Annali di Matematica Pura ed Applicata (1923 -) (2019) 198:61–81
https://doi.org/10.1007/s10231-018-0762-8

Transverse Kähler structures on central foliations of complex manifolds

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Received: 5 February 2018 / Accepted: 12 May 2018 / Published online: 26 May 2018

Abstract For a compact complex manifold, we introduce holomorphic foliations associated with certain abelian subgroups of the automorphism group. If there exists a transverse Kähler structure on such a foliation, then we obtain a nice differential graded algebra which is quasi-isomorphic to the de Rham complex and a nice differential bi-graded algebra which is quasi-isomorphic to the Dolbeault complex like the formality of compact Kähler manifolds. Moreover, under certain additional condition, we can develop Morgan's theory of mixed Hodge structures as similar to the study on smooth algebraic varieties.

Keywords Transverse Kähler structure · Central foliation · Basic cohomology · Basic Dolbeault cohomology · Mixed Hodge structure

Mathematics Subject Classification Primary 32Q55; Secondary 37F75 · 55P62 · 57N65 · 58A14

1 Introduction

When a connected Lie group H acts on a smooth manifold M local freely, we have a smooth foliation $\mathcal{F}$ whose leaves are H-orbits. In addition, if M is a complex manifold, H is a complex Lie group and the H-action is holomorphic, then the foliation $\mathcal{F}$ is holomorphic. We are interested in transverse complex geometry on a foliated manifold $(M, \mathcal{F})$. Denote

The first author was supported by JSPS Grant-in-Aid for Young Scientists (B) 16K17596.

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by $\Omega^*(M)$ the space of the differential forms on M. We say that $\omega \in \Omega^*(M)$ is *basic* if $i_{X_v}\omega = 0$ and $L_{X_v}\omega = 0$ for any $v \in \mathfrak{h}$, where X_v denotes the fundamental vector field generated by $v \in \mathfrak{h}$, and i_{X_v} and L_{X_v} are the interior product and the Lie derivation with X_v, respectively. For a holomorphic foliation $\mathcal{F}$ on a complex manifold M with the complex structure J, a transverse Kähler structure on $\mathcal{F}$ is a closed real basic $(1,1)$-form ω such that $\omega_p(v, Jv) \geq 0$ for any $v \in T_pM$ and $p \in M$, and the equality holds if and only if v sits in the subspace $T_p\mathcal{F}$ that consists of all vectors tangent to the leaf through p.

In this paper, we introduce an intrinsically defined holomorphic foliation for arbitrary compact complex manifold, that we call *canonical foliation*. Let M be a compact complex manifold. Let G_M be the identity component of the group of all biholomorphisms on M. Let T be a maximal compact torus of G_M and $\mathfrak{t}$ the Lie algebra of T. Let J be the complex structure on the Lie algebra of G_M. Put

$$\mathfrak{h}_M := \mathfrak{t} \cap J\mathfrak{t}$$

and denote by H_M the corresponding Lie subgroup of G_M. Then H_M acts on M local freely (see [13]). Moreover, H_M is a central subgroup in G_M and H_M does not depend on the choice of T (see Lemma 2.1). By the local freeness, for any connected subgroup $H \subset H_M$, we have the holomorphic foliation $\mathcal{F}_H$. We call $\mathcal{F}_H$ a *central foliation* associated with H and $\mathcal{F}_{H_M}$ the *canonical foliation*. If H is a compact complex torus, then the central foliation $\mathcal{F}_H$ associated with H gives a holomorphic principal Seifert bundle structure on a complex manifold M over the complex orbifold M/H (see [22]). Moreover, if the action of H is free, then such Seifert bundle is a holomorphic principal torus bundle over a complex manifold. Conversely, holomorphic principal torus bundle structure gives a holomorphic free complex torus action. Thus, a central foliation is a generalization of a holomorphic principal torus bundle.

The purpose of this paper is to study (non-Kähler) complex manifolds admitting a transverse Kähler structure on a central foliation $\mathcal{F}_H$. In particular, we study the de Rham and Dolbeault complexes of such complex manifolds. Typical examples are holomorphic principal torus bundles over compact Kähler manifolds. A Calabi–Eckmann manifold is a holomorphic principal torus bundle over $\mathbb{C}P^m \times \mathbb{C}P^n$, and its underlying smooth manifold is diffeomorphic to $S^{2m+1} \times S^{2n+1}$. Extending Calabi–Eckmann's construction, Meersseman constructed a large class of non-Kähler compact complex manifolds. Such complex manifolds are called *LVM manifolds* (see [18,19]). Every LVM manifold admits a transverse Kähler structure (see [18]). Among LVM manifolds with the canonical foliations, some are principal Seifert bundles. At that time the leaf space $M/\mathcal{F}_{H_M}$ is a projective toric variety [19]. However, there are many LVM manifolds with the canonical foliations which are not principal Seifert bundles.

We notice that our object appears in a certain non-Käher Hermitian manifold. A Vaisman manifold is a non-Käher locally conformal Kähler manifold with the nonzero parallel Lee form. There are many important non-Kähler manifolds which are Vaisman (e.g., Hopf manifolds, Kodaira-Thurston manifolds). On any Vaisman manifold, there exists a complex one-dimensional central foliation with a transverse Kähler structure which is canonically determined by its Vaisman structure.

For a complex manifold M with a holomorphic foliation $\mathcal{F}$, we consider the basic de Rham complex $\Omega^*_B(M)$, basic Dolbeault complex $\Omega^{*,*}_B(M)$, basic de Rham cohomology $H^*_B(M)$ and basic Dolbeault cohomology $H^{*,*}_B(M)$ for $\mathcal{F}$. If there exists a transverse Kähler form with respect to $\mathcal{F}$ and $\mathcal{F}$ is homologically oriented, then there is the Hodge decomposition

$$H^r_B(M, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_B(M),$$

$$\overline{H_B^{p,q}(M)} = H_B^{q,p}(M)$$

(see [8]).

Definition 1.1 • For a manifold M, a (de Rham) model of M is a differential graded algebra (shortly DGA) A^* such that A^* is quasi-isomorphic to the de Rham complex $\Omega^*(M)$, i.e., there exists a sequence of DGA homomorphisms

$$A^* \leftarrow C_1^* \rightarrow C_2^* \leftarrow \cdots \leftarrow C_n^* \rightarrow \Omega^*(M)$$

such that all the morphisms are quasi-isomorphisms (i.e., inducing cohomology isomorphisms).

• For a complex manifold M, a Dolbeault model of M is a differential bi-graded algebra (shortly DBA) $B^{*,*}$ such that $B^{*,*}$ is quasi-isomorphic to the Dolbeault complex $\Omega^{*,*}(M)$, i.e., there exists a sequence of DBA homomorphisms

$$B^{*,*} \leftarrow C_1^{*,*} \rightarrow C_2^{*,*} \leftarrow \cdots \leftarrow C_n^{*,*} \rightarrow \Omega^{*,*}(M)$$

such that all the morphisms are quasi-isomorphisms.

On a compact Kähler manifold M, the de Rham cohomology $H^*(M)$ with the trivial differential is a model of M (Formality [7]) and the Dolbeault cohomology $H^{*,*}(M)$ with the trivial differential is a Dolbeault model of M (Dolbeault Formality [21]). In this paper we prove:

Theorem 1.2 (See also Theorem 4.13) *Let M be a compact complex manifold. We assume that M admits a transverse Kähler structure on a complex k-dimensional central foliation $\mathcal{F}_H$. Then there exists a model A^* of M with a differential d and a Dolbeault model $B^{*,*}$ of M with a differential $\bar\partial$ satisfying the followings:*

(1) *Let W be a real $2k$-dimensional vector space with a direct sum decomposition $W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$ satisfying $\overline{W^{1,0}} = W^{0,1}$. As graded algebras, $A^* = H_B^*(M) \otimes \bigwedge W$. As bi-graded algebras, $B^{*,*} = H_B^{*,*}(M) \otimes \bigwedge(W^{1,0} \oplus W^{0,1})$. Here, the degree of an element in W is 1 and bi-degree of an element in $W^{1,0}$ (respectively, $W^{0,1}$) is $(1,0)$ (respectively, $(0,1)$).*

(2) *The differentials d and $\bar\partial$ are trivial on $H_B^*(M)$ and $H_B^{*,*}(M)$ respectively. $dW \subset H_B^2(M)$, $\bar\partial W^{1,0} \subset H_B^{1,1}(M)$ and $\bar\partial W^{0,1} \subset H_B^{0,2}(M)$.*

In [25], Tanré constructed a Dolbeault model for a holomorphic principal torus bundle over a compact Kähler manifold. The theorem above slightly generalizes the result of Tanré.

More precisely, a vector space W as in the theorem is a $2k$-dimensional subspace of $\Omega^1(M)^H$ such that the bilinear map $\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v} w \in \mathbb{R}$ is non-degenerate, where $\mathfrak{h}$ is the Lie algebra of H and $\Omega^1(M)^H$ is the H-invariant subspace of $\Omega^1(M)$. We can choose W to be closed under the complex structure of $\Omega^1(M)$. Then $W^{1,0}$ and $W^{0,1}$ are defined as $(1,0)$-part and $(0,1)$-part of $W \otimes \mathbb{C}$, respectively. The differential $W \rightarrow H_B^2(M)$ is given by $w \mapsto [dw]_B$, where $[dw]_B$ denotes the basic cohomology class represented by $dw \in \Omega_B^2(M)$. Similarly, the differentials $W^{1,0} \rightarrow H_B^{1,1}(M)$ and $W^{0,1} \rightarrow H_B^{0,2}(M)$ are given by $w \rightarrow [\bar\partial w]_B$, where $[\bar\partial w]_B$ denotes the basic Dolbeault cohomology class represented by $\bar\partial w \in \Omega_B^2(M) \otimes \mathbb{C}$.

By these results, we can construct explicit de Rham and Dolbeault models of Vaisman manifolds (see Sect. 6.3). Recently, similar de Rham models are also constructed in [5].

Definition 1.3 A central foliation $\mathcal{F}_H$ is *fundamental* if for any $w \in W$, $[dw]_B \in H_B^2(M)$ is represented by a closed basic $(1,1)$-form.

We prove:

Theorem 1.4 (See also Theorem 5.6) *Let M be a compact complex manifold. We assume that M admits a transverse Kähler structure on a fundamental central foliation $\mathcal{F}_H$. Then the de Rham cohomology of M admits an $\mathbb{R}$-mixed Hodge structure so that:*

(1) $H^1(M, \mathbb{C}) = H^1_{1,0} \oplus H^1_{0,1} \oplus H^1_{1,1}$
(2) $H^2(M, \mathbb{C}) = H^2_{2,0} \oplus H^2_{1,1} \oplus H^2_{0,2} \oplus H^2_{2,1} \oplus H^2_{1,2} \oplus H^2_{2,2}$

and Sullivan's minimal model of the complex valued de Rham complex admits the Morgan's bigrading [20].

As a consequence of this result, we can say that not every finitely generated group can be the fundamental group of a compact complex manifold admitting a transverse Kähler structure on a fundamental central foliation.

2 Central foliations

Let M be a compact complex manifold. In this section we define the *canonical* foliation and *central* foliations on M. Let G_M be the identity component of the group of all biholomorphisms on M. G_M is a complex Lie group (see [3]). Denote by $\mathfrak{g}_M$ the Lie algebra of G_M and by J the complex structure on $\mathfrak{g}_M$. Let T be a maximal compact torus of G_M and $\mathfrak{t}$ the Lie algebra of T. Put

$$\mathfrak{h}_M := \mathfrak{t} \cap J\mathfrak{t}$$

and denote by H_M the corresponding Lie subgroup of G_M. Then H_M acts on M local freely (see [13, Proposition 3.3]).

Lemma 2.1 *The following holds:*

(1) *Elements in $\mathfrak{h}_M$ centralize $\mathfrak{g}_M$.*
(2) *$\mathfrak{h}_M$ does not depend on the choice of T.*

Proof Since T is compact, $\mathfrak{g}_M$ is a unitary representation of T. In particular, $\mathfrak{g}_M$ is a unitary representation of H_M. However, H_M is a holomorphic subgroup of G_M and hence $\mathfrak{g}_M$ is a holomorphic representation of H_M. Therefore, $\mathfrak{g}_M$ is a trivial representation of H_M, showing Part (1).

Let T' be another maximal compact torus of G_M. Then, there exists $g \in G_M$ such that $gTg^{-1} = T'$ (see [12, Chapter XV, Section 3] for detail). Put

$$\mathfrak{h}' := \mathfrak{t}' \cap J\mathfrak{t}'.$$

Then, it follows from $gTg^{-1} = T'$ that $\mathrm{Ad}_g(\mathfrak{h}_M) = \mathfrak{h}'$. On the other hand, by (1), we have that Ad_g is the identity on $\mathfrak{h}_M$. Therefore, $\mathfrak{h}_M$ does not depend on the choice of T, proving (2). □

We remark that any $\mathbb{C}$-subspace $\mathfrak{h}$ of $\mathfrak{h}_M$ defines a holomorphic foliation $\mathcal{F}_H$ on M. We call $\mathcal{F}_H$ a *central foliation* on M and $\mathcal{F}_{H_M}$ the *canonical foliation* on M. It follows from Lemma 2.1 that the canonical foliation does not depend on the choice of T, that is, the canonical foliation is intrinsic to compact complex manifolds.

3 Hirsch extensions and minimal models

In this section, DGAs are defined over $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, if we do not specify. Let (A^*, d_A) be a DGA. Let k be an integer. For a linear map $\beta : V \to A^{k+1}$ with $d_A \circ \beta = 0$, we define a Hirsch extension (B, d_B) of A^* in degree k such that $B^* = A^* \otimes \bigwedge V$ with $\deg(v) = k$ for any $v \in V$, $d_B = d_A$ on A^* and $d_B = \beta$ on V. Defining the filtration on B^* by $F^p(B^*) = A^{*\geq p} \otimes \bigwedge V$, we have the spectral sequence $E^{*,*}$ with $E_2^{p,q} = H^p(A) \otimes \bigwedge^q V$. Consider the composition $q \circ \beta : V \to H^{k+1}(A^*)$ where $q : \ker d_A \to H^*(A^*)$ is the quotient map. The DGA structure of B^* is determined by the map $q \circ \beta$ (independent of a choice of β) [10, 10.2].

Lemma 3.1 *Let (A_1^*, d_{A_1}) and (A_2^*, d_{A_2}) be DGAs and $f : A_1^* \to A_2^*$ a quasi-isomorphism. Then for a Hirsch extension $A_1^* \otimes V$ (resp. $A_2^* \otimes V$), we have a Hirsch extension $A_2^* \otimes V$ (resp. $A_1^* \otimes V$) and quasi-isomorphism*

$$A_1^* \otimes V \to A_2^* \otimes V.$$

Proof In case $A_1^* \otimes V$ ($\beta_1 : V \to A_1^{k+1}$) is given. Consider the Hirsch extension $A_2^* \otimes V$ given by $\beta_2 = f \circ \beta_1 : V \to A_2^*$ and the homomorphism $f \otimes \mathrm{id} : A_1^* \otimes V \to A_2^* \otimes V$. Then we can easily show that $f \otimes \mathrm{id}$ induces an isomorphism on the E_2-term of the spectral sequence. Hence $f \otimes \mathrm{id}$ is a quasi-isomorphism.

In case $A_2^* \otimes V$ ($\beta_2 : V \to A_2^{k+1}$) is given. Since f is a quasi-isomorphism, we can take a linear map $\beta_1 : V \to A_1^*$ so that $d \circ \beta_1 = 0$ and $q \circ f \circ \beta_1 = q \circ \beta_2$. By the same argument as above, the Hirsch extension of A_2^* given by $\beta_2 : V \to A_1^{k+1}$ is identified with the one given by $f \circ \beta_1 : V \to A_1^{k+1}$. Under this identification, we have the homomorphism $f \otimes \mathrm{id} : A_1^* \otimes V \to A_2^* \otimes V$, and as in the first case, we can show that this homomorphism is a quasi-isomorphism. □

Definition 3.2 A DGA $\mathcal{M}^*$ is minimal if:

- $\mathcal{M}^0 = \mathbb{K}$.
- $\mathcal{M}^* = \bigcup \mathcal{M}_i^*$ for a sequence of sub-DGAs

$$\mathbb{K} = \mathcal{M}_0^* \subset \mathcal{M}_1^* \subset \dots$$

  such that $\mathcal{M}_{i+1}^*$ is a Hirsch extension of $\mathcal{M}_i^*$.
- $d\mathcal{M}^* \subset \mathcal{M}^+ \cdot \mathcal{M}^+$ where $M^+ = \bigoplus_{j>0} \mathcal{M}^j$.

We say that a DGA $\mathcal{M}^*$ is k-minimal if $\mathcal{M}^*$ is minimal and $\bigoplus_{j>k} \mathcal{M}^j \subset \mathcal{M}^+ \cdot \mathcal{M}^+$. Equivalently, each extension in a sequence for $\mathcal{M}^*$ has degree at most k.

Definition 3.3 Let A^* be a DGA with $H^0(A^*) = \mathbb{K}$.

- A minimal DGA $\mathcal{M}^*$ is a *minimal model* of A^* if there is a quasi-isomorphism $\mathcal{M} \to A^*$.
- A k-quasi-isomorphism $\mathcal{M}^* \to A^*$ is a homomorphism of DGAs that induces an isomorphism $H^j(\mathcal{M}^*) \cong H^j(A^*)$ for $j \leq k$ and an injection $H^{k+1}(\mathcal{M}^*) \hookrightarrow H^{k+1}(A^*)$. A k-minimal DGA $\mathcal{M}^*$ is the *k-minimal model* of A^* if there is a k-quasi-isomorphism $\mathcal{M}^* \to A^*$.

Theorem 3.4 [24] *For a DGA A^* with $H^0(A^*) = \mathbb{K}$, a minimal model and a k-minimal model exist, and each of them is unique up to DGA isomorphism.*

The minimal models give the following "de Rham homotopy theory". We shall state it but omit the details. See [7,10,20,24] for the details.

Theorem 3.5 *Let M be a compact smooth manifold. Consider the de Rham complex $\Omega^*(M)$ as a DGA. Then*

- *The 1-minimal model of $\Omega^*(M)$ is the dual to the Lie algebra of the nilpotent completion of $\pi_1(M)$.*
- *If M is simply connected, then the minimal model of $\Omega^*(M)$ determines the real homotopy type of M.*

4 Models for transverse Kähler torus actions

In this section, we give a model and Dolbeault model of a complex manifold equipped with a central foliation.

4.1 Models for compact Lie group actions

The following result is well-known (see [9] for example).

Proposition 4.1 *Let M be a compact manifold and K a compact connected Lie group. Assume that K acts on M. Then the inclusion*

$$\Omega^*(M)^K \subset \Omega(M)$$

induces a cohomology isomorphism.

Let M be a complex manifold. Let $(\Omega^{*,*}(M), \bar{\partial})$ be the Dolbeault complex of M. Suppose that a group K acts on M as biholomorphisms. Then the space $\Omega^{*,*}(M)^K$ of K-invariant differential forms is a subcomplex of $\Omega^{*,*}(M)$.

Proposition 4.2 *Let M be a compact complex manifold and K a connected compact Lie group. Assume that K acts on M as biholomorphisms and the induced action on the Dolbeault cohomology is trivial. Then the inclusion*

$$\Omega^{*,*}(M)^K \subset \Omega^{*,*}(M)$$

induces an isomorphism on Dolbeault cohomology.

Proof Let $d\mu$ be the normalized Haar measure of K. Define the linear map

$$I : \Omega^{*,*}(M) \ni \omega \mapsto \int_{g\in K} g^*\omega d\mu \in \Omega^{*,*}(M)^K.$$

Then I commutes with Dolbeault operator $\bar{\partial}$, that is, I induces a bi-graded module homomorphism $H^{*,*}(M) \to H(\Omega^{*,*}(M)^K)$. Since I is the identity on $\Omega^{*,*}(M)^K$, the composition

$$H(\Omega^{*,*}(M)^K) \to H^{*,*}(M) \to H\left(\Omega^{*,*}(M)^K\right)$$

of the homomorphisms induced by the inclusion and I is the identity. Therefore, the homomorphism $H(\Omega^{*,*}(M)^K) \to H^{*,*}(M)$ induced by the inclusion $\Omega^{*,*}(M)^K \subset \Omega^{*,*}(M)$ is injective.

Since the induced action on the Dolbeault cohomology is trivial, for a $\bar{\partial}$-closed form $\omega \in \Omega^{*,*}(M)$ and any $g \in K$, there exists $\theta_g \in \Omega^{*,*}(M)$ such that

$$\omega - g^*\omega = \bar{\partial}\theta_g.$$

By using Green operator, we can take θ_g smoothly on K. Integrating by $d\mu$, we have

$$\omega - I\omega = \bar{\partial} \int_{g\in K} \theta_g d\mu.$$

Hence the inclusion

$$\Omega^{*,*}(M)^K \subset \Omega^{*,*}(M)$$

induces a surjection on Dolbeault cohomology. □

Corollary 4.3 *Let M be a compact complex manifold and K a connected compact Lie group acting on M as biholomorphisms. Let H be a dense Lie subgroup of K such that H is a complex Lie group and the restricted action of K to H on M is holomorphic. Then, the inclusion $\Omega^{*,*}(M)^K \subset \Omega(M)$ induces an isomorphism on Dolbeault cohomology.*

Proof By Proposition 4.2, we only need to know that the representation of K on $H^{*,*}(M)$ is trivial under the assumptions of this proposition. Since K acts on M as biholomorphisms, the representation of K on $H^{*,*}(M)$ is $\mathbb{C}$-linear. Since K is compact, there exists a Hermitian inner product on $H^{*,*}(M)$ that is invariant under K.

Consider the restricted representation $H \to GL(H^{*,*}(M))$. Since H is a complex Lie group and the restricted action of K to H on M is holomorphic, this representation is holomorphic [16]. On the other hand, by the same argument as above, this representation is unitary. Therefore, the representation of H on $H^{*,*}(M)$ is trivial. Since H is dense in K, the representation of K on $H^{*,*}(M)$ is also trivial. The proposition is proved. □

4.2 Models for torus actions

Let T be a compact torus and H a connected Lie subgroup (not necessary to be closed in T). Let M be a paracompact smooth manifold equipped with an action of T. In this section, we suppose that the restricted action of T to H on M is local free. Denote by $\mathfrak{t}$ and $\mathfrak{h}$ the Lie algebras of T and H respectively.

Lemma 4.4 *There exists a $\mathfrak{h}$-valued 1-form ω on M such that*

(1) $i_{X_v}\omega = v$ *for all* $v \in \mathfrak{h}$,
(2) ω *is* T*-invariant.*

Proof Since T is compact and M is paracompact, it follows from the slice theorem that there exists a locally finite open covering $\mathcal{U} = \{U_\lambda\}_\lambda$ such that each U_λ is T-equivariantly diffeomorphic to $T \times_{T_\lambda} V_\lambda$ via φ_λ, where T_λ is a closed subgroup of T and V_λ is a representation space of T_λ. Let $\pi : T \times_{T_\lambda} V_\lambda \to T/T_\lambda$ be the map induced by the first projection $T \times V_\lambda \to T$. Since the action of H on M is local free, we have that $\mathfrak{h} \cap \mathfrak{t}_\lambda = 0$. Therefore, there exists a $\mathfrak{h}$-valued 1-form ω_λ on T/T_λ that satisfies the conditions (1) and (2). Since π and φ_λ are T-invariant, the pull-back $(\pi \circ \varphi_\lambda)^*\omega_\lambda$ that is a $\mathfrak{h}$-valued 1-form on U_λ also satisfies the conditions (1) and (2).

Let $\{\rho_\lambda\}$ be a partition of unity subordinate to the open covering $\mathcal{U}$. Averaging ρ_λ with the normalized Haar measure on T, we may assume that every ρ_λ is T-invariant. Then the 1-form

$$\omega := \sum_\lambda \rho_\lambda (\pi \circ \varphi_\lambda)^*\omega_\lambda$$

on M satisfies the condition (1) and (2), as required. □

Since the H-action is local free, the H-action induces the foliation $\mathcal{F}$ whose leaves are H-orbits of M. Denote by T' the closure of H.

Lemma 4.5 $\Omega^*(M)^{T'} = \Omega^*(M)^H$.

Proof Since $H \subset T'$, we have the inclusion $\Omega^*(M)^{T'} \subset \Omega^*(M)^H$. For $g \in T'$, take a sequence $\{g_i\}_{i=1,\ldots}$ of elements in H so that $\lim_{i\to\infty} g_i = g$. Then we have

$$g^*\omega = \lim_{i\to\infty} g_i^*\omega = \omega$$

for any $\omega \in \Omega^*(M)^H$, showing the opposite inclusion $\Omega^*(M)^{T'} \supset \Omega^*(M)^H$. The lemma is proved. □

Consider the basic forms

$$\Omega^*_B(M) = \left\{\omega \in \Omega^*(M) \mid i_{X_v}\omega = L_{X_v}\omega = 0,\ \forall v \in \mathfrak{h}\right\}.$$

We want to construct a finite-dimensional subspace $W \subset \Omega^1(M)^H$ such that

- $dW \subset \Omega^2_B(M)$,
- the bilinear map $\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v}w \in \mathbb{R}$ is non-degenerate.

To do this, take a $\mathfrak{h}$-valued 1-form ω as in Lemma 4.4. For a basis $v_1, \ldots, v_k$ of $\mathfrak{h}$, we may write $\omega = \sum_{i=1}^k w_i \otimes v_i$ with 1-forms $w_1, \ldots, w_k$. We claim that $dw_i \in \Omega^2_B(M)$. Since w_i is T-invariant, by Cartan formula we have

$$0 = L_{X_v}w_i = di_{X_v}w_i + i_{X_v}dw_i = i_{X_v}dw_i$$

for $v \in \mathfrak{h}$ because $i_{X_v}w_i$ is constant on M. By Cartan formula again,

$$L_{X_v}dw_i = di_{X_v}dw_i + i_{X_v}ddw_i = di_{X_v}dw_i.$$

This together with $i_{X_v}dw_i = 0$ yields that $dw_i \in \Omega^*_B(M)$. Then $W = \langle w_1, \ldots, w_k\rangle$ is a desired space.

Proposition 4.6 *We have the decomposition*

$$\Omega^*(M)^H = \Omega^*_B(M) \otimes \bigwedge W.$$

Proof For $\omega \in \Omega^*_B(M)$, the condition $L_{X_v}\omega = 0$ for all $v \in \mathfrak{h}$ implies that $\omega \in \Omega^*(M)^H$. Since $W \subset \Omega^1(M)^H$ and $\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v}w \in \mathbb{R}$ is non-degenerate, we have the inclusion

$$\Omega^*_B(M) \otimes \bigwedge W \subset \Omega^*(M)^H.$$

We will show that $\Omega^*(M)^H \subset \Omega^*_B(M) \otimes \bigwedge W$. We say that $\omega \in \Omega^*(M)^H$ is of q-type if for any $v_1, \ldots, v_q \in \mathfrak{h}$ we have

$$i_{X_{v_1}} \ldots i_{X_{v_q}}\omega = 0.$$

If $\omega \in \Omega^*(M)^H$ is of 1-type, $\omega \in \Omega^*_B(M)$. Suppose that $\omega \in \Omega^*(M)^H$ is of q-type for some $q \geq 2$. Then for any $v, v_1, \ldots, v_{q-1} \in \mathfrak{h}$, we have that

$$i_{X_v}i_{X_{v_1}} \ldots i_{X_{v_{q-1}}}\omega = 0$$

and

$$L_{X_v}i_{X_{v_1}} \ldots i_{X_{v_{q-1}}}\omega = i_{X_{v_1}} \ldots i_{X_{v_{q-1}}}L_{X_v}\omega = 0.$$

Therefore, we have that

$$i_{X_{v_1}} \dots i_{X_{v_{q-1}}} \omega \in \Omega^*_B(M).$$

Take a basis $v_1, \dots v_k$ of $\mathfrak{h}$ and the dual basis $w_1, \dots, w_k$ of W given by $W \subset \Omega^1(M)^H$ and $\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v} w \in \mathbb{R}$. Then for $\omega \in \Omega^*(M)^H$ of q-type, we can see that the form

$$\omega' = \omega - \sum_{i_1 < i_2 < \cdots < i_{q-1}} (i_{X_{v_{i_1}}} \dots i_{X_{v_{i_{q-1}}}} \omega) \wedge w_{i_1} \wedge \cdots \wedge w_{i_{q-1}}.$$

is of $(q-1)$-type. It turns out that $\omega - \omega' \in \Omega^*_B(M) \otimes \bigwedge^{q-1} W$. Since ω' is of $(q-1)$-type, applying the same argument eventually, we have that

$$\omega \in \bigoplus_{0 \le j \le q-1} \Omega^*_B(M) \otimes \bigwedge^j W,$$

showing the inclusion $\Omega^*(M)^H \subset \Omega^*_B(M) \otimes \bigwedge W$. The proposition is proved. □

By Propositions 4.1, 4.6 and Lemma 4.5, we have the following result.

Corollary 4.7 *The inclusion*

$$\Omega^*_B(M) \otimes \bigwedge W \to \Omega^*(M)$$

induces a cohomology isomorphism.

Proposition 4.8 *Suppose that* $\dim M = n + k$. *Then,* $H^{n+k}(M) \cong H^n_B(M)$. *In particular,* $\mathcal{F}$ *is homologically oriented if* M *is compact and oriented.*

Proof By Proposition 4.1 and Lemma 4.5, we can choose a representative α of an element in $H^{n+k}(M)$ so that α sits in $\Omega^{n+k}(M)^H$. By Proposition 4.6, there uniquely exists $\beta \in \Omega^n_B(M)$ such that $\alpha = \beta \wedge w_1 \wedge \cdots \wedge w_k$. Conversely, for $\beta \in \Omega^n_B(M)$, $\alpha := \beta \wedge w_1 \wedge \cdots \wedge w_k \in \Omega^n(M)^H$. Thanks to the degrees, α and β both are automatically closed. Therefore, it suffices to show that α is exact if and only if β is exact (in the sense of basic).

Let $\alpha' \in \Omega^{n+k-1}(M)^H$ such that $d\alpha' = \alpha$. By Proposition 4.6, we can write

$$\alpha' = \beta' \wedge w_1 \wedge \cdots \wedge w_k + \sum_{i=1}^{k} \beta_i \wedge w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_k$$

with $\beta' \in \Omega^{n-1}_B(M)$ and $\beta_i \in \Omega^n_B(M)$ for $i = 1, \dots, k$. Then, it follows from $dw_j \in \Omega^2_B(M)$ that $\alpha = d\alpha' = d\beta' \wedge w_1 \wedge \cdots \wedge w_k$. In particular, $\beta = d\beta'$.

To see the converse, let $\beta' \in \Omega^{n-1}_B(M)$ such that $d\beta' = \beta$. Then

$$d(\beta' \wedge w_1 \wedge \cdots \wedge w_k) = \beta \wedge w_1 \wedge \cdots \wedge w_k + (-1)^{n-1} \beta' \wedge d(w_1 \wedge \cdots \wedge w_k).$$

Since $\beta' \wedge dw_j = 0$ by the degree, we have that

$$\alpha = d(\beta' \wedge w_1 \wedge \cdots \wedge w_k),$$

showing the equivalence of exactness between α and β. The proposition is proved. □

4.3 Models for transverse Kähler torus actions

Let M be a compact complex manifold and T a compact torus acting on M as biholomorphisms. Let H be a dense Lie subgroup of T such that H is a complex Lie group and the restricted action of T to H on M is holomorphic and local free. Then we have a holomorphic central foliation $\mathcal{F}$ on M whose leaves are H-orbits. As before, let $\Omega^*_B(M)$ denote the space of basic differential forms with respect to $\mathcal{F}$. Since M is a complex manifold and the H-action is holomorphic, $\Omega^1(M)^H$ and $\Omega^1_B(M)$ both are complex vector spaces.

Proposition 4.9 *There exists a $\mathbb{C}$-subspace W of $\Omega^1(M)^H$ such that*

- *$dW \subset \Omega^2_B(M)$ and*
- *$\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v} w \in \mathbb{R}$ is non-degenerate.*

Proof By Lemma 4.4, there exists a $\mathfrak{h}$-valued 1-form $w \in \Omega^1(M) \otimes \mathfrak{h}$ on M such that $i_{X_v} w = v$ for all $v \in \mathfrak{h}$ and H-invariant. Let $v_1, \dots, v_k$ be a $\mathbb{C}$-basis of $\mathfrak{h}$ and $J_\mathfrak{h}$ the complex structure on $\mathfrak{h}$. Then $v_1, \dots, v_k, J_\mathfrak{h} v_1, \dots, J_\mathfrak{h} v_k$ form a $\mathbb{R}$-basis of $\mathfrak{h}$. There exist $w_1, \dots, w_k, w_{k+1}, \dots, w_{2k} \in \Omega^1(M)^H$ such that

$$w = \sum_{i=1}^{k} w_i \otimes v_i + \sum_{j=1}^{k} w_{k+j} \otimes J_\mathfrak{h} v_j.$$

For $i = 1, \dots, k$, we define $w'_i \in \Omega^1(M)^H$ to be $w'_i = -w_i \circ J$, where J denotes the complex structure on M. We define an H-invariant $\mathfrak{h}$-valued 1-form

$$w' = \sum_{i=1}^{k} w_i \otimes v_i + \sum_{j=1}^{k} w'_j \otimes J_\mathfrak{h} v_j.$$

It follows that $i_{X_v} w' = v$ for all $v \in \mathfrak{h}$ by definition of w' immediately. The subspace $W = \langle w_1, \dots, w_k, w'_1, \dots, w'_k \rangle$ of $\Omega^1(M)^H$ is closed under J. It follows from the Cartan formula that $dW \subset \Omega^2_B(M)$ immediately. Therefore, W is a desired space, proving the proposition. □

Let W be a $\mathbb{C}$-subspace of $\Omega^1_B(M)$ as in Proposition 4.9. Then $W \otimes \mathbb{C}$ is decomposed into (1, 0)-part $W^{1,0}$ and (0, 1)-part $W^{0,1}$. By tensoring $\mathbb{C}$ with $\Omega^*(M) \otimes \bigwedge W$, we have the DBA

$$\Omega^{*,*}_B(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1})$$

with the Dolbeault operator $\bar\partial$.

By Propositions 4.2, 4.6 and Lemma 4.5, we have the following result.

Corollary 4.10 *We have an injection*

$$\Omega^{*,*}_B(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1}) \to \Omega^{*,*}(M)$$

which induces a cohomology isomorphism.

We consider the bi-graded bi-differential algebra (BBA) $(\Omega^{*,*}_B(M), \partial_B, \bar\partial_B)$. Put $d^c = \sqrt{-1}(\bar\partial_B - \partial_B)$. Then d^c is a differential on $\Omega^*_B(M)$. We say that the $\partial_B \bar\partial_B$-lemma holds if

$$\ker \partial_B \cap \ker \bar\partial_B \cap \operatorname{im} d = \operatorname{im} \partial_B \bar\partial_B.$$

If the $\partial_B \bar{\partial}_B$-lemma holds, then we have the quasi-isomorphisms

$$(\ker d^c, d) \to (\Omega^*_B(M), d),$$
$$(\ker d^c, d) \to (H^*_B(M), 0),$$
$$(\ker \partial_B, \bar{\partial}_B) \to (\Omega^{*,*}_B(M), \bar{\partial}_B)$$

and

$$(\ker \partial_B, \bar{\partial}_B) \to (H^{*,*}_B(M), 0)$$

(see [7]).

Proposition 4.11 *Suppose that the $\partial_B \bar{\partial}_B$-lemma holds. Then there exist quasi-isomorphisms*

$$(\ker d^c \otimes \bigwedge W, d) \to (\Omega^*_B(M) \otimes \bigwedge W, d),$$
$$(\ker d^c \otimes \bigwedge W, d) \to (H^*_B(M) \otimes \bigwedge W, d),$$
$$(\ker \partial_B \otimes \bigwedge (W^{1,0} \oplus W^{0,1}), \bar{\partial}') \to (\Omega^{*,*}_B(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1}), \bar{\partial})$$

and

$$(\ker \partial_B \otimes \bigwedge (W^{1,0} \oplus W^{0,1}), \bar{\partial}') \to (H^{*,*}_B(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1}), \bar{\partial}).$$

Here, $\bar{\partial}'$ is a differential such that $(\bar{\partial}' - \bar{\partial})w$ is ∂_B-exact for any $w \in W^{1,0} \oplus W^{0,1}$ and $\bar{\partial}'\alpha = \bar{\partial}\alpha$ for any $\alpha \in \ker \partial_B$.

Proof This follows from Lemma 3.1 immediately. □

Theorem 4.12 (see [7,8]) *Let M be a compact manifold with a homologically oriented (that is, $H^{\mathrm{codim}\,\mathcal{F}}_B(M) \neq 0$) transversely Kähler foliation $\mathcal{F}$. Then for the BBA $(\Omega^{*,*}_B(M), \partial_B, \bar{\partial}_B)$, the $\partial_B \bar{\partial}_B$-lemma holds.*

This together with Propositions 4.8 and 4.11 implies the following result.

Theorem 4.13 *Assume that the central foliation $\mathcal{F}$ admits a transversely Kähler structure. Then the DGAs $\Omega^*(M)$ and $H^*_B(M) \otimes \bigwedge W$ (resp. DBAs $\Omega^{*,*}(M)$ and $H^{*,*}_B(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1})$) are quasi-isomorphic.*

5 Mixed Hodge structures

The purpose of this section is to show that the cohomology and minimal model of a complex manifold equipped with a special transverse Kähler structure on a central foliation admits a certain bigrading. We begin with basic notions and facts.

5.1 Mixed Hodge structures

Let V be an $\mathbb{R}$-vector space. An $\mathbb{R}$-*Hodge structure* of weight n on an $\mathbb{R}$-vector space V is a finite decreasing filtration F^* on $V_{\mathbb{C}} = V \otimes \mathbb{C}$ such that

$$F^p(V_{\mathbb{C}}) \oplus \overline{F^{n+1-p}(V_{\mathbb{C}})} = V_{\mathbb{C}}$$

for each p. Equivalently, there exists a finite bigrading

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V_{p,q}$$

such that

$$\overline{V_{p,q}} = V_{q,p}.$$

An $\mathbb{R}$-*mixed Hodge structure* on V is a pair (W_*, F^*) such that:

(1) W_* is an increasing filtration which is bounded below,
(2) F^* is a decreasing filtration on $V_{\mathbb{C}}$ such that the filtration on $Gr_n^W V_{\mathbb{C}}$ induced by F^* is an $\mathbb{R}$-Hodge structure of weight n.

We call W_* the *weight filtration* and F^* the *Hodge filtration*. If there exists a finite bigrading

$$V_{\mathbb{C}} = \bigoplus V_{p,q}$$

satisfying

$$\overline{V_{p,q}} = V_{q,p},$$

$W_n(V_{\mathbb{C}}) = \bigoplus_{p+q\le n} V_{p,q}$ and $F_r(V_{\mathbb{C}}) = \bigoplus_{p\ge r} V_{p,q}$ for any n, p, q, r, then we say that an $\mathbb{R}$-mixed Hodge structure (W_*, F^*) is $\mathbb{R}$-split.

Even if an $\mathbb{R}$-mixed Hodge structure (W_*, F^*) is not $\mathbb{R}$-split, we can obtain a canonical bigrading of (W_*, F^*).

Proposition 5.1 [20, Proposition 1.9] *Let (W_*, F^*) be an $\mathbb{R}$-mixed Hodge structure on an $\mathbb{R}$-vector space V. Define $V_{p,q} = R_{p,q} \cap L_{p,q}$ where $R_{p,q} = W_{p+q}(V_{\mathbb{C}}) \cap F^p(V_{\mathbb{C}})$ and $L_{p,q} = W_{p+q}(V_{\mathbb{C}}) \cap \overline{F^q(V_{\mathbb{C}})} + \sum_{i\ge 2} W_{p+q-i}(V_{\mathbb{C}}) \cap \overline{F^{q-i+1}(V_{\mathbb{C}})}$. Then we have the bigrading $V_{\mathbb{C}} = \bigoplus V_{p,q}$ such that $\overline{V_{p,q}} = V_{q,p}$ modulo $\bigoplus_{r+s<p+q} V_{r,s}$, $W_n(V_{\mathbb{C}}) = \bigoplus_{p+q\le n} V_{p,q}$ and $F^r(V_{\mathbb{C}}) = \bigoplus_{p\ge r} V_{p,q}$.*

We say that the bigrading in this proposition is the *canonical bigrading* of an $\mathbb{R}$-mixed Hodge structure (W_*, F^*).

We notice that this bigrading gives an equivalence of the category of $\mathbb{R}$-mixed Hodge structures on V and bigradings $V_{\mathbb{C}} = \bigoplus V_{p,q}$ such that $(\bigoplus_{p+q\le i} V_{p,q}) \cap V$ is a real structure of $\bigoplus_{p+q\le i} V_{p,q}$ and $\overline{V_{p,q}} = V_{q,p}$ modulo $\bigoplus_{r+s<p+q} V_{r,s}$ (see [20, Proposition 1.11]).

5.2 Morgan's mixed Hodge diagrams

In [6], Deligne proves that the real cohomology of a smooth algebraic variety over $\mathbb{C}$ admits a canonical $\mathbb{R}$-mixed Hodge structure. The following is Morgan's reformulation of Deligne's technique for studying the mixed Hodge theory on Sullivan's minimal models.

Definition 5.2 [20, Definition 3.5] An $\mathbb{R}$-*mixed Hodge diagram* is a pair of filtered $\mathbb{R}$-DGA (A^*, W_*) and bifiltered $\mathbb{C}$-DGA (E^*, W_*, F^*) and filtered DGA map $\phi : (A^*_{\mathbb{C}}, W_*) \to (E^*, W_*)$ such that:

(1) ϕ induces an isomorphism $\phi^* : {}_W E_1^{*,*}(A^*_{\mathbb{C}}) \to {}_W E_1^{*,*}(E^*)$ where ${}_W E_*^{*,*}(\cdot)$ is the spectral sequence for the decreasing filtration $W^* = W_{-*}$.
(2) The differential d_0 on ${}_W E_0^{*,*}(E^*)$ is strictly compatible with the filtration induced by F.
(3) The filtration on ${}_W E_1^{p,q}(E^*)$ induced by F is an $\mathbb{R}$-Hodge structure of weight q on $\phi^*({}_W E_1^{*,*}(A^*))$.

Now, Deligne's $\mathbb{R}$-mixed Hodge structure is described by the following way.

Theorem 5.3 [20, Theorem 4.3] *Let* $\{(A^*, W_*), (E^*, W_*, F^*), \phi\}$ *be an* $\mathbb{R}$*-mixed Hodge diagram. Define the filtration* W'_* *on* $H^r(A^*)$ *(resp.* $H^r(E^*)$*) as* $W'_i H^r(A^*) = W_{i-r}(H^r(A^*))$ *(resp.* $W'_i H^r(E^*) = W_{i-r}(H^r(E^*))$*). Then the filtrations* W'_* *and* F^* *on* $H^r(E^*)$ *give an* $\mathbb{R}$*-mixed Hodge on* $\phi^*(H^r(A^*))$*.*

Example 5.4 Let H^* be a graded commutative $\mathbb{R}$-algebra. We suppose that for any p, q, H^p admits an $\mathbb{R}$-Hodge structure $H^p \otimes \mathbb{C} = \bigoplus_{s+t=p} H^{s,t}$ of weight p and the multiplication $H^p \times H^q \to H^{p+q}$ is a morphism of Hodge structures. Let V be an $\mathbb{R}$-vector space with a linear map $\beta : V \to H^2$. We suppose that V admits an $\mathbb{R}$-Hodge structure $V \otimes \mathbb{C} = \bigoplus_{s+t=2} V^{s,t}$ of weight 2 and $\beta : V \to H^2$ is a morphism of Hodge structure. (e.g., $\beta(V) \subset H^{1,1}$.)

Under these assumptions, regarding H^* as a DGA with trivial differential, we consider the Hirsch extension $A^* = H^* \otimes \bigwedge V$. Define the increasing filtration $W_* A^*$ as

$$W_k A^q = \bigoplus_{l \le k} H^{q-l} \otimes \bigwedge^l V$$

and decreasing filtration $F^* A^*_{\mathbb{C}}$ as the Hodge filtration for the Hodge structure on $(H^p \otimes \mathbb{C}) \otimes \bigwedge^q (V \otimes \mathbb{C})$. Then for any p, q, we have:

- ${}_W E_0^{-p,q}(A^*_{\mathbb{C}}) = (H^{q-2p} \otimes \mathbb{C}) \otimes \bigwedge^p (V \otimes \mathbb{C})$ and d_0 is trivial.
- ${}_W E_1^{-p,q}(A^*_{\mathbb{C}}) = (H^{q-2p} \otimes \mathbb{C}) \otimes \bigwedge^p (V \otimes \mathbb{C})$ and clearly F induces the Hodge structure of weight q.

Thus $\{(A^*, W_*), (A^*_{\mathbb{C}}, W_*, F^*), \mathrm{id} : A^*_{\mathbb{C}} \to A^*_{\mathbb{C}}\}$ is an $\mathbb{R}$-mixed Hodge diagram.

We can easily check that for the canonical bigrading $H^r(A^*_{\mathbb{C}}) = \bigoplus H^r_{p,q}$ of the $\mathbb{R}$-mixed Hodge structure as in Theorem 5.3, we have

(1) $H^1(A^*_{\mathbb{C}}) = H^1_{1,0} \oplus H^1_{0,1} \oplus H^1_{1,1}$.
(2) $H^2(A^*_{\mathbb{C}}) = H^2_{2,0} \oplus H^2_{1,1} \oplus H^2_{0,2} \oplus H^2_{2,1} \oplus H^2_{1,2} \oplus H^2_{2,2}$.

Morgan's result on Sullivan's minimal models of $\mathbb{R}$-mixed Hodge diagrams is the following.

Theorem 5.5 [20, Sections 6, 8] *Let* $\{(A^*, W_*), (E^*, W_*, F^*), \phi\}$ *be an* $\mathbb{R}$*-mixed Hodge diagram. Then the minimal model (resp.* 1*-minimal model)* $\mathcal{M}^*$ *of the DGA* E^* *with a quasi-isomorphism (resp.* 1*-quasi-isomorphism)* $\phi : \mathcal{M}^* \to E^*$ *satisfies the following conditions:*

- $\mathcal{M}^*$ *admits a bigrading*

$$\mathcal{M}^* = \bigoplus_{p,q \ge 0} \mathcal{M}^*_{p,q}$$

such that $\mathcal{M}^*_{0,0} = \mathcal{M}^0 = \mathbb{C}$ *and the product and the differential are of type* $(0, 0)$.
- *For some real structure of* $\mathcal{M}^*$*, the bigrading* $\bigoplus_{p,q \ge 0} \mathcal{M}^*_{p,q}$ *induces an* $\mathbb{R}$*-mixed Hodge structure.*
- *Consider the canonical bigrading* $H^r(E^*) = \bigoplus V_{p,q}$ *for the* $\mathbb{R}$*-mixed Hodge structure as in Theorem* 5.3*. Then* $\phi^* : H^r(\mathcal{M}^*) \to H^r(E^*)$ *sends* $H^r(\mathcal{M}^*_{p,q})$ *to* $V_{p,q}$*.*

5.3 Mixed Hodge diagrams for transverse Kähler structures on central foliations

Let M be a compact complex manifold. We assume that M admits a transverse Kähler structure on a central foliation $\mathcal{F}_H$. Let $(\Omega_B^{*,*}(M), \partial_B, \bar{\partial}_B)$ be the BBA of basic differential forms associated with $\mathcal{F}_H$. The *basic Bott-Chern cohomology* $H_{B,BC}^{*,*}(M)$ is defined to be

$$H_{B,BC}^{*,*}(M) = \frac{\mathrm{Ker}\,\partial_B \cap \mathrm{Ker}\,\bar{\partial}_B}{\mathrm{Im}\,\partial_B\bar{\partial}_B}.$$

Then we have $\overline{H_{B,BC}^{p,q}(M)} = H_{B,BC}^{q,p}(M)$ and the natural algebra homomorphisms

$$\mathrm{Tot}^* H_{B,BC}^{*,*}(M) \to H_B^*(M, \mathbb{C})$$

and

$$H_{B,BC}^{*,*}(M) \to H_B^{*,*}(M).$$

By $\partial_B\bar{\partial}_B$-Lemma, these maps are isomorphisms (see [7, Remark 5.16]). Thus, we have the Hodge decomposition

$$H_B^r(M, \mathbb{C}) = \bigoplus_{p+q=r} H_B^{p,q}(M)$$

and

$$\overline{H_B^{p,q}(M)} = H_B^{q,p}(M).$$

We remark that this decomposition does not depend on the choice of a transverse Kähler structure.

Under the assumptions as in Theorem 4.13, we consider the model $\mathcal{A}^* = H_B^*(M) \otimes \bigwedge W$ as in Theorem 4.13. We suppose that $\mathcal{F}_H$ is fundamental as in Definition 1.3. we can obtain the mixed Hodge diagram $\{(A^*, W_*), (A_{\mathbb{C}}^*, W_*, F^*), \mathrm{id} : A_{\mathbb{C}}^* \to A_{\mathbb{C}}^*\}$ as in Example 5.4. Finally we obtain the following statement.

Theorem 5.6 *Let M be a compact complex manifold. We assume that M admits a transverse Kähler structure on a fundamental central foliation $\mathcal{F}_H$. Consider the minimal model $\mathcal{M}$ (resp. 1-minimal model) of $A_{\mathbb{C}}^*(M)$ with a quasi-isomorphism (resp. 1-quasi-isomorphism) $\phi : \mathcal{M} \to A_{\mathbb{C}}^*(M)$. Then we have:*

(1) *For each r, the real de Rham cohomology $H^r(M, \mathbb{R})$ admits an $\mathbb{R}$-mixed Hodge structure such that*
- $H^1(M, \mathbb{C}) = H^1_{1,0} \oplus H^1_{0,1} \oplus H^1_{1,1}$
- $H^2(M, \mathbb{C}) = H^2_{2,0} \oplus H^2_{1,1} \oplus H^2_{0,2} \oplus H^2_{2,1} \oplus H^2_{1,2} \oplus H^2_{2,2}$

where $H^r(M, \mathbb{C}) = \bigoplus H^r_{p,q}$ is the canonical bigrading.

(2) *$\mathcal{M}^*$ admits a bigrading*

$$\mathcal{M}^* = \bigoplus_{p,q\geq 0} \mathcal{M}^*_{p,q}$$

such that $\mathcal{M}^*_{0,0} = \mathcal{M}^0 = \mathbb{C}$ and the product and the differential are of type $(0, 0)$.

(3) *For some real structure of $\mathcal{M}^*$, the bigrading $\bigoplus_{p,q\geq 0} \mathcal{M}^*_{p,q}$ induces an $\mathbb{R}$-mixed Hodge structure.*

(4) *The induced map $\phi^* : H^r(\mathcal{M}^*) \to H^r(M, \mathbb{C})$ sends $H^r(\mathcal{M}^*_{p,q})$ to $H^r_{p,q}$.*

In this theorem, for the 1-minimal model $\mathcal{M}$ with a 1-quasi-isomorphism $\phi : \mathcal{M} \to A^*_{\mathbb{C}}(M)$, we have:

- $H^1(\mathcal{M}^*) = H^1(\mathcal{M}^*_{1,0}) \oplus H^1(\mathcal{M}^*_{0,1}) \oplus H^1(\mathcal{M}^*_{1,1})$
- $H^2(\mathcal{M}^*) = H^2(\mathcal{M}^*_{2,0}) \oplus H^2(\mathcal{M}^*_{1,1}) \oplus H^2(\mathcal{M}^*_{0,2}) \oplus H^2(\mathcal{M}^*_{2,1}) \oplus H^2(\mathcal{M}^*_{1,2}) \oplus H^2(\mathcal{M}^*_{2,2})$.

By Theorem 3.5, we can translate this condition to certain condition on the Lie algebra of the nilpotent completion of the fundamental group $\pi_1(M)$ as [20, Theorem 9.4] . We obtain:

Theorem 5.7 *Let M be a compact complex manifold. We assume that M admits a transverse Kähler structure on a fundamental central foliation $\mathcal{F}_H$. Then the Lie algebra of the nilpotent completion of the fundamental group $\pi_1(M)$ is isomorphic to $\mathcal{F}(H)/\mathcal{I}$ such that*

- *H is a $\mathbb{C}$-vector space with a bigrading $H = H_{-1,0} \oplus H_{0,-1} \oplus H_{-1,-1}$*
- *$\mathcal{I}$ is a Homogeneous ideal of the free bi-graded Lie algebra generated by H such that $\mathcal{I}$ has generators of types $(-1,-1)$, $(-1,-2)$, $(-2,-1)$ and $(-2,-2)$ only.*

As a consequence, the Lie algebra of the nilpotent completion of the fundamental group $\pi_1(M)$ is determined by $\pi_1(M)/\Gamma_5$ where Γ_5 is the fifth term of the lower central series of $\pi_1(M)$ [20, Corollary 9.5]. Thus, we can say that not every finitely generated group can be the fundamental group of a compact complex manifold with transverse Kähler structure on a fundamental central foliation.

6 Examples and applications

6.1 Simple examples

Example 6.1 Consider the product $S^{1,2n-1} = S^1 \times S^{2n-1}$ of a circle and a $(2n-1)$-dimensional sphere equipped with a complex structure so that there exists a special transverse Kähler structure on a one-dimensional central foliation $\mathcal{F}_H$. Then, by our results, $\Omega^*(S^{1,2n-1})$ is quasi-isomorphic to the DGA $A^* = H^*_B(S^{1,2n-1}) \otimes \bigwedge W$. By $\dim H^1(S^{1,2n-1}) = 1$ and $H^1(S^{1,2n-1}, \mathbb{C}) = H^{1,0}_B(S^{1,2n-1}) \oplus H^{0,1}_B(S^{1,2n-1}) \oplus \ker d|_W$, we have $H^{1,0}_B(S^{1,2n-1}) \oplus H^{0,1}_B(S^{1,2n-1}) = 0$ and $\dim \ker d|_W = 1$. By $\dim H^2(S^{1,2n-1}) = 0$, the differential $d : W \to H^2_B(S^{1,2n-1})$ is surjective and hence $\dim H^2_B(S^{1,2n-1}) = 1$. Take $W = \langle x, y \rangle$ so that $dx \neq 0$ in $H^2_B(S^{1,2n-1})$ and $dy = 0$. We have $H^2_B(S^{1,2n-1}) = \langle dx \rangle$. Since $dx \in H^2_B(S^{1,2n-1})$ must contain transverse Kähler form, we have $(dx)^i \neq 0$ for any $i \leq n-1$. Inductively we can easily compute $H^{2i}_B(S^{1,2n-1}) = \langle (dx)^i \rangle$ and $H^{2i-1}_B(S^{1,2n-1}) = 0$ for $2 \leq i \leq n-1$.

Consider the Hodge decomposition

$$H^r_B(S^{1,2n-1}, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_B(S^{1,2n-1}).$$

Then we have $H^{i,i}_B(S^{1,2n-1}) = \langle (dx)^i \rangle$ for any $i \leq n-1$ and $H^{p,q}_B(S^{1,2n-1}) = 0$ for $p \neq q$. Take the decomposition $W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$ with $W^{1,0} = \langle z \rangle$. Then we have $dz = c dx$ for some $c \in \mathbb{C}$. Thus we have $\bar{\partial} z = c dx$ and $\bar{\partial} \bar{z} = 0$. Hence $\Omega^{*,*}(S^{1,2n-1})$ is quasi-isomorphic to the DBA

$$B^{*,*} = \langle 1, dx, \ldots, (dx)^{n-1} \rangle \otimes \bigwedge \langle z, \bar{z} \rangle.$$

Thus every complex structure on $S^{1,2n-1}$ with a transverse Kähler structure on a one-dimensional fundamental central foliation $\mathcal{F}_H$ has same basic Betti, basic Hodge and Hodge numbers. There are many such complex structures; see Example 6.9.

Example 6.2 Consider the product $S^{3,3} = S^3 \times S^3$ of two three-dimensional spheres equipped with a complex structure so that there exists a transverse Kähler structure on a one-dimensional central foliation $\mathcal{F}_H$. Then, by our results, $\Omega^*(S^{3,3})$ is quasi-isomorphic to the DGA $A^* = H^*_B(S^{3,3}) \otimes \bigwedge W$. By $H^1(S^{3,3}) = 0$ and $H^2(S^{3,3}) = 0$, we have $H^1_B(S^{3,3}) = 0$ and the differential $d : W \to H^2_B(S^{3,3})$ is bijective. Take $W = \langle x, y\rangle$. Then $H^2_B(S^{3,3}) = \langle dx, dy\rangle$. By $\dim H^3(S^{3,3}) = 2$, just two of the elements

$$d(x \wedge dx) = dx \wedge dx, d(y \wedge dy) = dy \wedge dy, d(x \wedge dy) = -d(y \wedge dx) = dx \wedge dy$$

are equal to 0. Take x, y so that $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy \neq 0$. Since the codimension of $\mathcal{F}_H$ is 4, we have $\dim H^4_B(S^{3,3}) = 1$ and thus $H^4_B(S^{3,3}) = \langle dx \wedge dy\rangle$. Thus we have

$$H^*_B(S^{3,3}) = \bigwedge \langle dx, dy\rangle = \langle 1, dx, dy, dx \wedge dy\rangle.$$

Consider the Hodge decomposition

$$H^r_B(S^{3,3}, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_B(S^{3,3}).$$

Then, by $H^{1,1}_B(S^{3,3}) \neq 0$ and $\dim H^2_B(S^{3,3}) = 2$, we have that $H^{2,0}_B(S^{3,3}) = H^{0,2}_B(S^{3,3}) = 0$. Thus $H^{1,1}_B(S^{3,3}) = \mathbb{C}\langle dx, dy\rangle$. Take the decomposition $W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$ with $W^{1,0} = \langle \alpha + \sqrt{-1}\beta\rangle$. Now we have

$$\bar{\partial}(\alpha + \sqrt{-1}\beta) = d\alpha + \sqrt{-1}d\beta.$$

and

$$\bar{\partial}(\alpha - \sqrt{-1}\beta) = 0.$$

By $\langle x, y\rangle = \langle \alpha, \beta\rangle$, we have

$$d\alpha \wedge d\beta \neq 0 \in H^4_B(S^{3,3}, \mathbb{C}) = H^{2,2}_B(S^{3,3}).$$

Hence $\Omega^{*,*}(S^{3,3})$ is quasi-isomorphic to the DBA

$$B^{*,*} = \langle 1, d\alpha, d\beta, d\alpha \wedge d\beta\rangle \otimes \bigwedge \langle \alpha + \sqrt{-1}\beta, \alpha - \sqrt{-1}\beta\rangle.$$

We compute

$$H^{1,0}(S^{3,3}) = H^{2,0}(S^{3,3}) = H^{3,0}(S^{3,3}) = H^{0,2}(S^{3,3}) = H^{0,3}(S^{3,3}) = 0$$

and

$$\dim H^{0,1}(S^{3,3}) = \dim H^{2,1}(S^{3,3}) = \dim H^{1,2}(S^{3,3}) = 1.$$

Thus every complex structure on $S^{3,3}$ with a transverse Kähler structure on a one-dimensional central foliation $\mathcal{F}_H$ has same basic Betti, basic Hodge and Hodge numbers. Such complex manifolds are constructed as LVM manifolds associated with complex numbers $(\lambda_1, \ldots, \lambda_5)$ with certain conditions (see [19, Section 5]).

Example 6.3 Consider the product $S^{1,3} = S^1 \times S^3$ (resp. $S^{3,3} = S^3 \times S^3$) equipped with a complex structure so that there exists a transverse Kähler structure on a one-dimensional central foliation $\mathcal{F}_{H_1}$ (resp. $\mathcal{F}_{H_2}$). Then the product $S^{1,3} \times S^{1,3}$ has the natural complex structure so that there exists a special transverse Kähler structure on a two-dimensional central foliation $\mathcal{F}_{H_1 \times H_1}$. The Künneth formula allows us to compute the basic Betti, basic Hodge and Hodge numbers. By Künneth formula we have

$$\dim H_B^i(S^{1,3} \times S^{1,3}) = \begin{cases} 1 & i = 0, 4, \\ 2 & i = 2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\dim H_B^{p,q}(S^{1,3} \times S^{1,3}) = \begin{cases} 1 & p = q = 0, 2, \\ 2 & p = q = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\dim H^{p,q}(S^{1,3} \times S^{1,3}) = \begin{cases} 1 & (p,q) = (0,0), (4,4), (0,2), (4,2), \\ 2 & (p,q) = (0,1), (4,3), (1,2), (3,2), \\ 4 & (p,q) = (2,2), \\ 0 & \text{otherwise}. \end{cases}$$

Now we consider the complex one-dimensional torus $S^{1,1} = S^1 \times S^1$ and the central foliation $\mathcal{F}_{S^{1,1}}$ on $S^{1,1}$. Then the product $S^{1,1} \times S^{3,3}$ has the natural complex structure so that there exists a transverse Kähler structure on a two-dimensional central foliation $\mathcal{F}_{S^{1,1} \times H_2}$. By Künneth formula we have

$$\dim H_B^i(S^{1,1} \times S^{3,3}) = \begin{cases} 1 & i = 0, 4, \\ 2 & i = 2, \\ 0 & \text{otherwise} \end{cases}$$
$$= \dim H_B^i(S^{1,3} \times S^{1,3}),$$

$$\dim H_B^{p,q}(S^{1,1} \times S^{3,3}) = \begin{cases} 1 & p = q = 0, 2, \\ 2 & p = q = 1, \\ 0 & \text{otherwise} \end{cases}$$
$$= \dim H_B^{p,q}(S^{1,3} \times S^{1,3})$$

but

$$\dim H^{p,q}(S^{1,1} \times S^{3,3}) \neq \dim H^{p,q}(S^{1,3} \times S^{1,3})$$

for some p, q. Indeed, $\dim H^{1,0}(S^{1,1} \times S^{3,3}) = 1$ but $\dim H^{1,0}(S^{1,3} \times S^{1,3}) = 0$. Thus, in general, the Hodge numbers depend on a complex structure.

6.2 Nilmanifolds

Let N be a simply connected nilpotent Lie group. We suppose that N admits a lattice Γ, i.e., cocompact discrete subgroup. A compact homogeneous space $\Gamma\backslash N$ is called a *nilmanifold*. It is known that a nilmanifold admits a Kähler structure if and only if it is a torus (see [2,11]).

Denote by $\mathfrak{n}$ the Lie algebra of N. Let J be an endomorphism of $\mathfrak{n}$ satisfying $J \circ J = -\mathrm{id}$ and $[JA, JB] = [A, B]$ for any $A, B \in \mathfrak{n}$. Then J induces a complex structure on $\Gamma\backslash N$. Such complex structure is called *abelian*. We assume that $\mathfrak{n}$ is non-abelian and 2-step, i.e., $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$. Let C be the center of N and $\psi : N \to N/C$ the quotient map. Then we have the holomorphic principal torus bundle

$$T \hookrightarrow \Gamma\backslash N \to M$$

where T and M are complex tori $\Gamma \cap C\backslash C$ and $M = \psi(\Gamma)\backslash\psi(N)$ respectively. Let $\mathfrak{c}$ be the sub-algebra of $\mathfrak{n}$ corresponding to C. Consider the complex $\bigwedge \mathfrak{n}^*$ of left-N-invariant differential forms. Take $W \subset \bigwedge^1 \mathfrak{n}^*$ which is dual to $\mathfrak{c}$. Then we have $dW \subset \Omega^{1,1}(\Gamma\backslash N)$. Thus, in this case, $\Gamma\backslash N$ admits a transverse Kähler structure on the fundamental central foliation $\mathcal{F}_C$.

We study the properties of nilmanifolds admitting special transverse Kähler structures on fundamental central foliations.

Proposition 6.4 *Let $\Gamma\backslash N$ be a nilmanifold with a* (*not necessarily left-invariant*) *complex structure J. We assume that M admits a transverse Kähler structure on a k-dimensional central foliation $\mathcal{F}_H$. Suppose that $\mathcal{F}_H$ is regular, i.e., H is compact and the H-action is free. Then $\Gamma\backslash N$ is biholomorphic to a holomorphic principal torus bundle over a complex torus. In particular, $\Gamma\backslash N$ is 2-step nilmanifold* (see [23]).

Proof By the assumption, $\Gamma\backslash N$ admits a holomorphic principal torus H bundle structure $\Gamma\backslash N \to B$ so that the base space is a compact Kähler manifold. Since $\Gamma\backslash N$ is an aspherical manifold with $\pi_1(\Gamma\backslash N) \cong \Gamma$, B is a compact aspherical manifold such that $\pi_1(B)$ is a finitely generated nilpotent group. By results in [1,2,11], B is a complex torus. Thus $\Gamma\backslash N$ is a holomorphic principal torus bundle over a complex torus. □

We are interested in the non-regular case.

Proposition 6.5 *Let $\Gamma\backslash N$ be a nilmanifold with a* (*not necessarily left-invariant*) *complex structure J. We assume that $\Gamma\backslash N$ admits a transverse Kähler structure on a fundamental central foliation $\mathcal{F}_H$. If H is complex one-dimensional, then $\Gamma\backslash N$ is diffeomorphic to a 2-step nilmanifold.*

Proof Let M be a compact complex n-dimensional manifold which admits a special transverse Kähler structure on a k-dimensional central foliation $\mathcal{F}_H$. Then we have an isomorphism

$$H^{2n}(M, \mathbb{C}) \cong H_B^{n-k,n-k}(M) \otimes \bigwedge^{2k} W_{\mathbb{C}}.$$

Hence, for the mixed Hodge structure as in Theorem 5.6, $H^{2n}(M, \mathbb{C})$ is generated by elements of bi-degree $(n + k, n + k)$.

Consider nilmanifold $\Gamma\backslash N$. Then the DGA $\bigwedge \mathfrak{n}^*$ is the minimal model of $\Omega^*(\Gamma\backslash N)$ (see [11]). If $\Gamma\backslash N$ admits a special transverse Kähler structure on a central foliation $\mathcal{F}_H$, then by Theorem 5.6, the minimal model $\bigwedge \mathfrak{n}^*_{\mathbb{C}}$ of $\Omega^*(\Gamma\backslash N)$ admits a bigrading $\bigwedge \mathfrak{n}^*_{\mathbb{C}} = \bigoplus \mathcal{M}^*_{p,q}$. Denote $\mathcal{M}^*_w = \bigoplus_{p+q=w} \mathcal{M}^*_{p,q}$ and $m_\omega = \dim \mathcal{M}^1_w$. Since $\dim \mathcal{M}^1 = \dim \mathfrak{n}^*_{\mathbb{C}} = 2n$, we have $\sum_{W\geq 1} m_W = 2n$. Since we have $H^{2n}(M, \mathbb{C}) = \bigwedge^{2n} \mathfrak{n}^*_{\mathbb{C}} = \bigwedge^{2n} \bigoplus_W \mathcal{M}^1_W$, we have $\sum_{w\geq 1} w m_w = 2n + 2k$. Let $k = 1$. Then $\sum_{w\geq 2}(w-1)m_w = 2$ and hence we have $m_2 = 2$ and $m_i = 0$ for $i \leq 3$, or $m_2 = 0$, $m_3 = 1$ and $m_i = 0$ for $i \leq 4$. We can say $d\mathcal{M}_1 = 0$ and $\bigwedge \mathfrak{n}^*_{\mathbb{C}} = \bigwedge \mathcal{M}^1_1 \otimes \bigwedge V$ with $dV \subset \bigwedge^2 \mathcal{M}^1_1$. This implies that $\mathfrak{n}$ is 2-step. □

We suggest the following problem.

Problem 6.6 For $s \geq 3$ and $k \geq 2$, does there exist a s-step nilmanifold admitting a special transverse Kähler structure on a k-dimensional non-regular central foliation $\mathcal{F}_H$?

6.3 Vaisman manifolds

Let (M, J) be a compact complex manifold with a Hermitian metric g. We consider the fundamental form $\omega = g(-, J-)$ of g. The metric g is locally conformal Kähler (LCK) if we have a closed 1-form θ (called the Lee form) such that $d\omega = \theta \wedge \omega$. It is known that if $\theta \neq 0$ and θ is non-exact, then (M, J) does not admit a Kähler structure. Let ∇ be the Levi–Civita connection of g. A LCK metric g is a Vaisman metric if $\nabla\theta = 0$.

If g is Vaisman, then the following holds (see [27,28]):

- Let A and B be the dual vector fields of 1-forms θ and $-\theta \circ J$ with respect to g, respectively. Then $A = JB$, $L_A J = 0$, $L_B J = 0$, $L_A g = 0$, $L_B g = 0$ and $[A, B]$=0.
- The holomorphic vector field $B - \sqrt{-1}A$ gives a holomorphic foliation $\mathcal{F}$.
- The basic form $d(\theta \circ J)$ is a transverse Kähler structure.
- We denote by $\mathrm{Aut}_0(M, g)$ the identity component of the group of holomorphic isometries, by $\mathfrak{h}$ the abelian sub-algebra $\langle A, B\rangle$ of the Lie algebra of $\mathrm{Aut}_0(M, g)$ and by H the connected Lie subgroup of $\mathrm{Aut}_0(M, g)$ which corresponds to $\mathfrak{h}$. Let T be the closure of H in $\mathrm{Aut}_0(M, g)$. Then T is a torus.

Thus a compact Vaisman manifold M admits a transverse Kähler structure on the one-dimensional fundamental central foliation $\mathcal{F}_H$. Hence, taking $W = \langle\theta, \theta \circ J\rangle$ our results can be applied to a compact Vaisman manifold. The cohomology of the DGA

$$A^* = H_B^*(M) \otimes \bigwedge\langle\theta, \theta \circ J\rangle$$

is isomorphic to the de Rham cohomology of M and the cohomology of DBA

$$B^{*,*} = H_B^{*,*}(M) \otimes \bigwedge\langle\theta + \sqrt{-1}\theta \circ J, \theta - \sqrt{-1}\theta \circ J\rangle$$

is isomorphic to the Dolbeault cohomology of M. We can easily compute

$$H^1(M, \mathbb{C}) = H_B^1(M) \oplus \langle\theta\rangle = H_B^{1,0}(M) \oplus H_B^{0,1}(M) \oplus \langle\theta\rangle.$$

This implies a well-known fact that the first Betti number of a compact Vaisman manifold is odd (see [27]). We have the mixed Hodge structure

$$H^1(M, \mathbb{C}) = H^1_{1,0} \oplus H^1_{0,1} \oplus H^1_{1,1}$$

with $\dim H^1_{1,1} = 1$ as in Theorem 5.6. We notice that Vaisman metrics are closely related to Sasakian structures. We can also obtain nice de Rham models of Sasakian manifolds like the above DGA (see [26]) and we can develop Morgan's mixed Hodge theory on Sasakian manifolds (see [15]).

Since we have $\bar{\partial}(\theta + \sqrt{-1}\theta \circ J) = \sqrt{-1}d(\theta \circ J)$ and $\bar{\partial}(\theta - \sqrt{-1}\theta \circ J) = 0$, we can easily obtain an isomorphism of DGA

$$A^* \otimes \mathbb{C} \cong \mathrm{Tot}^* B^{*,*}.$$

Hence, by Theorem 4.13, we have the following (cf. [27, Theorem 3.5]).

Corollary 6.7 *Let M be a compact complex manifold. We suppose that M admits a Vaisman metric. Then the two DGAs $(\Omega^*(M)\otimes\mathbb{C}, d)$ and $(\Omega^*(M)\otimes\mathbb{C}, \bar\partial)$ are quasi-isomorphic. In particular, there exists an isomorphism between the complex valued de Rham cohomology and the Dolbeault cohomology.*

Remark 6.8 On compact Kähler manifold M, by the $\partial\bar\partial$-lemma, two DGAs $(\Omega^*(M)\otimes\mathbb{C}, d)$ and $(\Omega^*(M)\otimes\mathbb{C}, \bar\partial)$ are quasi-isomorphic (see [21]).

Example 6.9 Let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be complex numbers so that $0 < |\lambda_n| \le \cdots \le |\lambda_1| < 1$. A *primary Hopf manifold* M_Λ is the quotient of $\mathbb{C}^n - \{0\}$ by the group generated by the transformation $(z_1, \dots, z_n) \mapsto (\lambda_1 z_1, \dots, \lambda_n z_n)$. It is known that any M_Λ admits a Vaisman metric (see [14]). For any Λ, M_Λ is diffeomorphic to $S^{1,2n-1} = S^1 \times S^{2n-1}$. On the other hand, the complex structure on M_Λ varies. If $\lambda_n = \cdots = \lambda_1$, then M_Λ is a holomorphic principal torus bundle over $\mathbb{C}P^{n-1}$. Otherwise, any holomorphic principal torus bundle structure over $\mathbb{C}P^{n-1}$ does not exist on M_Λ. By Example 6.1 and the above arguments, we can obtain explicit representatives of de Rham, Dolbeault, basic de Rham and Basic Dolbeault cohomologies of M_Λ by using a Vaisman metric on M_Λ.

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