Annali di Matematica (2018) 197:1429–1444
https://doi.org/10.1007/s10231-018-0731-2

Infinitely many sign-changing solutions for a nonlocal problem

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Received: 28 July 2017 / Accepted: 5 February 2018 / Published online: 17 February 2018

Abstract In this paper, we consider the following general nonlocal problem

$$\begin{cases} -\mathcal{L}_K u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $s \in (0,1)$ with $2s < N$ and $\mathcal{L}_K$ is a nonlocal integrodifferential operator of fractional Laplacian type. We obtain the existence of infinitely many sign-changing solutions by combining critical point theory and invariant sets of descending flow.

Keywords Nonlocal problem · Infinitely many sign-changing solutions · Integrodifferential operator · Invariant set of descending flow

Mathematics Subject Classification 35R11 · 35J50 · 35Q55

1 Introduction

In the present paper, we are concerned with the following general nonlocal equation

$$\begin{cases} -\mathcal{L}_K u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

This work is supported partially by NSFC (11361078, 11661083 and 11771385) and NSFY of Yunnan Province (2015HB028), China.

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where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. The nonlocal integrodifferential operator $\mathcal{L}_K$ is defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))\, K(y)dy,\ \ x \in \mathbb{R}^N, \tag{1.2}$$

where $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ is a function with the following properties:

(K_1) $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\{|x|^2, 1\}$,
(K_2) there exists $\delta > 0$ such that $K(x) \geq \delta |x|^{-(N+2s)}$, $\forall x \in \mathbb{R}^N \backslash \{0\}$, where $s \in (0, 1)$ satisfying $N > 2s$.

It seems that Eq. (1.1) was first studied by Servadei and Valdinoci [28]. Observe that if the kernel $K(y) = |y|^{-(N+2s)}$, then $\mathcal{L}_K = (-\Delta)^s$, and hence (1.1) turns into a fractional elliptic equation

$$\begin{cases} (-\Delta)^s = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

where $(-\Delta)^s$ is the so-called fractional Laplacian operator which can be equivalently represented as (see [15, Lemma 3.2])

$$(-\Delta)^s u(x) = -\frac{C(N, s)}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy,\ \ \forall x \in \mathbb{R}^N, \tag{1.4}$$

where $C(N, s)$ is a constant depends on N and s. Different from the operator $-\Delta$, the fractional Laplacian operator $(-\Delta)^s$ is nonlocal.

Recently, a great attention has been focused on the study of nonlocal operators of elliptic type. From a physical point of view, nonlocal operators play a crucial rule in describing several different physical phenomena, such as in the anomalous diffusion [1,25], in the dynamics of the dislocation of atoms in crystals [17], in the fractional quantum mechanics [20], in the flow in porous media [36].

To overcome the difficulties brought by the nonlocal feature of fractional Laplacian, Caffarelli and Silvestre developed a powerful extension method in [12], which allows us to transform the nonlocal equation (1.3) into a local problem settled on $\mathbb{R}^N_+$. However, we do not know wether the Caffarelli–Silvestre extension method can be applied to the general integrodifferential operator $\mathcal{L}_K$ or not. So the nonlocal feature of the integrodifferential operator brought some difficulties to applications of variational methods to (1.1). Many of these additional difficulties have been overcome in Refs. [28,30]. Based on the variational settings established by Servadei and Valdinoci, the existence and multiplicity of nontrivial solutions of (1.1) have been investigated recently in some works. See, for example, [9,29,31, 32,35]. We refer to the books [10,11,16] and the references therein for more results related to nonlocal elliptic equations with integrodifferential operators.

From a mathematical point of view, the existence sign-changing solution is an interesting and important aspect in the studies of PDEs. When $K(y) = |y|^{-(N+2s)}$ with $s = 1$, problem (1.1) turns into the classical semilinear elliptic equation

$$\begin{cases} -\Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

It is well known that, in their celebrated paper, Ambrosetti and Rabinowitz obtained a positive and a negative solution of (1.5). The existence of the third solution was established by Wang [37]. In Ref. [13], Castro, Cossio and Neuberger proved that the third solution of (1.5) obtained in [37] changes sign only once. The more information about solutions was obtained in Bartsch and Wang [6], and they showed the existence of sign-changing solution. In Ref. [8], Bartsch, Weth and Willem showed that (1.5) possesses a least energy sign-changing solution. For a related result, we refer to Ref. [7]. In [23], Liu and Sun developed the theory of invariant sets of descending flow, which is powerful in studying the existence and multiplicity of sign-changing solutions of elliptic equations (see, e.g., [4,22]).

However, there are few works on the existence and multiplicity of sign-changing solutions of (1.1). For the special case (1.3), Chang and wang [14] obtain the existence and multiplicity of sign-changing solution via applying the Caffarelli–Silvestre extension method and invariant sets of descending flow. Very recently, by combining constraint variational method and quantitative deformation Lemma (Ref. [38]), we verify that (1.1) possesses one least energy sign-changing solution u_0 in [18]. Moreover, we showed that the energy of u_0 is strictly larger than the ground state energy.

It is natural to ask how about the multiplicity of sign-changing solutions of (1.1)? As far as we know, such a problem has not been considered before. In this paper, we are concerned with the existence of infinitely many sign-changing solutions of (1.1). We assume $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and satisfies

- (f_1) $f(x,t) = o(|t|)$ as $t \to 0$, uniformly for $x \in \bar{\Omega}$.
- (f_2) There exist $c_1, c_2 > 0$ and $p \in (2,\ 2_s^*)$ such that $|f(x,t)| \leq c_1 + c_2|t|^{p-1}$, where $2_s^* := \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent.
- (f_3) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \bar{\Omega} \times \mathbb{R}$.
- (f_4) there exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(x,u) \leq u f(x,u),\ \forall x \in \Omega, |u| \geq R,$$

where $F(x,t) = \int_0^t f(x,r)dr$.

Theorem 1.1 *If the assumptions (f_1)–(f_4) hold, then Eq.* (1.1) *has infinitely many sign-changing solutions.*

Assumption (f_4) is the well-known Ambrosetti–Rabinowitz condition (AR for short) , which was originally introduced by Ambrosetti and Rabinowitz [3], and they obtained the existence and multiplicity of nontrivial solutions of Eq. (1.5) under superlinear and subcritical growth conditions. A lot of works concerning superlinear elliptic boundary value problem have been researched under this usual (AR) condition (see, e.g., [38] and the references therein). The role of (AR) condition is to guarantee the boundedness of the Palais–Smale sequences of the energy functional associated with the problem, which is a crucial ingredient in the applications of critical point theory.

The (AR) condition is a superlinear growth assumption on the nonlinearity f. Indeed, (f_4) implies that for some $C_1, C_2 > 0$

$$F(x,u) \geq C_1|u|^{\mu} - C_2,\ \forall (x,u) \in (\bar{\Omega} \times \mathbb{R}). \tag{1.6}$$

However, there are many functions which are superquadratic at infinity, but do not satisfy the (AR) condition. Obviously, the condition

(f_5) $\lim\limits_{t \to +\infty} \frac{F(x,t)}{|t|^2} = +\infty,$ for any $(x,t) \in (\bar{\Omega} \times \mathbb{R})$.

is weaker than (AR) condition. Moreover, condition (f_5) characterizes the nonlinearity F to be superquadratic at infinity. It is easy to see that the function

$$f(x,t) = t\log(1+|t|) \tag{1.7}$$

verifies condition (f_5) but does not satisfy (1.6), not to mention (f_4).

In order to study the superlinear problem (1.5), Jeanjean introduced the following assumption on f in [19]

(f_6) there exists $\gamma \geq 1$ such that for any $x \in \Omega$, for any $t_1, t \in \mathbb{R}$ with $0 < t_1 \leq t$,

$$\mathcal{F}(x,t_1) \leq \gamma \mathcal{F}(x,t),$$

where $\mathcal{F}(x,t) = \frac{1}{2}f(x,t)t - F(x,t)$.

It is easy to see that the function defined in (1.7) also satisfies condition (f_6)

Without assuming (AR) the corresponding functional may possess unbounded Palais–Smale sequences. In recent years, condition (f_6) was often applied to consider the existence of nontrivial solutions for the superlinear problems without the (AR) condition, for example, see [2,21,26] and references therein. To overcome this difficulty, Miyagaki and Souto considered (1.6) and adapted some monotonicity arguments used by Struwe and Tarantello [34] and Schechter and Zou [27]. Our second main result can be stated as follows.

Theorem 1.2 *If the assumptions (f_1)–(f_3), (f_5) and (f_6) hold, then the problem* (1.1) *has infinitely many sign-changing solutions.*

According to Lemma 2.3 of [21], condition (f_6) is weaker than the following assumption:

(f_7) $\frac{f(\cdot,t)}{t}$ is increasing in $t \geq 0$ and decreasing in $t \leq 0$.

However, both (f_5) and (f_6) are global conditions, and therefore, they are not very satisfactory. For this reason, we replace condition (f_6) with the following condition introduced by Liu [21]:

(f_8) there exists $t_0 > 0$ such that $\frac{f(\cdot,t)}{t}$ is increasing in $t \geq t_0$ and decreasing in $t \leq -t_0$.

Under this kind of assumptions, we can also obtain a similar result:

Theorem 1.3 *If the assumptions (f_1)–(f_3), (f_5) and (f_8) hold, then the problem* (1.1) *has infinitely many sign-changing solutions.*

Here, we only provide the proof of Theorem 1.3 since the proof of Theorem 1.1 and Theorem 1.2 is similar with that of Theorem 1.3.

The paper is organized as follows. In Sect. 2, we collect some necessary preliminary observation. In Sect. 3, we will be devoted to the proof of main theorem. Through the paper, we make use of following notations: $C, C_0, C_1, \ldots$ for positive constants (possibly different from line to line) , $|\cdot|_p$ for the norm in $L^p(\Omega)$, $\|\cdot\|$ for the norm in E, "$\to$" for the strong convergence and "$\rightharpoonup$" for the weak convergence. We use B_R that denotes the open ball in E, that is $B_R = \{u \in E | \|u\| < R\}$.

2 Preliminaries

To prove our theorems, we recall the variational setting corresponding to the problem (1.1) (see [28,30]). Set

$$X := \Big\{u : \mathbb{R}^N \to \mathbb{R} |\ u \text{ is Lebesgue measurable, } u|_\Omega \in L^2(\Omega)$$
$$\text{and } \int_Q |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y < \infty\Big\},$$

where $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$. The space X is endowed with the norm defined as

$$\|u\|_X = |u|_2 + \left(\int_Q |u(x) - u(y)|^2 K(x-y) \mathrm{d}x\mathrm{d}y \right)^{1/2},$$

where

$$[u]_X = \left(\int_Q |u(x) - u(y)|^2 K(x-y) \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

It is easy to check that $\| \cdot \|_X$ is a norm on X. Then, we define

$$E := \left\{ u \in X : \ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

Also we have the Poincare type inequality: There exists a constant $C > 0$ such that

$$|u|_2 \leq C[u]_X$$

for all $u \in E$ (see [28,30]). Therefore, the norm

$$\begin{aligned} \|u\| := [u]_X &= \left(\int_Q |u(x) - u(y)|^2 K(x-y) \mathrm{d}x\mathrm{d}y \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y) \mathrm{d}x\mathrm{d}y \right)^{1/2} \end{aligned}$$

is an equivalent norm on E and $(E, \| \cdot \|)$ is a Hilbert space(see [28, Lemma 7]) with scalar product

$$(u, v) = \int_{\mathbb{R}^{2N}} (u(x) - u(y)) (v(x) - v(y)) K(x-y) \mathrm{d}x\mathrm{d}y.$$

Note that $C_0^\infty(\Omega)$ is dense in E and the norm $\| \cdot \|$ involves the interaction between Ω and $\mathbb{R}^N \setminus \Omega$. For reader's convenience, we recall some propositions which will be key ingredients in the proof.

Proposition 2.1 ([10]) *The embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [1, 2_s^*]$ and compact for $r \in [1, 2_s^*)$.*

Define the best fractional critical Sobolev constant in the embedding $E \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ as

$$S_0 := \inf_{u \in E, |u|_{L^{2_s^*}(\Omega)} \neq 0} \frac{\|u\|^2}{|u|_{2_s^*}^2}. \tag{2.1}$$

We observe that problem (1.1) has a variational structure, and as a matter of fact, its solutions can be searched as critical points of the energy functional $I : E \to \mathbb{R}$ defined as follows:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y) \mathrm{d}x\mathrm{d}y - \int_\Omega F(x, u(x)) \mathrm{d}x.$$

We would also note that $I \in C^1(E, \mathbb{R})$, and for $u \in E$ and $\phi \in E$, there holds

$$\langle I'(u), \phi \rangle = \int_{\mathbb{R}^{2N}} (u(x) - u(y)) (\phi(x) - \phi(y)) K(x-y) \mathrm{d}x\mathrm{d}y - \int_\Omega f(x, u(x)) \phi(x) \mathrm{d}x. \tag{2.2}$$

Now, consider the following eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.3}$$

Proposition 2.2 (see [30]) *Let $s \in (0,1)$, $N > 2s$, Ω be an open, bounded subset of $\mathbb{R}^N$, and let $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ be a function satisfying assumptions (K_1) and (K_2). Then*

(1) *Equation (2.3) admits an eigenvalue λ_1 that is positive, simple and that can be characterized as follows:*

$$\lambda_1 = \min_{u \in E,\, |u|_{L^2(\Omega)}=1} \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y \tag{2.4}$$

or, equivalently,

$$\lambda_1 = \min_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y}{\int_\Omega u^2(x)\mathrm{d}x}.$$

(2) *There exists a nonnegative function $e_1 \in E$ that is an eigenfunction corresponding to λ_1, attaining the minimum in (2.4); that is, $|e_1|_{L^2(\Omega)} = 1$ and*

$$\lambda_1 = \int_{\mathbb{R}^N \times \mathbb{R}^N} |e_1(x) - e_1(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y.$$

(3) *The set of the eigenvalues of Eq. (2.3) consists of a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with*

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \lambda_{k+1} \le \cdots$$

and

$$\lambda_k \to +\infty \text{ as } k \to +\infty.$$

Moreover, for any $k \in \mathbb{N}$, the eigenvalues can be characterized as follows:

$$\lambda_{k+1} = \min_{u \in X_k^\perp,\, |u|_{L^2(\Omega)}=1} \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y$$
$$\lambda_{k+1} = \min_{u \in X_k^\perp \setminus \{0\}} \frac{\int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y}{\int_\Omega u^2(x)\mathrm{d}x}, \tag{2.5}$$

where $X_k := span\{e_1, e_2, \ldots e_k\}$.

(4) *For any $k \in \mathbb{N}$, there exists a function $e_{k+1} \in X_k^\perp$ that is an eigenfunction corresponding to λ_{k+1}, attaining the minimum in (2.5); that is, $|e_{k+1}|_{L^2(\Omega)} = 1$ and*

$$\lambda_{k+1} = \int_{\mathbb{R}^N \times \mathbb{R}^N} |e_{k+1}(x) - e_{k+1}(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y.$$

(5) *The sequence $\{e_k\}_{k\in\mathbb{N}}$ of eigenfunctions corresponding to λ_k is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of E.*

Lemma 2.1 *Suppose conditions $(f_1) - -(f_3)$, (f_5) and (f_8) hold. Then, I satisfies (PS) condition at any level $c > 0$.*

Proof Let $\{u_n\} \subset E$ be a $(PS)_c$ sequence of I, that is $I(u_n) \to c$ and $I'(u_n) \to 0$. We claim that $\{u_n\}$ is bounded in E. Assume by contradiction that $\{u_n\}$ is unbounded in E. Setting $w_n = \frac{u_n}{\|u_n\|}$, up to a subsequence, we may assume that there exist $w \in E$ such that

$$w_n \rightharpoonup w \ in \ E, \quad w_n \to w \ in \ L^p(\Omega), \ p \in [1, 2_s^*), \quad w_n(x) \to w(x) \, a.e. \ x \in \Omega.$$

In the sequel, we will consider the following two cases separately.

Case 1: $w = 0$.

In this case, let $t_n \in [0, 1]$ such that

$$I(t_n u_n) = \max_{t \in [0,1]} I(t u_n).$$

On the one hand, the unboundedness of $\{u_n\}$ implies that, for any given $M > 0$, there exists $N > 0$ such that

$$\frac{M}{\|u_n\|} \in (0, 1)\ , \ \ n \geq N.$$

Denote $\bar{w}_n = (4M)^{\frac{1}{2}} w_n$, by the Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} \int_\Omega F(x, \bar{w}_n) \mathrm{d}x = 0.$$

Then for n large enough, we have

$$I(t_n u_n) \geq I(\bar{w}_n) = \frac{1}{2}\|\bar{w}_n\|^2 - \int_\Omega F(x, \bar{w}_n) \mathrm{d}x \geq M.$$

This implies

$$\lim_{n \to \infty} I(t_n u_n) = +\infty.$$

On the other hand, from condition (f_8), we know that $\mathcal{F}(x, t)$ is increasing in $t \geq t_0$ and decreasing in $t \leq -t_0$. Noting that $0 < 1 + \sup_{(x,t) \in \Omega \times [-t_0, t_0]} \mathcal{F}(x, t) - \inf_{(x,t) \in \Omega \times [-t_0, t_0]} \mathcal{F}(x, t) := C_1 < +\infty$, we have

$$\mathcal{F}(x, t_1) \leq \mathcal{F}(x, t) + C_1\ , \ \forall x \in \Omega, \ \ |t_1| \leq |t|.$$

Moreover, since $I(0) = 0$, we have $t_n \in (0, 1)$, thus $\frac{\mathrm{d}}{\mathrm{d}t} I(t u_n)|_{t=t_n} = 0$. Then

$$\begin{aligned}
I(t_n u_n) &= I(t_n u_n) - \frac{1}{2}\langle I'(t_n u_n), t_n u_n \rangle \\
&= \frac{1}{2} \int_\Omega f(x, t_n u_n) t_n u_n \mathrm{d}x - \int_\Omega F(x, t_n u_n) \mathrm{d}x \\
&= \int_\Omega \mathcal{F}(x, t_n u_n) \mathrm{d}x \\
&\leq \int_\Omega \left(\mathcal{F}(x, u_n) + C_1\right) \mathrm{d}x \\
&= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n \rangle + C_1 |\Omega| \\
&\leq C,
\end{aligned}$$

contradicts with the fact that $\lim_{n \to \infty} I(t_n u_n) = +\infty$.

Case 2: $w \neq 0$.

Set $\Omega' = \{x \in \Omega | w(x) \neq 0\}$. From the definition of w_n, we know $|u_n(x)| \to +\infty$ a.e. $x \in \Omega'$. By the unboundedness of $\{\|u_n\|\}$ and $I(u_n) \le C$, we know $\frac{I(u_n)}{\|u_n\|^2} \to 0$, that is

$$\frac{1}{2} - \int_{\Omega'} \frac{F(x, u_n)}{\|u_n\|^2} \mathrm{d}x - \int_{\Omega \backslash \Omega'} \frac{F(x, u_n)}{\|u_n\|^2} \mathrm{d}x = o_n(1). \tag{2.6}$$

Condition (f_5) and the Fatou lemma imply that

$$\lim_{n\to\infty} \int_{\Omega'} \frac{F(x, u_n)}{\|u_n\|^2} \mathrm{d}x = \lim_{n\to\infty} \int_{\Omega'} \frac{F(x, u_n)}{|u_n|^2} \cdot |w_n|^2 \mathrm{d}x \to +\infty. \tag{2.7}$$

Additionally, by (f_5) and $F(x, 0) = 0$, there exist $t_2 > 0$ and $C_1 > 0$ such that

$$F(x, t) > C_1, \ \forall x \in \bar{\Omega}, \ |t| > t_2.$$

By continuity of F,

$$F(x, t) \ge \min_{(x,t)\in\bar{\Omega}\times[-t_2,t_2]} F(x, t), \ \forall (x, t) \in \Omega \times [-t_2, t_2].$$

Then, it follows that $F(x, t) \ge -C'$, $\forall (x, t) \in \bar{\Omega} \times \mathbb{R}$. Thus, it is easy to see that

$$\lim_{n\to\infty} \int_{\Omega\backslash\Omega'} \frac{F(x, u_n)}{\|u_n\|^2} \mathrm{d}x \ge - \lim_{n\to\infty} \frac{C'}{\|u_n\|^2} |\Omega \backslash \Omega'| = 0.$$

This together with (2.7) contradicts with (2.6). Thus, we have proved that $\{u_n\}$ is bounded in E, up to a subsequence, and we can assume that there exists $u \in E$ such that

$$u_n \rightharpoonup u \ in \ E, \quad u_n \to u \ in \ L^p(\Omega), \ p \in [1, 2_s^*), \quad u_n(x) \to u(x) \, a.e. \ x \in \Omega.$$

Since

$$\begin{aligned} &\|u_n - u\|^2 \\ &= \langle I'(u_n) - I'(u), u_n - u\rangle + \int_\Omega (f(x, u_n) - f(x, u))(u_n - u)\mathrm{d}x \end{aligned}$$

and the Lebesgue dominant convergence theorem shows that

$$\begin{aligned} &\left| \int_\Omega (f(x, u_n) - f(x, u))(u_n - u)\mathrm{d}x \right| \\ &\le \left(\int_\Omega |u_n - u|^{2_s^*} \right)^{\frac{1}{2_s^*}} \left(\int_\Omega |f(x, u_n) - f(x, u)|^{\frac{2_s^*}{2_s^*-1}} \right)^{\frac{2_s^*-1}{2_s^*}} \\ &\to 0, \end{aligned}$$

we can easily verify that $u_n \to u$ in E, that is I satisfies $(PS)_c$ condition. □

3 Proof of the main result

3.1 Proof of the Theorem 1.1

Define an operator $A : E \to E$ as follows

$$Au := (-\mathcal{L}_K u)^{-1} \circ h(u), \ \forall u \in E,$$

where $h(u) := f(x, u)$. For $u \in E$ fixed, we consider the functional

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y - \int_\Omega F(x,u)\mathrm{d}x,\ \forall v \in E.$$

It is easy to prove that $J \in C^1(E, \mathbb{R})$, coercive, bounded below, weakly lower semicontinuous, and strictly convex in E. Therefore, by [24, Theorem 1.1], $J(v)$ admits a unique global minimizer $v = Au$, and $v = Au$ is the unique solution to the problem

$$-\mathcal{L}_K v = f(x,u),\ \forall u \in E.$$

That is to say,

$$\int_{\mathbb{R}^{2N}} (v(x) - v(y))\,(\varphi(x) - \varphi(y))\,K(x-y)\mathrm{d}x\mathrm{d}y = \int_\Omega f(x,u)\varphi\mathrm{d}x,\ \forall \varphi \in E. \tag{3.1}$$

Lemma 3.1 (1) *A is continuous and maps bounded sets into bounded sets.*
(2) $\langle I'(u), u - Au\rangle = \|u - Au\|^2$
(3) $\|I'(u)\| \le \|u - Au\|$

Proof (1) Let $\{u_n\} \subset E$ such that $u_n \to u$ in E. Denote $v_n = Au_n$ and $v = Au$. By (3.1), it follows that

$$\int_{\mathbb{R}^{2N}} (v_n(x) - v_n(y))\,(w(x) - w(y))\,K(x-y)\mathrm{d}x\mathrm{d}y = \int_\Omega f(x,u_n(x))w(x)\mathrm{d}x,\ \forall w \in E. \tag{3.2}$$

$$\int_{\mathbb{R}^{2N}} (v(x) - v(y))\,(w(x) - w(y))\,K(x-y)\mathrm{d}x\mathrm{d}y = \int_\Omega f(x,u(x))w(x)\mathrm{d}x,\ \forall w \in E. \tag{3.3}$$

By $(f_1)--(f_2)$, (3.2) and (3.3), we obtain

$$\begin{aligned}\|v_n - v\|^2 &= \int_{\mathbb{R}^{2N}} (v_n(x) - v_n(y) - v(x) + v(y))^2\,K(x-y)\mathrm{d}x\mathrm{d}y\\ &= \int_\Omega (f(x,u_n(x)) - f(x,u(x)))\,(v_n - v)\mathrm{d}x\\ &\le \left(\int_\Omega |v_n - v|^{2_s^*}\mathrm{d}x\right)^{\frac{1}{2_s^*}} \left(\int_\Omega |f(x,u_n) - f(x,u)|^{\frac{2_s^*}{2_s^*-1}}\mathrm{d}x\right)^{\frac{2_s^*-1}{2_s^*}}\\ &\le C\|v_n - v\| \left(\int_\Omega |f(x,u_n) - f(x,u)|^{\frac{2_s^*}{2_s^*-1}}\mathrm{d}x\right)^{\frac{2_s^*-1}{2_s^*}}.\end{aligned}$$

Hence,

$$\|v_n - v\| \le C\left(\int_\Omega |f(x,u_n) - f(x,u)|^{\frac{2_s^*}{2_s^*-1}}\mathrm{d}x\right)^{\frac{2_s^*-1}{2_s^*}}.$$

By Lebesgue dominated convergence theorem, we get that

$$\|v_n - v\| \to 0,\ \text{as } n \to +\infty,$$

which implies that A is continuous on E.

By taking $w = Au \in E$ in (3.2), Proposition 2.1, we obtain

$$\begin{aligned}\|A(u)\|^2 &= \int_{\mathbb{R}^{2N}} (Au(x) - Au(y))^2 K(x-y)\mathrm{d}x\mathrm{d}y = \int_{\Omega} f(x,u)Audx \\ &\leq C\left(\int_{\Omega} |Au||u|\mathrm{d}x + \int_{\Omega} |u|^{p-1}|Au|\mathrm{d}x\right) \\ &\leq C\|Au\|\left(\|u\| + \|u\|^{p-1}\right).\end{aligned}$$

Therefore, $\|Au\| \leq C(\|u\| + \|u\|^{p-1})$; this implies that A maps bounded sets into bounded sets.

(2) Taking $w = u - Au \in E$ into (3.3), we have

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} (Au(x) - Au(y))(u(x) - Au(x) - u(y) + Au(y)) K(x-y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\Omega} f(x,u)(u - Au)\mathrm{d}x,\end{aligned} \tag{3.4}$$

thus

$$\begin{aligned}&\langle I'(u), u - Au\rangle \\ &= \int_{\mathbb{R}^{2N}} (u(x) - u(y))(u(x) - Au(x) - u(y) + Au(y)) K(x-y)\mathrm{d}x\mathrm{d}y \\ &\quad - \int_{\Omega} f(x,u)(u - Au)\mathrm{d}x \\ &= \int_{\mathbb{R}^{2N}} (u(x) - Au(x) - u(y) + Au(y))^2 K(x-y)\mathrm{d}x\mathrm{d}y \\ &= \|u - Au\|^2\end{aligned}$$

(3) Using again (3.3), $\forall w \in E$, we deduce

$$\begin{aligned}|\langle I'(u), w\rangle| &= |\int_{\mathbb{R}^{2N}} (u(x) - u(y))(w(x) - w(y)) K(x-y)\mathrm{d}x\mathrm{d}y - \int_{\Omega} f(x,u)wdx| \\ &= |\int_{\mathbb{R}^{2N}} (u(x) - Au(x) - u(y) + Au(y))(w(x) - w(y)) K(x-y)\mathrm{d}x\mathrm{d}y| \\ &\leq \|u - Au\|\|w\|,\end{aligned}$$

which implies that $\|I'(u)\| \leq \|u - Au\|$. □

Define $P^+ = \{u \in X : u \geq 0\}$, $P^- = \{u \in X : u \leq 0\}$. For an arbitrary $\varepsilon > 0$, we define

$$P_\varepsilon^+ = \{u \in X : \mathrm{dist}(u, P^+) < \varepsilon\} \quad \text{and} \quad P_\varepsilon^- = \{u \in X : \mathrm{dist}(u, P^-) < \varepsilon\},$$

where $\mathrm{dist}(u, P^\pm) = \inf_{v \in P^\pm} \|v - u\|$.

Lemma 3.2 *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $A(\partial(P_\varepsilon^-)) \subset P_\varepsilon^-$, $A(\partial(P_\varepsilon^+)) \subset P_\varepsilon^+$.*

Proof By (f_1) and (f_2), for each $\delta > 0$, there exists $C_\delta > 0$ such that

$$|f(x,t)| \leq \delta|t| + C_\delta|t|^{p-1}, \forall t \in \mathbb{R}.$$

Taking $w = v^+$ in (3.3) and using the Hölder inequality, we have

$$\begin{aligned}
\|v^+\|^2 &= \int_{\mathbb{R}^{2N}} \left(v^+(x) - v^+(y)\right)^2 K(x-y)\mathrm{d}x\mathrm{d}y \\
&\le \int_{\mathbb{R}^{2N}} \left(v^+(x) - v^+(y)\right)^2 K(x-y)\mathrm{d}x\mathrm{d}y - 2\int_{\mathbb{R}^{2N}} v^+(x)v^-(y)K(x-y)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^{2N}} (v(x) - v(y))\left(v^+(x) - v^+(y)\right) K(x-y)\mathrm{d}x\mathrm{d}y \\
&= \int_{\Omega} f(x,u)v^+\mathrm{d}x \\
&\le \int_{\Omega} f(x,u^+)v^+\mathrm{d}x \\
&\le \int_{\Omega} \delta\left(u^+v^+ + C_\delta (u^+)^{p-1}v^+\right)\mathrm{d}x \\
&\le \delta|u^+|_2|v^+|_2 + C_\delta|u^+|_p^{p-1}|v^+|_p.
\end{aligned}$$

Set $u \in E$ and $v = Au$, for any $p \in [2, 2_s^*]$, there exists $C_p > 0$ such that

$$|u^\pm|_p = \inf_{v\in P^\mp} |v-u|_p \le C_p \inf_{v\in P^\mp} \|v-u\| = C_p \mathrm{dist}(u, P^\mp). \tag{3.5}$$

It is clear that $\mathrm{dist}(v, P^-) \le \|v^+\|$. Consequently,

$$\begin{aligned}
\mathrm{dist}(v, P^-)\|v^+\| &\le \|v^+\|^2 \\
&\le \delta|u^+|_2|v^+|_2 + C_\delta|u^+|_p^{p-1}|v^+|_p \\
&\le C\left(\delta \mathrm{dist}(u, P^-) + C_p(\mathrm{dist}(u, P^-))^{p-1}\right)\|v^+\|.
\end{aligned} \tag{3.6}$$

Therefore

$$\mathrm{dist}(Au, P^-) \le C\left(\delta \mathrm{dist}(u, P^-) + C_p(\mathrm{dist}(u, P^-))^{p-1}\right).$$

So, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$

$$\mathrm{dist}(Au, P^-) \le \frac{1}{2}\delta \mathrm{dist}(u, P^-),\ \forall u \in \partial(P_\varepsilon^-).$$

In particulary, we have $A(\partial(P_\varepsilon^-)) \subset P_\varepsilon^-$. Similarly, $A(\partial(P_\varepsilon^+)) \subset P_\varepsilon^+$. □

Since A is merely continuous, we would first construct a locally Lipschitz continuous operator B which inherits the properties of A. Similar with Lemma 2.1 in [5], we have the following lemma.

Lemma 3.3 *There exists a locally Lipschitz continuous odd operator $B : E\backslash\mathcal{K} \to E$ such that*

(1) $B(\partial(P_\varepsilon^-)) \subset P_\varepsilon^-$, $B(\partial(P_\varepsilon^+)) \subset P_\varepsilon^+$;
(2) $\frac{1}{2}\|u - Bu\| \le \|u - Au\| \le 2\|u - Bu\|$;
(3) $\langle I'(u), u - Bu\rangle \ge \frac{1}{2}\|u - Au\|^2$;
(4) $\|I'(u)\| \le 2\|u - Bu\|$;

where $\mathcal{K} = \{u \in E | I'(u) = 0\}$.

Lemma 3.4 *Suppose that N is a symmetric closed neighborhood of $\mathcal{K}_c := \{u \in E | I'(u) = 0$ and $I(u) = c\}$. Then, there exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon' < \varepsilon_1$, there exists a continuous map $\sigma : [0, 1] \times E \to E$ satisfying:*

(1) $\sigma(0, u) = u, \forall u \in E$.
(2) $\sigma(t, u) = u, \forall t \in [0, 1], I(u) \notin [c - \varepsilon', c + \varepsilon']$.
(3) $\sigma(t, -u) = -\sigma(t, u), \forall (t, u) \in [0, 1] \times E$.
(4) $\sigma(1, I^{c+\varepsilon} \backslash N) \subset I^{c-\varepsilon}$.
(5) $\sigma(t, \overline{P_\varepsilon^+}) \subset P_\varepsilon^+, \sigma(t, \overline{P_\varepsilon^-}) \subset P_\varepsilon^-$.

In particular, if N is a symmetric closed neighborhood of $\mathcal{K}_c \backslash W$, where $W = P_\varepsilon \cup -P_\varepsilon$, then there exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ there exists a continuous map $\eta : E \to E$ such that

(6) $\eta(-u) = -\eta(u), \forall u \in E$.
(7) $\eta|_{I^{c-2\varepsilon}} = id$.
(8) $\eta(I^{c+\varepsilon} \backslash (N \cup W)) \subset I^{c-\varepsilon}$.
(9) $\eta(\overline{P_\varepsilon^+}) \subset P_\varepsilon^+, \eta(\overline{P_\varepsilon^-}) \subset P_\varepsilon^-$.

Proof For $\delta > 0$ sufficiently small, let $N(\delta) = \{u \in X | d(u, \mathcal{K}_c) < \delta\} \subset N$. Since I satisfies the (PS) condition, there exist constants $\varepsilon_1, b_0 > 0$ such that

$$\|I'(u)\| \geq b_0, \forall u \in I^{-1}([c - \varepsilon_1, c + \varepsilon_1]) \backslash N\left(\frac{\delta}{2}\right).$$

By Lemma 3.3, there exists $b > 0$ such that $\langle I'(u), \frac{u - Bu}{\|u - Bu\|}\rangle \geq b > 0$ for $u \in I^{-1}([c - \varepsilon_1, c + \varepsilon_1]) \backslash N(\frac{\delta}{2})$. Assume $\varepsilon_1 < \min\{\frac{1}{4} b\delta, \varepsilon_0\}$.

Define two even Lipschitz continuous functions $g, p : E \to [0, 1]$ such that

$$g(u) = \begin{cases} 0, & u \in N(\frac{\delta}{4}), \\ 1, & u \notin N(\frac{\delta}{2}), \end{cases}$$

$$p(u) = \begin{cases} 0, & u \notin I^{-1}([c - \varepsilon', c + \varepsilon']), \\ 1, & u \in I^{-1}([c - \varepsilon, c + \varepsilon]). \end{cases}$$

Set $\Psi(u) = g(u)p(u)\frac{u - Bu}{\|u - Bu\|}$, then the initial value problem

$$\begin{cases} \frac{d\tau(t,u)}{dt} = -\Psi(\tau(t, u)), \\ \tau(0, u) = u, \end{cases}$$

has unique a solution $\tau(t, u)$ and τ is continuous about u. Set $[0, T(u)]$ is the maximal interval of existence to τ. Then $\sigma(t, u) = \tau(\frac{2\varepsilon}{b} t, u)$ is what we need. In fact, we can verify (1)-(3) as usual. For (4), let $u \in I^{c+\varepsilon} \backslash N$. If $I(\tau(t, u)) > c - \varepsilon$ for each $t \in [0, \frac{2\varepsilon}{b}]$, then $p(\tau) = 1$. And if there exists $t_0 \in [0, \frac{2\varepsilon}{b}]$ such that $\tau(t_0, u) \in N(\frac{\delta}{2})$, then $\frac{\delta}{2} \leq \|\tau(t_0, u) - u\| \leq \int_0^{t_0} \|\tau\prime(s, u)\| ds \leq t_0 \leq \frac{2\varepsilon}{b} < \frac{\delta}{2}$, which is a contradiction. Therefore, $I(\sigma(1, u)) = I(\tau(\frac{2\varepsilon}{b}, u)) \leq I(u) - \frac{2\varepsilon}{b} \int_0^1 \langle I'(\tau(s, u)), \Psi(\tau(s, u))\rangle ds \leq c + \varepsilon - \frac{2\varepsilon}{b} b \leq c - \varepsilon$. For (5), we can immediately verify it since $B(\partial P_\varepsilon^+) \subset P_\varepsilon^+, B(\partial P_\varepsilon^-) \subset P_\varepsilon^-$.

In particular, it is normal to verify that $\eta(u) := \sigma(1, u)$ satisfies (6)–(9). □

Proof of Theorem 1.3

Let $\lambda_i, i = 1, 2, \ldots$ be the ith eigenvalue of (2.3) and e_i be the eigenfunction corresponding to λ_i, $X_j = span\{e_1, e_2, \ldots, e_j\}$. Denote

$$M = \{u \in E \mid \frac{1}{4}\|u\|^2 > \int_\Omega F(x, u)\mathrm{d}x\} \cup B_\rho,$$

where $\rho > 0$ such that

$$\{u \in E \mid \frac{1}{4}\|u\|^2 = \int_\Omega F(x, u)\mathrm{d}x\} \cap \partial B_\rho \neq \phi.$$

Note that, $\forall \varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x, u)| \leq \varepsilon|u|^2 + C_\varepsilon|u|^p.$$

From the arbitrary of ε and the definition of λ_i, which is defined in Proposition 2.2, for all $u \in \partial M \cap X_{j-1}^\perp$, we deduce

$$\begin{aligned}\int_\Omega F(x, u)\mathrm{d}x &\leq C\int_\Omega |u|^p dx\\ &\leq \left(\int_\Omega |u|^2\mathrm{d}x\right)^{\frac{p\theta}{2}}\left(\int_\Omega |u|^{2_s^*}\mathrm{d}x\right)^{\frac{p(1-\theta)}{2_s^*}}\\ &\leq C\lambda_j^{-\frac{p\theta}{2}}\|u\|^{p\theta}\|u\|^{p(1-\theta)}\\ &= C\lambda_j^{-\frac{p\theta}{2}}\left(\int_\Omega F(x, u)\mathrm{d}x\right)^{\frac{p}{2}},\end{aligned} \tag{3.7}$$

where $\theta \in (0, 1)$ satisfying $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2_s^*}$. Hence,

$$\int_\Omega F(x, u)\mathrm{d}x \geq C\lambda_j^{\frac{p\theta}{p-2}}.$$

Therefore, for $u \in \partial M \cap X_{j-1}^\perp$, there holds

$$\begin{aligned}I(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(x, u)\mathrm{d}x\\ &\geq \frac{1}{4}\int_\Omega F(x, u)\mathrm{d}x\\ &\geq C\lambda_j^{\frac{p\theta}{p-2}}.\end{aligned}$$

That is,

$$\inf_{u\in\partial M\cap X_{j-1}^\perp} I(u) \geq C\lambda_j^{\frac{p\theta}{p-2}} \to +\infty,\ \ j \to +\infty.$$

Choose R_j large enough such that $I(u) < 0$, for $u \in X_j \backslash B_{R_j}$. Define

$$c_j = \inf_{D\in\Gamma_j}\sup_{u\in D\backslash W} I(u),$$

where

$$\Gamma_j = \left\{H(X_{j+1} \cap B_{R_{j+1}}) \,\middle|\, H \in C(X_{j+1} \cap B_{R_{j+1}}, E),\ H\ is\ \text{odd},\ H\big|_{X_{j+1}\cap\partial B_{R_{j+1}}} = id\right\}.$$

Now, we claim

$$\forall D \in \Gamma_j,\ j \geq 2,\ (D\backslash W) \cap X_{j-1}^{\perp} \cap \partial M \neq \phi.$$

Indeed, for $D = H(X_{j+1} \cap B_{R_{j+1}})$, where $H \in C(X_{j+1} \cap B_{R_{j+1}}, X)$, H is odd and $H|_{\partial B_{R_{j+1}} \cap X_{j+1}} = id$. Let Ô$= \{u \in X_{j+1} \cap B_{R_{j+1}} | H(u) \in \text{int}M\}$ and O be the connected component of Ô containing 0. Then, O is a bounded symmetric neighborhood of 0 in X_{j+1} and $O \cap X_{j+1} \cap \partial B_{R_{j+1}} = \emptyset$. By Borsuk's theorem,

$$\gamma(\partial O) = j + 1 \ \text{ and } \ H(\partial O) \subset \partial M,$$

where $\gamma(\partial O)$ denote the genus of ∂O, one can refer to [33] for more properties of genus. Define $h : W \cap \partial M \to \mathbb{R}$ by $h(u) = \int_\Omega F(x, u^+)\text{d}x - \int_\Omega F(x, u^-)\text{d}x$, then h is an odd continuous map. If $0 \in h(W \cap \partial M)$, that is there exists $u \in W \cap \partial M$ such that $\int_\Omega F(x, u^+)\text{d}x = \int_\Omega F(x, u^-)\text{d}x$. On one hand, for $u \in W$, we have $\int_\Omega F(x, u^+)\text{d}x = \int_\Omega F(x, u^-)\text{d}x \leq C\varepsilon$. On the other hand, for $u \in \partial M$, there exists $C > 0$ such that $\int_\Omega F(x, u)\text{d}x \geq C > 0$, which is a contradiction when ε is small enough. Therefore, $\gamma(\partial M \cap W) = 1$. Thus, $\gamma((H(\partial O)\backslash W) \cap \partial M) \geq j + 1 - 1 = j$, which is contradict to $\text{codim}(X_{j-1}^{\perp}) = j - 1 < j$. So $H(\partial O)\backslash W \cap \partial M \cap X_{j-1}^{\perp} \neq \emptyset$, and $H(\partial O)\backslash W \subset D\backslash W$; thus, the claim is proved. Then,

$$c_j \geq \inf_{u \in \partial M \cap X_{j-1}^{\perp}} I(u) \geq C\lambda_j^{\frac{p\theta}{p-2}} \to +\infty.$$

Finally, we prove that $\mathcal{K}_{c_j}\backslash W \neq \phi$, $j \geq 2$. Otherwise, by Lemma 3.4, there exists $\varepsilon > 0$ and an odd continuous map $\eta : E \to E$ such that

$$\eta|_{I^{c_j - 2\varepsilon}} = id\,,\ \ \eta(I^{c_j+\varepsilon}\backslash W) \subset I^{c_j - \varepsilon}\,,\ \ \eta(\overline{P_\varepsilon^{\pm}}) \subset P_\varepsilon^{\pm}.$$

For the ε above, there exists $D \in \Gamma_j$ such that $\sup_{u \in D\backslash W} I(u) < c_j + \varepsilon$, that is $D\backslash W \subset I^{c_j+\varepsilon}$. On the one hand, denote $U = \eta(D)$, and it is easy to verify that $U \in \Gamma_j$ and $c_j \leq \sup_{u \in U\backslash W} I(u)$. On the other hand,

$$U\backslash W = \eta(D)\backslash W \subset (\eta(D\backslash W) \cup \eta(W))\backslash W \subset \eta(D\backslash W)\backslash W \subset \eta(I^{c_j+\varepsilon}\backslash W) \subset I^{c_j - \varepsilon}.$$

Therefore,

$$c_j \leq \sup_{u \in U\backslash W} I \leq c_j - \varepsilon,$$

which is a contradiction. Thus, we have completed the proof of Theorem 1.3.

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