It satisfies $L^1 \subseteq L^{1,\infty} \subseteq L^0$, with all inclusions continuous.

The dilation operator E_t for $t > 0$ is defined, for each $f \in X$, by $E_t(f)(s) := f(st)$ for $-1 \le st \le 1$ and zero in other cases. The operator $E_t \colon X \to X$ is bounded with $\|E_t\|_{X\to X} \le \max\{t, 1\}$. The *lower* and *upper Boyd indices* of X are defined, respectively, by

$$\underline{\alpha}_X := \sup_{0<t<1} \frac{\log \|E_{1/t}\|_{X\to X}}{\log t} \quad \text{and} \quad \overline{\alpha}_X := \inf_{1<t<\infty} \frac{\log \|E_{1/t}\|_{X\to X}}{\log t},$$

[2, Definition III.5.12]. They satisfy $0 \le \underline{\alpha}_X \le \overline{\alpha}_X \le 1$. Note that $\underline{\alpha}_{L^p} = \overline{\alpha}_{L^p} = 1/p$.

We recall a technical fact from the theory of r.i. spaces that will be often used; see, for example [21, Proposition 2.b.3].

Lemma 2.1 *Let X be a r.i. space such that $0 < \alpha < \underline{\alpha}_X \le \overline{\alpha}_X < \beta < 1$. Then, there exist p, q satisfying $1/\beta < p < q < 1/\alpha$ such that $L^q \subseteq X \subseteq L^p$ with continuous inclusions.*

An important role will be played by the Marcinkiewicz space $L^{2,\infty}(-1, 1) = L^{2,\infty}$, also known as weak-$L^2$, [2, Definition IV.4.1]. It consists of those $f \in L^0$ satisfying

$$f^*(t) \le \frac{M}{t^{1/2}}, \quad 0 < t \le 2, \tag{2.1}$$

for some constant $M > 0$. Consider the function $1/\sqrt{1-x^2}$ on $(-1, 1)$. Since its decreasing rearrangement $(1/\sqrt{1-x^2})^*$ is the function $t \mapsto 2/t^{1/2}$, it follows that $1/\sqrt{1-x^2}$ belongs to $L^{2,\infty}$. Actually, for any r.i. space X it is the case that $1/\sqrt{1-x^2} \in X$ if and only if $L^{2,\infty} \subseteq X$. Consequently, $L^{2,\infty}$ is the *smallest* r.i. space which contains $1/\sqrt{1-x^2}$. Note that $\underline{\alpha}_{L^{2,\infty}} = \overline{\alpha}_{L^{2,\infty}} = 1/2$.

For all of the above and further facts on r.i. spaces, see [2,21], for example.

3 Inversion of the finite Hilbert transform on r.i. spaces

In [18, Ch. 11], [26], [33, §4.3], a detailed study of the inversion of the finite Hilbert transform was undertaken for T acting on the spaces L^p whenever $1 < p < 2$ and $2 < p < \infty$. We study here the extension of those results to a larger class of spaces, namely the r.i. spaces. The restrictions on p indicated above for the L^p spaces can be formulated for r.i. spaces in terms of their Boyd indices, namely $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$ and $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$.

A result of Boyd [2, Theorem III.5.18] allows the extension of Riesz's classical theorem on the boundedness of the Hilbert transform H on the spaces $L^p(\mathbb{R})$, for $1 < p < \infty$, to a certain class of r.i. spaces. Indeed, since $Tf = \chi_{(-1,1)} H(f\chi_{(-1,1)})$, it follows for a r.i. space X with non-trivial lower and upper Boyd indices, that is, $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, that $T \colon X \to X$ boundedly; this is indicated by simply writing T_X. Since $\underline{\alpha}_{X'} = 1-\overline{\alpha}_X$ and $\overline{\alpha}_{X'} = 1-\underline{\alpha}_X$, the condition $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ implies that $0 < \underline{\alpha}_{X'} \le \overline{\alpha}_{X'} < 1$. Hence, $T_{X'} \colon X' \to X'$ is also bounded. The operator T is not continuous on L^1. However, due to a result of Kolmogorov [2, Theorem III.4.9(b)], $T \colon L^1 \to L^{1,\infty}$ is continuous. It follows from the Parseval formula in Proposition 3.1(b) that the restriction of the dual operator $T_X^*\colon X^* \to X^*$ of T_X to the closed subspace X' of X^* is precisely $-T_{X'}\colon X' \to X'$.

In the study of the operator T, an important role is played by the particular function $1/\sqrt{1-x^2}$, which belongs to each L^p, $1 \le p < 2$. The reason is that

$$T\Big(\frac{1}{\sqrt{1-x^2}}\Big)(t) = p.v.\frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}(x-t)}\,\mathrm{d}x = 0, \quad -1 < t < 1, \tag{3.1}$$

and, moreover, that if $T(f)(t) = 0$ for a.e. $t \in (-1,1)$ with f a function belonging to some space L^p, $1 < p < \infty$, then necessarily $f(x) = C/\sqrt{1-x^2}$ for some constant $C \in \mathbb{C}$, [33, §4.3 (14)]. Combining this observation with Lemma 2.1, it follows for every r.i. space X satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, that T_X is either injective or $\dim(\mathrm{Ker}(T_X)) = 1$. Recall that $L^{2,\infty}$ is the *smallest* r.i. space containing the function $1/\sqrt{1-x^2}$, that is, $1/\sqrt{1-x^2} \in X$ if and only if $L^{2,\infty} \subseteq X$.

The Parseval and Poincaré-Bertrand formulae are important tools for studying the finite Hilbert transform in the spaces L^p, $1 < p < \infty$ [33, §4.3]. It should be noted that a result of Love is essential in order to have a sharp version of the Poincaré–Bertrand formula [22]. The validity of both of these formulae can be extended to the setting of r.i. spaces.

Proposition 3.1 *Let X be a r.i. space satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$.*

(a) *Let $f \in L^1$ satisfy $fT_{X'}(g) \in L^1$ for all $g \in X'$. Then, for every $g \in X'$, the function $gT(f) \in L^1$ and*

$$\int_{-1}^{1} fT_{X'}(g) = -\int_{-1}^{1} gT(f).$$

(b) *The Parseval formula holds for the pair X and X', that is,*

$$\int_{-1}^{1} fT_{X'}(g) = -\int_{-1}^{1} gT_X(f), \quad f \in X, g \in X'.$$

(c) *The Poincaré–Bertrand formula holds for the pair X and X'; that is, for all $f \in X$ and $g \in X'$, we have*

$$T(gT_X(f) + fT_{X'}(g)) = (T_X(f))(T_{X'}(g)) - fg, \quad a.e.$$

Proof (a) Assume first that $f \in L^\infty$. By Lemma 2.1, there exists $1 < q < \infty$ satisfying $L^q \subseteq X$, so that $X' \subseteq L^{q'}$. Then,

$$\int_{-1}^{1} fT_{X'}(g) = -\int_{-1}^{1} gT_X(f) = -\int_{-1}^{1} gT(f), \quad g \in X',$$

via the Parseval formula for the pair L^q and $L^{q'}$ [18, Sect. 11.10.8], [33, Sect. 4.2, 4.3], because $f \in L^\infty \subseteq L^q$ and $g \in X' \subseteq L^{q'}$.

Now let $f \in L^1$ be a general function satisfying the assumption of (a). Define $A_n := |f|^{-1}([0,n])$ and $f_n := f\chi_{A_n} \in L^\infty$ for $n \in \mathbb{N}$. Then, $\lim_n f_n = f$ in L^1. It follows from Kolmogorov's theorem that $\lim_n T(f_n) = T(f)$ in $L^{1,\infty}$. Since the inclusion $L^{1,\infty} \subseteq L^0$ is continuous, we can conclude that $\lim_n T(f_n) = T(f)$ in measure. Accordingly, by passing to a subsequence if necessary, we may assume that $\lim_n T_X(f_n) = \lim_n T(f_n) = T(f)$ pointwise a.e.

Fix $g \in X'$. Given any $A \in \mathcal{B}$, the dominated convergence theorem ensures that

$$\lim_n f_n T_{X'}(g\chi_A) = fT_{X'}(g\chi_A), \quad \text{in } L^1, \tag{3.2}$$

as $|f_n T_{X'}(g\chi_A)| \le |f T_{X'}(g\chi_A)|$ pointwise for $n \in \mathbb{N}$ and because $f T_{X'}(g\chi_A) \in L^1$ by assumption. For each $n \in \mathbb{N}$, the first part of this proof applied to $f_n \in L^\infty \subseteq X$ yields $\int_{-1}^1 f_n T_{X'}(g\chi_A) = -\int_{-1}^1 (g\chi_A) T_X(f_n)$. It follows from (3.2) that

$$\lim_n \int_A g T_X(f_n) = \lim_n \int_{-1}^1 (g\chi_A) T_X(f_n)$$
$$= -\lim_n \int_{-1}^1 f_n T_{X'}(g\chi_A) = -\int_{-1}^1 f T_{X'}(g\chi_A).$$

Since this holds for all sets $A \in \mathcal{B}$ and since $\lim_n g T_X(f_n) = gT(f)$ pointwise a.e., we can conclude that both $gT(f) \in L^1$ and

$$\lim_n g T_X(f_n) = gT(f), \quad \text{in } L^1; \tag{3.3}$$

see, for example, [20, Lemma 2.3]. This and (3.2) with $A := (-1, 1)$ ensure that $\int_{-1}^1 f T_{X'}(g) = -\int_{-1}^1 g T(f)$. So, (a) is established.

(b) Given any $f \in X$ and $g \in X'$, Hölder's inequality ensures that $f T_{X'}(g) \in L^1$. So, part (b) follows from (a).

(c) Fix $f \in X$ and $g \in X'$. The proof of part (a) shows that there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq L^\infty \subseteq X$ satisfying the conditions:

(i) $\lim_n f_n = f$ and $\lim_n T_X(f_n) = T_X(f)$ pointwise a.e., as well as
(ii) $\lim_n f_n T_{X'}(g) = f T_{X'}(g)$ in L^1 and $\lim_n g T_X(f_n) = g T_X(f)$ in L^1;

see (3.2) with $A := (-1, 1)$ and (3.3), respectively. Condition (ii) implies that

$$\lim_n T(g T_X(f_n) + f_n T_{X'}(g)) = T(g T_X(f) + f T_{X'}(g)) \tag{3.4}$$

in $L^{1,\infty}$ (via Kolmogorov's theorem) and hence, in L^0. On the other hand, condition (i) implies that

$$\lim_n \big((T_X(f_n))(T_{X'}(g)) - f_n g\big) = (T_X(f))(T_{X'}(g)) - fg \tag{3.5}$$

pointwise a.e. As in the proof of part (a), select $1 < q < \infty$ such that $L^q \subseteq X$. Since $f_n \in L^\infty \subseteq L^q$ for $n \in \mathbb{N}$ and $g \in X' \subseteq L^{q'}$, the Poincaré–Bertrand formula for the pair L^q and $L^{q'}$ gives, for each $n \in \mathbb{N}$, that

$$T(g T_X(f_n) + f_n T_{X'}(g)) = (T_X(f_n))(T_{X'}(g)) - f_n g, \quad \text{a.e.,} \tag{3.6}$$

with the identities holding outside a null set which is independent of $n \in \mathbb{N}$. In view of (3.4) and (3.5), take the limit of both sides of (3.6) in L^0 to obtain the identity $T(g T_X(f) + f T_{X'}(g)) = (T_X(f))(T_{X'}(g)) - fg$ in L^0. This is precisely the Poincaré–Bertrand formula for $f \in X$ and $g \in X'$. □

We can now extend certain results obtained in [26], [33, §4.3] for the spaces L^p with $1 < p < 2$ to the larger family of r.i. spaces satisfying $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$.

For each $f \in X$, define pointwise the measurable function

$$(\widehat{T}_X(f))(x) := \frac{-1}{\sqrt{1-x^2}} T_X(\sqrt{1-t^2} f(t))(x), \quad \text{a.e. } x \in (-1, 1). \tag{3.7}$$

Theorem 3.2 *Let X be a r.i. space satisfying* $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$.

(a) $\mathrm{Ker}(T_X)$ *is the one-dimensional subspace of* X *spanned by the function* $1/\sqrt{1-x^2}$.
(b) *The linear operator* $\widehat{T}_X$ *defined by* (3.7) *maps* X *boundedly into* X *and satisfies* $T_X\widehat{T}_X = I_X$ *(the identity operator on* X*). Moreover,*

$$\int_{-1}^{1} (\widehat{T}_X(f))(x)\, dx = 0, \quad f \in X. \tag{3.8}$$

(c) *The operator* $T_X : X \to X$ *is surjective.*
(d) *The identity* $\widehat{T}_X T_X = I_X - P_X$ *holds, with* P_X *the bounded projection given by*

$$f \mapsto P_X(f) := \left(\frac{1}{\pi}\int_{-1}^{1} f(t)\, dt\right)\frac{1}{\sqrt{1-x^2}}, \quad f \in X. \tag{3.9}$$

(e) *The operator* $\widehat{T}_X$ *is an isomorphism onto its range* $R(\widehat{T}_X)$*. Moreover,*

$$R(\widehat{T}_X) = \left\{ f \in X : \int_{-1}^{1} f(x)dx = 0 \right\}. \tag{3.10}$$

(f) *The following decomposition of* X *holds (with* $\langle\cdot\rangle$ *denoting linear span):*

$$X = \left\{ f \in X : \int_{-1}^{1} f(x)dx = 0 \right\} \oplus \left\langle \frac{1}{\sqrt{1-x^2}} \right\rangle = R(\widehat{T}_X) \oplus \left\langle \frac{1}{\sqrt{1-x^2}} \right\rangle. \tag{3.11}$$

Proof (a) Since $1/2 < \underline{\alpha}_X$ we have $L^{2,\infty} \subseteq X$ and so $1/\sqrt{1-x^2} \in X$. Accordingly, $\langle\frac{1}{\sqrt{1-x^2}}\rangle \subseteq \mathrm{Ker}(T_X)$. Conversely, let $f \in \mathrm{Ker}(T_X)$. By Lemma 2.1, there is $1 < p < 2$ such that $f \in L^p$. As noted prior to Proposition 3.1, this implies that $f(x) = c/\sqrt{1-x^2}$ for some $c \in \mathbb{C}$.

(b) Via Lemma 2.1, there exist $1 < p < q < 2$ such that $1/q < \underline{\alpha}_X \le \overline{\alpha}_X < 1/p$ and $L^q \subseteq X \subseteq L^p$. Consider the weight function $\rho(x) := 1/\sqrt{1-x^2}$ on $(-1, 1)$. Appealing to results on boundedness of the Hilbert transform on weighted L^p spaces, T is bounded from the weighted space $L^p((-1, 1), \rho)$ into itself and from the weighted space $L^q((-1, 1), \rho)$ into itself, [16, Ch. 1, Theorem 4.1]. This is equivalent to the fact that

$$f \mapsto \widehat{T}(f) := \frac{-1}{\sqrt{1-x^2}} T_X\big(\sqrt{1-x^2} f(x)\big),$$

is well defined on L^p and bounded as an operator from L^p into L^p and from L^q into L^q. The condition on the indices $1/q < \underline{\alpha}_X \le \overline{\alpha}_X < 1/p$ allows us to apply Boyd's interpolation theorem [21, Theorem 2.b.11], to conclude that $\widehat{T}$ maps X boundedly into X. According to (3.7), note that $\widehat{T}_X$ is the operator $\widehat{T} : X \to X$.

To establish $T_X\widehat{T}_X = I_X$, choose $1 < p < 2$ such that $X \subseteq L^p$. It follows from (2.7) on p. 46 of [26] that $T_{L^p}\widehat{T}_{L^p} = I_{L^p}$. Let $f \in X \subseteq L^p$. Since all three operators T_X, $\widehat{T}_X$ and I_X map X into X, it follows that $T_X(\widehat{T}_X(f)) = f = I_X(f)$.

To establish (3.8), let $f \in X \subseteq L^p$, with $1 < p < 2$ as above. Then, (3.8) follows from the validity of (3.8) in L^p; see (2.6) on p. 46 of [26].

(c) Follows immediately from $T_X\widehat{T}_X = I_X$.

(d) Since $(1/\pi)\int_{-1}^{1} \mathrm{d}x/\sqrt{1-x^2} = 1$, it follows that P_X as given in (3.9) is indeed a linear projection from X onto the one-dimensional subspace $\langle\frac{1}{\sqrt{1-x^2}}\rangle \subseteq X$. The boundedness of P_X is a consequence of Hölder's inequality (applied to $f = \mathbf{1} \cdot f$ with $\mathbf{1} \in X'$ and $f \in X$ fixed), namely

$$\|P_X(f)\|_X \le \frac{1}{\pi}\left\|\frac{1}{\sqrt{1-x^2}}\right\|_X \|\mathbf{1}\|_{X'}\|f\|_X.$$

To verify that $P_X = I_X - \widehat{T}_X T_X$, fix $f \in X$. Then, $T_X \widehat{T}_X = I_X$ implies the identity $T_X(I_X - \widehat{T}_X T_X)(f) = 0$, that is,

$$(I_X - \widehat{T}_X T_X)(f) \in \mathrm{Ker}(T_X).$$

According to part (a), there exists $c \in \mathbb{C}$ such that

$$(I_X - \widehat{T}_X T_X)(f) = \frac{c}{\sqrt{1-x^2}}. \tag{3.12}$$

But, $\int_{-1}^{1} \widehat{T}_X(T_X(f))(x)\,\mathrm{d}x = 0$ (by (3.8)) and so (3.12) implies that

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = c\int_{-1}^{1} \mathrm{d}x/\sqrt{1-x^2} = c\pi,$$

that is, $c = (1/\pi)\int_{-1}^{1} f(x)\,\mathrm{d}x$. So, again by (3.12), we have established that $(I_X - \widehat{T}_X T_X)(f) = P_X(f)$. Since $f \in X$ is arbitrary, it follows that $I_X - \widehat{T}_X T_X = P_X$.

(e) The identity $T_X \widehat{T}_X = I_X$ implies that $\widehat{T}_X$ is injective. So, $\widehat{T}_X : X \to R(\widehat{T})$ is a linear bijection.

To verify (3.10), suppose $f \in X$ satisfies $\int_{-1}^{1} f(x)\,\mathrm{d}x = 0$, i.e., $P_X(f) = 0$. Then, the identity $\widehat{T}_X T_X = I_X - P_X$ shows that $f = \widehat{T}_X(h)$ with $h := T_X(f) \in X$, i.e., $f \in R(\widehat{T}_X)$. Conversely, suppose that $f = \widehat{T}_X(g) \in R(\widehat{T}_X)$ for some $g \in X$. Then, $g = T_X(f)$ as $T_X \widehat{T}_X = I_X$. Accordingly,

$$f = \widehat{T}_X(g) = \widehat{T}_X T_X(f) = I_X(f) - P_X(f) = f - P_X(f)$$

and so $P_X(f) = 0$. It is then clear from (3.9) that $\int_{-1}^{1} f(x)\,\mathrm{d}x = 0$, i.e., f belongs to the right-side of (3.10). This establishes (3.10).

Since the linear functional $f \mapsto \varphi_1(f) := \int_{-1}^{1} f(x)\,\mathrm{d}x$, for $f \in X$, belongs to X^*, as $\mathbf{1} \in X' \subseteq X^*$, it follows via (3.10) that $R(\widehat{T}_X) = \mathrm{Ker}(\varphi_1)$ and hence, $R(\widehat{T}_X)$ is a *closed* subspace of X. Accordingly, $\widehat{T}_X : X \to R(\widehat{T}_X)$ is a Banach space isomorphism.

(f) The identity $\widehat{T}_X T_X + P_X = I_X$ shows that each $f \in X$ has the form $f = \widehat{T}_X(T_X(f)) + P_X(f)$ with $\widehat{T}_X(T_X(f)) \in R(\widehat{T}_X)$ and, via (3.9), $P_X(f) \in \langle 1/\sqrt{1-x^2}\rangle$. So, it remains to show that the decomposition in (3.11) is a *direct sum*. To this effect, let $h \in R(\widehat{T}_X) \cap \langle 1/\sqrt{1-x^2}\rangle$, in which case $h = \widehat{T}_X(f)$ for some $f \in X$ and $h = c/\sqrt{1-x^2}$ for some $c \in \mathbb{C}$, that is, $\widehat{T}_X(f) = c/\sqrt{1-x^2}$. Integrating both sides of this identity over $(-1, 1)$ and appealing to (3.8) show that $c = 0$. Hence, $h = 0$. □

Next, we extend certain results obtained in [26], [33, §4.3], for the spaces L^p with $2 < p < \infty$, to the larger family of r.i. spaces X satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$. Then, $1/2 < \underline{\alpha}_{X'} \le \overline{\alpha}_{X'} < 1$ and so $1/\sqrt{1-x^2} \in X'$. Hence, for every $f \in X$, the function $f(x)/\sqrt{1-x^2} \in L^1$. Accordingly, we can define pointwise the measurable function

$$(\check{T}_X(f))(x) := -\sqrt{1-x^2}\, T\Big(\frac{f(t)}{\sqrt{1-t^2}}\Big)(x), \quad \text{a.e. } x \in (-1, 1). \tag{3.13}$$

Theorem 3.3 *Let X be a r.i. space satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$.*

(a) *The operator $T_X : X \to X$ is injective.*

(b) *The linear operator* $\check{T}_X$ *defined by* (3.13) *is bounded from* X *into* X *and satisfies* $\check{T}_X T_X = I_X$.
(c) *The identity* $T_X \check{T}_X = I_X - Q_X$ *holds, with* Q_X *the bounded projection given by*

$$f \in X \mapsto Q_X(f) := \left(\frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} \, dx \right) \mathbf{1}. \tag{3.14}$$

(d) *The range of* T_X *is the closed subspace of* X *given by*

$$R(T_X) = \left\{ f \in X : \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = 0 \right\} = \mathrm{Ker}(Q_X). \tag{3.15}$$

Moreover, $\check{T}_X$ *is an isomorphism from* $R(T_X)$ *onto* X.
(e) *The following decomposition of* X *holds:*

$$X = \left\{ f \in X : \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = 0 \right\} \oplus \langle \mathbf{1} \rangle = R(T_X) \oplus \langle \mathbf{1} \rangle. \tag{3.16}$$

Proof (a) Since $\overline{\alpha}_X < 1/2$, we have that $X \subsetneqq L^{2,\infty}$ and so $1/\sqrt{1-x^2} \notin X$. Hence, T_X is injective; see the discussion after (3.1).

(b) Via Lemma 2.1 there exist $2 < p < q < \infty$ such that $1/q < \underline{\alpha}_X \le \overline{\alpha}_X < 1/p$ and $L^q \subseteq X \subseteq L^p$. Consider the weight function $\rho(x) := \sqrt{1-x^2}$ on $(-1, 1)$. Appealing again to results on boundedness of the Hilbert transform on weighted L^p spaces, T is bounded from the weighted space $L^p((-1,1),\rho)$ into itself and from the weighted space $L^q((-1,1),\rho)$ into itself, [16, Ch. 1 Theorem 4.1]. This is equivalent to the fact that

$$f \mapsto \check{T}(f) := -\sqrt{1-x^2}\, T\Big(\frac{f(x)}{\sqrt{1-x^2}}\Big),$$

is well defined on L^p and bounded as an operator from L^p into L^p and from L^q into L^q. The condition on the indices $1/q < \underline{\alpha}_X \le \overline{\alpha}_X < 1/p$ allows us to apply Boyd's interpolation theorem [21, Theorem 2.b.11], to deduce that $\check{T}$ maps X boundedly into X. According to (3.13), note that $\check{T}_X$ is the operator $\check{T} \colon X \to X$.

To establish $\check{T}_X T_X = I_X$, recall that $X \subseteq L^p$. It follows from (2.10) on p. 48 of [26] that $\check{T}_{L^p} T_{L^p} = I_{L^p}$. Let $f \in X \subseteq L^p$. Since all three operators T_X, $\check{T}_X$ and I_X map X into X, it follows that $\check{T}_X(T_X(f)) = f = I_X(f)$.

(c) It is routine to check that Q_X is a linear projection onto the one-dimensional space $\langle \mathbf{1} \rangle$. Since $g(x) = 1/\sqrt{1-x^2} \in X'$, the boundedness of Q_X follows from (3.14) via Hölder's inequality, namely

$$\|Q_X(f)\|_X \le \frac{1}{\pi} \|g\|_{X'} \|\mathbf{1}\|_X \|f\|_X, \quad f \in X.$$

To establish the identity $T_X \check{T}_X = I_X - Q_X$, choose $2 < p < \infty$ such that $X \subseteq L^p$. It follows from (2.11) on p. 48 of [26] that $T_{L^p} \check{T}_{L^p} = I_{L^p} - Q_{L^p}$. Let $f \in X \subseteq L^p$. Since all four operators T_X, $\check{T}_X$, Q_X and I_X map X into X, it follows that $T_X(\check{T}_X(f)) = f - Q_X(f) = (I_X - Q_X)(f)$.

(d) Using the identities $\check{T}_X T_X = I_X$ and $T_X \check{T}_X = I_X - Q_X$, one can argue as on p. 48 of [26] to verify the identity (3.15). In particular, since Q_X is bounded, it follows that $R(T_X) = \mathrm{Ker}(Q_X)$ is a *closed* subspace of X. It is clear from $\check{T}_X T_X = I_X$ that $\check{T}_X$ maps $R(T_X)$ onto X and also that $\check{T}_X$ restricted to $R(T_X)$ is injective, i.e., $\check{T}_X \colon R(T_X) \to X$ is a linear bijection and bounded. By the open mapping theorem, $\check{T}_X \colon R(T_X) \to X$ is actually a Banach space isomorphism.

(e) As Q_X is a bounded projection, we have $X = \mathrm{Ker}(Q_X) \oplus R(Q_X)$. But, $\mathrm{Ker}(Q_X) = R(T_X)$ by part (d) and $R(Q_X) = \langle \mathbf{1} \rangle$ by part (c). The direct sum decomposition (3.16) is then immediate. $\square$

Remark 3.4 Let X be a r.i. space satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$ or $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$. Then, $T_X \colon X \to X$ is a Fredholm operator, that is, $\dim(\mathrm{Ker}(T_X)) < \infty$, the range $R(T_X)$ is a closed subspace of X and $\dim(X/R(T_X)) < \infty$. This holds when $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ because $\dim(\mathrm{Ker}(T_X)) = 1$ and T_X is surjective; see Theorem 3.2(a), (c). The operator T_X is also Fredholm when $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$ because it is injective, $R(T_X)$ is closed in X and $\dim(X/R(T_X)) = 1$; see (a), (d), (e) of Theorem 3.3.

A consequence of Theorems 3.2 and 3.3 is the possibility to extend the results in [18, Ch. 11], [26], [33, §4.3], concerning the *inversion* of the airfoil equation

$$(T(f))(t) = p.v.\frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{x-t}\,\mathrm{d}x = g(t), \quad \text{a.e. } t \in (-1,1), \tag{3.17}$$

within the class of L^p-spaces for $1 < p < \infty$, $p \neq 2$ (with $g \in L^p$ given), to the significantly larger class of r.i. spaces X whose Boyd indices satisfy $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$ or $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$.

Corollary 3.5 *Let X be a r.i. space.*

(a) *Suppose that $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ and $g \in X$ is fixed. Then, all solutions $f \in X$ of the airfoil equation* (3.17) *are given by*

$$f(x) = \frac{-1}{\sqrt{1-x^2}}\, T_X\left(\sqrt{1-t^2}\,g(t)\right)(x) + \frac{\lambda}{\sqrt{1-x^2}}, \quad \text{a.e. } x \in (-1,1), \tag{3.18}$$

with $\lambda \in \mathbb{C}$ arbitrary.

(b) *Suppose that $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$ and $g \in X$ satisfies $\int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}}dx = 0$. Then, there is a unique solution $f \in X$ of the airfoil equation* (3.17), *namely*

$$f(x) := -\sqrt{1-x^2}\, T_X\left(\frac{g(t)}{\sqrt{1-t^2}}\right)(x), \quad \text{a.e. } x \in (-1,1).$$

Proof (a) In this case, $1/\sqrt{1-x^2} \in X$. Given any $\lambda \in \mathbb{C}$ define the function

$$f(x) := \frac{-1}{\sqrt{1-x^2}}\, T_X\left(\sqrt{1-t^2}\,g(t)\right)(x) + \frac{\lambda}{\sqrt{1-x^2}} = \widehat{T}_X(g)(x) + \frac{\lambda}{\sqrt{1-x^2}}.$$

Then, the identities $T_X\widehat{T}_X(g) = g$ and $T_X(\lambda/\sqrt{1-x^2}) = 0$ (see Theorem 3.2) imply that $T_X(f) = g$.

Conversely, suppose that $f \in X$ satisfies $T_X(f) = g$. It follows from $\widehat{T}_X T_X = I_X - P_X$ that $f - P_X(f) = \widehat{T}_X(g)$. By (3.9), there exists $\lambda \in \mathbb{C}$ such that $P_X(f) = \lambda/\sqrt{1-x^2}$ and hence, $f = \widehat{T}_X(g) + \frac{\lambda}{\sqrt{1-x^2}}$. So, all solutions of the airfoil equation are indeed given by (3.18).

(b) Define $f(x) := -\sqrt{1-x^2}\, T(g(t)/\sqrt{1-t^2})(x) = \check{T}_X(g)$. By Theorem 3.3(c), we have

$$T_X(f) = T_X\check{T}_X(g) = g - Q_X(g).$$

But, the hypothesis on $g \in X$ implies, via (3.15), that $g \in \mathrm{Ker}(Q_X)$ and so $T_X(f) = g$. The uniqueness of the solution f is immediate as T_X is injective by Theorem 3.3(a). $\square$

Remark 3.6 The conditions $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$ or $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ are not always satisfied, e.g., if $X = L^{2,q}$ with $1 \le q \le \infty$. There also exist r.i. spaces X such that $\underline{\alpha}_X < 1/2 < \overline{\alpha}_X$; see [2, pp. 177–178].

4 Extension of the finite Hilbert transform on r.i. spaces

The finite Hilbert transform $T\colon L^1 \to L^{1,\infty}$ has the property that $T(L^1) \nsubseteq L^1$. Hence, for any r.i. space X we necessarily have $T(L^1) \nsubseteq X$. On the other hand, if X satisfies $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, then $T(X) \subseteq X$ continuously. Do there exist any other B.f.s.' $Z \subseteq L^1$ such that $X \subsetneqq Z$ and $T(Z) \subseteq X$? As a consequence of Theorems 3.2 and 3.3, for those r.i. spaces X satisfying $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ or $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$, the answer is shown to be negative; see Theorem 4.7.

The proof of the following result uses important facts from the theory of vector measures, namely a theorem of Talagrand concerning L^0-valued measures and the Dieudonné–Grothendieck theorem for bounded vector measures.

Proposition 4.1 *Let X be a r.i. space satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$. Let $f \in L^1$. The following conditions are equivalent.*

(a) $T(f\chi_A) \in X$ *for every* $A \in \mathcal{B}$.
(b) $\sup_{A\in\mathcal{B}} \|T(f\chi_A)\|_X < \infty$.
(c) $T(h) \in X$ *for every* $h \in L^0$ *with* $|h| \le |f|$ *a.e.*
(d) $\sup_{|h|\le|f|} \|T(h)\|_X < \infty$.
(e) $T(\theta f) \in X$ *for every* $\theta \in L^\infty$ *with* $|\theta| = 1$ *a.e.*
(f) $\sup_{|\theta|=1} \|T(\theta f)\|_X < \infty$.
(g) $fT_{X'}(g) \in L^1$ *for every* $g \in X'$.

Moreover, if any one of (a)–(g) *is satisfied, then*

$$\sup_{A\in\mathcal{B}} \big\|T(\chi_A f)\big\|_X \le \sup_{|\theta|=1} \big\|T(\theta f)\big\|_X \le \sup_{|h|\le|f|} \big\|T(h)\big\|_X \le 4 \sup_{A\in\mathcal{B}} \big\|T(\chi_A f)\big\|_X. \tag{4.1}$$

Proof (a)⇒(b). Consider the X-valued, finitely additive measure

$$\nu\colon A \mapsto T(f\chi_A), \quad A \in \mathcal{B}. \tag{4.2}$$

Let $J_X\colon X \to L^0$ denote the natural continuous linear embedding. Then, the composition $J_X \circ \nu\colon \mathcal{B} \to L^0$ is σ-additive. To establish this, let $A_n \downarrow \emptyset$ in $\mathcal{B}$. Then, $\lim_n f\chi_{A_n} = 0$ in L^1 and hence, $\lim_n T(f\chi_{A_n}) = 0$ in $L^{1,\infty}$ by Kolmogorov's theorem. Since $L^{1,\infty} \subseteq L^0$ continuously, we also have $\lim_n T(f\chi_{A_n}) = 0$ in L^0. Consequently, $\lim_n (J_X \circ \nu)(A_n) = 0$ in L^0, which verifies the σ-additivity of $J_X \circ \nu$.

It follows from a result of Talagrand, [31, Theorem B], that there exist a non-negative function $\Psi_0 \in L^0$ and a σ-additive vector measure $\mu_0\colon \mathcal{B} \to L^2$ such that

$$(J_X \circ \nu)(A) = \Psi_0 \cdot \mu_0(A), \quad A \in \mathcal{B},$$

where $\Psi_0 \cdot \mu_0(A)$ is the pointwise product of two functions in L^0. Define $B_0 := \Psi_0^{-1}(\{0\})$. Then, $\Psi := \Psi_0 + \chi_{B_0} \in L^0$ is strictly positive. Consider the L^2-valued vector measure

$$\mu\colon A \mapsto \chi_{(-1,1)\setminus B_0} \cdot \mu_0(A), \quad A \in \mathcal{B}.$$

For every $A \in \mathcal{B}$, we claim that $(J_X \circ \nu)(A) = \Psi \cdot \mu(A)$. This follows from

$$\begin{aligned}\Psi \cdot \mu(A) &= (\Psi_0 + \chi_{B_0}) \cdot \chi_{(-1,1)\setminus B_0} \cdot \mu_0(A)\\ &= \chi_{(-1,1)\setminus B_0} \cdot \Psi_0 \cdot \mu_0(A)\\ &= \chi_{(-1,1)\setminus B_0} \cdot (J_X \circ \nu)(A) + \chi_{B_0} \cdot (J_X \circ \nu)(A)\\ &= (J_X \circ \nu)(A),\end{aligned}$$

where we have used $\chi_{B_0} \cdot (J_X \circ \nu)(A) = \chi_{B_0} \cdot \Psi_0 \cdot \mu(A) = 0$.

Set $B_n := \{x \in (-1, 1) : (n-1) < 1/\Psi(x) \leq n\}$, for $n \in \mathbb{N}$. Then, the subset

$$\left\{\chi_{B_n \cap B}/\Psi : n \in \mathbb{N}, B \in \mathcal{B}\right\} \tag{4.3}$$

of $L^\infty \subseteq X' \subseteq X^*$ is total for X. To verify this, let $g \in X$ satisfy

$$\int_{-1}^{1} g(x)\chi_{B_n \cap B}(x)/\Psi(x)\,\mathrm{d}x = 0, \quad n \in \mathbb{N}, B \in \mathcal{B}.$$

Then, for every $n \in \mathbb{N}$, the function $(g\chi_{B_n}/\Psi) \in X \subseteq L^1$ is 0 a.e. Since $1/\Psi$ is strictly positive on $(-1, 1) = \cup_{n=1}^{\infty} B_n$, we have $g = 0$ a.e. This implies that the subset (4.3) of X^* is total for X.

Fix $n \in \mathbb{N}$ and $B \in \mathcal{B}$. Then, the scalar-valued set function $A \mapsto \langle \nu(A), \chi_{B_n \cap B}/\Psi\rangle$, for $A \in \mathcal{B}$, is σ-additive. Indeed, as $\nu(A) \in X$ and $(\chi_{B_n \cap B}/\Psi) \in L^\infty \subseteq X'$, we have, for each $A \in \mathcal{B}$, that

$$\begin{aligned}\langle \nu(A), \chi_{B_n \cap B}/\Psi\rangle &= \int_{-1}^{1} \nu(A)(x)\chi_{B_n \cap B}(x)/\Psi(x)\,\mathrm{d}x\\ &= \int_{-1}^{1} \mu(A)(x)\chi_{B_n \cap B}(x)\,\mathrm{d}x = \langle \mu(A), \chi_{B_n \cap B}\rangle,\end{aligned}$$

which implies the desired σ-additivity because μ is σ-additive as an L^2-valued vector measure and $\chi_{B_n \cap B} \in L^2$. Consequently, each $\mathbb{C}$-valued, σ-additive measure $A \mapsto \langle \nu(A), \chi_{B_n \cap B}/\Psi\rangle$ on $\mathcal{B}$, for $n \in \mathbb{N}$, has bounded range. Recalling that the subset (4.3) of X^* is total for X, the Dieudonné–Grothendieck theorem [12, Corollary I.3.3] implies that ν has bounded range in X. Hence, (b) is established.

(b)$\Rightarrow$(c). The *semivariation* $\|\nu\|(\cdot)$ of the bounded, finitely additive, X-valued measure ν defined in (4.2) satisfies both

$$\|\nu\|(A) = \sup\left\{\|T(\chi_A f s)\|_X : s \in \operatorname{sim}\mathcal{B},\ |s| \leq 1\right\}, \quad A \in \mathcal{B},$$

and

$$\sup_{B \in \mathcal{B}, B \subseteq A} \|\nu(B)\|_X \leq \|\nu\|(A) \leq 4 \sup_{B \in \mathcal{B}, B \subseteq A} \|\nu(B)\|_X, \quad A \in \mathcal{B},$$

[12, p. 2 and Proposition I.1.11]. Thus, for $s \in \operatorname{sim}\mathcal{B}$ with $s \neq 0$,

$$\|T(fs)\|_X \leq \left(4 \sup_{A \in \mathcal{B}} \|T(f\chi_A)\|_X\right) \cdot \sup_{|x|<1} |s(x)| < \infty \tag{4.4}$$

because $|s| \leq \sup_{|x|<1} |s(x)|$ pointwise on $(-1, 1)$, [12, p. 6]. To obtain (c) from (4.4), take any $h \in L^0$ with $|h| \leq |f|$ a.e. Then, $h = f\varphi$ for some $\varphi \in L^0$ with $|\varphi| \leq 1$ a.e. Select a sequence $\{s_n\}_{n=1}^{\infty} \subseteq \operatorname{sim}\mathcal{B}$ such that $|s_n| \leq |\varphi|$ on $(-1, 1)$ for all $n \in \mathbb{N}$ and $s_n \to \varphi$

uniformly on $(-1, 1)$ as $n \to \infty$. Then, the sequence $\{T(fs_n)\}_{n=1}^{\infty}$ is Cauchy in X as (4.4) yields

$$\|T(fs_j) - T(fs_k)\|_X \leq \left(4 \sup_{A \in \mathcal{B}} \|T(f\chi_A)\|_X\right) \cdot \sup_{|x|<1} |s_j(x) - s_k(x)|$$

for all $j, k \in \mathbb{N}$. Accordingly, $\{T(fs_n)\}_{n=1}^{\infty}$ has a limit in X, say g. Since the natural inclusion $X \subseteq L^{1,\infty}$ is continuous, we have $\lim_n T(fs_n) = g$ in $L^{1,\infty}$. On the other hand, since $\lim_n fs_n = f\varphi$ in L^1, Kolmogorov's theorem gives $\lim_n T(fs_n) = T(f\varphi)$ in $L^{1,\infty}$. Thus, $T(h) = T(f\varphi) = g$ as elements of L^0. In particular, $T(h) \in X$ as $g \in X$. So, (c) is established.

(c)⇒(d). Clearly, (c)⇒(a) and we already know that (a)⇒(b). Thus, the previous arguments also imply the inequality

$$\sup_{|h| \leq |f|} \|T(h)\|_X \leq 4 \sup_{A \in \mathcal{B}} \|T(f\chi_A)\|_X. \tag{4.5}$$

To see this, consider any $h \in L^0$ with $|h| \leq |f|$ a.e. Select φ and $\{s_n\}_{n=1}^{\infty} \subseteq \operatorname{sim} \mathcal{B}$ as in the previous paragraph. Then, (4.4) yields

$$\begin{aligned}
\|T(h)\|_X &= \lim_n \|T(fs_n)\|_X \\
&\leq \left(4 \sup_{A \in \mathcal{B}} \|T(f\chi_A)\|_X\right) \sup_{n \in \mathbb{N}} \sup_{|x|<1} |s_n(x)| \\
&= \left(4 \sup_{A \in \mathcal{B}} \|T(f\chi_A)\|_X\right) \sup_{|x|<1} |\varphi(x)| \\
&\leq 4 \sup_{A \in \mathcal{B}} \|T(f\chi_A)\|_X.
\end{aligned}$$

(d)⇒(f)⇒(e) Clear.

(e)⇒(a) Fix $A \in \mathcal{B}$. Since $|\chi_A \pm \chi_{(-1,1)\setminus A}| = 1$, it follows from (e) that both

$$T(f\chi_A) + T(f\chi_{(-1,1)\setminus A}) = T(f(\chi_A + \chi_{(-1,1)\setminus A})) \in X$$

and

$$T(f\chi_A) - T(f\chi_{(-1,1)\setminus A}) = T(f(\chi_A - \chi_{(-1,1)\setminus A})) \in X.$$

These two identities imply that $T(f\chi_A) \in X$.

(d)⇒(g). Fix $g \in X'$. Given $n \in \mathbb{N}$ define $A_n := |f|^{-1}([0, n])$ and set $f_n := f\chi_{A_n} \in L^\infty \subseteq X$. Since $|f_n| \uparrow |f|$ pointwise on $(-1, 1)$, the monotone convergence theorem yields

$$\int_{-1}^{1} |f(x)| \cdot |(T_{X'}(g))(x)| \, dx = \lim_n \int_{-1}^{1} |f_n(x)| \cdot |(T_{X'}(g))(x)| \, dx. \tag{4.6}$$

Select $\theta_1, \theta_2 \in L^\infty$ with $|\theta_1| = 1$ and $|\theta_2| = 1$ pointwise such that $|f| = \theta_1 f$ and $|T_{X'}(g)| = \theta_2 T_{X'}(g)$ pointwise. In particular, $|f_n| = \theta_1 f_n$ pointwise for all $n \in \mathbb{N}$. Then, Parseval formula (cf. Proposition 3.1(b)), Hölder's inequality and condition (d) ensure, for every $n \in \mathbb{N}$, that

$$\begin{aligned}\int_{-1}^{1} |f_n(x)| \cdot |(T_{X'}(g))(x)| \, \mathrm{d}x &= \int_{-1}^{1} \theta_1(x)\theta_2(x) f_n(x)(T_{X'}(g))(x) \, \mathrm{d}x \\ &= -\int_{-1}^{1} (T_X(\theta_1\theta_2 f_n))(x) g(x) \, \mathrm{d}x \\ &\le \|T_X(\theta_1\theta_2 f_n)\|_X \|g\|_{X'} \\ &\le \sup_{|h| \le |f|} \|T(h)\|_X \|g\|_{X'} < \infty.\end{aligned}$$

This inequality and (4.6) imply that (g) holds.

(g)$\Rightarrow$(a). Fix any $A \in \mathcal{B}$. Then, $(f\chi_A)T_{X'}(g) \in L^1$ for every $g \in X'$ by assumption. Apply Proposition 3.1(a) to $f\chi_A$ in place of f to obtain that $gT(f\chi_A) \in L^1$ for all $g \in X'$. Accordingly, $T(f\chi_A) \in X'' = X$, which establishes (a).

The equivalences (a)–(g) are thereby established.

Suppose now that any one of (a)–(g) is satisfied. The second inequality of (4.1) is clear. For the left-hand inequality, fix $A \in \mathcal{B}$. Then, $T(f\chi_A) = (1/2)(T(\theta_1 f) + T(\theta_2 f))$, where $\theta_1 = 1$ and $\theta_2 = \chi_A - \chi_{(-1,1)\setminus A}$ satisfy $|\theta_1| = 1$ and $|\theta_2| = 1$. Accordingly,

$$\|T(f\chi_A)\|_X \le (1/2)(\|T(\theta_1 f)\|_X + \|T(\theta_2 f)\|_X) \le \sup_{|\theta|=1} \|T(\theta f)\|_X.$$

Finally, the last inequality in (4.1) is precisely (4.5). □

Another consequence of Theorems 3.2 and 3.3 is that membership of a given r.i. space X is completely determined by the finite Hilbert transform in X.

Proposition 4.2 *Let X be a r.i. space satisfying either $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ or $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$. Let $f \in L^1$. The following conditions are equivalent.*

(a) $f \in X$.
(b) $T(f\chi_A) \in X$ *for every* $A \in \mathcal{B}$.
(c) $T(f\theta) \in X$ *for every* $\theta \in L^\infty$ *with* $|\theta| = 1$ *a.e.*
(d) $T(h) \in X$ *for every* $h \in L^0$ *with* $|h| \le |f|$ *a.e.*

Proof The three conditions (b), (c) and (d) are equivalent by Proposition 4.1.

(a)$\Rightarrow$(b). Clear as $T : X \to X$ is bounded.

(b)$\Rightarrow$(a). By Proposition 4.1, we have $fT_{X'}(g) \in L^1$ for every $g \in X'$, which we shall use to obtain (a).

Assume that $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, in which case $0 < \underline{\alpha}_{X'} \le \overline{\alpha}_{X'} < 1/2$. This enables us to apply Theorem 3.3(c), with X' in place of X, to the operator $T_{X'}$. So, for any $\psi \in X'$, it follows that $\psi = T_{X'}(\check{T}_{X'}(\psi)) + c\mathbf{1}$ with $c := (1/\pi)\int_{-1}^{1}(\psi(x)/\sqrt{1-x^2}) \, \mathrm{d}x$. Define $g := \check{T}_{X'}(\psi) \in X'$. Then, $fT_{X'}(\check{T}_{X'}(\psi)) \in L^1$ and hence, $f\psi - cf = fT_{X'}(\check{T}_{X'}(\psi))$ belongs to L^1. But, $cf \in L^1$ as $f \in L^1$ by assumption. So, $f\psi \in L^1$, from which it follows that $f \in X'' = X$ as $\psi \in X'$ is arbitrary. Thus, (a) holds.

Consider the remaining case when $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$. Then, $1/2 < \underline{\alpha}_{X'} \le \overline{\alpha}_{X'} < 1$. We apply Theorem 3.2(c) with X' in place of X, to conclude that $T_{X'} : X' \to X'$ is surjective. So, given any $\psi \in X'$, there exists $g \in X'$ with $\psi = T_{X'}(g)$. It follows that $f\psi = fT_{X'}(g) \in L^1$. Since $\psi \in X'$ is arbitrary, we may conclude that $f \in X'' = X$. Hence, (a) again holds. □

Even though T_X is not an isomorphism, Theorems 3.2 and 3.3 imply the impossibility of extending (continuously) the finite Hilbert transform $T_X : X \to X$ to any genuinely larger domain space within L^1 while still maintaining its values in X; see Theorem 4.7. This is in contrast to the situation for the Fourier transform operator acting in the spaces $L^p(\mathbb{T})$, $1 < p < 2$; see Introduction.

We first require an important technical construction. Define

$$[T, X] := \left\{ f \in L^1 : T(h) \in X, \ \forall |h| \le |f| \right\}.$$

If $f \in [T, X]$, then $f \in L^1$ and $T(h) \in X$ for every $h \in L^0$ with $|h| \le |f|$. Hence, Proposition 4.1 implies that

$$\|f\|_{[T,X]} := \sup_{|h| \le |f|} \|T(h)\|_X < \infty, \quad f \in [T, X]. \tag{4.7}$$

The properties of $[T, X]$ are established via a series of steps, with the aim of showing that it is a B.f.s.

First, the functional $f \mapsto \|f\|_{[T,X]}$ is compatible with the lattice structure in the following sense: if $f_1, f_2 \in [T, X]$ satisfy $|f_1| \le |f_2|$, then $\|f_1\|_{[T,X]} \le \|f_2\|_{[T,X]}$. This is because $\{h : |h| \le |f_1|\} \subseteq \{h : |h| \le |f_2|\}$. The same argument shows that $[T, X]$ is an ideal in L^1. In particular, $X \subseteq [T, X]$.

It is routine to verify that if $\alpha \in \mathbb{C}$ and $f \in [T, X]$, then $\alpha f \in [T, X]$ and $\|\alpha f\|_{[T,X]} = |\alpha| \cdot \|f\|_{[T,X]}$.

To verify the subadditivity of $\|\cdot\|_{[T,X]}$ we use the following Freudenthal-type decomposition: If $h, f_1, f_2 \in L^1$ with $|h| \le |f_1 + f_2|$, then there exist h_1, h_2 such that $h = h_1 + h_2$ and $|h_1| \le |f_1|, |h_2| \le |f_2|$; this follows from [34, Theorem 91.3] applied in L^1. Using this fact, given $f_1, f_2 \in [T, X]$, it follows that $f_1 + f_2 \in [T, X]$ and

$$\begin{aligned}
\|f_1 + f_2\|_{[T,X]} &= \sup \left\{ \|T(h)\|_X : |h| \le |f_1 + f_2| \right\} \\
&= \sup \left\{ \|T(h_1) + T(h_2)\|_X : |h| \le |f_1 + f_2|, h = h_1 + h_2, |h_i| \le |f_i| \right\} \\
&\le \sup \left\{ \|T(h_1)\|_X : |h_1| \le |f_1| \right\} + \sup \left\{ \|T(h_2)\|_X : |h_2| \le |f_2| \right\} \\
&= \|f_1\|_{[T,X]} + \|f_2\|_{[T,X]}.
\end{aligned}$$

So, $[T, X]$ is a vector space and $\|\cdot\|_{[T,X]}$ is a lattice seminorm on $[T, X]$.

Let $\|f\|_{[T,X]} = 0$. Then, $T(h) = 0$ in X for every $h \in L^0$ with $|h| \le |f|$. Suppose that $f \ne 0$. Then, there exists $A \in \mathcal{B}$ with $|A| > 0$ such that $f\chi_A \in L^\infty$ and $f(x)\chi_A(x) \ne 0$ for every $x \in A$. Choose two disjoint sets $A_1, A_2 \in \mathcal{B} \cap A$ with $|A_j| > 0$, $j = 1, 2$, and define $h_j := f\chi_{A_j}$, $j = 1, 2$. Then, $h_j \in L^\infty \subseteq X$ satisfies $|h_j| \le |f|$ and so $T_X(h_j) = T(h_j) = 0$ for $j = 1, 2$. That is, $h_1, h_2 \in \mathrm{Ker}(T_X)$. Since h_1, h_2 are linearly independent elements in X, it follows that $\dim(\mathrm{Ker}(T_X)) \ge 2$. But, this contradicts the fact that T_X is either injective or its kernel is one-dimensional; see the discussion after (3.1). Hence, $f = 0$. So, we have shown that $[T, X]$ is a normed function space.

The following result is a Parseval-type formula that will be needed in the sequel.

Lemma 4.3 *Let X be a r.i. space satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$. Then*

$$\int_{-1}^{1} f T_{X'}(g) = -\int_{-1}^{1} g T(f), \quad f \in [T, X], \quad g \in X'.$$

Proof Given $f \in [T, X] \subseteq L^1$, it follows from the definition of $[T, X]$ and Proposition 4.1 that $f T_{X'}(g) \in L^1$ for every $g \in X'$. The desired formula is then immediate from Proposition 3.1(a). ◻

Lemma 4.4 *Let X be a r.i. space satisfying $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$. Then, the normed function space $[T, X]$ is complete.*

Proof Let $f_n \in [T, X]$, for $n \in \mathbb{N}$, satisfy

$$\sum_{n=1}^{\infty} \|f_n\|_{[T,X]} < \infty.$$

This implies, for every choice of h_n with $|h_n| \leq |f_n|$, that

$$\sum_{n=1}^{\infty} \|T(h_n)\|_X < \infty. \tag{4.8}$$

(A) Let $h \in [T, X] \subseteq L^1$. As $|h|\chi_{(-1,0)} \leq |h|$, we have that $T(|h|\chi_{(-1,0)}) \in X$. If $0 < t < 1$, then

$$\left|T\left(|h|\chi_{(-1,0)}\right)(t)\right| = \frac{1}{\pi}\int_{-1}^{0} \frac{|h(x)|}{|x-t|}\mathrm{d}x \geq \frac{1}{2\pi}\int_{-1}^{0} |h(x)|\,\mathrm{d}x,$$

since for $-1 < x < 0$ and $0 < t < 1$ we have $|x - t| \leq 2$. Consequently,

$$\|T\left(|h|\chi_{(-1,0)}\right)\|_X \geq \|T\left(|h|\chi_{(-1,0)}\right)\chi_{(0,1)}\|_X \geq \left(\frac{1}{2\pi}\int_{-1}^{0} |h(x)|\,\mathrm{d}x\right)\|\chi_{(0,1)}\|_X.$$

In a similar way, as $|h|\chi_{(0,1)} \leq |h|$, we have that $T(|h|\chi_{(0,1)}) \in X$. If $-1 < t < 0$, then

$$T\left(|h|\chi_{(0,1)}\right)(t) = \frac{1}{\pi}\int_{0}^{1} \frac{|h(x)|}{x-t}\mathrm{d}x \geq \frac{1}{2\pi}\int_{0}^{1} |h(x)|\,\mathrm{d}x,$$

since for $-1 < t < 0$ and $0 < x < 1$ we have $0 \leq x - t \leq 2$. Consequently,

$$\|T\left(|h|\chi_{(0,1)}\right)\|_X \geq \|T\left(|h|\chi_{(0,1)}\right)\chi_{(-1,0)}\|_X \geq \left(\frac{1}{2\pi}\int_{0}^{1} |h(x)|\,\mathrm{d}x\right)\|\chi_{(-1,0)}\|_X.$$

Applying (4.8) with $h_n := |f_n|\chi_{(-1,0)}$ and $h_n := |f_n|\chi_{(0,1)}$, it follows, from the previous bounds for $h = f_n$, that

$$\begin{aligned}\sum_{n=1}^{\infty}\|f_n\|_{L^1} &= \sum_{n=1}^{\infty}\left(\int_{-1}^{0}|f_n(x)|\,\mathrm{d}x + \int_{0}^{1}|f_n(x)|\,\mathrm{d}x\right)\\ &\leq \sum_{n=1}^{\infty} C\left(\|T\left(|f_n|\chi_{(-1,0)}\right)\|_X + \|T\left(|f_n|\chi_{(0,1)}\right)\|_X\right)\\ &\leq 2C\sum_{n=1}^{\infty}\|f_n\|_{[T,X]} < \infty,\end{aligned}$$

with $C := (2\pi)/\varphi_X(1)$, since $\|\chi_{(0,1)}\|_X = \|\chi_{(-1,0)}\|_X = \varphi_X(1)$. Hence, we have

$$\sum_{n=1}^{\infty} f_n =: f \in L^1 \tag{4.9}$$

with absolute convergence in L^1 and hence, also pointwise a.e.

(B) We now show that $f \in [T, X]$. Select $h \in L^0$ satisfying $|h| \leq |f|$. We need to prove that $T(h) \in X$. To this end, let $\varphi \in L^0$ satisfy $|\varphi| \leq 1$ and $h = \varphi f$. Then,

$$h = \varphi f = \sum_{n=1}^{\infty} \varphi f_n, \quad \text{a.e.}$$

The functions $h_n := \varphi f_n \in [T, X]$, for $n \in \mathbb{N}$, satisfy

$$\sum_{n=1}^{\infty} \|h_n\|_{[T,X]} \leq \sum_{n=1}^{\infty} \|f_n\|_{[T,X]} < \infty$$

due to the ideal property of $[T, X]$. We can apply the arguments in (A) to deduce that the series $\sum_{n=1}^{\infty} h_n$ converges (absolutely) in L^1 to h. Kolmogorov's theorem yields that the series $\sum_{n=1}^{\infty} T(h_n)$ converges to $T(h)$ in $L^{1,\infty}$.

On the other hand, since the series $\sum_{n=1}^{\infty} T(h_n)$ converges absolutely in X (see (4.8)), it is convergent, say to $g = \sum_{n=1}^{\infty} T(h_n)$ in X and hence, also in $L^{1,\infty}$. Accordingly, $T(h) = g$ and so $T(h) \in X$. This establishes that $f \in [T, X]$.

(C) It remains to show that $\sum_{n=1}^{\infty} f_n$ converges to f in the topology of $[T, X]$, that is, $\|f - \sum_{n=1}^{N} f_n\|_{[T,X]} \to 0$ as $N \to \infty$. Fix $N \in \mathbb{N}$. Let $h \in L^0$ satisfy

$$|h| \leq \left| f - \sum_{n=1}^{N} f_n \right| = \left| \sum_{n=N+1}^{\infty} f_n \right| \leq \sum_{n=N+1}^{\infty} |f_n|.$$

We can reproduce the argument used in (B) to deduce that

$$h = \sum_{n=N+1}^{\infty} h_n, \quad |h_n| \leq |f_n|, \quad n \geq N+1.$$

Then,

$$\|T(h)\|_X \leq \sum_{n=N+1}^{\infty} \|T(h_n)\|_X \leq \sum_{n=N+1}^{\infty} \|f_n\|_{[T,X]}.$$

That is, for each $N \in \mathbb{N}$, we have

$$\|f - \sum_{n=1}^{N} f_n\|_{[T,X]} = \sup_{|h| \leq |f - \sum_{n=1}^{N} f_n|} \|T(h)\|_X \leq \sum_{n=N+1}^{\infty} \|f_n\|_{[T,X]} \to 0,$$

which establishes the completeness of $[T, X]$. $\square$

We will require an alternate description of the norm $\|\cdot\|_{[T,X]}$ to that given in (4.7), namely

$$\|f\|_{[T,X]} = \sup_{\|g\|_{X'} \leq 1} \|f T_{X'}(g)\|_{L^1}, \quad f \in [T, X]. \tag{4.10}$$

To verify this, fix $f \in [T, X]$. Given $\varphi \in L^0$ with $|\varphi| \leq 1$, the function $\varphi f \in [T, X]$ as $|\varphi f| \leq |f|$. It follows from Lemma 4.3 (see also its proof) with φf in place of f, that $\varphi f T_{X'}(g) \in L^1$ for all $g \in X'$ (in particular, also $f T_{X'}(g) \in L^1$) and

$$\int_{-1}^{1} (\varphi f) T_{X'}(g) = -\int_{-1}^{1} g T(\varphi f), \quad g \in X'.$$

Since $\{\varphi f : \varphi \in L^0, |\varphi| \le 1\} = \{h \in L^0 : |h| \le |f|\}$, the previous formula yields (4.10) because (4.7) implies that

$$\|f\|_{[T,X]} = \sup_{|\varphi|\le 1} \|T(\varphi f)\|_X = \sup_{|\varphi|\le 1} \sup_{\|g\|_{X'}\le 1} \Big| \int_{-1}^{1} gT(\varphi f) \Big|$$
$$= \sup_{\|g\|_{X'}\le 1} \sup_{|\varphi|\le 1} \Big| \int_{-1}^{1} (\varphi f) T_{X'}(g) \Big| = \sup_{\|g\|_{X'}\le 1} \|f T_{X'}(g)\|_{L^1}.$$

Proposition 4.5 *Let X be a r.i. space satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$. Then, $[T, X]$ is a B.f.s.*

Proof In view of Lemma 4.4, it remains to establish that $[T, X]$ possesses the Fatou property.

Let $0 \le f \in L^0$ and $\{f_n\}_{n=1}^{\infty} \subseteq [T, X] \subseteq L^1$ be a sequence such that $0 \le f_n \le f_{n+1} \uparrow f$ pointwise a.e. with $\sup_n \|f_n\|_{[T,X]} < \infty$. In Step A of the proof of Lemma 4.4, it was shown that

$$\|h\|_{L^1} \le (4\pi\backslash\varphi_X(1))\|h\|_{[T,X]}, \quad h \in [T, X],$$

which ensures that also $\sup_n \|f_n\|_{L^1} < \infty$. Hence, via Fatou's lemma, $f \in L^1$. Moreover, the monotone convergence theorem together with (4.10) applied to $f_n \in [T, X]$ for each $n \in \mathbb{N}$ yields

$$\sup_{\|g\|_{X'}\le 1} \int_{-1}^{1} |f T_{X'}(g)| = \sup_{\|g\|_{X'}\le 1} \sup_{n} \int_{-1}^{1} |f_n T_{X'}(g)|$$
$$= \sup_{n} \sup_{\|g\|_{X'}\le 1} \int_{-1}^{1} |f_n T_{X'}(g)| = \sup_{n} \|f_n\|_{[T,X]} < \infty.$$

In particular, $fT_{X'}(g) \in L^1$ for every $g \in X'$ with $f \in L^1$. According to (c)$\Leftrightarrow$(g) in Proposition 4.1, we have $f \in [T, X]$ and, via (4.10) and the previous identity, that $\|f\|_{[T,X]} = \sup_n \|f_n\|_{[T,X]}$. So, we have established that $[T, X]$ has the Fatou property. □

The optimality property of the B.f.s. $[T, X]$ relative to T_X can now be formulated.

Theorem 4.6 *Let X be a r.i. space satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$. Then, $[T, X]$ is the largest B.f.s. containing X to which $T_X : X \to X$ has a continuous, linear, X-valued extension.*

Proof Let $Z \subseteq L^1$ be any B.f.s. with $X \subseteq Z$ such that T_X has a continuous, linear extension $T : Z \to X$. Fix $f \in Z$. Then, for each $h \in L^0$ with $|h| \le |f|$, we have $h \in Z$ and

$$\|T(h)\|_X \le \|T\|_{op}\|h\|_Z \le \|T\|_{op}\|f\|_Z,$$

where $\|T\|_{op}$ is the operator norm of $T : Z \to X$. Then, $f \in [T, X]$ and so the space $[T, X]$ contains Z continuously. Due to the boundedness of $T_X : X \to X$, we have that

$$\|f\|_{[T,X]} = \sup_{|h|\le|f|} \|T(h)\|_X \le \|T_X\|_{op}\|f\|_X, \quad f \in X,$$

and so $X \subseteq [T, X]$ continuously. By construction $T : [T, X] \to X$ and T is continuous. Hence, $[T, X]$ is the *largest* B.f.s. containing X to which $T_X : X \to X$ has a continuous, linear, X-valued extension. □

We can now prove the impossibility of extending $T_X : X \to X$.

Theorem 4.7 *Let X be a r.i. space satisfying either $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ or $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$. Then, the finite Hilbert transform $T_X \colon X \to X$ has no X-valued, continuous linear extension to any larger B.f.s.*

Proof According to Theorem 4.6, whenever $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, the space $[T, X]$ is the largest B.f.s. to which $T_X \colon X \to X$ can be continuously extended with $X \subseteq [T, X]$ continuously. So, it suffices to prove that $[T, X] = X$. But, this corresponds precisely to the equivalence in Proposition 4.2 between the condition (a), i.e., $f \in X$, and the condition (d), i.e., $T(h) \in X$ for all $h \in L^0$ with $|h| \le |f|$, which is the statement that $f \in [T, X]$. □

Recall that T_X is not an isomorphism. Nevertheless, Theorems 3.2 and 3.3 yield norms, in terms of the finite Hilbert transform, which are equivalent to the given norm in the corresponding r.i. space.

Corollary 4.8 *Let X be a r.i. space satisfying either $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ or $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2$. Then, there exists a constant $C_X > 0$ such that*

$$\begin{aligned}\frac{C_X}{4}\|f\|_X \le \sup_{A\in\mathcal{B}} \left\|T_X(\chi_A f)\right\|_X &\le \sup_{|\theta|=1} \left\|T_X(\theta f)\right\|_X \\ &\le \sup_{|h|\le|f|} \left\|T_X(h)\right\|_X \le \|T_X\|\cdot\|f\|_X,\end{aligned}$$

for every $f \in X$.

Proof The final inequality is clear from

$$\|T_X(h)\|_X \le \|T_X\|\cdot\|h\|_X \le \|T_X\|\cdot\|f\|_X$$

for every $f \in X$ and every $h \in L^0$ with $|h| \le |f|$.

It was shown in the proof of Theorem 4.7 that $[T, X] = X$. Hence, there exists a constant $C_X > 0$ such that

$$C_X\|f\|_X \le \sup_{|h|\le|f|} \|T_X(h)\|_X, \quad f \in X.$$

The remaining inequalities now follow from (4.1) which is applicable because if $f \in X$, then condition (c) in Proposition 4.1 is surely satisfied. □

Remark 4.9 The notion of the optimal domain $[T, X]$ is meaningful for a large family of operators acting on function spaces, as already commented in Introduction. Among them, in a much simpler situation, are the positive operators. For a thorough study of this topic, see, for example, [27] and the references therein.

5 The finite Hilbert transform on L^2

Theorems 3.2 and 3.3 are not applicable to $X = L^2$. Moreover, T_{L^2} is not Fredholm and no inversion formula is available. Nevertheless, it turns out that no extension of T_{L^2} is possible. A new approach is needed to establish this. Trying to use the results and techniques obtained for the cases $p \ne 2$ in an attempt to study the possible extension of $T_{L^2} \colon L^2 \to L^2$ is futile as shown by the following consideration. Let $X = L^p$ for $1 < p < 2$ and set $T_p := T_{L^p}$.

Since $\underline{\alpha}_X = \overline{\alpha}_X = 1/p \in (1/2, 1)$, we are in the setting of Theorem 3.2. The left inverse of T_p is the operator $\widehat{T}_p := \widehat{T}_{L^p}$, defined by (3.7), that is,

$$\widehat{T}_p(f)(x) := \frac{-1}{\sqrt{1-x^2}} T_p(\sqrt{1-t^2} f(t))(x), \quad \text{a.e. } x \in (-1, 1),$$

which maps L^p into L^p and is an isomorphism onto its range. We estimate from below the operator norm of $\widehat{T}_p$. Since $T_p(\sqrt{1-t^2})(x) = -x$, for $f := \mathbf{1}$, we obtain

$$\|\widehat{T}_p\| \geq \frac{\|x/\sqrt{1-x^2}\|_{L^p}}{\|\mathbf{1}\|_{L^p}} = \left(\frac{1}{2}\int_{-1}^{1} \frac{|x|^p}{(1-x^2)^{p/2}}\, \mathrm{d}x\right)^{1/p}$$

which goes to ∞ as $p \to 2^-$.

We denote by T_2 the finite Hilbert transform $T_{L^2} \colon L^2 \to L^2$. The norm $\|\cdot\|_{L^2}$ will simply be denoted by $\|\cdot\|_2$.

Lemma 5.1 *For every set $A \in \mathcal{B}$, we have*

$$\|T_2(\chi_A)\|_2 \geq \left(\int_0^\infty \frac{4\lambda}{e^{\pi\lambda}+1} d\lambda\right)^{1/2} |A|^{1/2}.$$

Proof We rely on a consequence of the Stein–Weiss formula for the distribution function of the Hilbert transform of a characteristic function, due to Laeng [19, Theorem 1.2]. Namely, for $A \subseteq \mathbb{R}$ with $|A| < \infty$, we have

$$|\{x \in A : |H(\chi_A)(x))| > \lambda\}| = \frac{2|A|}{\mathrm{e}^{\pi\lambda}+1}, \quad \lambda > 0.$$

For $A \in \mathcal{B}$, it follows from properties of the distribution function for $T_2(\chi_A)$ that

$$\begin{aligned}\|T_2(\chi_A)\|_2^2 &= \int_0^\infty 2\lambda \cdot |\{x \in (-1,1) : |T_2(\chi_A)(x)| > \lambda\}|\, \mathrm{d}\lambda \\ &\geq \int_0^\infty 2\lambda \cdot |\{x \in A : |H(\chi_A)(x)| > \lambda\}|\, \mathrm{d}\lambda \\ &= |A| \int_0^\infty \frac{4\lambda}{\mathrm{e}^{\pi\lambda}+1}\, \mathrm{d}\lambda.\end{aligned}$$

□

The approach we use for proving the impossibility of extending T_2 is to show that L^2 coincides with the B.f.s. $[T, L^2]$. For this, we need to compare the norm in L^2 with the norm in $[T, L^2]$.

Theorem 5.2 *For each function $\phi \in$ sim $\mathcal{B}$, we have*

$$\left(\int_0^\infty \frac{4\lambda}{e^{\pi\lambda}+1} d\lambda\right)^{1/2} \|\phi\|_2 \leq \sup_{|\theta|=1} \left\|T_2(\theta\phi)\right\|_2.$$

Proof In order to prove the claim, fix any simple function $\phi = \sum_{n=1}^N a_n \chi_{A_n}$, with $a_1, \ldots, a_N \in \mathbb{C}$ and pairwise disjoint sets $A_1, \ldots, A_N \in \mathcal{B}$ with $N \in \mathbb{N}$.

Let τ denote the product measure on $\Lambda := \{-1, 1\}^N$ for the uniform probability on $\{-1, 1\}$. Thus, given $\sigma \in \Lambda$ we have $\sigma = (\sigma_1, \ldots, \sigma_N)$ with $\sigma_n = \pm 1$ for $n = 1, \ldots, N$. Note that the coordinate projections

$$P_n : \sigma \in \Lambda \mapsto \sigma_n \in \{-1, 1\}, \quad n = 1, \ldots, N,$$

form an orthonormal set, i.e.,

$$\int_{\Lambda} P_j P_k \, \mathrm{d}\tau = \int_{\Lambda} \sigma_j \sigma_k \, \mathrm{d}\tau(\sigma) = \delta_{j,k}, \quad j, k = 1, \ldots, N. \tag{5.1}$$

The function $F : \Lambda \to [0, \infty)$ defined by

$$F(\sigma) := \left\| T_2 \left(\sum_{n=1}^{N} \sigma_n a_n \chi_{A_n} \right) \right\|_2, \quad \sigma \in \Lambda,$$

is bounded and measurable and so satisfies

$$\|F\|_{L^2(\tau)} \leq \|F\|_{L^\infty(\tau)}. \tag{5.2}$$

We now compute both of the norms in (5.2) explicitly.

Given $\sigma = (\sigma_n) \in \Lambda$, the measurable function defined on $(-1, 1)$ by

$$t \mapsto \theta_\sigma(t) := \chi_{(-1,1)\setminus(\cup_{n=1}^{N} A_n)}(t) + \sum_{n=1}^{N} \sigma_n \chi_{A_n}(t)$$

satisfies $|\theta_\sigma| = 1$ and

$$\theta_\sigma \phi = \sum_{n=1}^{N} \sigma_n a_n \chi_{A_n}.$$

Consequently,

$$T_2\big(\theta_\sigma \phi\big) = T_2\Big(\sum_{n=1}^{N} \sigma_n a_n \chi_{A_n} \Big),$$

from which it is clear that

$$\|F\|_{L^\infty(\tau)} = \sup_{\sigma \in \Lambda} \left\| T_2\Big(\sum_{n=1}^{N} \sigma_n a_n \chi_{A_n} \Big) \right\|_2 \leq \sup_{|\theta|=1} \big\| T_2(\theta\phi) \big\|_2. \tag{5.3}$$

Set $\beta := \big(\int_0^\infty \frac{4\lambda}{\mathrm{e}^{\pi\lambda}+1} \mathrm{d}\lambda \big)^{1/2}$. By Fubini's theorem, (5.1) and Lemma 5.1 it follows that

$$\begin{aligned}
\|F\|_{L^2(\tau)}^2 &= \int_{\Lambda} \left\| T_2\Big(\sum_{n=1}^{N} \sigma_n a_n \chi_{A_n} \Big) \right\|_2^2 \mathrm{d}\tau(\sigma) = \int_{\Lambda} \int_{-1}^{1} \left| \sum_{n=1}^{N} \sigma_n a_n T_2(\chi_{A_n})(t) \right|^2 \mathrm{d}t \, \mathrm{d}\tau(\sigma) \\
&= \int_{-1}^{1} \int_{\Lambda} \left| \sum_{n=1}^{N} \sigma_n a_n T_2(\chi_{A_n})(t) \right|^2 \mathrm{d}\tau(\sigma) \, \mathrm{d}t = \int_{-1}^{1} \sum_{n=1}^{N} \big| a_n T_2(\chi_{A_n})(t) \big|^2 \, \mathrm{d}t \\
&= \sum_{n=1}^{N} |a_n|^2 \big\| T_2(\chi_{A_n}) \big\|_2^2 \geq \beta^2 \sum_{n=1}^{N} |a_n|^2 |A_n| \\
&= \beta^2 \int_{-1}^{1} \left| \sum_{n=1}^{N} a_n \chi_{A_n}(t) \right|^2 \mathrm{d}t \\
&= \beta^2 \|\phi\|_2^2.
\end{aligned}$$

This inequality, together with (5.2) and (5.3), yields

$$\beta\|\phi\|_2 \leq \sup_{|\theta|=1} \left\|T_2(\theta\phi)\right\|_2.$$

Since the simple function ϕ is arbitrary, this establishes the result. □

Theorem 5.2 implies the impossibility of extending T_2. Note that this does not follow from Theorem 4.7 since L^2 does not satisfy the restriction on the Boyd indices.

Theorem 5.3 *The finite Hilbert transform $T_2\colon L^2 \to L^2$ has no continuous, L^2-valued extension to any genuinely larger B.f.s.*

Proof We follow the approach used for proving Theorem 4.7 to show that

$$L^2 = [T_2, L^2] := \left\{f \in L^1 : T_2(h) \in L^2,\ \forall |h| \leq |f|\right\}.$$

Note first note that

$$\beta\|\phi\|_2 \leq \sup_{|\theta|=1} \left\|T_2(\theta\phi)\right\|_2 \leq \sup_{|h|\leq|\phi|} \left\|T_2(h)\right\|_2, \quad \phi \in \operatorname{sim} \mathcal{B}. \tag{5.4}$$

The left-hand inequality is Theorem 5.2. The right-hand inequality is clear from (4.1).

Let $f \in [T, L^2]$. According to (5.4), for every $\phi \in \operatorname{sim} \mathcal{B}$ satisfying $|\phi| \leq |f|$ it follows that

$$\beta\|\phi\|_2 \leq \sup_{|h|\leq|f|} \left\|T_2(h)\right\|_2 = \|f\|_{[T,L^2]}.$$

Taking the supremum with respect to all such ϕ yields $\beta\|f\|_2 \leq \|f\|_{[T,L^2]}$. This implies that $f \in L^2$. Consequently, $[T, L^2] = L^2$ with equivalent norms. □

A further consequence of Theorem 5.2 leads to various equivalent norms, in terms of the operator T_2, to the standard norm $\|\cdot\|_2$ in L^2. As before, note that this does not follow from Corollary 4.8 since L^2 does not satisfy the restriction on the Boyd indices. Recall that $\beta := \left(\int_0^\infty \frac{4\lambda}{e^{\pi\lambda}+1} d\lambda\right)^{1/2}$.

Corollary 5.4 *For every $f \in L^2$, we have*

$$\frac{\beta}{4}\|f\|_2 \leq \sup_{A\in\mathcal{B}} \left\|T_2(\chi_A f)\right\|_2 \leq \sup_{|\theta|=1} \left\|T_2(\theta f)\right\|_2 \leq \sup_{|h|\leq|f|} \left\|T_2(h)\right\|_2 \leq \|f\|_2.$$

Proof The last inequality follows (since $\|\cdot\|_2$ is a lattice norm and $\|T_2\| = 1$, [23]) via

$$\|T_2(h)\|_2 \leq \|T_2\| \cdot \|h\|_2 \leq \|f\|_2, \quad |h| \leq |f|.$$

If $f \in L^2$, then surely (c) of Proposition 4.1 is satisfied with $X = L^2$. Hence, the second and third inequalities follow from (4.1).

Finally, in order to prove the first inequality, we begin by establishing, for $h, f \in L^2$ satisfying $|h| \leq |f|$, that

$$\sup_{|\theta|=1} \left\|T(\theta h)\right\|_2 \leq \sup_{|\tilde{\theta}|=1} \left\|T(\tilde{\theta} f)\right\|_2. \tag{5.5}$$

Fix θ with $|\theta| = 1$. Then, via Parseval formula, for some function $\tilde{\theta}_{f,g}$ satisfying $|\tilde{\theta}_{f,g}| = 1$, we have

$$\begin{aligned}
\|T_2(\theta h)\|_2 &= \sup_{\|g\|_2 \le 1} \left| \int_{-1}^{1} T_2(\theta h)(t) \cdot g(t)\, \mathrm{d}t \right| = \sup_{\|g\|_2 \le 1} \left| \int_{-1}^{1} \theta(t) h(t) \cdot T_2(g)(t)\, \mathrm{d}t \right| \\
&\le \sup_{\|g\|_2 \le 1} \int_{-1}^{1} |h(t)| \cdot |T_2(g)(t)|\, \mathrm{d}t \le \sup_{\|g\|_2 \le 1} \int_{-1}^{1} |f(t)| \cdot |T_2(g)(t)|\, \mathrm{d}t \\
&= \sup_{\|g\|_2 \le 1} \int_{-1}^{1} f(t)\tilde{\theta}_{f,g}(t) T_2(g)(t)\, \mathrm{d}t \le \sup_{\|g\|_2 \le 1} \left| \int_{-1}^{1} T_2(f\tilde{\theta}_{f,g})(t) g(t)\, \mathrm{d}t \right| \\
&\le \sup_{\|g\|_2 \le 1} \|T_2(f\tilde{\theta}_{f,g})\|_2 \|g\|_2 \\
&\le \sup_{|\tilde{\theta}|=1} \|T_2(f\tilde{\theta})\|_2.
\end{aligned}$$

Accordingly, (5.5) holds.

Fix $f \in L^2$. Then, Theorem 5.2, together with (4.1) and (5.5), gives, for $\phi \in \operatorname{sim} \mathcal{B}$ satisfying $|\phi| \le |f|$, that

$$\beta \|\phi\|_2 \le \sup_{|\theta|=1} \|T_2(\theta\phi)\|_2 \le \sup_{|\theta|=1} \|T_2(\theta f)\|_2 \le 4 \sup_{A \in \mathcal{B}} \|T_2(f\chi_A)\|_2.$$

Taking the supremum with respect to all such simple functions ϕ , we arrive at

$$\beta \|f\|_2 \le 4 \sup_{A \in \mathcal{B}} \|T_2(f\chi_A)\|_2.$$

□

From Corollary 5.4, we can deduce conditions, in terms of the finite Hilbert transform, for membership of L^2.

Corollary 5.5 *Given $f \in L^1$, the following conditions are equivalent.*

(a) $f \in L^2$.
(b) $T(f\chi_A) \in L^2$ *for every* $A \in \mathcal{B}$.
(c) $T(f\theta) \in L^2$ *for every* $\theta \in L^\infty$ *with* $|\theta| = 1$ *a.e.*
(d) $T(h) \in L^2$ *for every* $h \in L^0$ *with* $|h| \le |f|$ *a.e.*

Proof (b)⇔(c)⇔(d) follow from Proposition 4.1 with $X = L^2$.

(a)⇒(b) Clear as $T_2\colon L^2 \to L^2$ is bounded.

(b)⇒(a) For $X = L^2$, it follows that condition (b) of Proposition 4.1 holds, that is, $\gamma := \sup_{A \in \mathcal{B}} \|T(f\chi_A)\|_2 < \infty$. For each $n \in \mathbb{N}$ define $A_n := |f|^{-1}([0, n])$ and $f_n := f\chi_{A_n}$. Then,

$$\|T(f_n\chi_A)\|_2 = \|T(f\chi_{A \cap A_n})\|_2 \le \gamma, \quad A \in \mathcal{B}, n \in \mathbb{N},$$

which implies, via Corollary 5.4, that

$$\|f_n\|_2 \le \frac{4\gamma}{\beta}, \quad n \in \mathbb{N}.$$

Since $|f_n|^2 \uparrow |f|^2$ pointwise a.e. on $(-1, 1)$, from the monotone convergence theorem it follows that $f \in L^2$. This is condition (a). □

Remark 5.6 As commented in the Introduction the operator $T_2 \colon L^2 \to L^2$ is injective and has proper dense range. A detailed study of its range is carried out in Sections 3 and 4 of [26]. Let us highlight a somewhat unexpected result given there. Namely, for every $-1 < a < 1$, each function $f_a(x) := \chi_{(a,1)}(x)/\sqrt{1-x^2}$, for $x \in (-1, 1)$, which belongs to L^1, satisfies $T(f_a) \in L^2$ and

$$\|T(f_a)\|_2 = \left\| T\left(\frac{\chi_{(a,1)}(x)}{\sqrt{1-x^2}}\right)\right\|_2 = \frac{1}{\pi}\big(7\zeta(3)\big)^{1/2},$$

[26, Lemma 4.3 and Note 4.4]. Observe that $f_a \notin L^2$ for every $-1 < a < 1$. On the other hand, if X is a r.i. space satisfying $1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, then $K = \{f_a : -1 < a < 1\} \subseteq L^{2,\infty} \subseteq X$. Moreover, for every sequence $a_n \uparrow 1^-$ the sequence $\{f_{a_n}\}_{n=1}^{\infty}$ satisfies $0 \le f_{a_n} \downarrow 0$ pointwise. Assuming the absolute continuity of the norm $\|\cdot\|_X$, it follows in this case that $\lim_n T_X(f_{a_n}) = 0$ in X.

Remark 5.7 For r.i. spaces X satisfying the conditions of Theorem 4.7, namely

$$0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1/2 \quad \text{or} \quad 1/2 < \underline{\alpha}_X \le \overline{\alpha}_X < 1, \tag{5.6}$$

we know that the finite Hilbert transform $T_X \colon X \to X$ cannot be extended to a larger B.f.s. The proof is based on arguments from Fredholm operator theory, a deep factorization result of Talagrand on L^0-valued measures and on the construction of the largest domain space $[T, X]$. For r.i. spaces X with $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$ not satisfying the conditions (5.6), it is unknown in general when T_X is Fredholm and when not (for $X = L^2$, it is known that T_X is not Fredholm). So, the arguments used to prove Theorem 4.7 may apply to some further cases but surely not to all. The proof given in Theorem 5.3 for $X = L^2$ relies heavily on properties of the L^2-setting. Thus, it is difficult to extend to other spaces. The possibility of a related proof, at least for the spaces $L^{2,q}$ with $1 \le q \le \infty$ and $q \neq 2$, would require carefully looking at the “measure of level sets.” Many technical difficulties would be expected to arise in such an attempt, and still not all cases would be covered. Nevertheless, the class of r.i. spaces X having the property (5.6), together with $X = L^2$, is rather large and suggests that $[T, X] = X$ should hold for all r.i. spaces satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$.

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