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Sign-changing bubble tower solutions for the supercritical Hénon-type equations

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Abstract This paper deals with the following supercritical Hénon-type equation

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{p_\alpha - 1 - \varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha > -2$, $\varepsilon > 0$, $p_\alpha = \frac{N+2+2\alpha}{N-2}$, $N \geq 3$, Ω is a smooth bounded domain in $\mathbb{R}^N$ containing the origin. For $\varepsilon > 0$ small enough, it is shown that if α is not an even integer, the above problem has sign-changing bubble tower solutions, which blow up at the origin. It seems that this is the first existence result of sign-changing bubble tower solutions for the supercritical Hénon-type equation.

Keywords Sign-changing bubble tower solutions · Hénon-type equation · Lyapunov–Schmidt reduction

Mathematics Subject Classification Primary 35J60 ; Secondary 35J05 · 35J40

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1 Introduction and main results

In this paper, we consider the existence of sign-changing bubble tower solutions for the following supercritical Hénon-type equation

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p_\alpha - 1 - \varepsilon} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\alpha > -2$, $\varepsilon > 0$, $p_\alpha = \frac{N+2+2\alpha}{N-2}$, $N \geq 3$, Ω is a smooth bounded domain in $\mathbb{R}^N$ containing the origin.

When Ω is the unit ball $B_1(0)$ of $\mathbb{R}^N$, problem (1.1) becomes the well-known Hénon equation, i.e.,

$$\begin{cases} -\Delta u = |x|^\alpha u^p, \ u > 0, & \text{in } B_1(0), \\ u = 0, & \text{on } \partial B_1(0). \end{cases} \tag{1.2}$$

Problem (1.2) was proposed by Hénon in [15] when he studied rotating stellar structures, which has attracted a lot of interest in recent years. Ni [21] first considered (1.2) and proved that it possesses a positive radial solution when $p \in (1, p_\alpha)$. Due to the appearance of the weighted term $|x|^\alpha$, the classical moving plane method in [13] cannot be applied to problem (1.2) when $\alpha > 0$. Therefore it is quite natural to ask whether problem (1.2) with $\alpha > 0$ has non-radial solutions. Based on numerical results in [4], Smets, Su and Willem [27] obtained the existence of non-radial solutions for $1 < p < \frac{N+2}{N-2}$, when α is large enough. For $p = \frac{N+2}{N-2} - \varepsilon$, Cao and Peng [5] showed that the ground state solution is non-radial and blows up near the boundary of $B_1(0)$ as $\varepsilon \to 0$. Later on, Peng [22] constructed multiple boundary peak solutions for problem (1.2). When $p = \frac{N+2}{N-2}$, Serra [26] proved that problem (1.2) has a non-radial solution provided α is large enough. More recently, Wei and Yan [28] showed that there are infinitely many non-radial positive solutions for problem (1.2) with $\alpha > 0$. For other results related to the Hénon-type problems, see [1,2,6,16,24] and the references therein.

On the other hand, using the Pohozaev-type identity [25], we know that for $p \geq p_\alpha$ there are no nontrivial solutions to problem (1.2). So it seems more interesting whether there are solutions for $p \in (\frac{N+2}{N-2}, p_\alpha)$. When $p = p_\alpha - \varepsilon$ with small $\varepsilon > 0$, Gladiali and Grossi in [11] showed that there exists a solution concentrating at origin provided $0 < \alpha \leq 1$. By the results in [12], the same results still hold when α is not an even integer. In [17], the asymptotic behavior of the radial solutions obtained by Ni in [21] was analyzed as $\varepsilon \to 0^+$. More recently, Liu and Peng [19] constructed large number of peak solutions for (1.2) with $p = \frac{N+2}{N-2} + \varepsilon$. However, as far as we know, it seems that there are no results on existence of sign-changing solutions for (1.2) when $p \in (\frac{N+2}{N-2}, p_\alpha)$

Our purpose in the present paper is to construct sign-changing bubble tower solutions for (1.1) which blow up at the origin. It seems that this is the first existence result of sign-changing bubble tower solutions for (1.1).

The main result of this paper is as follows.

Theorem 1.1 *Assume that $N \geq 3$, $\alpha > -2$ is not an even integer, then for any $k \in \mathbb{N}^+$, there exists $\varepsilon_k > 0$ such that for $\varepsilon \in (0, \varepsilon_k)$, problem (1.1) has a sign-changing bubble tower solution u_ε with exactly k nodal sets in Ω.*

Remark 1.2 When $\alpha = 0$ and Ω has some symmetry property, problem (1.1) has been studied in [3] and [23]. Our results do not need any symmetry property of Ω. Further more, compared with the classical Hénon equation where $\alpha > 0$, our result covers the more general case $\alpha > -2$.

Remark 1.3 If Ω is the unit ball of $\mathbb{R}^N$, then Theorem 1.1 holds for all $\alpha > -2$. Actually, we can construct a sign-changing radial bubble tower solution u_ε. Considering the transformation $\mathrm{w}(s) = u(r),\ \ r = s^{\frac{2}{\alpha+2}}$, problem (1.1) can be changed into the following problem

$$\begin{cases} -\mathrm{w}'' - \frac{M-1}{s}\mathrm{w}' = \frac{4}{(2+\alpha)^2}|\mathrm{w}|^{\frac{M+2}{M-2}-\varepsilon}\mathrm{w}, & \text{in } (0,1), \\ \mathrm{w}'(0) = \mathrm{w}(1) = 0, \end{cases} \tag{1.3}$$

where $M = \frac{2(N+\alpha)}{2+\alpha}$. When M is an integer, problem (1.3) was studied in [3]. However, problem (1.3) can be dealt with in a similar way if M is not an integer.

Let us outline the main idea to prove Theorem 1.1. To do this, we introduce a few notations first. For $x \in \mathbb{R}^N$ and $\mu > 0$, set

$$U_\mu(x) = C_{\alpha,N}\left(\frac{\mu^{\frac{2+\alpha}{2}}}{\mu^{2+\alpha} + |x|^{2+\alpha}}\right)^{\frac{N-2}{2+\alpha}},\ \ C_{\alpha,N} = ((N+\alpha)(N-2))^{\frac{N-2}{4+2\alpha}}.$$

It is well known from [12,14] that $U_\mu(x)$ are the only radial solutions of

$$-\Delta u = |x|^\alpha u^{p_\alpha}, u > 0 \text{ in } \mathbb{R}^N. \tag{1.4}$$

We define the following Emden–Fowler-type transformation

$$v(y,\Theta) = \mathcal{T}(u)(y,\Theta) = \left(\frac{p_\alpha - 1}{2}\right)^{\frac{2}{p_\alpha-1}} r^{\frac{N-2}{2}} u(r,\Theta), \tag{1.5}$$

where

$$r = e^{-\frac{p_\alpha-1}{2}y}, \Theta \in \mathbb{S}^{N-1}.$$

Define

$$D = \{(y,\Theta) \in \mathbb{R}\times\mathbb{S}^{N-1} : (e^{-\frac{p_\alpha-1}{2}y}, \Theta) \in \Omega\}.$$

After these changes of variables, problem (1.1) becomes

$$\begin{cases} L(v) = \sigma_\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon y}|v|^{p_\alpha-1-\varepsilon}v & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases} \tag{1.6}$$

where

$$\sigma_\varepsilon = \left(\frac{p_\alpha - 1}{2}\right)^{\frac{2\varepsilon}{p_\alpha-1}}$$

and

$$L(v) = -v'' + \frac{(2+\alpha)^2}{4}v - \left(\frac{p_\alpha-1}{2}\right)^2 \Delta_{\mathbb{S}^{N-1}}v.$$

The energy functional corresponding to problem (1.6) is

$$I_\varepsilon(v) = \frac{1}{2}\int_D \left(|v'|^2 + \frac{(2+\alpha)^2}{4}|v|^2\right)\mathrm{d}y\mathrm{d}\Theta + \frac{1}{2}\left(\frac{p_\alpha-1}{2}\right)^2\int_D |\nabla_{\mathbb{S}^{N-1}}v|^2\mathrm{d}y\mathrm{d}\Theta$$
$$- \frac{\sigma_\varepsilon}{p_\alpha+1-\varepsilon}\int_D e^{-\frac{2+\alpha}{2}\varepsilon y}|v|^{p_\alpha+1-\varepsilon}\mathrm{d}y\mathrm{d}\Theta.$$

It is easy to see that

$$I_\varepsilon(v) = \left(\frac{p_\alpha - 1}{2}\right)^{\frac{p_\alpha+3}{p_\alpha-1}} J_\varepsilon(u), \tag{1.7}$$

where

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p_\alpha + 1 - \varepsilon}\int_\Omega |x|^\alpha |u|^{p_\alpha+1-\varepsilon} dx.$$

We observe that $W(y)$ is the unique solution of the problem

$$\begin{cases} W'' - \frac{(2+\alpha)^2}{4} W + W^{p_\alpha} = 0 \ \text{ in } \mathbb{R}, \\ W'(0) = 0, \quad W(y) > 0, \\ W(y) \to 0 \ \text{ as } \ y \to \pm\infty, \end{cases} \tag{1.8}$$

where

$$W(y) = \gamma_{\alpha,N} \frac{e^{-\frac{2+\alpha}{2} y}}{\left(1 + e^{-\frac{(2+\alpha)^2}{N-2} y}\right)^{\frac{N-2}{2+\alpha}}}, \quad \gamma_{\alpha,N} = \left(\frac{(2+\alpha)^2(N+\alpha)}{N-2}\right)^{\frac{N-2}{4+2\alpha}}.$$

We denote the function $\mathrm{PU}_\mu := U_\mu + R_\mu$, which is the projection onto $H_0^1(\Omega)$ of the function U_μ, that is,

$$\begin{cases} -\Delta \mathrm{PU}_\mu = |x|^\alpha U_\mu^{p_\alpha} & \text{in } \Omega, \\ \mathrm{PU}_\mu = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, we have

$$R_\mu = -C_{\alpha,N}\mu^{\frac{N-2}{2}} H(x,0) + O\left(\mu^{\frac{N+2+2\alpha}{2}}\right),$$

where $H(x,0)$ is the Robin function.
For given $\Lambda_i > 0,\ i = 1, 2, \ldots, k$, set

$$\begin{aligned} \xi_1 &= -\frac{1}{2+\alpha}\log\varepsilon + \frac{2}{2+\alpha}\log\Lambda_1, \\ \xi_{i+1} - \xi_i &= -\frac{2}{2+\alpha}\log\varepsilon - \frac{2}{2+\alpha}\log\Lambda_{i+1}, \quad i = 1, 2, \ldots, k-1. \end{aligned} \tag{1.9}$$

Let us write

$$W_i(y) = W(y - \xi_i), \quad V_i(y) = W_i(y) + \Pi_i(y), \quad V(y) = \sum_{i=1}^{k} (-1)^i V_i(y), \tag{1.10}$$

where

$$\Pi_i(y) = \mathcal{T}(R_{\mu_i}), \ \mu_i = e^{-\frac{p_\alpha - 1}{2}\xi_i}.$$

We will prove Theorem 1.1 by verifying the following result.

Theorem 1.4 *Suppose that $\alpha > -2$ is not an even integer. Then for any integer $k \geq 1$, there exists $\varepsilon_k > 0$ such that for $\varepsilon \in (0, \varepsilon_k)$, problem (1.6) has a pair of solutions v_ε and $-v_\varepsilon$ of the form*

$$v_\varepsilon = V + \phi_\varepsilon,$$

where $\|\phi_\varepsilon\|_{L^\infty} \to 0$, as $\varepsilon \to 0$.

Remark 1.5 Using the Emden–Fowler-type transformation (1.5), we can give the explicit expression of solution to problem (1.1), that is,

$$u_\varepsilon(x) = C_{\alpha,N} \sum_{i=1}^{k} (-1)^i \left(\frac{M_i^{\frac{2+\alpha}{2}} \varepsilon^{\frac{(2+\alpha)(2i-1)}{2(N-2)}}}{M_i^{2+\alpha} \varepsilon^{\frac{(2+\alpha)(2i-1)}{N-2}} + |x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}} (1 + o(1)),$$

where $M_i, i = 1, 2, \ldots, k$ are some certain positive constants (see (4.3)) and $o(1) \to 0$ uniformly on compact subsets of Ω as $\varepsilon \to 0$.

The proof of Theorem 1.4 is motivated by [7,23]. More precisely, we will use the Lyapunov–Schmidt reduction argument to prove Theorem 1.4, which reduces the construction of the solutions to a finite-dimensional variational problem. As a final remark, we point out that bubble tower concentration phenomena have been observed in [3,7,8,10,18,20,23] near the critical Sobolev exponent, i.e., $\alpha = 0$. However, as far as we know, there are no such results for $\alpha \neq 0$.

This paper is organized as follows. In Sect. 2, we give some basic estimates and asymptotic expansion. In Sect. 3, we will carry out the finite-dimensional reduction argument and the main results will be proved in Sect. 4.

2 Energy expansion

In this section, we give some estimates and asymptotic expansion used in the later sections.

Lemma 2.1 *For fixed $\delta > 0$ and $\delta < \Lambda_i < \delta^{-1}, i = 1, 2, \ldots, k$, we have the following estimates:*

$$\int_D |V|^{p_\alpha+1} \mathrm{d}y\mathrm{d}\Theta = k\omega_{N-1} \int_{\mathbb{R}} W^{p_\alpha+1} + o(1), \tag{2.1}$$

$$\int_D \left(|V|^{p_\alpha+1} - |V|^{p_\alpha+1-\varepsilon}\right) \mathrm{d}y\mathrm{d}\Theta = k\omega_{N-1}\varepsilon \int_{\mathbb{R}} W^{p_\alpha+1} \log W + o(\varepsilon), \tag{2.2}$$

$$\int_D y|V|^{p_\alpha+1} \mathrm{d}y\mathrm{d}\Theta = \left(\sum_{\ell=1}^{k} \xi_\ell\right) \omega_{N-1} \int_{\mathbb{R}} W^{p_\alpha+1} + o(1), \tag{2.3}$$

$$\int_{D_\ell} W_i^{p_\alpha} W_j \mathrm{d}y\mathrm{d}\Theta = o(\varepsilon), \quad i \neq \ell, \tag{2.4}$$

$$\int_{D_\ell} W_\ell^{p_\alpha} W_j \mathrm{d}y\mathrm{d}\Theta = a_3 e^{-\frac{2+\alpha}{2}|\xi_\ell - \xi_j|} + o(\varepsilon), \quad j \neq \ell, \tag{2.5}$$

$$\int_{D_\ell} \left(|V_\ell|^{p_\alpha+1} - |V|^{p_\alpha+1} + (p_\alpha+1) V_\ell^{p_\alpha} \sum_{j\neq\ell} (-1)^{\ell+j} V_j \right) \mathrm{d}y\mathrm{d}\Theta = o(\varepsilon), \tag{2.6}$$

$$\int_{D_\ell} \left(W_\ell^{p_\alpha} - V_\ell^{p_\alpha}\right) V_j \mathrm{d}y\mathrm{d}\Theta = o(\varepsilon), \quad j \neq \ell, \tag{2.7}$$

where $a_3 = \gamma_{\alpha,N}\omega_{N-1} \int_{\mathbb{R}} e^{-\frac{2+\alpha}{2}y} W^{p_\alpha}$, $D_\ell = \{(y, \Theta) \in D : \eta_\ell \le y < \eta_{\ell+1}\}$, $\eta_1 = 0$, $\eta_\ell = \frac{\xi_{\ell-1}+\xi_\ell}{2}$, $\ell = 2, \ldots, k$, $\eta_{k+1} = +\infty$.

Proof The results are similar to Lemma 4.4 in [23], we omit the details. □

Next, we will calculate the asymptotic expansions of the energy functional $I_\varepsilon(V)$.

Proposition 2.2 *For any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, we have the following asymptotic expansion*

$$I_\varepsilon(V) = ka_0 + ka_1\varepsilon - \frac{k^2}{2}a_4\varepsilon \log \varepsilon + \varepsilon\Psi_k(\Lambda) + \varepsilon R_\varepsilon(\Lambda), \tag{2.8}$$

where

$$\Psi_k(\Lambda) = ka_4 \log \Lambda_1 + \frac{a_2 H(0,0)}{\Lambda_1^2} + \sum_{\ell=2}^{k} (a_3\Lambda_\ell - (k-\ell+1)a_4 \log \Lambda_\ell)$$

and $R_\varepsilon \to 0$, as $\varepsilon \to 0$ uniformly in C^1-norm on the set of Λ_i's with $\delta < \Lambda_i < \delta^{-1}$, $i = 1, 2, \ldots, k$. Here a_i, $i = 0, 1, \ldots, 4$, are given by

$$\begin{cases} a_0 = \dfrac{2+\alpha}{2(N+\alpha)}\left(\dfrac{p_\alpha - 1}{2}\right)^{\frac{p_\alpha+3}{p_\alpha-1}} \displaystyle\int_{\mathbb{R}^N} |x|^\alpha U_1^{p_\alpha+1}, \\ a_1 = \dfrac{\omega_{N-1}}{p_\alpha+1}\left(\displaystyle\int_{\mathbb{R}} W^{p_\alpha+1}\log W - \dfrac{1}{p_\alpha+1}\int_{\mathbb{R}} W^{p_\alpha+1} - \dfrac{2}{p_\alpha-1}\log\dfrac{p_\alpha-1}{2}\int_{\mathbb{R}} W^{p_\alpha+1}\right), \\ a_2 = \dfrac{2+\alpha}{2(N+\alpha)}\left(\dfrac{p_\alpha-1}{2}\right)^{\frac{p_\alpha+3}{p_\alpha-1}} C_{\alpha,N}\displaystyle\int_{\mathbb{R}^N} |x|^\alpha U_1^{p_\alpha}, \\ a_3 = \gamma_{\alpha,N}\omega_{N-1}\displaystyle\int_{\mathbb{R}} e^{-\frac{2+\alpha}{2}y} W^{p_\alpha}, \\ a_4 = \dfrac{\omega_{N-1}}{p_\alpha+1}\displaystyle\int_{\mathbb{R}} W^{p_\alpha+1}. \end{cases}$$

Proof The proof is standard, and we only give a sketch here.
Note that

$$\begin{aligned} I_\varepsilon(V) =& I_0(V) + \frac{1}{p_\alpha+1}\int_D |V|^{p_\alpha+1} - \frac{\sigma_\varepsilon}{p_\alpha+1-\varepsilon}\int_D e^{-\frac{2+\alpha}{2}\varepsilon y}|V|^{p_\alpha+1-\varepsilon} \\ =& I_0(V) - \frac{1}{p_\alpha+1}\int_D \left(e^{-\frac{2+\alpha}{2}\varepsilon y} - 1\right)|V|^{p_\alpha+1} + ka_1\varepsilon + o(\varepsilon). \end{aligned}$$

It follows from Lemma 2.1 that

$$\begin{aligned} \frac{1}{p_\alpha+1}\int_D \left(e^{-\frac{2+\alpha}{2}\varepsilon y} - 1\right)|V|^{p_\alpha+1} &= -\frac{(2+\alpha)\varepsilon}{2(p_\alpha+1)}\int_D y|V|^{p_\alpha+1} + o(\varepsilon) \\ &= -\varepsilon\frac{a_4(2+\alpha)}{2}\sum_{j=1}^{k}\xi_j + o(\varepsilon). \end{aligned}$$

It is easy to check that

$$I_0(V) - \sum_{i=1}^{k} I_0(V_i) = \frac{1}{p_\alpha+1}\int_D\left(\sum_{i=1}^{k} V_i^{p_\alpha+1} - |V|^{p_\alpha+1}\right) + \sum_{i,j=1,i>j}^{k}(-1)^{i+j}\int_D W_i^{p_\alpha}V_j.$$

Since $\Pi_j = O(e^{-\frac{2+\alpha}{2}\xi_j}) = O(\varepsilon^{\frac{3}{2}})$, $j \geq 2$, from Lemma 2.1, we have

$$\begin{aligned} I_0(V) &- \sum_{i=1}^{k} I_0(V_i) \\ &= \frac{1}{p_\alpha+1}\sum_{\ell=1}^{k}\int_{D_\ell}\left(V_\ell^{p_\alpha+1} - |V|^{p_\alpha+1} + (p_\alpha+1)\sum_{j<\ell}(-1)^{\ell+j}W_\ell^{p_\alpha}V_j\right) + o(\varepsilon) \\ &= -\sum_{\ell=1}^{k}\sum_{j>\ell}(-1)^{\ell+j}\int_{D_\ell}W_\ell^{p_\alpha}W_j + o(\varepsilon) \\ &= a_3\sum_{\ell=1}^{k-1}e^{-\frac{2+\alpha}{2}|\xi_{\ell+1}-\xi_\ell|} + o(\varepsilon). \end{aligned}$$

Next, we estimate $I_0(V_i)$, $i = 1, 2, \dots, k$.
Recall that

$$V_i = W_i + \Pi_i, \quad \Pi_i(y) = \mathcal{T}(R_{\mu_i}),\ \mu_i = e^{-\frac{p_\alpha-1}{2}\xi_i}$$

and

$$I_0(V_i) = \left(\frac{p_\alpha-1}{2}\right)^{\frac{p_\alpha+3}{p_\alpha-1}} J_0(PU_{\mu_i}).$$

Thus, we find

$$\begin{aligned} J_0(PU_{\mu_i}) = &\frac{2+\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}|x|^\alpha U_1^{p_\alpha+1} \\ &+\frac{2+\alpha}{2(N+\alpha)}C_{\alpha,N}H(0,0)\mu_i^{N-2}\int_{\mathbb{R}^N}|x|^\alpha U_1^{p_\alpha} + O(\mu_i^N). \end{aligned}$$

Since $\mu_i = e^{-\frac{p_\alpha-1}{2}\xi_i}$, we find

$$\sum_{i=1}^{k} I_0(V_i) = ka_0 + a_2H(0,0)e^{-(2+\alpha)\xi_1} + o(\varepsilon).$$

Hence, we can deduce

$$\begin{aligned} I_\varepsilon(V) = ka_0 + ka_1\varepsilon + a_2H(0,0)e^{-(2+\alpha)\xi_1} + a_3\sum_{\ell=1}^{k-1}e^{-\frac{2+\alpha}{2}|\xi_{\ell+1}-\xi_\ell|} \\ + a_4\varepsilon\frac{2+\alpha}{2}\sum_{\ell=1}^{k}\xi_\ell + o(\varepsilon). \end{aligned}$$

By the definition of ξ_i, $i = 1, 2, \dots, k$, we can obtain (2.8) immediately and the proof of Proposition 2.2 is concluded. □

3 The finite-dimensional reduction

In this section, we perform the finite-dimensional procedure, which reduces problem (1.6) to a finite-dimensional problem on $\mathbb{R}_+$.

For given $\xi_i,\ i=1,2,\ldots,k$, let

$$\|\phi\|_* = \sup_{(y,\Theta)\in D}\left(\sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}\right)^{-1}|\phi(y,\Theta)|,$$

where $\sigma>0$ is a small constant. We denote $\mathcal{C}_*$ by the continuous function space defined on D with finite norm defined as above.

Define

$$\tilde{Z}_i(x)=\mu_i\frac{\partial U_{\mu_i}}{\partial\mu_i},\qquad \mu_i=e^{-\frac{p_\alpha-1}{2}\xi_i},\ i=1,2,\ldots,k.$$

Then, $\tilde{Z}_i(x)$ solves

$$-\Delta\tilde{Z}_i(x)=p_\alpha U_{\mu_i}^{p_\alpha-1}\tilde{Z}_i(x)\ \text{ in } \mathbb{R}^N.$$

Let $P\tilde{Z}_i$ be the projection onto $H_0^1(\Omega)$ of the function $\tilde{Z}_i(x)$, that is,

$$\begin{cases}-\Delta P\tilde{Z}_i=p_\alpha U_{\mu_i}^{p_\alpha-1}\tilde{Z}_i(x) & \text{in } \Omega,\\ P\tilde{Z}_i=0 & \text{on } \partial\Omega.\end{cases}$$

Set

$$Z_i(y,\Theta)=\mathcal{T}(P\tilde{Z}_i)(y,\Theta).$$

Then, Z_i satisfies

$$\begin{cases}L(Z_i)=p_\alpha W_i^{p_\alpha-1}W_i' & \text{in } D,\\ Z_i=0 & \text{on } \partial D.\end{cases}$$

First, we consider the following linear problem

$$\begin{cases}\mathbb{L}_\varepsilon(\phi)=h+\sum_{j=1}^{k}c_jZ_j & \text{in } D,\\ \phi=0 & \text{on } \partial D,\\ \int_D Z_i\phi\,\mathrm{d}y\mathrm{d}\Theta=0,\ \ i=1,2,\ldots,k,\end{cases}\tag{3.1}$$

where c_i, $i=1,2,\ldots,k$, are some constants and

$$\mathbb{L}_\varepsilon(\phi)=L(\phi)-(p_\alpha-\varepsilon)\sigma_\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon y}|V|^{p_\alpha-1-\varepsilon}\phi.$$

Lemma 3.1 *Assume that there are sequences $\varepsilon_n\to0$ and points $0<\xi_1^n<\xi_2^n<\cdots<\xi_k^n$ with*

$$\xi_1^n\to\infty,\quad \min_{1\le i\le k-1}(\xi_{i+1}^n-\xi_i^n)\to+\infty,\quad \xi_k^n=o(\varepsilon_n^{-1}),$$

such that ϕ_n solves (3.1) for scalars c_i^n and h_n with $\|h_n\|_*\to0$, then $\lim_{n\to\infty}\|\phi_n\|_*=0$.

Proof We will first show that

$$\lim_{n\to\infty}\|\phi_n\|_{L^\infty}=0.$$

Arguing by contradiction, we may assume that $\|\phi_n\|_{L^\infty}=1$. Multiplying (3.1) by Z_ℓ^n and integrating by parts, we find

$$\sum_{i=1}^{k}c_i^n\int_D Z_i^nZ_\ell^n\mathrm{d}y\mathrm{d}\Theta=\int_D\mathbb{L}_{\varepsilon_n}(Z_\ell^n)\phi_n\mathrm{d}y\mathrm{d}\Theta-\int_D h_nZ_\ell^n\mathrm{d}y\mathrm{d}\Theta.$$

Note that

$$\int_D Z_i^n Z_\ell^n \mathrm{d}y \mathrm{d}\Theta = C\delta_{i\ell} + o(1)$$

where $\delta_{i\ell}$ is the Kronecker's delta function. This defines an almost diagonal system in the c_i^n's as $n \to \infty$.

Thus, we have

$$\sum_{i=1}^{k} c_i^n \int_D Z_i^n Z_\ell^n = \int_D \left[L(Z_\ell^n) - (p_\alpha - \varepsilon_n)\sigma_{\varepsilon_n} e^{-\frac{2+\alpha}{2}\varepsilon_n y} |V|^{p_\alpha - 1 - \varepsilon_n} Z_\ell^n \right] \phi_n - \int_D h_n Z_\ell^n. \tag{3.2}$$

But

$$L(Z_\ell^n) = p_\alpha W^{p_\alpha - 1}(y - \xi_\ell^n) W'(y - \xi_\ell^n),$$

by the dominated convergence theorem, we know that $\lim_{n\to\infty} c_i^n = 0$. Assume that $(y_n, \Theta_n) \in D$ is such that $|\phi_n(y_n, \Theta_n)| = 1$, we claim that there is an $\ell \in \{1, \ldots, k\}$ and a fixed $R > 0$, such that $|\xi_\ell^n - y_n| \le R$ for n large enough. Otherwise, we can suppose that $|\xi_\ell^n - y_n| \to +\infty$ as $n \to +\infty$ for any $\ell = 1, 2, \ldots, k$. Then either $|y_n| \to +\infty$ or $|y_n|$ is bounded. Assume first that $|y_n| \to +\infty$.

Define

$$\tilde{\phi}_n(y, \Theta) = \phi_n(y + y_n, \Theta).$$

By the standard elliptic regularity theory, we may assume that $\tilde{\phi}_n$ converges uniformly over compact sets to a function $\tilde{\phi}$. Set $\tilde{\psi} = \mathcal{T}^{-1}(\tilde{\phi})$, then we have

$$\Delta\tilde{\psi} = 0 \quad \text{in} \quad \mathbb{R}^N \backslash \{0\}.$$

Due to $\|\tilde{\phi}_n\|_{L^\infty} = 1$, we see that $|\tilde{\psi}(x)| \le |x|^{-\frac{N-2}{2}}$. Hence, $\tilde{\psi}$ can extend smoothly to 0 to be a harmonic function in $\mathbb{R}^N$ with this decay condition. So, $\tilde{\phi} = 0$ gives a contradiction. The fact that $|y_n|$ cannot be bounded can be handled in similar way. Thus, there exists an integer $\ell \in \{1, \ldots, k\}$ and a positive number $R > 0$ such that for n large enough, $|y_n - \xi_\ell^n| \le R$.

Define again

$$\tilde{\phi}_n(y, \Theta) = \phi_n(y + \xi_\ell^n, \Theta).$$

Thus, $\tilde{\phi}_n$ converges uniformly over compact sets to a function $\tilde{\phi}$. Set again that $\tilde{\psi} = \mathcal{T}^{-1}(\tilde{\phi})$. Hence, $\tilde{\psi}$ is a nontrivial solution of

$$\Delta\tilde{\psi} + p_\alpha |x|^\alpha U_1^{p_\alpha - 1}\tilde{\psi} = 0 \quad \text{in} \quad \mathbb{R}^N \backslash \{0\}.$$

Moreover, $|\tilde{\psi}(x)| \le C|x|^{-\frac{N-2}{2}}$. Therefore, we obtain a classical solution in $\mathbb{R}^N \backslash \{0\}$ decaying at infinity. It follows from [12] that it equals a linear combination of the $\{\tilde{Z}_i\}$ provided that α is not an even integer. However, the orthogonality conditions imply $\tilde{\phi} = 0$. This is again a contradiction. Thus, we can deduce that $\lim_{n\to\infty} \|\phi_n\|_{L^\infty} = 0$.

Next we shall establish that

$$\lim_{n\to\infty} \|\phi_n\|_* \to 0.$$

Now we see that (3.1) possesses the following form

$$-\phi_n'' + \frac{(2+\alpha)^2}{4}\phi_n - \left(\frac{p_\alpha - 1}{2}\right)^2 \Delta_{\mathbb{S}^{N-1}}\phi_n = g_n, \tag{3.3}$$

where

$$g_n = h_n + (p_\alpha - \varepsilon_n)\sigma_\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon_n y}|V|^{p_\alpha - 1 - \varepsilon_n}\phi_n + \sum_{i=1}^{n} c_i^n Z_i^n.$$

If $0 < \sigma < \min\{p_\alpha - 1, 1\}$, we find

$$|g_n(y)| \le \theta_n \sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i^n|} \text{ with } \theta_n \to 0.$$

Choosing $C > 0$ large enough, we see that

$$\varphi_n(y) = C\theta_n \sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i^n|}$$

is a supersolution of (3.3), and $-\varphi_n(y)$ will be a subsolution of (3.3). Thus,

$$|\phi_n| \le C\theta_n \sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i^n|}.$$

The following proposition is a direct consequence of Proposition 1 in [9] combining with Lemma 3.1.

Proposition 3.2 *There exist positive numbers* $\varepsilon_0, \delta_0, R_0$, *such that if*

$$R_0 < \xi_1, \quad R_0 < \min_{i=1,\dots,k-1}(\xi_{i+1} - \xi_i), \quad \xi_k < \frac{\delta_0}{\varepsilon}, \tag{3.4}$$

then for all $0 < \varepsilon < \varepsilon_0$ *and* $h \in \mathcal{C}_*$, *problem* (3.1) *has a unique solution* $\phi = T_\varepsilon(h)$. *Moreover, there exists* $C > 0$ *such that*

$$\|T_\varepsilon(h)\|_* \le C\|h\|_*, \quad |c_i| \le C\|h\|_*.$$

For later purposes, we need to understand the differentiability of the operator T_ε on the variables ξ_i. We will use the notation $\xi = (\xi_1, \xi_2, \dots, \xi_k)$. We also consider the space $L(\mathcal{C}_*)$ of the linear operator of $\mathcal{C}_*$. We have the following result.

Proposition 3.3 *Under the same assumptions of Proposition 3.2, the map* $\xi \to T_\varepsilon$ *with values in* $L(\mathcal{C}_*)$ *is of class* C^1. *Besides, there is a constant* $C > 0$ *such that*

$$\|D_\xi T_\varepsilon\|_{L(\mathcal{C}_*)} \le C$$

uniformly on the vectors ξ *satisfying* (3.4).

Proof Fix $h \in \mathcal{C}_*$, and let $\phi = T_\varepsilon(h)$. We are interested in studying the differentiability of ϕ with respect to ξ_ℓ for $\ell = 1, 2, \dots, k$. Recall that ϕ satisfies

$$\begin{cases} \mathbb{L}_\varepsilon(\phi) = h + \sum_{j=1}^{k} c_j Z_j & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \\ \int_D Z_i \phi \, dy d\Theta = 0, \quad i = 1, 2, \dots, k, \end{cases}$$

for certain constants c_i. Differentiating the above equation with respect to ξ_ℓ, $\ell = 1, \dots, k$. Define $Y = \partial_{\xi_\ell}\phi$ and $d_i = \partial_{\xi_\ell} c_i$, we find

$$\begin{cases} \mathbb{L}_\varepsilon(Y) = (p_\alpha - \varepsilon)\sigma_\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon y}(\partial_{\xi_\ell}|V|^{p_\alpha - 1 - \varepsilon})\phi + c_\ell \partial_{\xi_\ell} Z_\ell + \sum_{j=1}^{k} d_j Z_j & \text{in } D, \\ Y = 0 & \text{on } \partial D, \\ \int_D (Y Z_i + \phi \partial_{\xi_\ell} Z_i) dy d\Theta = 0, \quad i = 1, 2, \dots, k. \end{cases}$$

Set $\chi = Y - \sum_{i=1}^{k} b_i Z_i$, where the constants b_i satisfy

$$\sum_{i=1}^{k} b_i \int_D Z_i Z_j \mathrm{d}y\mathrm{d}\Theta = 0, \quad j \neq \ell,$$
$$\sum_{i=1}^{k} b_i \int_D Z_i Z_\ell \mathrm{d}y\mathrm{d}\Theta = -\int_D \phi \partial_{\xi_\ell} Z_\ell \mathrm{d}y\mathrm{d}\Theta.$$

This is also an almost diagonal system and $Y = \chi + \sum_{j=1}^{k} b_j Z_j$, where$\int_D \chi Z_j \mathrm{d}y\mathrm{d}\Theta = 0,\ j = 1, 2, \ldots, k$. Moreover, it is easy to see that χ satisfies

$$\begin{cases} \mathbb{L}_\varepsilon(\chi) = g + \sum_{j=1}^{k} d_j Z_j & \text{in } D, \\ \chi = 0 & \text{on } \partial D, \\ \int_D \chi Z_j \mathrm{d}y\mathrm{d}\Theta = 0,\ j = 1, 2, \ldots, k, \end{cases}$$

where

$$g = (p_\alpha - \varepsilon)\sigma_\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon y}(\partial_{\xi_\ell}|V|^{p_\alpha - 1 - \varepsilon})\phi + c_\ell \partial_{\xi_\ell} Z_\ell - \sum_{j=1}^{k} b_j \mathbb{L}_\varepsilon(Z_j).$$

Then, we find

$$\chi = T_\varepsilon(g)$$

and

$$\partial_{\xi_\ell}\phi = T_\varepsilon(g) + \sum_{j=1}^{k} b_j Z_j.$$

By Proposition 3.2, we find

$$\|T_\varepsilon(g)\|_* \le C\|g\|_*.$$

Since

$$\|g\|_* \le C\left(\|\phi\|_* + |c_\ell| + \sum_{j=1}^{k} |b_j|\right)$$

and

$$|b_i| \le C\|\phi\|_*, \quad |c_i| \le C\|h\|_*, \quad \|\phi\|_* \le C\|h\|_*.$$

Thus, we can obtain that $\|\partial_{\xi_\ell}\phi\|_* \le C\|h\|_*$, and $\partial_{\xi_\ell}\phi$ depends continuously on ξ for this norm. □

Now we consider

$$\begin{cases} L(V + \phi) - \sigma_\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon y}|V + \phi|^{p_\alpha - 1 - \varepsilon}(V + \phi) = \sum_{j=1}^{k} c_j Z_j \quad \text{in } D, \\ \phi = 0 \quad \text{on } \partial D, \\ \int_D Z_i \phi \mathrm{d}y\mathrm{d}\Theta = 0, \quad i = 1, 2, \ldots, k. \end{cases} \tag{3.5}$$

In order to solve problem (3.5), we rewrite it as

$$\begin{cases} \mathbb{L}_\varepsilon(\phi) = N_\varepsilon(\phi) + R_\varepsilon + \sum_{j=1}^{k} c_j Z_j \quad \text{in } D, \\ \phi = 0 \quad \text{on } \partial D, \\ \int_D Z_i \phi \mathrm{d}y \mathrm{d}\Theta = 0, \quad i = 1, 2, \ldots, k, \end{cases} \tag{3.6}$$

where

$$N_\varepsilon(\phi) = \sigma_\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon y} \left(|V + \phi|^{p_\alpha - 1 - \varepsilon}(V + \phi) - |V|^{p_\alpha - 1 - \varepsilon} V - (p_\alpha - \varepsilon)|V|^{p_\alpha - 1 - \varepsilon}\phi \right)$$

and

$$R_\varepsilon = \sigma_\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon y} |V|^{p_\alpha - 1 - \varepsilon} V - \sum_{i=1}^{k} (-1)^i W_i^{p_\alpha}.$$

Let us fix a large number $M > 0$, ξ satisfies the following conditions

$$\xi_1 > \frac{1}{2} \log \frac{1}{M\varepsilon}, \quad \min_{1 \le i \le k-1} (\xi_{i+1} - \xi_i) > \log \frac{1}{M\varepsilon}, \quad \xi_k < k \log \frac{1}{M\varepsilon}. \tag{3.7}$$

In order to prove that (3.6) is uniquely solvable in the set that $\|\phi\|_*$ is small, we need to estimate R_ε and $N_\varepsilon(\phi)$.

Lemma 3.4 *If* $N \ge 3$, *then*

$$\begin{aligned} &\|N_\varepsilon(\phi)\|_* \le C\|\phi\|_*^{\min\{p_\alpha - \varepsilon, 2\}}, \\ &\left\| \frac{\partial N_\varepsilon(\phi)}{\partial \phi} \right\|_* \le C\|\phi\|_*^{\min\{p_\alpha - 1 - \varepsilon, 1\}}. \end{aligned} \tag{3.8}$$

Proof Since

$$|N_\varepsilon(\phi)| \le \begin{cases} C|\phi|^{p_\alpha - \varepsilon}, & p_\alpha - 1 \le 1, \\ C|V|^{p_\alpha - 2 - \varepsilon}\phi^2 + C|\phi|^{p_\alpha - \varepsilon}, & p_\alpha - 1 > 1. \end{cases}$$

First, we consider the case $p_\alpha - 1 \le 1$.

$$\begin{aligned} |N_\varepsilon(\phi)| &\le C\|\phi\|_*^{p_\alpha - \varepsilon} \left(\sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y - \xi_i|} \right)^{p_\alpha - \varepsilon} \\ &\le C\|\phi\|_*^{p_\alpha - \varepsilon} \left(\sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y - \xi_i|} \right). \end{aligned}$$

where we have used the fact that

$$\sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y - \xi_i|} \le C.$$

Thus, the result follows.

Now we show the result holds for $p_\alpha - 1 > 1$.

$$|N_\varepsilon(\phi)| \le C\|\phi\|_*^2 |V|^{p_\alpha-2-\varepsilon}\left(\sum_{i=1}^k e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}\right)^2 + C\|\phi\|_*^{p_\alpha-\varepsilon}\left(\sum_{i=1}^k e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}\right)^{p_\alpha-\varepsilon}$$

$$\le C\left(\|\phi\|_*^{p_\alpha-\varepsilon} + \|\phi\|_*^2\right)\left(\sum_{i=1}^k e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}\right).$$

Thus,

$$\|N_\varepsilon(\phi)\|_* \le C\|\phi\|_*^{\min\{p_\alpha-\varepsilon,2\}}.$$

The other terms can be estimated similarly, and the proof of the lemma is completed. □

Lemma 3.5 *If $N \ge 3$, then*

$$\|R_\varepsilon\|_* \le C\varepsilon^{\frac{1+\tau}{2}}, \quad \|\partial_\xi R_\varepsilon\|_* \le C\varepsilon^{\frac{1+\tau}{2}}, \tag{3.9}$$

where $\tau > 0$ is a small constant.

Proof We give here the proof of the first one only. The second one can be obtained similarly. Note that

$$\begin{aligned} R_\varepsilon = &(\sigma_\varepsilon - 1)e^{-\frac{2+\alpha}{2}\varepsilon y}|V|^{p_\alpha-1-\varepsilon}V + e^{-\frac{2+\alpha}{2}\varepsilon y}\left(|V|^{p_\alpha-1-\varepsilon}V - |V|^{p_\alpha-1}V\right) \\ &+ |V|^{p_\alpha-1}V\left(e^{-\frac{2+\alpha}{2}\varepsilon y} - 1\right) + |V|^{p_\alpha-1}V - \sum_{i=1}^k(-1)^i V_i^{p_\alpha} \\ &+ \sum_{i=1}^k(-1)^i V_i^{p_\alpha} - \sum_{i=1}^k(-1)^i W_i^{p_\alpha} \\ =: &\, J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Recalling that

$$V = \sum_{i=1}^k(-1)^i V_i, \quad 0 \le V_i \le W_i.$$

Thus, we find

$$|J_1| \le C\varepsilon e^{-\frac{2+\alpha}{2}\varepsilon y}|V|^{p_\alpha-\varepsilon} \le C\varepsilon\sum_{i=1}^k e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|},$$

$$|J_2| \le C\varepsilon|\log V||V|^{p_\alpha-1} \le C\varepsilon\sum_{i=1}^k e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|},$$

$$|J_3| = \left|\left(e^{-\frac{2+\alpha}{2}\varepsilon y} - 1\right)|V|^{p_\alpha-1}V\right| \le C\varepsilon y|V|^{p_\alpha} \le C\varepsilon\sum_{i=1}^k e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}.$$

Next we estimate J_4 and J_5.
Define

$$\chi_\ell = \frac{\xi_{\ell-1} + \xi_\ell}{2}, \quad \ell = 1, 2, \ldots, k+1, \text{ where } \xi_0 = \inf_{(y,\Theta)\in D}|y|, \ \xi_{k+1} = +\infty.$$

Thus, for $\chi_\ell \leq y < \chi_{\ell+1}$, we have

$$\begin{aligned}
|J_4| &= \left| |V|^{p_\alpha-1}V - \sum_{i=1}^{k}(-1)^i V_i^{p_\alpha} \right| \leq C V_\ell^{p_\alpha-1}\left(\sum_{j\neq\ell} V_j\right) \\
&\leq C \sum_{j\neq\ell} e^{-\frac{2+\alpha}{2}(p_\alpha-1)|y-\xi_\ell|} e^{-\frac{2+\alpha}{2}|y-\xi_j|} \\
&\leq C e^{-\frac{2+\alpha}{2}\sigma|y-\xi_\ell|} \sum_{j\neq\ell} e^{-\frac{2+\alpha}{2}(p_\alpha-\sigma-1)|y-\xi_\ell|} e^{-\frac{2+\alpha}{2}|y-\xi_j|} \\
&\leq C e^{-\frac{2+\alpha}{2}\sigma|y-\xi_\ell|} \sum_{j\neq\ell} e^{-\frac{(2+\alpha)(1+\tau)}{4}|\xi_\ell-\xi_{\ell-1}|} \\
&\leq C\varepsilon^{\frac{1+\tau}{2}} \sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}
\end{aligned}$$

and

$$\begin{aligned}
|J_5| &= \left| \sum_{i=1}^{k}\left(V_i^{p_\alpha} - W_i^{p_\alpha}\right) \right| \leq C \sum_{i=1}^{k} W_i^{p_\alpha-1}|\Pi_i| \\
&\leq C R_{\mu_1}\left(e^{-\frac{p_\alpha-1}{2}y}, \Theta\right) \sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}(p_\alpha-1)|y-\xi_i|} e^{-\frac{2+\alpha}{2}y}, \ \mu_1 = e^{-\frac{p_\alpha-1}{2}\xi_1} \\
&\leq C\varepsilon^{\frac{1+\tau}{2}} \sum_{i=1}^{k} e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}.
\end{aligned}$$

Therefore, $\|R_\varepsilon\|_* \leq C\varepsilon^{\frac{1+\tau}{2}}$ and the results follow. □

The next proposition enables us to reduce the problem of finding a solution for (1.6) to a finite-dimensional problem.

Proposition 3.6 *Suppose that condition* (3.7) *holds. Then there exists a positive constant* C *such that, for* $\varepsilon > 0$ *small enough, problem* (3.6) *admits a unique solution* $\phi = \phi(\xi)$*, which satisfies*

$$\|\phi\|_* \leq C\varepsilon^{\frac{1+\tau}{2}}.$$

Moreover, $\phi(\xi)$ *is of class* C^1 *on* ξ *with the* $\|\cdot\|_*$*-norm, and*

$$\|D_\xi\phi\|_* \leq C\varepsilon^{\frac{1+\tau}{2}},$$

where $\tau > 0$ *is a small constant.*

Proof Define

$$A_\varepsilon(\phi) := T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon),$$

then we know that problem (3.6) is equivalent to the fixed point problem $\phi = A_\varepsilon(\phi)$. We will use the contraction mapping theorem to solve it.

Set

$$E_\rho = \{\phi \in \mathcal{C}_* : \|\phi\|_* \le \rho \varepsilon^{\frac{1+\tau}{2}}\},$$

where $\rho > 0$ will be fixed later.
We will show that A_ε is a contraction map from E_ρ to E_ρ.

In fact, for $\varepsilon > 0$ small enough, we find

$$\|A_\varepsilon(\phi)\|_* \le C\|N_\varepsilon(\phi) + R_\varepsilon\|_* \le C\left((\rho\varepsilon)^{\min\{p_\alpha-\varepsilon,2\}} + \varepsilon^{\frac{1+\tau}{2}}\right) \le \rho\varepsilon^{\frac{1+\tau}{2}},$$

provided ρ is chosen large enough, but independent of ε.
Thus, A_ε maps E_ρ into itself. Moreover,

$$\begin{aligned}|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)| \le& |\partial_\phi N_\varepsilon(t\phi_1 + (1-t)\phi_2)||\phi_1 - \phi_2| \\ \le& C\left(\rho\varepsilon^{\frac{1+\tau}{2}}\right)^{\min\{p_\alpha-1-\varepsilon,1\}} |\phi_1 - \phi_2|.\end{aligned}$$

Hence,

$$\begin{aligned}\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \le& C\left(\rho\varepsilon^{\frac{1+\tau}{2}}\right)^{\min\{p_\alpha-1-\varepsilon,1\}} \|\phi_1 - \phi_2\|_* \\ \le& \frac{1}{2}\|\phi_1 - \phi_2\|_*.\end{aligned}$$

Thus, there is a unique $\phi \in E_\rho$, such that $\phi = A_\varepsilon(\phi)$.

Now we consider the differentiability of $\xi \to \phi(\xi)$.
Let

$$B(\xi, \phi) = \phi - T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon).$$

First, we have $B(\xi, \phi(\xi)) = 0$. Let us write

$$D_\phi B(\xi, \phi)[\psi] = \psi - T_\varepsilon(\psi D_\phi N_\varepsilon(\phi)) = \psi + M(\psi),$$

where

$$M(\psi) = -T_\varepsilon(\psi D_\phi N_\varepsilon(\phi)).$$

From (3.8), we find

$$\|M(\psi)\|_* \le C\varepsilon^{\frac{1+\tau}{2}\min\{p_\alpha-1-\varepsilon,1\}}\|\psi\|_*.$$

Thus, the linear operator $D_\phi B(\varepsilon, \phi)$ is invertible in $\mathcal{C}_*$ with uniformly bounded inverse depending continuously on its parameters. Differentiating with respect to ξ, we deduce

$$D_\xi B(\xi, \phi) = -D_\xi T_\varepsilon[N_\varepsilon(\phi) + R_\varepsilon] - T_\varepsilon[D_\xi N_\varepsilon(\xi, \phi) + D_\xi R_\varepsilon],$$

where all these expressions depend continuously on their parameters.
By the implicit function theorem, we see that $\phi(\xi)$ is of class C^1 and

$$D_\xi\phi = -\left(D_\phi B(\xi, \phi)\right)^{-1}[D_\xi B(\xi, \phi)].$$

Thus,

$$\|D_\xi(\phi)\|_* \le C\left(\|N_\varepsilon(\phi) + R_\varepsilon\|_* + \|D_\xi N_\varepsilon(\xi, \phi)\|_* + \|D_\xi R_\varepsilon\|_*\right) \le C\varepsilon^{\frac{1+\tau}{2}}.$$

The proof of Proposition 3.6 is concluded. □

4 Proof of the main result

In this section, we will prove Theorem 1.1. As deduced in the introduction, we need to verify Theorem 1.4. To do this, we will choose ξ such that $V+\phi$ is a solution of (1.6), where ϕ is the map obtained in Proposition 3.6.
Recall that

$$I_\varepsilon(v)=\frac{1}{2}\int_D\left(|v'|^2+\frac{(2+\alpha)^2}{4}|v|^2\right)\mathrm{d}y\mathrm{d}\Theta+\frac{1}{2}\left(\frac{p_\alpha-1}{2}\right)^2\int_D|\nabla_{\mathbb{S}^{N-1}}v|^2\mathrm{d}y\mathrm{d}\Theta \\ -\frac{\sigma_\varepsilon}{p_\alpha+1-\varepsilon}\int_D e^{-\frac{2+\alpha}{2}\varepsilon y}|v|^{p_\alpha+1-\varepsilon}\mathrm{d}y\mathrm{d}\Theta. \tag{4.1}$$

Define

$$K_\varepsilon(\xi)=I_\varepsilon(V+\phi).$$

It is now well known that if ξ is a critical point of $K_\varepsilon(\xi)$, then $V+\phi$ is a solution of (1.6). Next, we will prove that $K_\varepsilon(\xi)$ has a critical point. To this end, we need the next lemma, which is important in finding the critical point of K_ε.

Lemma 4.1 *The following expansion holds*

$$K_\varepsilon(\xi)=I_\varepsilon(V)+O(\varepsilon^{1+\tau}), \tag{4.2}$$

where $O(\varepsilon^{1+\tau})$ is uniformly in the C^1-sense on the vectors ξ satisfying (3.4).

Proof Using the Taylor expansion

$$F(u+v)=F(u)+dF(u)[v]+\int_0^1(1-t)d^2F(u+tv)[v,v]dt$$

and the fact that $\nabla I_\varepsilon(V+\phi)[\phi]=0$, we have

$$\begin{aligned}I_\varepsilon(V+\phi)-I_\varepsilon(V)&=\int_0^1\nabla^2 I_\varepsilon(V+t\phi)[\phi,\phi]tdt\\&=\int_0^1\left(\int_D(N_\varepsilon(\phi)+R_\varepsilon)\phi+(p_\alpha-\varepsilon)\sigma_\varepsilon\int_D e^{-\frac{2+\alpha}{2}\varepsilon y}\right.\\&\quad\left.\times\left(|V|^{p_\alpha-1-\varepsilon}-|V+t\phi|^{p_\alpha-1-\varepsilon}\right)\phi^2\right)tdt.\end{aligned}$$

Since $\|\phi\|_*\le C\varepsilon^{\frac{1+\tau}{2}}$, we find that

$$\begin{aligned}\int_D|(N_\varepsilon(\phi)+R_\varepsilon)\phi|&\le C(\|N_\varepsilon(\phi)\|_*+\|R_\varepsilon\|_*)\|\phi\|_*\int_D\left(\sum_{i=1}^k e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}\right)^2\\&\le C(\|N_\varepsilon(\phi)\|_*+\|R_\varepsilon\|_*)\|\phi\|_*=O(\varepsilon^{1+\tau}),\end{aligned}$$

and

$$\begin{aligned}&\int_D\left||V|^{p_\alpha-1-\varepsilon}-|V+t\phi|^{p_\alpha-1-\varepsilon}\right|\phi^2\\&\le C\|\phi\|_*^2\int_D\left(\sum_{i=1}^k e^{-\frac{2+\alpha}{2}\sigma|y-\xi_i|}\right)^2\\&\le C\|\phi\|_*^2.\end{aligned}$$

Thus,

$$I_\varepsilon(V+\phi) = I_\varepsilon(V) + O(\varepsilon^{1+\tau}).$$

Differentiating with respect to ξ_ℓ, we see that

$$\begin{aligned}
&\partial_{\xi_\ell}\left(I_\varepsilon(V+\phi) - I_\varepsilon(V)\right)\\
&= \int_0^1 \int_D \partial_{\xi_\ell}\left[(N_\varepsilon(\phi) + R_\varepsilon)\phi\right] t dt\\
&\quad + (p_\alpha - \varepsilon)\sigma_\varepsilon \int_0^1 \int_D e^{-\frac{2+\alpha}{2}\varepsilon y}\partial_{\xi_\ell}\left[\left(|V|^{p_\alpha - 1-\varepsilon} - |V+t\phi|^{p_\alpha-1-\varepsilon}\right)\phi^2\right] t dt.
\end{aligned}$$

In a similar way, we have that

$$\partial_{\xi_\ell} I_\varepsilon(V+\phi) = \partial_{\xi_\ell} I_\varepsilon(V) + O(\varepsilon^{1+\tau}).$$

Thus, the result follows. □

Proof of Theorem 1.4 Recalling that

$$\begin{aligned}
\xi_1 &= -\frac{1}{2+\alpha}\log\varepsilon + \frac{2}{2+\alpha}\log\Lambda_1,\\
\xi_{i+1} - \xi_i &= -\frac{2}{2+\alpha}\log\varepsilon - \frac{2}{2+\alpha}\log\Lambda_{i+1}, \quad i = 1, 2, \dots, k-1,
\end{aligned}$$

where $\delta < \Lambda_i < \frac{1}{\delta}$, $\delta > 0$ is a fixed constant. To simplify the notation, we denote $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_k)$. Thus, it is sufficient to find a critical point of the function

$$\widetilde{K}_\varepsilon(\Lambda) = \varepsilon^{-1}\left(K_\varepsilon(\xi(\Lambda)) - ka_0\right).$$

From Lemma 4.1 and Proposition 2.2, we have

$$\widetilde{K}_\varepsilon(\Lambda) = \Psi_k(\Lambda) + ka_1 - \frac{k^2}{2}a_4\log\varepsilon + o(1),$$

where the term $o(1)$ goes to 0 uniformly as $\varepsilon \to 0$.

It is easy to see that the function

$$\Lambda_1 \to ka_4\log\Lambda_1 + \frac{a_2 H(0,0)}{\Lambda_1^2}$$

has a stable minimum point $\Lambda_1^* = \left(\frac{2a_2H(0,0)}{ka_4}\right)^{\frac{1}{2}}$ on $(0, +\infty)$, and for $i = 2, \dots, k$, the function

$$\Lambda_i \to a_3\Lambda_i - (k-i+1)a_4\log\Lambda_i$$

also has a stable minimum point $\Lambda_i^* = \frac{(k-i+1)a_4}{a_3}$ on $(0, +\infty)$. Thus, the function $\Psi_k(\Lambda)$ has a stable minimum point $\Lambda^* = (\Lambda_1^*, \dots, \Lambda_k^*)$. Therefore, for ε small enough, there exists a critical point $\Lambda^\varepsilon = (\Lambda_1^\varepsilon, \dots, \Lambda_k^\varepsilon)$ of the function $\widetilde{K}_\varepsilon(\Lambda)$, such that $\Lambda_i^\varepsilon \to \Lambda_i^*$ as $\varepsilon \to 0$ for $i = 1, 2, \dots, k$.

For the Λ_i^ε $(i = 1, \dots, k)$ obtained above, let

$$\xi_1^\varepsilon = \frac{2}{2+\alpha}\log\frac{\Lambda_1^\varepsilon}{\varepsilon^{\frac{1}{2}}}, \quad \xi_i^\varepsilon = \frac{2}{2+\alpha}\log\frac{\Lambda_1^\varepsilon}{\Lambda_2^\varepsilon\dots\Lambda_i^\varepsilon\varepsilon^{\frac{2i-1}{2}}}, \quad i = 2, 3, \dots, k.$$

Hence, $\xi^\varepsilon = (\xi_1^\varepsilon, \ldots, \xi_k^\varepsilon)$ is a critical point of $K_\varepsilon(\xi)$ and $V + \phi(\xi^\varepsilon)$ is a solution of (1.6). □

Proof of Theorem 1.1 Note that $\Lambda_i^\varepsilon = \Lambda_i^* + o(1), i = 1, 2, \ldots, k$ as $\varepsilon \to 0$. Then

$$\xi_1^\varepsilon = \frac{2}{2+\alpha} \log \frac{\Lambda_1^*}{\varepsilon^{\frac{1}{2}}} + o(1),$$
$$\xi_i^\varepsilon = \frac{2}{2+\alpha} \log \frac{\Lambda_1^*}{\Lambda_2^* \ldots \Lambda_i^* \varepsilon^{\frac{2i-1}{2}}} + o(1), \quad i = 2, 3, \ldots, k.$$

Using the fact that $e^{-\frac{p_\alpha - 1}{2}\xi_i^\varepsilon} = M_i \varepsilon^{\frac{2i-1}{N-2}}(1 + o(1)), i = 1, \ldots, k$, where

$$M_1 = \left(\frac{1}{\Lambda_1^*}\right)^{\frac{2}{N-2}}, \quad M_i = \left(\frac{\Lambda_2^* \ldots \Lambda_i^*}{\Lambda_1^*}\right)^{\frac{2}{N-2}}, \ i = 2, \ldots, k. \tag{4.3}$$

Thus, by the transformation (1.5), we find

$$u_\varepsilon(x) = C_{\alpha,N} \sum_{i=1}^{k} (-1)^i \left(\frac{M_i^{\frac{2+\alpha}{2}} \varepsilon^{\frac{(2+\alpha)(2i-1)}{2(N-2)}}}{M_i^{2+\alpha} \varepsilon^{\frac{(2+\alpha)(2i-1)}{N-2}} + |x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}} (1 + o(1)),$$

where $o(1) \to 0$ uniformly on compact subsets of Ω as $\varepsilon \to 0$.
Let

$$\hat{u}_\varepsilon(x) = \sum_{i=1}^{k} (-1)^i \left(\frac{M_i^{\frac{2+\alpha}{2}} \varepsilon^{\frac{(2+\alpha)(2i-1)}{2(N-2)}}}{M_i^{2+\alpha} \varepsilon^{\frac{(2+\alpha)(2i-1)}{N-2}} + |x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}}$$
$$= \sum_{i=1}^{k} (-1)^i \left(\frac{1}{M_i^{\frac{2+\alpha}{2}} \varepsilon^{\frac{(2+\alpha)(2i-1)}{2(N-2)}} + M_i^{-\frac{2+\alpha}{2}} \varepsilon^{-\frac{(2+\alpha)(2i-1)}{2(N-2)}} |x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}}.$$

Hence,

$$u_\varepsilon(x) = C_{\alpha,N} \hat{u}_\varepsilon(x)(1 + o(1)). \tag{4.4}$$

Set $S_\varepsilon^j = \{x \in \mathbb{R}^N : |x| = \varepsilon^{\frac{2j-1}{N-2}}\}, j = 1, 2, \ldots, k$, and choose a compact subset $K \subset \Omega$ such that, for ε small enough, $S_\varepsilon^j \subset K$ for $j = 1, 2, \ldots, k$.
Then, for $x \in S_\varepsilon^j$, we have

$$\hat{u}_\varepsilon(x) = \sum_{i=1}^{k} (-1)^i \left(\frac{1}{M_i^{\frac{2+\alpha}{2}} \varepsilon^{\frac{(2+\alpha)(2i-1)}{2(N-2)}} + M_i^{-\frac{2+\alpha}{2}} \varepsilon^{\frac{(2+\alpha)(4j-2i-1)}{2(N-2)}}} \right)^{\frac{N-2}{2+\alpha}}$$
$$= \varepsilon^{-\frac{2j-1}{2}} \sum_{i=1}^{k} (-1)^i \left(\frac{1}{M_i^{\frac{2+\alpha}{2}} \varepsilon^{\frac{(2+\alpha)(i-j)}{(N-2)}} + M_i^{-\frac{2+\alpha}{2}} \varepsilon^{\frac{(2+\alpha)(j-i)}{(N-2)}}} \right)^{\frac{N-2}{2+\alpha}}$$
$$= (-1)^j \varepsilon^{-\frac{2j-1}{2}} \left(\frac{1}{\left(M_j^{\frac{2+\alpha}{2}} + M_j^{-\frac{2+\alpha}{2}}\right)^{\frac{N-2}{2+\alpha}}} + o(1) \right).$$

Thus, for $\varepsilon > 0$ small enough, $(-1)^j \hat{u}_\varepsilon > 0$ on S_ε^j, $j = 1, 2, \ldots, k$, which implies that $(-1)^j u_\varepsilon > 0$ on S_ε^j. Therefore, u_ε has at least k nodal domains $\Omega_1, \ldots, \Omega_k$ such that Ω_i contains the sphere S_ε^i.

Next we show that, for ε small enough, u_ε has at most k nodal sets. Thanks to Proposition 2.2, Lemma 4.1, (1.7) and (1.10), we have

$$J_\varepsilon(PU_{\mu_i}) \to \frac{(2+\alpha)}{2(N+\alpha)} \int_{\mathbb{R}^N} |x|^\alpha U_1^{p_\alpha+1}, \quad i = 1, 2, \ldots, k, \quad \text{as } \varepsilon \to 0 \tag{4.5}$$

and

$$J_\varepsilon(u_\varepsilon) \to \frac{(2+\alpha)k}{2(N+\alpha)} \int_{\mathbb{R}^N} |x|^\alpha U_1^{p_\alpha+1}, \quad \text{as } \varepsilon \to 0. \tag{4.6}$$

Argue by contradiction, we can assume that there exists another nodal domain denoted by Ω_{k+1}. If $\alpha > 0$, we find that

$$\left(\int_{\Omega_{k+1}} |u_\varepsilon|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \leq C \int_{\Omega_{k+1}} |x|^\alpha |u_\varepsilon|^{p_\alpha+1-\varepsilon}. \tag{4.7}$$

Hence,

$$\left(\int_{\Omega_{k+1}} |u_\varepsilon|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \leq C \|u_\varepsilon\|_{L^\infty(\Omega_{k+1})}^{\frac{2\alpha}{N-2}-\varepsilon} \int_{\Omega_{k+1}} |u_\varepsilon|^{\frac{2N}{N-2}}.$$

By (4.4), we see that $\|u_\varepsilon\|_{L^\infty(\Omega_{k+1})} \leq C$. Thus, $\int_{\Omega_{k+1}} |u_\varepsilon|^{\frac{2N}{N-2}} \geq C > 0$, which implies $J_\varepsilon(u_\varepsilon) > \frac{(2+\alpha)k}{2(N+\alpha)} \int_{\mathbb{R}^N} |x|^\alpha U_1^{p_\alpha+1}$. This is a contradiction with (4.6). If $-2 < \alpha < 0$, by Hardy inequality, we obtain that $\int_\Omega |x|^\alpha |u|^{p_\alpha+1} \leq C\left(\int_\Omega |\nabla u|^2\right)^{\frac{p_\alpha+1}{2}}$. Similar to the case $\alpha = 0$ in [23], we still have that $J_\varepsilon(u_\varepsilon) > \frac{(2+\alpha)k}{2(N+\alpha)} \int_{\mathbb{R}^N} |x|^\alpha U_1^{p_\alpha+1}$ and the proof of Theorem 1.1 is finished. □

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