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Sweeping processes with prescribed behavior on jumps

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Abstract We present a generalized formulation of sweeping process where the behavior of the solution is prescribed at the jump points of the driving moving set. An existence and uniqueness theorem for such formulation is proved. As a consequence we derive a formulation and an existence/uniqueness theorem for sweeping processes driven by an arbitrary BV moving set, whose evolution is not necessarily right continuous. Applications to the play operator of elastoplasticity are also shown.

Keywords Sweeping processes · Evolution variational inequalities · Play operator · Convex sets · Functions of bounded variation

Mathematics Subject Classification 34A60 · 49J52 · 34G25 · 47J20

1 Introduction

Sweeping processes are a class of evolution differential inclusions introduced by J. J. Moreau in a series of articles [54–57] which culminated in the celebrated paper [60], originating a research that is still active. The original formulation introduced in [54,57] reads as follows. Let $\mathcal{H}$ be a real Hilbert space, $a, b \in \mathbb{R}$, $a < b$, and, for every time $t \in [a, b]$, let $\mathcal{C}(t)$

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be a given nonempty, closed and convex subset of $\mathcal{H}$ such that the mapping $t \longmapsto \mathcal{C}(t)$ is Lipschitz continuous when the family of closed subsets of $\mathcal{H}$ is endowed with the Hausdorff metric. One has to find a Lipschitz continuous function $y : [a, b] \longrightarrow \mathcal{H}$ such that

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b], \tag{1.1}$$

$$-y'(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mathcal{L}^1 - \text{a.e.} t \in [a, b], \tag{1.2}$$

$$y(a) = y_0, \tag{1.3}$$

y_0 being a prescribed point in $\mathcal{C}(a)$. Here $\mathcal{L}^1$ is the Lebesgue measure, and $N_{\mathcal{C}(t)}$ is the exterior normal cone to $\mathcal{C}(t)$ at $y(t)$ (all the precise definitions will be given in Sect. 2). When the interior of $\mathcal{C}(t)$ is nonempty, the process defined by (1.1)–(1.3) has a nice and useful geometrical–mechanical interpretation which we recall from [60]: “the moving point $u(t)$ remains at rest as long as it happens to lie in the interior of $\mathcal{C}(t)$; when caught up with the boundary of the moving set, it can only proceed in an inward normal direction, as if pushed by this boundary, so as to go on belonging to $\mathcal{C}(t)$.” Moreau was originally motivated by plasticity and friction dynamics (cf. [57,59,61]), but now sweeping processes have found applications to nonsmooth mechanics (see, e.g., [43,53,62]), to economics (cf., e.g., [25,31,33]), to electrical circuits (see, e.g., [1–3,10]), to crowd motion modeling (cf., e.g., [26,45–49]), and to other fields (see, e.g., the references in the recent paper [72]).

In [60] the formulation (1.1)–(1.3) is extended to the case when the mapping $t \longmapsto \mathcal{C}(t)$ is of bounded variation and right continuous in the following natural way: it is proved that there is a unique $y \in BV^r([a, b]; \mathcal{H})$, the space of right continuous $\mathcal{H}$-valued functions of bounded variation, such that there exist a positive measure μ and a μ-integrable function $v : [a, b] \longrightarrow \mathcal{H}$ satisfying

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b], \tag{1.4}$$

$$\mathrm{D}y = v\mu, \tag{1.5}$$

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b], \tag{1.6}$$

$$y(a) = y_0, \tag{1.7}$$

where $\mathrm{D}y$ denotes distributional derivative of y, which is a measure since $y \in BV$. A relevant particular case is provided by the case when $\mathcal{C}(t) = u(t) - \mathcal{Z}$, with $u \in BV^r([0, T]; \mathcal{H})$, and $\mathcal{Z} \subseteq \mathcal{H}$ closed, convex and nonempty, namely the sweeping process driven by a set with constant shape. The resulting solution operator $\mathsf{P} : BV^r([a, b]; \mathcal{H}) \longrightarrow BV^r([a, b]; \mathcal{H})$ associating with u the unique function y satisfying (1.4)–(1.7) with $\mathcal{C}(t) = u(t) - \mathcal{Z}$ is called *vector play operator* (see, e.g., [37–39,63,65]) and has an important role in elastoplasticity and hysteresis (cf., e.g., [12,36,38,50,51,76]).

The theoretical analysis of problem (1.4)–(1.7) has been expanded in various directions: the case of $\mathcal{C}$ continuous was first dealt in [53], and in [52] the application of external forces is also considered; the nonconvex case has been studied in several papers, e.g., [5,6,8,15,17, 20,23,29,32,69–71,73–75]; for stochastic versions see, e.g., [7,13,14,18], while periodic solutions can be found in [16]. The continuous dependence properties of various sweeping problems are investigated, e.g., in [11,34,35,41,42,60,64–67], and the control problems are studied, e.g., in [21,22,24].

When the moving set $\mathcal{C}$ jumps, the geometrical interpretation of the sweeping process (1.4)–(1.7) has to be revisited by analyzing the behavior of the solution y at jump points t of $\mathcal{C}$: at such points it can be showed (cf. [60]) that $y(t) = y(t+) = \mathrm{Proj}_{\mathcal{C}(t)}(u(t-))$, where Proj is the classical projection operator; thus, y instantaneously moves from $\mathcal{C}(t-)$ to $\mathcal{C}(t) = \mathcal{C}(t+)$ along the shortest path which allows to satisfy the constraint (1.4). Although this is a very natural requirement of formulation (1.4)–(1.7), this is not the only one: let us see, for instance, two cases where a different behavior can be prescribed at the jump points of $\mathcal{C}$.

First let us consider the problem of extending the formulation (1.4)–(1.7) to the case of arbitrary moving sets $\mathcal{C}$ of bounded variation, not necessarily right continuous. Of course one could replace $\mathcal{C}(t)$ by $\mathcal{C}(t+)$ in the differential inclusions (1.6), but this would not take into account of the position of $\mathcal{C}(t)$ at a jump point t: it would be very natural, instead, to expect that for such t one should find $y(t+) = \mathrm{Proj}_{\mathcal{C}(t+)}(\mathrm{Proj}_{\mathcal{C}(t)}(y(t-)))$.

Another situation is given by the play operator P. The play operator is *rate independent*, i.e., $\mathsf{P}(u \circ \psi) = \mathsf{P}(u) \circ \psi$ for every increasing surjective reparametrization of time $\psi : [a, b] \longrightarrow [a, b]$; therefore, if one reparametrizes the "input function" $u \in BV^r([a, b]\,; \mathcal{H})$ by connecting its jumps with segments, one could expect that the output y would behave as if u traversed the segment joining $u(t-)$ and $u(t+)$ with "infinite velocity" at every jump points t, so that it would be very natural to expect that $y(t+)$ would be the final point of the play operator driven by the segment $(1 - \sigma)u(t-) + \sigma u(+)$, $\sigma \in [0, 1]$. Nevertheless it can be proved (cf. [65, Section 5.3] and [40]) that in general the solution $y = \mathsf{P}(u)$ provided by (1.4)–(1.7) does not satisfy this property.

The aim of this paper is to provide a general formulation which takes into account all of these possible different behaviors at jumps point of the driving moving set $\mathcal{C}$. To be more precise if $t \longmapsto \mathcal{C}(t)$ is right continuous of bounded variation and if S denotes the set of discontinuity point of $\mathcal{C}$, then we prove that there exists a unique function $y \in BV^r([a, b]\,; \mathcal{H})$ such that $y(t) \in \mathcal{C}(t) = \mathcal{C}(t+)$ for every $t \in [a, b]$ and

$$\mathrm{D}y = v\mu, \tag{1.8}$$

$$v(t) + N_{\mathcal{C}(t)}(y(t)) \ni 0 \quad \text{for } \mu\text{-a.e. } t \in [a, b] \setminus S, \tag{1.9}$$

$$y(t+) = g_t(y(t-)) \quad \forall t \in S, \tag{1.10}$$

$$y(0) = y_0, \tag{1.11}$$

where $g_t : \mathcal{C}(t-) \longrightarrow \mathcal{C}(t)$ is a family of functions prescribing the behavior of y at every jump point t of $\mathcal{C}$. Thus, we are imposing a sort of family of "initial conditions at the jump points of the datum" (but we actually consider a more general situation) which make the concept of solution different (actually, more general) than the one of the classical sweeping processes where the normality condition $v(t) + N_{\mathcal{C}(t)}(y(t)) \ni 0$ is required on the whole interval $[a, b]$: yet, we still have a unique solution. The case of the arbitrary moving set $\mathcal{C}$, not necessarily continuous, can be immediately deduced by taking $g_t = \mathrm{Proj}_{\mathcal{C}(t+)} \circ \mathrm{Proj}_{\mathcal{C}(t)}$, the "double projection".

The paper is organized as follows. In the next section we present some technical preliminaries and in Sect. 3 we state our main result. This result will be proved in Sect. 4, and finally in the last Sect. 5 we present some applications and consequences of our main results.

2 Preliminaries

In this section we recall the main definitions and tools needed in the paper. The set of integers greater than or equal to 1 will be denoted by $\mathbb{N}$. Given an interval I of the real line $\mathbb{R}$, if $\mathscr{B}(I)$ indicates the family of Borel sets in I, $\mu : \mathscr{B}(I) \longrightarrow [0, \infty]$ is a measure, $p \in [1, \infty]$, and E is a Banach space, then the space of E-valued functions which are p-integrable with respect to μ will be denoted by $L^p(I, \mu; E)$ or simply by $L^p(\mu; E)$. We do not identify two functions which are equal μ-almost everywhere. The one dimensional Lebesgue measure is denoted by $\mathcal{L}^1$, and the Dirac delta at a point $t \in \mathbb{R}$ is denoted by δ_t. For the theory of integration of vector valued functions we refer, e.g., to [44, Chapter VI].

2.1 Functions with values in a metric space

In this subsection we assume that

$$(X, d) \text{ is a complete metric space,} \tag{2.1}$$

where we admit that d is an extended metric, i.e., X is a set and $d : X \times X \longrightarrow [0, \infty]$ satisfies the usual axioms of a distance, but may take on the value ∞. The notion of completeness remains unchanged. The general topological notions of interior, closure and boundary of a subset $A \subseteq X$ will be, respectively, denoted by $\mathrm{int}(A)$, $\mathrm{cl}(A)$ and ∂A. We also set $d(x, A) := \inf_{a \in A} d(x, a)$. If (Y, d_Y) is a metric space, then the continuity set of a function $f : Y \longrightarrow X$ is denoted by $\mathrm{Cont}(f)$, while $\mathrm{Discont}(f) := T \smallsetminus \mathrm{Cont}(f)$. The set of continuous X-valued functions defined on Y is denoted by $C(Y; X)$. For $S \subseteq Y$ we write $\mathrm{Lip}(f, S) := \sup\{d(f(s), f(t))/d_Y(t, s) \ :\ s, t \in S,\ s \neq t\}$, $\mathrm{Lip}(f) := \mathrm{Lip}(f, Y)$, the Lipschitz constant of f, and $Lip(Y; X) := \{f : Y \longrightarrow X \ :\ \mathrm{Lip}(f) < \infty\}$, the set of X-valued Lipschitz continuous functions on Y. If E is a Banach space with norm $\|\cdot\|_E$ and $S \subseteq Y$, we set $\|f\|_{\infty,S} := \sup_{t \in S} \|f(t)\|_E$ for every function $f : Y \longrightarrow E$. We recall now the notion of BV function with values in a metric space (see, e.g., [4,77]).

Definition 2.1 Given an interval $I \subseteq \mathbb{R}$, a function $f : I \longrightarrow X$, and a subinterval $J \subseteq I$, the *(pointwise) variation of f on J* is defined by

$$\mathrm{V}(f, J) := \sup\left\{\sum_{j=1}^{m} d(f(t_{j-1}), f(t_j)) \ :\ m \in \mathbb{N},\ t_j \in J\ \forall j,\ t_0 < \cdots < t_m\right\}.$$

If $\mathrm{V}(f, I) < \infty$ we say that f *is of bounded variation on I* and we set $BV(I; X) := \{f : I \longrightarrow X \ :\ \mathrm{V}(f, I) < \infty\}$.

It is well known that the completeness of X implies that every $f \in BV(I; X)$ admits one-sided limits $f(t-)$, $f(t+)$ at every point $t \in I$, with the convention that $f(\inf I-) := f(\inf I)$ if $\inf I \in I$, and $f(\sup I+) := f(\sup I)$ if $\sup I \in I$, and that $\mathrm{Discont}(f)$ is at most countable. We set $BV^{\mathrm{r}}(I; X) := \{f \in BV(I; X) \ :\ f(t) = f(t+) \quad \forall t \in I\}$ and if I is bounded we have $Lip(I; X) \subseteq BV(I; X)$.

2.2 Convex sets in Hilbert spaces

Throughout the remainder of the paper we assume that

$$\begin{cases} \mathcal{H} \text{ is a real Hilbert space with inner product } (x, y) \longmapsto \langle x, y \rangle \\ \|x\| := \langle x, x \rangle^{1/2} \end{cases}, \tag{2.2}$$

and we endow $\mathcal{H}$ with the natural metric defined by $d(x, y) := \|x - y\|$, $x, y \in \mathcal{H}$. We set

$$\mathscr{C}_{\mathcal{H}} := \{\mathcal{K} \subseteq \mathcal{H} \ :\ \mathcal{K} \text{ nonempty, closed and convex}\}.$$

If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $x \in \mathcal{H}$, then $\mathrm{Proj}_{\mathcal{K}}(x)$ is the projection on $\mathcal{K}$, i.e., $y = \mathrm{Proj}_{\mathcal{K}}(x)$ is the unique point such that $d(x, \mathcal{K}) = \|x - y\|$, and it is also characterized by the two conditions

$$y \in \mathcal{K}, \quad \langle x - y, v - y \rangle \leq 0 \quad \forall v \in \mathcal{K}.$$

If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $x \in \mathcal{K}$, then $N_{\mathcal{K}}(x)$ denotes the *(exterior) normal cone of* $\mathcal{K}$ *at* x:

$$N_{\mathcal{K}}(x) := \{u \in \mathcal{H} \ : \ \langle u, v - x\rangle \leq 0 \ \forall v \in \mathcal{K}\} = \mathrm{Proj}_{\mathcal{K}}^{-1}(x) - x. \tag{2.3}$$

It is well known that the multivalued mapping $x \longmapsto N_{\mathcal{K}}(x)$ is *monotone*, i.e., $\langle u_1 - u_2, x_1 - x_2\rangle \geq 0$ whenever $x_j \in \mathcal{K}$, $u_j \in N_{\mathcal{K}}(x_j)$, $j = 1, 2$ (see, e.g., [9, Exemple 2.8.2, p.46]). We endow the set $\mathscr{C}_{\mathcal{H}}$ with the Hausdorff distance. Here we recall the definition.

Definition 2.2 The *Hausdorff distance* $\mathrm{d}_{\mathcal{H}} : \mathscr{C}_{\mathcal{H}} \times \mathscr{C}_{\mathcal{H}} \longrightarrow [0, \infty]$ is defined by

$$\mathrm{d}_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) := \max\left\{\sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \ \sup_{b \in \mathcal{B}} d(b, \mathcal{A})\right\}, \quad \mathcal{A}, \mathcal{B} \in \mathscr{C}_{\mathcal{H}}.$$

The metric space $(\mathscr{C}_{\mathcal{H}}, \mathrm{d}_{\mathcal{H}})$ is complete (cf. [19, Theorem II-14, Section II.3.14, p. 47]).

2.3 Differential measures

We recall that a *$\mathcal{H}$-valued measure on I* is a map $\mu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ such that $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ whenever (B_n) is a sequence of mutually disjoint sets in $\mathscr{B}(I)$. The *total variation of* μ is the positive measure $\|\mu\| : \mathscr{B}(I) \longrightarrow [0, \infty]$ defined by

$$\|\mu\|(B) := \sup\left\{\sum_{n=1}^{\infty} \|\mu(B_n)\| \ : \ B = \bigcup_{n=1}^{\infty} B_n, \ B_n \in \mathscr{B}(I), \ B_h \cap B_k = \varnothing \text{ if } h \neq k\right\}.$$

The vector measure μ is said to be *with bounded variation* if $\|\mu\|(I) < \infty$. In this case the equality $\|\mu\| := \|\mu\|(I)$ defines a norm on the space of measures with bounded variation (see, e.g., [28, Chapter I, Section 3]).

If $\nu : \mathscr{B}(I) \longrightarrow [0, \infty]$ is a positive bounded Borel measure and if $g \in L^1(I, \nu; \mathcal{H})$, then $g\nu$ will denote the vector measure defined by $g\nu(B) := \int_B g \, \mathrm{d}\nu$ for every $B \in \mathscr{B}(I)$. In this case $\|g\nu\|(B) = \int_B \|g(t)\| \, \mathrm{d}\nu$ for every $B \in \mathscr{B}(I)$ (see [28, Proposition 10, p. 174]). Moreover a vector measure μ is called *ν-absolutely continuous* if $\mu(B) = 0$ whenever $B \in \mathscr{B}(I)$ and $\nu(B) = 0$.

Assume that $\mu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ is a vector measure with bounded variation and let $f : I \longrightarrow \mathcal{H}$ and $\phi : I \longrightarrow \mathbb{R}$ be two *step maps with respect to* μ, i.e. ,there exist $f_1, \ldots, f_m \in \mathcal{H}$, $\phi_1, \ldots, \phi_m \in \mathcal{H}$ and $A_1, \ldots, A_m \in \mathscr{B}(I)$ mutually disjoint such that $\|\mu\|(A_j) < \infty$ for every j and $f = \sum_{j=1}^{m} \mathbb{1}_{A_j} f_j, , \phi = \sum_{j=1}^{m} \mathbb{1}_{A_j} \phi_j$, where $\mathbb{1}_S$ is the characteristic function of a set S, i.e., $\mathbb{1}_S(x) := 1$ if $x \in S$ and $\mathbb{1}_S(x) := 0$ if $x \notin S$. For such step functions we define $\int_I \langle f, \mu\rangle := \sum_{j=1}^{m} \langle f_j, \mu(A_j)\rangle \in \mathbb{R}$ and $\int_I \phi \, \mathrm{d}\mu := \sum_{j=1}^{m} \phi_j \mu(A_j) \in \mathcal{H}$. If $St(\|\mu\|; \mathcal{H})$ (resp. $St(\|\mu\|)$) is the set of $\mathcal{H}$-valued (resp. real valued) step maps with respect to μ, then the maps $St(\|\mu\|; \mathcal{H}) \longrightarrow \mathcal{H} : f \longmapsto \int_I \langle f, \mu\rangle$ and $St(\|\mu\|) \longrightarrow \mathcal{H} : \phi \longmapsto \int_I \phi \, \mathrm{d}\mu$ are linear and continuous when $St(\|\mu\|; \mathcal{H})$ and $St(\|\mu\|)$ are endowed with the L^1-seminorms $\|f\|_{L^1(\|\mu\|;\mathcal{H})} := \int_I \|f\| \, \mathrm{d}\|\mu\|$ and $\|\phi\|_{L^1(\|\mu\|)} := \int_I |\phi| \, \mathrm{d}\|\mu\|$. Therefore, they admit unique continuous extensions $\mathsf{I}_\mu : L^1(\|\mu\|; \mathcal{H}) \longrightarrow \mathbb{R}$ and $\mathsf{J}_\mu : L^1(\|\mu\|) \longrightarrow \mathcal{H}$, and we set

$$\int_I \langle f, \mathrm{d}\mu\rangle := \mathsf{I}_\mu(f), \quad \int_I \phi\mu := \mathsf{J}_\mu(\phi), \quad f \in L^1(\|\mu\|; \mathcal{H}), \quad \phi \in L^1(\|\mu\|).$$

If ν is bounded positive measure and $g \in L^1(\nu; \mathcal{H})$, arguing first on step functions, and then taking limits, it is easy to check that $\int_I \langle f, \mathrm{d}(g\nu)\rangle = \int_I \langle f, g\rangle \, \mathrm{d}\nu$ for every $f \in L^\infty(\mu; \mathcal{H})$. The following results (cf., e.g., [28, Section III.17.2–3, pp. 358–362]) provide a connection between functions with bounded variation and vector measures which will be implicitly used in the paper.

Theorem 2.1 *For every $f \in BV(I;\mathcal{H})$ there exists a unique vector measure of bounded variation $\mu_f : \mathscr{B}(I) \longrightarrow \mathcal{H}$ such that*

$$\mu_f(]c,d[) = f(d-) - f(c+), \quad \mu_f([c,d]) = f(d+) - f(c-),$$
$$\mu_f([c,d[) = f(d-) - f(c-), \quad \mu_f(]c,d]) = f(d+) - f(c+).$$

whenever $c < d$ and the left hand side of each equality makes sense. Conversely, if $\mu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ is a vector measure with bounded variation, and if $f_\mu : I \longrightarrow \mathcal{H}$ is defined by $f_\mu(t) := \mu([\inf I, t[\cap I)$, then $f_\mu \in BV(I;\mathcal{H})$ and $\mu_{f_\mu} = \mu$.

Proposition 2.1 *Let $f \in BV(I;\mathcal{H})$, let $g : I \longrightarrow \mathcal{H}$ be defined by $g(t) := f(t-)$, for $t \in \operatorname{int}(I)$, and by $g(t) := f(t)$, if $t \in \partial I$, and let $V_g : I \longrightarrow \mathbb{R}$ be defined by $V_g(t) := \mathrm{V}(g, [\inf I, t] \cap I)$. Then $\mu_g = \mu_f$ and $\| \mu_f \| = \mu_{V_g} = \mathrm{V}(g, I)$.*

The measure μ_f is called *Lebesgue–Stieltjes measure* or *differential measure* of f. Let us see the connection with the distributional derivative. If $f \in BV(I;\mathcal{H})$ and if $\overline{f} : \mathbb{R} \longrightarrow \mathcal{H}$ is defined by

$$\overline{f}(t) := \begin{cases} f(t) & \text{if } t \in I \\ f(\inf I) & \text{if } \inf I \in \mathbb{R}, t \notin I, t \le \inf I, \\ f(\sup I) & \text{if } \sup I \in \mathbb{R}, t \notin I, t \ge \sup I \end{cases} \tag{2.4}$$

then, as in the scalar case, it turns out (cf. [65, Section 2]) that $\mu_f(B) = \mathrm{D}\overline{f}(B)$ for every $B \in \mathscr{B}(\mathbb{R})$, where $\mathrm{D}\overline{f}$ is the distributional derivative of $\overline{f}$, i.e.,

$$-\int_{\mathbb{R}} \varphi'(t)\overline{f}(t)\,\mathrm{d}t = \int_{\mathbb{R}} \varphi\,\mathrm{d}\mathrm{D}\overline{f} \quad \forall \varphi \in C_c^1(\mathbb{R};\mathbb{R}),$$

$C_c^1(\mathbb{R};\mathbb{R})$ being the space of real continuously differentiable functions on $\mathbb{R}$ with compact support. Observe that $\mathrm{D}\overline{f}$ is concentrated on I: $\mathrm{D}\overline{f}(B) = \mu_f(B \cap I)$ for every $B \in \mathscr{B}(I)$, hence in the remainder of the paper, if $f \in BV(I,\mathcal{H})$ then we will simply write

$$\mathrm{D}f := \mathrm{D}\overline{f} = \mu_f, \quad f \in BV(I;\mathcal{H}), \tag{2.5}$$

and from the previous discussion it follows that

$$\|\mathrm{D}f\| = \|\mathrm{D}f\|(I) = \|\mu_f\| = \mathrm{V}(f, I) \quad \forall f \in BV^{\mathrm{r}}(I;\mathcal{H}). \tag{2.6}$$

3 Main result

We are now in position to state the main theorem of the paper.

Theorem 3.1 *Assume that $-\infty < a < b < \infty$, $\mathcal{C} \in BV^{\mathrm{r}}([a,b];\mathscr{C}_{\mathcal{H}})$, $y_0 \in \mathcal{C}(a)$, $S \subseteq]a,b]$, and that for every $t \in S$ we are given a function $g_t : \mathcal{C}(t-) \longrightarrow \mathcal{C}(t)$ such that $\mathrm{Lip}(g_t) \le 1$ and*

$$\sum_{t \in S} \|g_t - Id\|_{\infty,\mathcal{C}(t-)} < \infty. \tag{3.1}$$

Then there exists a unique $y \in BV^{\mathrm{r}}([a,b];\mathcal{H})$ such that there is a measure $\mu : \mathscr{B}([a,b]) \longrightarrow [0,\infty[$ and a function $v \in L^1(\mu;\mathcal{H})$ for which

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a,b], \tag{3.2}$$

$$\mathrm{D}y = v\mu, \tag{3.3}$$

$$- v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a,b] \smallsetminus S, \tag{3.4}$$

$$y(t) = g_t(y(t-)) \quad \forall t \in S, \tag{3.5}$$

$$y(a) = y_0. \tag{3.6}$$

Moreover if $a \le s < t \le b$ *we have*

$$\mathrm{V}(y,[s,t]) \le \mathrm{V}(\mathcal{C},[s,t]) + \sum_{r \in S \cap [s,t]} \left(\|g_r - Id\|_{\infty,\mathcal{C}(r-)} - d_{\mathcal{H}}(\mathcal{C}(r-),\mathcal{C}(r)) \right). \tag{3.7}$$

Finally if $y_{0,j} \in \mathcal{C}(a)$, $j = 1,2$ *and* y_j *is the only function such that there is a measure* $\mu_j : \mathscr{B}([a,b[) \longrightarrow [0,\infty[$ *and a function* $v_j \in L^1(\mu;\mathcal{H})$ *for which* (3.2)-(3.6) *hold with* y, v, μ, y_0 *replaced, respectively, by* $y_j, v_j, \mu_j, y_{0,j}$*, then*

$$t \longmapsto \|y_1(t) - y_2(t)\|^2 \text{ is nonincreasing.}$$

In Sect. 5 we will show a series of results that can be deduced from Theorem 3.1.

4 Proofs

We start by recalling the existence and uniqueness result of classical sweeping processes due to J. J. Moreau (cf. [60]).

Theorem 4.1 *If* $-\infty < a < b < \infty$, $\mathcal{C} \in BV^{\mathrm{r}}([a,b[\,;\mathscr{C}_{\mathcal{H}})$*, and* $y_0 \in \mathcal{C}(a)$*, then there exists a unique* $y \in BV^{\mathrm{r}}([a,b[\,;\mathcal{H})$ *such that there is a measure* $\mu : \mathscr{B}([a,b[) \longrightarrow [0,\infty[$ *and a function* $v \in L^1(\mu;\mathcal{H})$ *for which*

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a,b[\,, \tag{4.1}$$

$$\mathrm{D}y = v\mu, \tag{4.2}$$

$$- v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in \,]a,b[\,, \tag{4.3}$$

$$y(a) = y_0. \tag{4.4}$$

Moreover

$$\mathrm{V}(y,[s,t]) \le \mathrm{V}(\mathcal{C},[s,t]) \quad \forall s,t \in [a,b[\,,\ s < t. \tag{4.5}$$

Finally if $y_{0,j} \in \mathcal{C}(a)$, $j = 1,2$ *and* y_j *is the only function such that there is a measure* $\mu_j : \mathscr{B}([a,b]) \longrightarrow [0,\infty[$ *and a function* $v_j \in L^1(\mu;\mathcal{H})$ *for which* (4.1)–(4.4) *hold with* y, v, μ, y_0 *replaced, respectively, by* $y_j, v_j, \mu_j, y_{0,j}$*, then*

$$t \longmapsto \|y_1(t) - y_2(t)\|^2 \text{ is nonincreasing.} \tag{4.6}$$

Concerning Theorem 4.1 let us observe that the existence and uniqueness of the solution y is proved in [60, Proposition 3b], while formula (4.5) is proved in [60, Proposition 2c]. Finally the last statement is proved in [60, Proposition 2b].

It is important to compare Theorem 4.1 to our main result, Theorem 3.1: it is possible to see that Theorem 3.1 includes the statement of Theorem 4.1 when $g_t : \mathcal{C}(t-) \longrightarrow \mathcal{C}(t)$ is given by the projection onto $\mathcal{C}(t)$, since in this case $\|g_t - Id\|_{\infty,\mathcal{C}(t-)} \le d_{\mathcal{H}}(\mathcal{C}(t-),\mathcal{C}(t))$. The goal of our main theorem is to allow for different prescribed behaviors at points $t \in S$, and the proof will be based on a suitable combination of Theorem 4.1 with some explicit applications of the maps g_t.

In the following Lemma we provide an integral formulation of the sweeping process.

Lemma 4.1 *Assume that* $-\infty < a < b < \infty$, $\mu : \mathscr{B}([a,b[) \longrightarrow [0,\infty[$ *is a measure,* $B \in \mathscr{B}([a,b[)$, *and* $\mu(B) \neq 0$. *If* $\mathcal{C} \in BV^r([a,b[\,;\mathscr{C}_{\mathcal{H}})$, $v \in L^1(\mu;\mathcal{H})$, $y \in BV^r([a,b[\,;\mathcal{H})$, *and* $y(t) \in \mathcal{C}(t)$ *for every* $t \in [a,b[$, *then the following two conditions are equivalent.*

(i) $-v(t) \in N_{\mathcal{C}(t)}(y(t))$ *for* μ*-a.e.* $t \in B$.

(ii) $\int_B \langle y(t) - z(t), v(t)\rangle \,\mathrm{d}\mu(t) \le 0$ *for every* μ*-measurable* $z : [0,T] \longrightarrow \mathcal{H}$ *such that* $z(t) \in \mathcal{C}(t)$ *for every* $t \in B$.

Proof Let us start by assuming that (i) holds and let $z : [a,b[\longrightarrow \mathcal{H}$ be a μ-measurable function such that $z(t) \subseteq \mathcal{C}(t)$ for every $t \in B$. Then it follows that

$$\langle y(t) - z(t), v(t)\rangle \le 0 \quad \forall t \in \mathrm{B},$$

and integrating over B we infer condition (ii). Now assume that (ii) is satisfied and observe that $\mathfrak{V} = \{(t, [t,t+h]\cap B) \;:\; h > 0,\ t \in B\}$ is a μ-Vitali relation covering B according to the definition given in [30, Section 2.8.16, p. 151]. Recall that if $f \in L^1(\mu, B; \mathcal{H})$ then there exists a μ-zero measure set Z such that $f(B \smallsetminus Z)$ is separable (see, e.g., [44, Property M11, p. 124]), therefore from (the proof) of [30, Corollary 2.9.9., p. 156] it follows that

$$\lim_{h\searrow 0} \frac{1}{\mu([t,t+h]\cap B)} \int_{[t,t+h]\cap B} \|f(\tau) - f(t)\|_E \,\mathrm{d}\mu(\tau) = 0 \quad \text{for } \mu\text{-a.e. } t \in B. \tag{4.7}$$

In [30] the points s satisfying (4.7) are called *(right)* μ-Lebesgue points of f on B with respect to the μ-Vitali relation $\mathfrak{V}$ covering B. Let L be the set of μ-Lebesgue points for $\tau \longmapsto v(\tau)$ on B with respect to $\mathfrak{V}$, fix $t \in L$, and choose $\zeta_t \in \mathcal{C}(t)$ arbitrarily. Since $y \in BV^r([a,b[\,;\mathcal{H})$, we have that

$$\begin{aligned}&\frac{1}{\mu([t,t+h]\cap B)} \int_{[t,t+h]\cap B} |\langle y(\tau), v(\tau)\rangle - \langle y(t), v(t)\rangle| \,\mathrm{d}\tau \\ &\quad\le \frac{1}{\mu([t,t+h]\cap B)} \int_{[t,t+h]\cap B} (\|y\|_\infty \|v(\tau) - v(t)\| + \|v\|_\infty \|y(\tau) - y(t)\|) \,\mathrm{d}\tau,\end{aligned} \tag{4.8}$$

thus

$$\lim_{h\searrow 0} \frac{1}{\mu([t,t+h]\cap B)} \int_{[t,t+h]\cap B} \langle y(\tau), v(\tau)\rangle \,\mathrm{d}\mu(\tau) = \langle y(t), v(t)\rangle. \tag{4.9}$$

If $\zeta : [a,b[\longrightarrow \mathcal{H}$ is defined by $\zeta(\tau) := \mathrm{Proj}_{\mathcal{C}(\tau)}(\zeta_t)$, $\tau \in [a,b[$, then we have (see, e.g., [53, Proposition 4.7, p. 26]

$$\begin{aligned}\|\zeta(\tau) - \zeta(\sigma)\| &\le 2(d(\zeta_t, \mathcal{C}(\tau)) + d(\zeta_t, \mathcal{C}(\sigma))d_{\mathcal{H}}(\mathcal{C}(\tau), \mathcal{C}(\sigma)) \\ &\le 2\,\mathrm{V}(\mathcal{C}, [a,b])d_{\mathcal{H}}(\mathcal{C}(\tau), \mathcal{C}(\sigma))\end{aligned}$$

for all $\sigma, \tau \in [a,b[$, therefore $\zeta \in BV^r([a,b[\,;\mathcal{H})$, $\zeta(t) = \zeta_t$, and using the same argument of (4.8)–(4.9) we get that t is a (right) μ-Lebesgue points of $\tau \longmapsto \langle \zeta(\tau), v(\tau)\rangle$ on B with respect to $\mathfrak{V}$. Since $\zeta(\tau) \in \mathcal{C}(\tau)$ for every $\tau \in [a,b[$, the function $z(\tau) := \mathbb{1}_{[a,b[\cap[t,t+h]}(\tau)\zeta(\tau) + \mathbb{1}_{[a,b[\smallsetminus[t,t+h]}(\tau)y(\tau)$, $\tau \in [a,b[$, is well defined for every sufficiently small $h > 0$ and $z(\tau) \in \mathcal{C}(\tau)$ for every $\tau \in [a,b[$, and thus we can take z in condition (ii) and we get

$$\int_{[t,t+h]\cap B} \langle y(\tau), v(\tau)\rangle \,\mathrm{d}\mu(\tau) \le \int_{[t,t+h]\cap B} \langle \zeta(\tau), v(\tau)\rangle \,\mathrm{d}\mu(\tau).$$

Dividing this inequality by $\mu([t,t+h]\cap B)$ and taking the limit as $h \searrow 0$ we get $\langle y(t) - \zeta_t, v(t)\rangle \le 0$. Therefore, as $\mu(L) = 0$, we have proved that

$$\langle y(t) - \zeta, v(t)\rangle \le 0 \quad \forall \zeta \in \mathcal{C}(t), \quad \text{for } \mu\text{-a.e. } t \in B,$$

i.e., condition (i) holds. □

Now it is convenient to recall the notion of normalized arc length parametrization for a metric-space-valued curve, provided by the following proposition (cf., e.g., [30,65])

Proposition 4.1 *Assume that* (2.1) *is satisfied and that* $f \in BV([a,b]\,; X)$. *We define* $\ell_f : [a,b] \longrightarrow [a,b]$ *by*

$$\ell_f(t) := \begin{cases} a + \dfrac{b-a}{\mathrm{V}(f,[a,b])}\mathrm{V}(f,[a,t]) & \text{if } \mathrm{V}(f,[a,b]) \neq 0 \\ a & \text{if } \mathrm{V}(f,[a,b]) = 0 \end{cases} \qquad t \in [a,b]\,.$$

If $X = \mathcal{H}$, *a Hilbert space, and* $f \in BV^{\mathrm{r}}([a,b]\,;\mathcal{H})$, *then there is a unique* $\widetilde{f} \in Lip([a,b]\,;\mathcal{H})$ *such that* $\mathrm{Lip}(\widetilde{f}) \le \mathrm{V}(f,[a,b])/(b-a)$ *and*

$$f = \widetilde{f} \circ \ell_f, \tag{4.10}$$

$$\widetilde{f}(\ell_f(t-)(1-\lambda) + \ell_f(t)\lambda) = (1-\lambda)f(t-) + \lambda f(t) \;\; \forall t \in [a,b]\,,\ \forall \lambda \in [0,1]\,. \tag{4.11}$$

The normalization factor in Definition 4.1 is not necessary in the proofs of our theorems, but it is consistent and simplifies the statement of Theorem 5.4.

In the following lemma we show that we can take $\mu = \mathrm{D}\ell_{\mathcal{C}}$, and in this case we have an explicit bound for the density of $\mathrm{D}y$ with respect to $\mathrm{D}\ell_{\mathcal{C}}$.

Lemma 4.2 *Assume that* $-\infty < a < b < \infty$, $\mathcal{C} \in BV^{\mathrm{r}}([a,b[\,;\mathscr{C}_{\mathcal{H}})$, $y_0 \in \mathcal{C}(c)$, *and* y *is the only function such that there is a measure* $\mu : \mathscr{B}([a,b[) \longrightarrow [0,\infty[$ *and a function* $v \in L^1(\mu;\mathcal{H})$ *for which* (4.1)–(4.4) *hold. Then there is a unique (up to* $\mathrm{D}\ell_{\mathcal{C}}$*-equivalence)* $w \in L^1(\mathrm{D}\ell_{\mathcal{C}};\mathcal{H})$ *such that*

$$\mathrm{D}y = w\,\mathrm{D}\ell_{\mathcal{C}}, \tag{4.12}$$

and we have

$$-\,w(t) \in N_{\mathcal{C}(t)}(y(t)) \;\; \text{for } \mathrm{D}\ell_{\mathcal{C}}\text{-a.e. } t \in \,]a,b[\,, \tag{4.13}$$

$$\|w(t)\| \le \mathrm{V}(\mathcal{C},[a,b])/(b-a) \;\; \text{for } \mathrm{D}\ell_{\mathcal{C}}\text{-a.e. } t \in [a,b[\,. \tag{4.14}$$

Proof The existence of a measure $\mu : \mathscr{B}([a,b[) \longrightarrow [0,\infty[$ and a function $v \in L^1(\mu;\mathcal{H})$ such that (4.1)–(4.4) hold is guaranteed by Theorem 4.1, which, by estimate (4.5), also implies that $\mathrm{D}y$ is absolutely continuous with respect to $\mathrm{D}\ell_{\mathcal{C}}$ and the existence and uniqueness of w is a consequence of the vectorial Radon–Nikodym theorem [44, Corollary 4.2, Section VII.4, p. 204]. From Lemma 4.1 we infer that that for every μ-measurable function $z : [a,b[\longrightarrow \mathcal{H}$ such that $z(t) \in \mathcal{C}(t)$ for every $t \in [a,b[$ we have that

$$\begin{aligned} \int_{[a,b[} \langle y(t) - z(t), w(t)\rangle \,\mathrm{d}\mathrm{D}\ell_{\mathcal{C}}(t) &= \int_{[a,b[} \langle y(t) - z(t), \mathrm{d}(w\,\mathrm{D}\ell_{\mathcal{C}})(t)\rangle \\ &= \int_{[a,b[} \langle y(t) - z(t), \mathrm{d}\mathrm{D}y(t)\rangle = \int_{[a,b[} \langle y(t) - z(t), \mathrm{d}(v\mu)(t)\rangle \\ &= \int_{[a,b[} \langle y(t) - z(t), v(t)\rangle \,\mathrm{d}\mu(t) \le 0, \end{aligned}$$

therefore using again Lemma 4.1 we get (4.13). Finally using [30, Corollary 2.9.9., p. 156] as in formula (4.7) in the proof of Lemma 4.1, and exploiting estimate (4.5), we get that for $\mathrm{D}\ell_{\mathcal{C}}$-a.e. $t \in \,]a, b[$ we have

$$
\begin{aligned}
\|w(t)\| &= \lim_{h \searrow 0} \frac{1}{\mathrm{D}\ell_{\mathcal{C}}([t-h, t+h])} \int_{[t-h,t+h]} \|w(\tau)\| \, \mathrm{d}\mathrm{D}\ell_{\mathcal{C}}(\tau) \\
&= \lim_{h \searrow 0} \frac{|\mathrm{D}y|([t-h, t+h])}{\mathrm{D}\ell_{\mathcal{C}}([t-h, t+h])} = \lim_{h \searrow 0} \frac{\mathrm{V}(\mathcal{C}, [a, b])\, \mathrm{V}(y, [t-h, t+h])}{(b-a)\, \mathrm{V}(\mathcal{C}, [t-h, t+h])} \\
&\le \frac{\mathrm{V}(\mathcal{C}, [a, b])}{b-a},
\end{aligned}
$$

and the lemma is completely proved, since $\mathrm{D}\ell_{\mathcal{C}}(\{a\}) = 0$. □

Remark 4.1 The previous Lemma can also be proved by using the representation formula [68, Formula (69)].

Now we can prove our main result in the particular case when the behavior of the solution is prescribed on a finite number of points.

Lemma 4.3 *Assume that* $-\infty < a < b < \infty$, $\mathcal{C} \in BV^r([a, b[\,; \mathscr{C}_{\mathcal{H}})$, $y_0 \in \mathcal{C}(a)$, $S \subseteq \,]a, b[$, *and that for every* $t \in S$ *we are given* $g_t : \mathcal{H} \longrightarrow \mathcal{C}(t)$ *such that*

$$
\sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} < \infty. \tag{4.15}
$$

If $F \subseteq S$ *is a finite set, then there exists a unique* $y \in BV^r([a, b[\,; \mathcal{H})$ *such that there are a measure* $\mu : \mathscr{B}([a, b[) \longrightarrow [0, \infty[$ *and a function* $v \in L^1(\mu; \mathcal{H})$ *satisfying*

$$
\begin{aligned}
& y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b[\,, && (4.16)\\
& \mathrm{D}y = v\mu, && (4.17)\\
& -v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b[\smallsetminus F, && (4.18)\\
& y(t) = g_t(y(t-)) \quad \forall t \in F, && (4.19)\\
& y(a) = y_0. && (4.20)
\end{aligned}
$$

Moreover, if $a \le s < t \le b$, *we have*

$$
\mathrm{V}(y, [s, t]) \le \mathrm{V}(\mathcal{C}, [s, t]) + \sum_{r \in F \cap [s,t]} \left(\|g_r - Id\|_{\infty, \mathcal{C}(r-)} - d_{\mathcal{H}}(\mathcal{C}(r-), \mathcal{C}(r)) \right), \tag{4.21}
$$

one can take

$$
\mu = \mathrm{D}\ell_{\mathcal{C}} + \sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} \delta_t, \tag{4.22}
$$

and for such μ *the function* v *satisfying* (4.17)–(4.18) *is unique up to* μ*-equivalence and we have*

$$
\|v(t)\| \le \max\{1, \mathrm{V}(\mathcal{C}, [a, b])/(b-a)\} \quad \text{for } \mu\text{-a.e. } t \in [a, b[\,. \tag{4.23}
$$

Proof It is not restrictive to assume that $\|g_t - Id\|_{\infty, \mathcal{C}(t-)} \ne 0$ for every $t \in S$. Let us set $t_0 := a$, $t_n := b$ and suppose that $F = \{t_1, \ldots, t_{n-1}\}$ for some $n \in \mathbb{N}$ with $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Let us call $\mathcal{C}_j$ the restriction of $\mathcal{C}$ to the interval $[t_{j-1}, t_j]$. Observe that we have

$$\ell_{\mathcal{C}_j}(t) = t_{j-1} + \frac{t_j - t_{j-1}}{\mathrm{V}(\mathcal{C}, [t_{j-1}, t_j])} \mathrm{V}(\mathcal{C}, [t_{j-1}, t])$$
$$= t_{j-1} + \frac{t_j - t_{j-1}}{\mathrm{V}(\mathcal{C}, [t_{j-1}, t_j])} \frac{\mathrm{V}(\mathcal{C}, [a, b])}{b - a} \left(\ell_{\mathcal{C}}(t) - \ell_{\mathcal{C}}(t_{j-1})\right)$$

therefore

$$\mathrm{D}\ell_{\mathcal{C}_j} = \frac{t_j - t_{j-1}}{\mathrm{V}(\mathcal{C}, [t_{j-1}, t_j])} \frac{\mathrm{V}(\mathcal{C}, [a, b])}{b - a} \mathrm{D}\ell_{\mathcal{C}} \tag{4.24}$$

and by applying Theorem 4.1 and Lemma 4.2 we get that for every $j \in \{1, \ldots, n\}$ there is a unique $y^j \in BV^r([t_{j-1}, t_j[\,; \mathcal{H})$ and a unique $v^j \in L^1(\mathrm{D}\ell_{\mathcal{C}}, [t_{j-1}, t_j[\,; \mathcal{H})$ such that

$$y^j(t) \in \mathcal{C}(t) \quad \forall t \in [t_{j-1}, t_j[\,, \tag{4.25}$$
$$\mathrm{D}y^j = v^j \, \mathrm{D}\ell_{\mathcal{C}} \quad \text{on } \mathscr{B}([t_{j-1}, t_j[), \tag{4.26}$$
$$-v^j(t) \in N_{\mathcal{C}(t)}(y^j(t)) \quad \text{for } \mathrm{D}\ell_{\mathcal{C}}\text{-a.e. } t \in [t_{j-1}, t_j[\,, \tag{4.27}$$
$$y^j(t_{j-1}) = \begin{cases} y_0 & \text{if } j = 1 \\ g_{t_{j-1}}(y^{j-1}(t_{j-1}-)) & \text{if } j \in \{2, \ldots, n\} \end{cases}, \tag{4.28}$$

and, using (4.14) and (4.24), we have

$$\|v^j(t)\| \le \frac{t_j - t_{j-1}}{\mathrm{V}(\mathcal{C}, [t_{j-1}, t_j])} \frac{\mathrm{V}(f, [a, b])}{b - a} \frac{\mathrm{V}(\mathcal{C}, [t_{j-1}, t_j])}{t_j - t_{j-1}}$$
$$= \frac{\mathrm{V}(f, [a, b])}{b - a} \quad \text{for } \mathrm{D}\ell_{\mathcal{C}}\text{-a.e. } t \in \,]t_{j-1}, t_j[\,. \tag{4.29}$$

Now we define $y : [a, b[\, \longrightarrow \mathcal{H}$ by setting

$$y(t) := \sum_{j=1}^{n} \mathbb{1}_{[t_{j-1}, t_j[}(t) y^j(t) \tag{4.30}$$

and $\mu : \mathscr{B}([a, b[) \longrightarrow [0, \infty[$ by (4.22). Observe that y is right continuous and satisfies (4.16), (4.19), and (4.20). Moreover μ is a positive measure, $\mu(\{a\}) = 0$, and thanks to right continuity of y we have that

$$\mathrm{V}(y, [s, t]) = \sum_{j=1}^{n} \|\mathrm{D}y\| ([t_{j-1}, t_j[\cap [s, t])$$
$$= \sum_{j=1}^{n} \mathrm{V}(y,]t_{j-1}, t_j[\cap [s, t]) + \sum_{t_j \in F \cap [s,t]} \|y(t_j) - y(t_j-)\|$$
$$= \sum_{j=1}^{n} \mathrm{V}(y^j,]t_{j-1}, t_j[\cap [s, t]) + \sum_{F \cap [s,t]} \|g_{t_j}(y^j(t_j-)) - y^j(t_j-)\|$$
$$\le \mathrm{V}(\mathcal{C}, [s, t]) + \sum_{t \in F \cap [s,t]} \|g_t - Id\|_{\infty, \mathcal{C}(t-)},$$

hence y has bounded variation and (4.21) holds. Now we consider a set $B \in \mathscr{B}([a, b[)$ such that $\mu(B) = 0$. For every $j \in \{1, \ldots, n\}$ we have $0 = \mu(B \cap \,]t_{j-1}, t_j[) = \mathrm{D}\ell_{\mathcal{C}}(B \cap \,]t_{j-1}, t_j[)$, thus from (4.26) we infer that $\mathrm{D}y^j(B \cap \,]t_{j-1}, t_j[) = 0$, and, since $y = y^j$ on $]t_{j-1}, t_j[$, it follows that $\mathrm{D}y = \mathrm{D}y^j$ on $\mathscr{B}(]t_{j-1}, t_j[)$, so that $\mathrm{D}y(B \cap \,]t_{j-1}, t_j[) = 0$. We also have that $0 = \mu(B \cap \{t_j\}) = \|g_{t_j} - Id\|_{\infty, \mathcal{C}(t_j-)}\delta_{t_j}(B)$ for every $j \in \{1, \ldots, n-1\}$, thus $\mathrm{D}y(\{t_j\}) = y(t_j) - y(t_j-) = g_{t_j}(y(t_j-)) - y((t_j-)) = 0$. Therefore, we infer that $\mathrm{D}y(B) = 0$ whenever $B \in \mathscr{B}([a, b[)$ and $\mu(B) = 0$, so that $\mathrm{D}y$ is μ-absolutely continuous and by the vectorial Radon–Nikodym theorem [44, Corollary 4.2, Section VII.4, p. 204] there exists a unique (up to μ-equivalence) function $v \in L^1(\mu; \mathcal{H})$ such that $\mathrm{D}y = v\mu$. It follows that on $\mathscr{B}(\left]t_{j-1}, t_j\right[)$ we have $\mathrm{D}y^j = \mathrm{D}y = v\mu = v\,\mathrm{D}\ell_{\mathcal{C}}$, thus $v = v^j$ for $\mathrm{D}\ell_{\mathcal{C}}$-a.e. $t \in \left]t_{j-1}, t_j\right[$ and v satisfies (4.18) thanks to (4.27). Let us also observe that for every $t \in F$ we have

$$y(t) - y(t-) = \mathrm{D}y(\{t\}) = \int_{\{t\}} v \,\mathrm{d}\mu = \left(\mathrm{D}\ell_{\mathcal{C}}(\{t\}) + \|g_t - Id\|_{\infty, \mathcal{C}(t-)}\right) v(t) \quad \forall t \in F$$

therefore

$$\|v(t)\| = \frac{\|y(t) - y(t-)\|}{\mathrm{D}\ell_{\mathcal{C}}(\{t\}) + \|g_t - Id\|_{\infty, \mathcal{C}(t-)}} \le \frac{\|g_t(y(t-)) - y(t-)\|}{\|g_t - Id\|_{\infty, \mathcal{C}(t-)}} \le 1 \quad \forall t \in F, \tag{4.31}$$

thus formula (4.23) follows from (4.29). The uniqueness of y is a consequence of its construction. ◻

We will need the following weak compactness theorem for measures [27, Theorem 5, p. 105], which we state in a form which is suitable to our purposes.

Theorem 4.2 *Let $I \subseteq \mathbb{R}$ be an interval and let M be a subset of the vector space of measures $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ with bounded variation endowed with the norm $\|\nu\| := \|\nu\|(I)$. Assume that M is bounded. Then M is weakly sequentially precompact if and only if there is a bounded positive measure $\mu : \mathscr{B}(I) \longrightarrow [0, \infty[$ such that*

$$\forall \varepsilon > 0\, \exists \delta > 0 \quad : \quad \left(B \in \mathscr{B}(I),\ \mu(B) < \delta \implies \sup_{\nu \in M} \|\nu\|(B) < \varepsilon\right). \tag{4.32}$$

It is worthwhile to mention that the above theorem is concerned with the notion of weak convergence of measures (in duality with the space of linear continuous functionals on the space of measures) and not with weak-* convergence of measures (in duality with continuous functions). Anyway, if one does not want an equivalence but only an implication, it is clearly true that the same condition above also provides weakly-* sequential precompactness of M.

Theorem 4.2 is stated in [27, Theorem 5, p. 105] as a topological precompactness result. An inspection in the proof easily shows that this is actually a sequential precompactness theorem, since an isometric isomorphism reduces it to the well-known Dunford–Pettis weak sequential precompactness theorem in $L^1(\mu; \mathcal{H})$ (see, e.g., [27, Theorem 1, p. 101]).

We are now in position to prove the following theorem which immediately implies our main result Theorem 3.1.

Theorem 4.3 *Assume that $-\infty < a < b < \infty$, $\mathcal{C} \in BV^{\mathrm{r}}([a, b[\,; \mathscr{C}_{\mathcal{H}})$, $y_0 \in \mathcal{C}(a)$, $S \subseteq \left]a, b\right[$, and that for every $t \in S$ we are given $g_t : \mathcal{C}(t-) \longrightarrow \mathcal{C}(t)$ such that $\mathrm{Lip}(g_t) \le 1$ and*

$$\sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} < \infty. \tag{4.33}$$

Then there exists a unique $y \in BV^{\mathrm{r}}([a, b[\,; \mathcal{H})$ such that there is a measure $\mu : \mathscr{B}([a, b[) \longrightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ such that

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b[\,, \tag{4.34}$$

$$\mathrm{D}y = v\mu, \tag{4.35}$$

$$-\, v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \textit{for } \mu\textit{-a.e. } t \in [a, b[\smallsetminus S, \tag{4.36}$$

$$y(t) = g_t(y(t-)) \quad \forall t \in S, \tag{4.37}$$

$$y(a) = y_0. \tag{4.38}$$

Moreover (3.7) *holds whenever* $a \le s < t \le b$. *Finally the function* $t \longmapsto \|y_1(t) - y_2(t)\|^2$ *is nonincreasing whenever* $y_{0,j} \in \mathcal{C}(a)$, $j = 1, 2$, *and* y_j *is the only function such that there is a measure* $\mu_j : \mathscr{B}([a, b[) \longrightarrow [0, \infty[$ *and a function* $v_j \in L^1(\mu; \mathcal{H})$ *for which* (4.34)-(4.38) *hold with* y, v, μ, y_0 *replaced, respectively, by* $y_j, v_j, \mu_j, y_{0,j}$.

Proof We may assume again that $\|g_t - Id\|_{\infty, \mathcal{C}(t-)} \neq 0$ for every $t \in S$, thus from (4.33) it follows that S is at most countable, and we may assume it contains infinitely many elements, since the finite case is considered in Lemma 4.3. Let $\mu : \mathscr{B}([a, b[) \longrightarrow [0, \infty[$ be defined by

$$\mu = \mathrm{D}\ell_{\mathcal{C}} + \sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} \delta_t. \tag{4.39}$$

If $S = \{s_n \;:\; n \in \mathbb{N}\}$, we set $S_n := \{s_1, \ldots, s_n\}$ for every $n \in \mathbb{N}$ and from Lemma 4.3 it follows that there is a unique $y_n \in BV^{\mathrm{r}}([a, b[\,; \mathcal{H})$ and a unique $v_n \in L^1(\mu; \mathcal{H})$ satisfying

$$y_n(t) \in \mathcal{C}(t) \quad \forall t \in [a, b[\,, \tag{4.40}$$

$$\mathrm{D}y_n = v_n \mu, \tag{4.41}$$

$$-\, v_n(t) \in N_{\mathcal{C}(t)}(y_n(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b[\smallsetminus S_n, \tag{4.42}$$

$$y_n(t) = g_t(y_n(t-)) \quad \forall t \in S_n, \tag{4.43}$$

$$y_n(a) = y_0. \tag{4.44}$$

Moreover if $a \le s < t \le b$ we have

$$\mathrm{V}(y_n, [s, t]) \le \mathrm{V}(\mathcal{C}, [s, t]) + \sum_{r \in S \cap [s,t]} \big(\|g_r - Id\|_{\infty, \mathcal{C}(r-)} - d_{\mathcal{H}}(\mathcal{C}(r-), \mathcal{C}(r))\big) < \infty \tag{4.45}$$

and

$$\|v_n(t)\| \le \mathrm{V}(\mathcal{C}, [a, b])/(b - a) \quad \text{for } \mu\text{-a.e. } t \in [a, b[\,. \tag{4.46}$$

Recall that $S_{n+1} \smallsetminus S_n = \{s_{n+1}\}$ and assume that $S_n = \{t_1, \ldots, t_n\}$ with $t_{j-1} < t_j$ for every j, and that $t_{h-1} < s_{n+1} < t_h$ for some $h \in \{2, \ldots, n\}$ (the cases $s_{n+1} < t_1$ and $t_n < s_{n+1}$ are dealt with similarly). Then $y_{n+1}(t) = y_n(t)$ for every $t \in \left]a, s_{n+1}\right[$, while at $t \in \left[s_{n+1}, t_h\right[$ the distance between y_n and y_{n+1} can be estimated by using (4.6), Theorem 4.1, and (4.40)–(4.43) as follows:

$$\begin{aligned}
\|y_{n+1}(t) - y_n(t)\| &\le \|y_{n+1}(s_{n+1}) - y_n(s_{n+1})\| \\
&= \|g_{s_{n+1}}(y_{n+1}(s_{n+1}-)) - \mathrm{Proj}_{\mathcal{C}(s_{n+1})}(y_n(s_{n+1}-))\| \\
&= \|g_{s_{n+1}}(y_n(s_{n+1}-)) - \mathrm{Proj}_{\mathcal{C}(s_{n+1})}(y_n(s_{n+1}-)\| \\
&\le \|g_{s_{n+1}}(y_n(s_{n+1}-)) - y_n(s_{n+1}-)\| \\
&\quad + \|y_n(s_{n+1}-) - \mathrm{Proj}_{\mathcal{C}(s_{n+1})}(y_n(s_{n+1}-))\| \\
&\le \|g_{s_{n+1}} - Id\|_{\infty, \mathcal{C}(s_{n+1}-)} + d_{\mathcal{H}}(\mathcal{C}(s_{n+1}-), \mathcal{C}(s_{n+1})) \quad \forall t \in \left[s_{n+1}, t_h\right[.
\end{aligned} \tag{4.47}$$

If $h < n$ and $t \in \left[t_h, t_{h+1}\right[$, from (4.6), (4.40)–(4.43), and (4.47), we infer that

$$\begin{aligned}\|y_{n+1}(t) - y_n(t)\| &\le \|g_{t_h}(y_{n+1}(t_h-)) - g_{t_h}(y_n(t_h-))\| \\ &\le \|y_{n+1}(t_h-) - y_n(t_h-)\| \\ &= \lim_{t \to t_h-} \|y_{n+1}(t) - y_n(t)\| \\ &\le \|g_{s_{n+1}} - Id\|_{\infty, \mathcal{C}(s_{n+1}-)} + d_{\mathcal{H}}(\mathcal{C}(s_{n+1}-), \mathcal{C}(s_{n+1})),\end{aligned}$$

and iterating this procedure we get the same estimate for $t \in \left[t_k, t_{k+1}\right[$ and $h \le k < n$, thus

$$\|y_{n+1} - y_n\|_\infty \le \|g_{s_{n+1}} - Id\|_{\infty, \mathcal{C}(s_{n+1}-)} + d_{\mathcal{H}}(\mathcal{C}(s_{n+1}-), \mathcal{C}(s_{n+1})),$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \|y_n - y_{n+1}\|_\infty &\le \sum_{n=1}^{\infty} \Big(\|g_{s_{n+1}} - Id\|_{\infty, \mathcal{C}(s_{n+1}-)} + d_{\mathcal{H}}(\mathcal{C}(s_{n+1}-), \mathcal{C}(s_{n+1}))\Big) \\ &\le \sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} + \mathrm{V}(\mathcal{C}, [a, b[) < \infty\end{aligned}$$

i.e., y_n is uniformly Cauchy and there exists $y : [a, b[\longrightarrow \mathcal{H}$ such that

$$y_n \to y \quad \text{uniformly on } [a, b[\,. \tag{4.48}$$

Moreover $y \in BV([a, b[\,; \mathcal{H})$ and (3.7) holds by (4.45) and the semicontinuity of the variation w.r.t. to the pointwise convergence, and y satisfies (4.34) by virtue of the closedness of $\mathcal{C}(t)$. Conditions (4.37)–(4.38) are trivially satisfied because for every $t \in S$ the sequence $y_n(t)$ is eventually constant, equal to $g_t(y(t-))$. Let us observe that thanks to (4.46) we have that

$$\begin{aligned}|\mathrm{D}y_n|(B) &= \int_B \|v_n(t)\|\,\mathrm{d}\mu(t) \le \int_B \frac{\mathrm{V}(\mathcal{C}, [a, b])}{b - a}\,\mathrm{d}\mu(t) \\ &= \frac{\mathrm{V}(\mathcal{C}, [a, b])}{b - a}\mu(B) \quad \forall B \in \mathscr{B}([a, b[)\end{aligned}$$

hence by the Dunford-Pettis Theorem 4.2 for measures and from [35, Lemma 7.1] we infer that

$$\mathrm{D}y_n \rightharpoonup \mathrm{D}y \tag{4.49}$$

Since $\|v_n\|_{L^2(\mu;\mathcal{H})} \le \mathrm{V}(\mathcal{C}, [a, b])\sqrt{\mu([a, b[)}/(b - a) < \infty$, there exists $v \in L^2(\mu; \mathcal{H})$ such that, up to subsequences,

$$v_n \rightharpoonup v \quad \text{in } L^2(\mu; \mathcal{H}) \text{as } n \to \infty, \tag{4.50}$$

and thanks to (4.42) and Lemma 4.1 we have that

$$\int_{[a,b[\smallsetminus S} \langle y_n(t) - z(t), v_n(t)\rangle\,\mathrm{d}\mu(t) \le 0 \quad \forall n \in \mathbb{N}, \tag{4.51}$$

therefore taking the limit as $n \to \infty$ from (4.50) and (4.50) we infer that

$$\int_{[a,b[\smallsetminus S} \langle y(t) - z(t), v(t)\rangle\,\mathrm{d}\mu \le 0 \tag{4.52}$$

which, by Lemma 4.1, is equivalent to (4.36). Estimate (3.7) follows from (4.45) and from the lower semicontinuity of the variation. If $\phi : [a, b[\longrightarrow \mathcal{H}$ is an arbitrary bounded Borel function then $\mu \longmapsto \int_{[a,b[} \langle \phi(t), \mathrm{d}\mu(t) \rangle$ is a continuous linear functional on the space of measures with bounded variation and we have

$$\lim_{n \to \infty} \int_{[a,b[} \langle \phi(t), \mathrm{d}\mathrm{D}y_n(t) \rangle = \int_{[a,b[} \langle \phi(t), \mathrm{d}\mathrm{D}y(t) \rangle.$$

On the other hand we have

$$\int_{[a,b[} \langle \phi(t), \mathrm{d}\mathrm{D}y_n(t) \rangle = \int_{[a,b[} \langle \phi(t), v_n(t) \rangle \, \mathrm{d}\mu(t) \quad \forall n \in \mathbb{N},$$

thus taking the limit as $n \to \infty$ thanks to (4.49) and (4.50) we get

$$\int_{[a,b[} \langle \phi(t), \mathrm{d}\mathrm{D}y(t) \rangle = \int_{[a,b[} \langle \phi(t), v(t) \rangle \, \mathrm{d}\mu(t)$$

therefore the arbitrariness of ϕ yields $\mathrm{D}y = v\mu$ (cf., e.g., [28, Proposition 35, p. 326]). Then (4.35) is proved, and the existence part of the theorem is done. If $B \in \mathscr{B}([a, b[)$ then, by [58, Proposition 2] and by the monotonicity of the normal cone we have

$$\begin{aligned}\int_{B \setminus S} \mathrm{d}\mathrm{D}(\|y_1(\cdot) - y_2(\cdot)\|^2) &\leq 2 \int_{B \setminus S} \langle y_1 - y_2, \mathrm{d}\mathrm{D}(y_1 - y_2) \rangle \\ &= 2 \int_{B \setminus S} \langle y_1(t) - y_2(t), v_1(t) - v_2(t) \rangle \, \mathrm{d}\mu(t) \leq 0, \end{aligned} \tag{4.53}$$

while if $t \in S$ then we have

$$\begin{aligned}\mathrm{D}(\|y_1(\cdot) - y_2(\cdot)\|^2)(\{t\}) &= \|y_1(t) - y_2(t)\|^2 - \|y_1(t-) - y_2(t-)\|^2 \\ &= \|g_t(y_1(t-)) - g_t(y_2(t-))\| - \|y_1(t-) - y_2(t-)\|^2 \\ &\leq \|y_1(t-) - y_2(t-)\| - \|y_1(t-) - y_2(t-)\|^2 = 0, \end{aligned} \tag{4.54}$$

therefore for every $B \in \mathscr{B}([a, b[)$ we find

$$\begin{aligned}\mathrm{D}(\|y_1(\cdot) - y_2(\cdot)\|^2)(B) &= \mathrm{D}(\|y_1(\cdot) - y_2(\cdot)\|^2)(B \setminus S) + \mathrm{D}(\|y_1(\cdot) - y_2(\cdot)\|^2)(B \cap S) \\ &= \int_{B \setminus S} \mathrm{d}\mathrm{D}(\|y_1(\cdot) - y_2(\cdot)\|^2) + \sum_{t \in B \cap S} \mathrm{D}(\|y_1(\cdot) - y_2(\cdot)\|^2)(\{t\}) \leq 0\end{aligned}$$

which implies that $t \longmapsto \|y_1(t) - y_2(t)\|^2$ is nonincreasing and leads to the uniqueness of the solution. As a consequence the whole sequence y_n converges uniformly to y. □

5 Applications

In this section we discuss some consequences and particular cases of Theorem 3.1.

5.1 Sweeping processes with arbitrary *BV* driving set and prescribed behavior on jumps

We first consider the case of a sweeping processes with prescribed behavior on jumps, where the driving moving set is not assumed to be right continuous.

Theorem 5.1 *Assume that* $-\infty < a < b < \infty$, $\mathcal{C} \in BV([a,b]\,;\mathscr{C}_{\mathcal{H}})$, $y_0 \in \mathcal{H}$, *and that for every* $t \in \mathrm{Discont}(\mathcal{C}) \cup \{a\}$ *we are given* $g_t^l : \mathcal{C}(t-) \longrightarrow \mathcal{C}(t)$ *and* $g_t^r : \mathcal{C}(t) \longrightarrow \mathcal{C}(t+)$ *such that*

$$\mathrm{Lip}(g_t^l) \le 1, \quad \mathrm{Lip}(g_t^r) \le 1 \quad \forall t \in \mathrm{Discont}(\mathcal{C}) \smallsetminus \{a\} \tag{5.1}$$

and

$$\sum_{t \in \mathrm{Discont}(\mathcal{C})} \|g_t^l - Id\|_{\infty,\mathcal{C}(t-)} < \infty, \qquad \sum_{t \in \mathrm{Discont}(\mathcal{C})} \|g_t^r - Id\|_{\infty,\mathcal{C}(t)} < \infty. \tag{5.2}$$

Then there exists a unique $y \in BV([a,b]\,;\mathcal{H})$ *such that there is a measure* $\mu : \mathscr{B}([a,b]) \longrightarrow [0,\infty[$ *and a function* $v \in L^1(\mu;\mathcal{H})$ *for which*

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a,b]\,, \tag{5.3}$$

$$\mathrm{D}y = v\mu, \tag{5.4}$$

$$-\,v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in \mathrm{Cont}(\mathcal{C}), \tag{5.5}$$

$$y(t) = g_t^l((y(t-))), \quad y(t+) = g_t^r(g_t^l((y(t-))) \quad \forall t \in \mathrm{Discont}(\mathcal{C}) \smallsetminus \{a\}, \tag{5.6}$$

$$y(a) = g_a^l(y_0), \quad y(a+) = g_a^r(a)(y(a)). \tag{5.7}$$

Moreover if $a \le s < t \le b$ *then*

$$\mathrm{V}(y,[s,t]) \le \mathrm{V}(\mathcal{C},[s,t]) + \sum_{\sigma \in \mathrm{Discont}(\mathcal{C}) \cap [s,t]} \Big(\|g_\sigma^l - Id\|_{\infty,\mathcal{C}(\sigma-)} + \|g_\sigma^r - Id\|_{\infty,\mathcal{C}(\sigma)} + d_{\mathcal{H}}(\mathcal{C}(\sigma-),\mathcal{C}(\sigma)) + d_{\mathcal{H}}(\mathcal{C}(\sigma),\mathcal{C}(\sigma+))\Big).$$

Proof We can apply Theorem 3.1 with $g_t := g_t^r \circ g_t^l$ and find a unique function $\hat{y} \in BV^r([a,b]\,;\mathcal{H})$ such that there is a measure $\hat{\mu} : \mathscr{B}([a,b]) \longrightarrow [0,\infty[$ and a function $\hat{v} \in L^1(\mu;\mathcal{H})$ for which

$$\hat{y}(t) \in \mathcal{C}(t+) \quad \forall t \in [a,b] \tag{5.8}$$

$$\mathrm{D}\hat{y} = \hat{v}\hat{\mu}, \tag{5.9}$$

$$-\,\hat{v}(t) \in N_{\mathcal{C}(t+)}(\hat{y}(t)) = N_{\mathcal{C}(t)}(\hat{y}(t)) \quad \text{for } \hat{\mu}\text{-a.e. } t \in \mathrm{Cont}(\mathcal{C}), \tag{5.10}$$

$$\hat{y}(t) = g_t^r(g_t^l(\hat{y}(t-))) \quad \forall t \in \mathrm{Discont}(\mathcal{C}) \smallsetminus \{a\}, \tag{5.11}$$

$$\hat{y}(a) = g_a^r(y_0). \tag{5.12}$$

Then the theorem is satisfied if we take $y : [a,b] \longrightarrow \mathcal{H}$, $\mu : \mathscr{B}([a,b]) \longrightarrow [0,\infty[$, and $v : [a,b] \longrightarrow \mathcal{H}$ defined by

$$y(t) := \begin{cases} \hat{y}(t) & \text{if } t \in \mathrm{Cont}(\mathcal{C}) \\ g_t^l(\hat{y}(t-)) & \text{if } t \in \mathrm{Discont}(\mathcal{C}) \end{cases},$$

$\mu := \hat{\mu}$, and

$$v(t) := \begin{cases} \hat{v}(t) & \text{if } t \in \mathrm{Cont}(\mathcal{C}) \\ 0 & \text{if } t \in \mathrm{Discont}(\mathcal{C}) \text{ and } \hat{\mu}(\{t\}) = 0 \\ \dfrac{y(t+) - y(t-)}{\hat{\mu}(\{t\})} & \text{if } t \in \mathrm{Discont}(\mathcal{C}) \text{ and } \hat{\mu}(\{t\}) \ne 0 \end{cases}.$$

(observe that the first condition in (5.2) ensures that $y \in BV([a,b]\,;\mathcal{H})$). $\square$

Remark 5.1 Let us observe that we can actually prove a result which is more general than Theorem 5.1: indeed we can prescribe the behavior of the solution $y(t)$ also on a countable set of points t where $\mathcal{C}$ is continuous. In order to do that we need to assume that the families $g_t^l : \mathcal{C}(t-) \longrightarrow \mathcal{C}(t)$ and $g_t^r : \mathcal{C}(t) \longrightarrow \mathcal{C}(t+)$ are indexed by $t \in S$, where $S \subseteq [a,b]$, $\mathrm{Lip}(g_t^l) \le 1$ and $\mathrm{Lip}(g_t^r) \le 1$ for every $t \in S$, and $\sum_{t\in S} \|g_t^l - Id\|_{\infty,\mathcal{C}(t-)} < \infty$ and $\sum_{t\in S} \|g_t^r - Id\|_{\infty,\mathcal{C}(t)} < \infty$. Therefore it follows that if $\mathcal{C} \in BV([a,b]\,;\mathscr{C}_{\mathcal{H}})$, then there exists a unique $y \in BV([a,b]\,;\mathcal{H})$ such that there is a measure $\mu : \mathscr{B}([a,b]) \longrightarrow [0,\infty[$ and a function $v \in L^1(\mu;\mathcal{H})$ for which (5.3), (5.4), and (5.7) hold together with

$$-\,v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a,b] \smallsetminus S, \tag{5.13}$$

$$y(t) = g_t^l((y(t-))), \quad y(t+) = g_t^r(g_t^l((y(t-))) \quad \forall t \in S \smallsetminus \{a\}. \tag{5.14}$$

Observe that in this case S is a fortiori at most countable and y may jump even when $\mathcal{C}$ does not jump.

5.2 Sweeping processes with arbitrary BV driving moving set.

Another consequence of Theorem 5.1 is the existence and uniqueness theorem for sweeping processes with arbitrary BV driving moving set.

Theorem 5.2 *Assume that* $-\infty < a < b < \infty$*,* $\mathcal{C} \in BV([a,b]\,;\mathscr{C}_{\mathcal{H}})$ *and* $y_0 \in \mathcal{H}$*. Then there exists a unique* $y \in BV([a,b]\,;\mathcal{H})$ *such that there is a measure* $\mu : \mathscr{B}([a,b]) \longrightarrow [0,\infty[$ *and a function* $v \in L^1(\mu;\mathcal{H})$ *for which*

$$y(t) \in \mathcal{C}(t), \tag{5.15}$$

$$\mathrm{D}y = v\mu, \tag{5.16}$$

$$-\,v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \textit{for } \mu\textit{-a.e. } t \in \mathrm{Cont}(\mathcal{C}), \tag{5.17}$$

$$y(t) = \mathrm{Proj}_{\mathcal{C}(t)}(y(t-)), \quad y(t+) = \mathrm{Proj}_{\mathcal{C}(t+)}(y(t)) \quad \forall t \in \mathrm{Discont}(\mathcal{C}) \smallsetminus \{a\} \tag{5.18}$$

$$y(a) = \mathrm{Proj}_{\mathcal{C}(a)}(y_0), \quad y(a+) = \mathrm{Proj}_{\mathcal{C}(a+)(y(a))}\,. \tag{5.19}$$

Moreover $\mathrm{V}(y,[s,t]) \le \mathrm{V}(\mathcal{C},[s,t])$ *whenever* $a \le s < t \le b$*.*

Proof It is enough to apply Theorem 5.1 with $g_t^l := \mathrm{Proj}_{\mathcal{C}(t+)}$ and $g_t^r := \mathrm{Proj}_{\mathcal{C}(t)}$. □

5.3 The play operator

The play operator is the solution operator of the sweeping process driven by a moving set $\mathcal{C}(t)$ with constant shape, i.e., $\mathcal{C}(t) = u(t) - \mathcal{Z}$, where $u \in BV^{\mathrm{r}}([a,b]\,;\mathcal{H})$ and $\mathcal{Z} \in \mathscr{C}_{\mathcal{H}}$. In the following result we restate here the existence Theorem 4.1 in this particular case by using the integral formulation of Lemma 4.1 and we collect some other well-known results (see also [39], where the Young integral is used, and [35, Section 5] containing a slightly different integral formulation).

Theorem 5.3 *Assume that* $-\infty < a < b < \infty$*,* $\mathcal{Z} \in \mathscr{C}_{\mathcal{H}}$*,* $u \in BV^{\mathrm{r}}([a,b]\,;\mathcal{H})$ *and* $z_0 \in \mathcal{Z}$*. Then there exists a unique* $y \in BV^{\mathrm{r}}([a,b]\,;\mathcal{H})$ *such that*

$$y(t) \in u(t) - \mathcal{Z} \quad \forall t \in [a,b]\,, \tag{5.20}$$

$$\int_{[a,b]} \langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t)\rangle \le 0 \quad \textit{for every } \mu\textit{-measurable } z : [a,b] \longrightarrow \mathcal{Z}, \tag{5.21}$$

$$u(0) - y(0) = z_0. \tag{5.22}$$

The solution operator $\mathsf{P} : BV^r([a,b];\mathcal{H}) \times \mathcal{Z} \longrightarrow BV^r([a,b];\mathcal{H})$ *associating with* $(u, z_0) \in BV^r([a,b];\mathcal{H}) \times \mathcal{Z}$ *the unique function* $y = \mathsf{P}(u, z_0)$ *satisfying* (5.20)–(5.22) *is called* play operator *and it is* rate independent*, i.e.,*

$$\mathsf{P}(u \circ \psi, z_0) = \mathsf{P}(u, z_0) \circ \psi \quad \forall u \in BV^r([a,b];\mathcal{H}) \tag{5.23}$$

whenever $\psi \in C([a,b];[a,b])$ *is nondecreasing and surjective. We have that* $\mathsf{P}(Lip([a,b];\mathcal{H}) \times \mathcal{Z}) \subseteq Lip([a,b];\mathcal{H})$ *and if* $u \in Lip([a,b];\mathcal{H})$ *then* $\mathsf{P}(z_0, u) = y$ *is the unique function satisfying* (5.20), (5.22)*, and*

$$\left\langle z(t) - u(t) - y(t), y'(t) \right\rangle \le 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a,b]\,, \ \forall z \in \mathcal{Z}. \tag{5.24}$$

The restriction $\mathsf{P} : Lip([a,b];\mathcal{H}) \times \mathcal{Z} \longrightarrow Lip([a,b];\mathcal{H})$ *is continuous if* $Lip([a,b];\mathcal{H})$ *is endowed with both the* BV*-norm metric*

$$d_{BV}(v, w) := \|v - w\|_{\infty,[a,b]} + \mathrm{V}(v - w; [a,b]),$$

and the strict metric

$$d_s(v, w) := \|v - w\|_{L^1(\mathcal{L}^1;\mathcal{H})} + |\,\mathrm{V}(v; [a,b]) - \mathrm{V}(w; [a,b])|,$$

and the operator $\mathsf{P} : BV^r([a,b];\mathcal{H}) \times \mathcal{Z} \longrightarrow BV^r([a,b];\mathcal{H})$ *is its unique continuous extension w.r.t. the topology induced by* d_{BV}*.*

Let us observe that the integral formulation (5.20)–(5.22) is a direct consequence of Theorem 4.1 and Lemma 4.1. Formulas (5.23) and (5.24) are well known (see, e.g., [60, Section 3i, Section 3c] and [38, Proposition 3.9, p. 33]). The continuity of the restriction of P w.r.t. d_{BV} is proved in [38, Theorem 3.12, p. 34], while its continuity w.r.t $d_s(v, w)$ is proved in [65, Theorem 5.5]. The BV-norm continuity of P on $BV^r([a,b];\mathcal{H}) \times \mathcal{Z}$ is proved in [35, Theorem 3.3].

The operator $\mathsf{P} : BV^r([a,b];\mathcal{H}) \times \mathcal{Z} \longrightarrow BV^r([a,b];\mathcal{H})$ in general is not continuous w.r.t. to the strict metric d_s (cf. [65, Thereom 3.7]) but its restriction $\mathsf{P} : Lip([a,b];\mathcal{H}) \times \mathcal{Z} \longrightarrow Lip([a,b];\mathcal{H})$ can be continuously extended to another operator if $BV^r([a,b];\mathcal{H})$ is endowed with the strict topology in the domain, and with the L^1-topology in the codomain. Let us recall the precise result proved in [65, Thereom 3.7].

Theorem 5.4 *The operator* $\mathsf{P} : Lip([a,b];\mathcal{H}) \times \mathcal{Z} \longrightarrow Lip([a,b];\mathcal{H})$ *admits a unique continuous extension* $\overline{\mathsf{P}} : BV^r([a,b];\mathcal{H}) \times \mathcal{Z} \longrightarrow BV^r([a,b];\mathcal{H})$ *if* $BV^r([a,b];\mathcal{H})$ *is endowed with the strict topology* d_s *in the domain, and with the* $L^1(\mathcal{L}^1;\mathcal{H})$*-topology in the codomain. We have*

$$\overline{\mathsf{P}}(u, z_0) = \mathsf{P}(\widetilde{u}, z_0) \circ \ell_u \quad \forall u \in BV^r([a,b];\mathcal{H}), \tag{5.25}$$

where $\widetilde{u} \in Lip([a,b];\mathcal{H})$ *is the arc length reparametrization introduced in Proposition* 4.1*. In general* $\overline{\mathsf{P}} \neq \mathsf{P}$*.*

The meaning of the extension $\overline{\mathsf{P}}$ and of formula (5.25) is clear: we reparametrize by the arc length the function $u = \widetilde{u} \circ \ell_u$ and we apply the play operator to the Lipschitz reparametrization $\widetilde{u}$, which is a segment on the jump sets $[\ell_u(t-), \ell_u(t+)]$. Then we "throw away" the jump sets from $\mathsf{P}(\widetilde{u}, z_0)$ by reinserting ℓ_u and we obtain $\overline{\mathsf{P}}(u, z_0) = \mathsf{P}(\widetilde{u}, z_0) \circ \ell_u$. A further motivation to this procedure is the rate independence of P (but observe that ℓ_u is not continuous), and we could also say that we are filling in the jumps of u with a segment traversed with "infinite velocity". The continuity property of $\overline{\mathsf{P}}$ in Theorem 5.4 confirms this interpretation. Now we are going to show that $\overline{\mathsf{P}}(u, z_0)$ can also be obtained as the solution of a sweeping processes with a suitable prescribed behavior on jumps.

Theorem 5.5 *Assume that $\mathcal{Z} \in \mathscr{C}_{\mathcal{H}}$, $u \in BV^{r}([a,b]\,;\mathcal{H})$ and $z_0 \in \mathcal{Z}$. For every $x, y \in \mathcal{H}$ let* $\mathrm{seg}_{x,y} : [0,1] \longrightarrow \mathcal{H}$ *be defined by* $\mathrm{seg}_{x,y}(t) := (1-t)x + ty$, $t \in [0,1]$. *Then there exists a unique $y \in BV^{r}([a,b]\,;\mathcal{H})$ such that*

$$y(t) \in u(t) - \mathcal{Z} \quad \forall t \in [a,b] \tag{5.26}$$

$$\int_{\mathrm{Cont}(u)} \langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t)\rangle \le 0 \quad \textit{for every } \mu\textit{-measurable } z : [a,b] \longrightarrow \mathcal{Z}, \tag{5.27}$$

$$y(t) = \mathsf{P}(\mathrm{seg}_{u(t-),u(t)}, u(t-) - y(t-))(1) \quad \forall t \in \mathrm{Discont}(\mathcal{C}), \tag{5.28}$$

$$y(0) = y_0. \tag{5.29}$$

Moreover if $u \in BV^{r}([a,b]\,;\mathcal{H})$, $z_0 \in \mathcal{Z}$, $s, t \in [a,b]$, $s < t$, then we have

$$y = \mathsf{P}(\widetilde{u}, z_0) \circ \ell_u = \overline{\mathsf{P}}(u, z_0), \tag{5.30}$$

and

$$\mathrm{V}(\overline{\mathsf{P}}(u), [s,t]) \le \mathrm{V}(u, [s,t]). \tag{5.31}$$

Proof We set $\mathcal{C}(t) := u(t) - \mathcal{Z}$, $y_0 := u(0) - z_0$, $S := \mathrm{Discont}(u)$, and

$$g_t(x) := \mathsf{P}(\mathrm{seg}_{u(t-),u(t)}, u(t-) - x)(1), \quad t \in S.$$

If $t \in \mathrm{Discont}(u)$ and $x \in \mathcal{C}(t-)$ then $g_t(x)$ is the solution of the sweeping process driven by $\mathcal{K}_u(\tau) := (1-\tau)u(t-) - \tau u(t) - \mathcal{Z}$, $\tau \in [0,1]$, and having x as initial condition, thus from (4.5) we infer that $\mathrm{V}(\mathsf{P}(\mathrm{seg}_{u(t-),u(t)}, u(t-) - x), [0,1]) \le \mathrm{V}(\mathcal{K}_u, [0,1]) = \|u(t) - u(t-)\|$ and we have

$$\begin{aligned}
\sum_{t \in \mathrm{Discont}(u)} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} &= \sum_{t \in \mathrm{Discont}(u)} \sup_{x \in \mathcal{C}(t-)} \|g_t(x) - x\| \\
&= \sum_{t \in \mathrm{Discont}(u)} \sup_{x \in \mathcal{C}(t-)} \|\mathsf{P}(\mathrm{seg}_{u(t-),u(t)}, u(t-) - x)(1) - x\| \\
&\le \sum_{t \in \mathrm{Discont}(u)} \sup_{x \in \mathcal{C}(t-)} \mathrm{V}(\mathsf{P}(\mathrm{seg}_{u(t-),u(t)}, u(t-) - x), [0,1]) \\
&\le \sum_{t \in \mathrm{Discont}(u)} \|u(t) - u(t-)\| \le \mathrm{V}(u, [0,T])
\end{aligned}$$

which together with (3.7) implies (5.31). Moreover thanks to (4.6) we have that $\|g_t(x_1) - g_t(x_2)\| \le \|x_1 - x_2\|$ therefore we can apply Theorem 3.1 and infer the existence of a unique y satisfying (5.26)–(5.29). Now we show that (5.30) holds. Thanks to the chain rule [65, Theorem A.7] we have that $\mathrm{D}\overline{\mathsf{P}}(u, z_0) = w\, \mathrm{D}\ell_u$ with

$$w(t) = \begin{cases} (\mathsf{P}(\widetilde{u}), z_0)'(t) & \text{if } t \in \mathrm{Cont}(u) \\[2ex] \dfrac{\mathsf{P}(\widetilde{u}, z_0)(\ell_u(t)) - \mathsf{P}(\widetilde{u}, z_0)(\ell_u(t-))}{\ell_u(t) - \ell_u(t-))} & \text{if } t \in \mathrm{Discont}(u) \end{cases}$$

and if we set

$$N := \{\sigma \in [a,b] \,:\, \langle z - \widetilde{u}(\sigma) + \mathsf{P}(\widetilde{u}, z_0)(\sigma), (\mathsf{P}(\widetilde{u}, z_0))'(\sigma)\rangle > 0 \text{ for some } z \in \mathcal{Z}\}$$

from (5.24) we deduce that $\mathcal{L}^1(N) = 0$, hence, thanks to [65, Lemma A.5], we have that

$$\begin{aligned}&\mathrm{D}\ell_{\mathcal{C}}(\{t \in \mathrm{Cont}(\ell_{\mathcal{C}}) \ :\ \langle z - \widetilde{u}(\ell_u(t)) + \mathsf{P}(\widetilde{u}, z_0)(\ell_u(t)), (\mathsf{P}(\widetilde{u}), z_0)'(\ell_u(t))\rangle \\ &\quad > 0 \text{ for some } z \in \mathcal{Z}\}) \\ &\quad = \mathrm{D}\ell_{\mathcal{C}}(\{t \in \mathrm{Cont}(\ell_{\mathcal{C}}) \ :\ \ell_{\mathcal{C}}(t) \in N\}) = \mathcal{L}^1(N) = 0.\end{aligned}$$

This implies that if $z : [a, b] \longrightarrow \mathcal{H}$ is μ-measurable and $z(t) \in \mathcal{Z}$ for every $t \in [a, b]$, then

$$\begin{aligned}&\int_{\mathrm{Cont}(u)} \langle z(t) - u(t) + \mathsf{P}(\widetilde{u}, z_0)(\ell_u(t)), \mathrm{d}\mathrm{D}(\mathsf{P}(\widetilde{u}, z_0) \circ \ell_u)\rangle \\ &= \int_{\mathrm{Cont}(u)} \langle z(t) - u(t) + \mathsf{P}(\widetilde{u}, z_0)(\ell_u(t)), (\mathsf{P}(\widetilde{u}, z_0)'(\ell_u(t))\rangle \, \mathrm{d}\mathrm{D}\ell_{\mathcal{C}}(t) \le 0\end{aligned}$$

and (5.27) is satisfied. Finally (5.28) is trivially satisfied by $\overline{\mathsf{P}}(u, z_0)$. □

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