Annali di Matematica (2018) 197:1201–1214
https://doi.org/10.1007/s10231-017-0720-x

Analytic and Gevrey hypoellipticity for perturbed sums of squares operators

Antonio Bove[1] · **Gregorio Chinni**[1]

Received: 21 June 2017 / Accepted: 2 December 2017 / Published online: 11 December 2017

Abstract We prove a couple of results concerning pseudodifferential perturbations of differential operators being sums of squares of vector fields and satisfying Hörmander's condition. The first is on the minimal Gevrey regularity: if a sum of squares with analytic coefficients is perturbed with a pseudodifferential operator of order strictly less than its subelliptic index it still has the Gevrey minimal regularity. We also prove a statement concerning real analytic hypoellipticity for the same type of pseudodifferential perturbations, provided the operator satisfies to some extra conditions (see Theorem 1.2 below) that ensure the analytic hypoellipticity.

Keywords Sums of squares of vector fields · Analytic hypoellipticity · Gevrey hypoellipticity

Mathematics Subject Classification 35H10 · 35H20 (primary) · 35B65 · 35A20 · 35A27 (secondary)

1 Introduction and statement of the result

Let $X_j(x, D)$, $j = 1, \dots, N$, $N \in \mathbb{N}$, be real vector fields defined in an open subset of $U \subset \mathbb{R}^n$. We may suppose that the origin belongs to U and that the vector fields have real analytic coefficients defined in U. Let

$$P(x, D) = \sum_{j=1}^{N} X_j(x, D)^2, \tag{1.1}$$

and assume that the vector fields satisfy the Hörmander's condition:

✉ Antonio Bove
bove@bo.infn.it

Gregorio Chinni
gregorio.chinni@gmail.com

1 Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, Bologna, Italy

(H) The Lie algebra generated by the vector fields and their commutators has dimension n, equal to the dimension of the ambient space.

Hörmander proved in [11] that (H) is sufficient for C^∞ hypoellipticity and M. Derridj proved in [7] that Hörmander's condition is necessary if the coefficients of the vector fields are real analytic.

The operator P satisfies the *a priori* estimate

$$\|u\|_{1/r}^2 + \sum_{j=1}^{N} \|X_j u\|_0^2 \leq C \left(|\langle Pu, u\rangle| + \|u\|_0^2 \right), \tag{1.2}$$

which we call, for the sake of brevity, the "subelliptic estimate." Here $u \in C_0^\infty(U)$, $\|\cdot\|_0$ denotes the norm in $L^2(U)$ and $\|\cdot\|_s$ the Sobolev norm of order s in U. Since the vector fields satisfy condition (H), we denoted by r the length of the iterated commutator such that the vector fields, their commutators, their triple commutators etcetera up to the commutators of length r generate a Lie algebra of dimension equal to that of the ambient space.

The above estimate was proved first by Hörmander [11] for a Sobolev norm of order $r^{-1} + \varepsilon$ and up to order r^{-1} subsequently by Rothschild and Stein ([16]) as well as in a pseudodifferential context by Bolley et al. [4].

Basically using (1.2) Derridj and Zuily proved in [8] that any operator of the form (1.1) is Gevrey hypoelliptic of order r, i.e., that if u is a distribution on an open set U such that $Pu \in G^r(U)$ then $u \in G^r(U)$. In [1], a microlocal version of this has been proved and we refer to Sect. 2.4 for more details.

We recall in passing that a smooth function u defined in an open set $U \subset \mathbb{R}^n$ is of class Gevrey s if for every compact subset $K \Subset U$ there is a positive C_K such that for any multiindex α, $|\partial_x^\alpha u(x)| \leq C_K^{|\alpha|+1} \alpha!^s$ for $x \in K$. If $s = 1$ we obtain the real analytic functions.

The purpose of this note is to study the following problem: when the hypoellipticity properties of the operator P are preserved if we are willing to perturb it with an analytic pseudodifferential operator?

It is known (see [13], Theorems 22.4.14 as well as 22.4.15) that if we perturb a sum of squares with an arbitrary first-order operator, we may obtain a non-hypoelliptic operator. For instance if we consider $P(x, D) = D_1^2 + x_1^2 D_2^2$ in two variables and perturb it with a first-order operator, obtaining $\tilde{P}(x, D) = D_1^2 + x_1^2 D_2^2 + \alpha D_2$, we have a non-hypoelliptic operator if $\alpha = \pm 1$ or if α is a function assuming those values at the point of interest in the characteristic set.

In a sort of converse direction Stein [20], proved that if we consider Kohn's Laplacian, $\Box_b$, which is neither hypoelliptic nor analytic hypoelliptic, and perturb it with a non zero complex number, $\Box_b + \alpha$, $\alpha \in \mathbb{C} \setminus \{0\}$, we obtain an operator being both hypoelliptic and analytic hypoelliptic.

For higher-order operators, G. Métivier gave a result of analytic hypoellipticity provided certain conditions are satisfied on the lower-order terms (Levi conditions) in the paper [14].

For further details on (first order) differential perturbations, we refer to the papers [9] and [15]. For a pseudodifferential perturbation, we give, in "Appendix", a very brief account showing that the order of the perturbation does matter lest we have to impose extra conditions on the perturbing symbol.

These facts suggest that, if no other conditions are to be imposed on the perturbing operator, its order has to be strictly less than the subelliptic index of the sum of squares.

Before stating our result, we need some notation.

Write $\{X_i, X_j\}$ for the Poisson bracket of the symbols of the vector fields X_i, X_j:

$$\{X_i, X_j\}(x,\xi) = \sum_{\ell=1}^{n} \left(\frac{\partial X_i}{\partial \xi_\ell} \frac{\partial X_j}{\partial x_\ell} - \frac{\partial X_j}{\partial \xi_\ell} \frac{\partial X_i}{\partial x_\ell} \right)(x,\xi).$$

Definition 1.1 Fix a point $(x_0, \xi_0) \in \mathrm{Char}(P)$.[1] Consider all the iterated Poisson brackets $\{X_i, X_j\}$, $\{\{X_i, X_j\}, X_k\}$ etcetera.

We define $\nu(x_0, \xi_0)$ as the length of the shortest iterated Poisson bracket of the symbols of the vector fields which is nonzero at (x_0, ξ_0).

Now we have

Theorem 1.1 *Let P be as in (1.1) and denote by $Q(x, D)$ an analytic pseudodifferential operator defined in a conical neighborhood of the point $(x_0, \xi_0) \in \mathrm{Char}(P)$. If*

$$\mathrm{ord}(Q) < 2/\nu(x_0, \xi_0)$$

then $P + Q$ is $G^{\nu(x_0,\xi_0)}$ hypoelliptic at (x_0, ξ_0).

A few remarks are in order.

(a) Definition 1.1 as well as the regularity obtained in Theorem 1.1 microlocal. We say that an operator Q is G^s hypoelliptic at (x_0, ξ_0) if $(x_0, \xi_0) \notin WF_s(u)$ provided $(x_0, \xi_0) \notin WF_s(Qu)$. Here $WF_s(u)$ denotes the Gevrey s wave front set of the distribution u, i.e., the set of points in $T^*\mathbb{R}^n \setminus \{0\}$ where the distribution u is not (microlocally) Gevrey s.

(b) We stated Theorem 1.1 in the case of analytic coefficients, for the sake of simplicity. Actually one might assume some Gevrey regularity like we do in the following corollary.

Corollary 1.1 *Let V denote a neighborhood of the point x_0 and*

$$r = \sup_{x \in V, |\xi|=1} \nu(x, \xi).$$

Let moreover P be as above with G^r coefficients defined in V and $Q \in OPS^m_r(V)$ be a G^r pseudodifferential operator of order $m < 2/r$. Then $P + Q$ is G^r hypoelliptic at x_0.

A perturbation result for the analytic case can also be proved using the same ideas as for Theorem 1.1.

We make the following assumptions on the operator P in (1.1):

(1) Let $U \times \Gamma$ be a conic neighborhood of (x_0, ξ_0). There exists a real analytic function, $h(x, \xi)$, $h \colon U \times \Gamma \to [0, +\infty[$ such that $h(x_0, \xi_0) = 0$ and $h(x, \xi) > 0$ in $U \times \Gamma \setminus \{(x_0, \xi_0)\}$.

(2) There exist real analytic functions $\alpha_{jk}(x, \xi)$ defined in $U \times \Gamma$, such that

$$\{h(x,\xi), X_j(x,\xi)\} = \sum_{\ell=1}^{N} \alpha_{j\ell}(x,\xi) X_\ell(x,\xi), \tag{1.3}$$

for $j = 1, \dots, N$.

In [2] it was proved that if P, defined as in (1.1), satisfies (1), (2) then P is analytic hypoelliptic at (x_0, ξ_0).

[1] $\mathrm{Char}(P)$ denotes the characteristic variety of P, i.e., $\mathrm{Char}(P) = \{(x, \xi) \in T^*\mathbb{R}^n \setminus \{0\} \mid X_j(x, \xi) = 0,\ j = 1, \dots, N\}$. Here $X_j(x, \xi)$ is the symbol of the vector field X_j.

Theorem 1.2 *Let P be as in (1.1) and assume that (1) and (2) above are satisfied. Let Q be a real analytic pseudodifferential operator of order strictly less than $2/\nu(x_0, \xi_0)$, then $P + Q$ is analytic hypoelliptic at (x_0, ξ_0).*

We point out that the *ideal* statement of the above theorem would be one deducing analytic hypoellipticity of the perturbation from the analytic hypoellipticity of the operator, without any assumption but the order of the perturbation. Unfortunately this seems a much more difficult result to prove and it has been proved in the global case, for some classes of operators, by Chinni and Cordaro [6], and by Braun Rodrigues et al. [5].

Finally we say a few words about the method of proof. It consists in using the FBI transform and the subelliptic inequality on the FBI side obtained in [1]. To do that, we use a deformation technique of the Lagrangian manifold associated with the FBI transform, proposed by Grigis and Sjöstrand in [10].

2 Background on FBI and sums of squares

We are going to use a pseudodifferential and FIO (Fourier Integral Operators) calculus introduced by Grigis and Sjöstrand in the paper [10]. We recall below the main definitions and properties to make this paper self-consistent and readable. For further details, we refer to the paper [10], to the lecture notes [19], as well as to [12] and [17].

2.1 The FBI transform

We define the FBI transform of a temperate distribution u as

$$Tu(x, \lambda) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(x,y)} u(y) \mathrm{d}y,$$

where $\lambda \geq 1$ is a large parameter, φ is a holomorphic function such that $\det \partial_x \partial_y \varphi \neq 0$, $\operatorname{Im} \partial_y^2 \varphi > 0$.

Here ∂_x denotes the complex derivative with respect to the complex variable x.

Example 1 A typical phase function may be $\varphi(x, y) = \frac{i}{2}(x - y)^2$.

To the phase φ, there corresponds a weight function $\Phi(x)$, defined as

$$\Phi(x) = \sup_{y\in\mathbb{R}^n} - \operatorname{Im} \varphi(x, y), \qquad x \in \mathbb{C}^n.$$

We may take a slightly different perspective. Let us consider $(x_0, \xi_0) \in \mathbb{C}^{2n}$ and a real-valued real analytic function $\Phi(x)$ defined near x_0, such that Φ is strictly plurisubharmonic and

$$\frac{2}{i} \partial_x \Phi(x_0) = \xi_0.$$

Denote by $\psi(x, y)$ the holomorphic function defined near $(x_0, \bar{x}_0)$ by

$$\psi(x, \bar{x}) = \Phi(x). \tag{2.1}$$

Because of the plurisubharmonicity of Φ, we have

$$\det \partial_x \partial_y \psi \neq 0 \tag{2.2}$$

and

$$\operatorname{Re}\psi(x,\bar{y}) - \frac{1}{2}\left[\Phi(x) + \Phi(y)\right] \sim -|x-y|^2. \tag{2.3}$$

To end this section, we recall the definition of s—Gevrey wave front set of a distribution.

Definition 2.1 Let $(x_0, \xi_0) \in U \subset T^*\mathbb{R}^n \setminus 0$. We say that $(x_0, \xi_0) \notin WF_s(u)$ if there exist a neighborhood Ω of $x_0 - i\xi_0 \in \mathbb{C}^n$ and positive constants C_1, C_2 such that

$$|e^{-\lambda\Phi_0(x)} Tu(x,\lambda)| \leq C_1 e^{-\lambda^{1/s}/C_2},$$

for every $x \in \Omega$. Here T denotes the classical FBI transform, i.e., that using the phase function of Example 1.

2.2 Pseudodifferential operators

Let $\lambda \geq 1$ be a large positive parameter. We write

$$\tilde{D} = \frac{1}{\lambda} D, \qquad D = \frac{1}{i}\partial.$$

Denote by $q(x, \xi, \lambda)$ an analytic classical symbol and by $Q(x, \tilde{D}, \lambda)$ the formal classical pseudodifferential operator associated with q.

Using "Kuranishi's trick" one may represent $Q(x, \tilde{D}, \lambda)$ as

$$Qu(x,\lambda) = \left(\frac{\lambda}{2i\pi}\right)^n \int e^{2\lambda(\psi(x,\theta)-\psi(y,\theta))} \tilde{q}(x,\theta,\lambda) u(y) \mathrm{d}y \mathrm{d}\theta. \tag{2.4}$$

Here $\tilde{q}$ denotes the symbol of Q in the actual representation.

To realize the above operator, we need a prescription for the integration path.

This is accomplished by transforming the classical integration path via the Kuranishi change of variables and eventually applying Stokes theorem:

$$Q^\Omega u(x,\lambda) = \left(\frac{\lambda}{\pi}\right)^n \int_\Omega e^{2\lambda\psi(x,\bar{y})} \tilde{q}(x,\bar{y},\lambda) u(y) e^{-2\lambda\Phi(y)} L(\mathrm{d}y), \tag{2.5}$$

where $L(\mathrm{d}y) = (2i)^{-n} \mathrm{d}y \wedge \mathrm{d}\bar{y}$, the integration path is $\theta = \bar{y}$ and Ω is a small neighborhood of $(x_0, \bar{x}_0)$. We remark that $Q^\Omega u(x)$ is an holomorphic function of x.

Definition 2.2 Let Ω be an open subset of $\mathbb{C}^n$. We denote by $H_\Phi(\Omega)$ the space of all holomorphic functions $u(x, \lambda)$ such that for every $\varepsilon > 0$ and for every compact $K \subset\subset \Omega$, there exists a constant $C > 0$ such that

$$|u(x,\lambda)| \leq C e^{\lambda(\Phi(x)+\varepsilon)},$$

for $x \in K$ and $\lambda \geq 1$.

Remark 2.1 If $\tilde{q}$ is a classical symbol of order zero, Q^Ω is uniformly bounded as $\lambda \to +\infty$, from $H_\Phi(\Omega)$ into itself.

Remark 2.2 If the principal symbol is real, Q^Ω is formally self adjoint in $L^2(\Omega, e^{-2\lambda\Phi})$.

Remark 2.3 Definition (2.4) of (the realization of) a pseudodifferential operator on an open subset Ω of $\mathbb{C}^n$ is not the classical one. Via the Kuranishi trick it can be reduced to the classical definition. On the other hand, using the function ψ allows us to use a weight function not explicitly related to an FBI phase. This is useful since in the proof we deform the Lagrangian Λ_{Φ_0}, corresponding, e.g., to the classical FBI phase, and obtain a *deformed* weight function which is useful in the a priori estimate.

For future reference, we also recall that the identity operator can be realized as

$$I^{\Omega}u(x,\lambda) = \left(\frac{\lambda}{\pi}\right)^n \int_{\Omega} e^{2\lambda\psi(x,\bar{y})} i(x,\bar{y},\lambda) e^{-2\lambda\Phi(y)} u(y,\lambda) L(dy), \tag{2.6}$$

for a suitable analytic classical symbol $i(x,\xi,\lambda)$. Moreover, we have the following estimate (see [10] and [18])

$$\| I^{\Omega}u - u \|_{\Phi - d^2/C} \le C' \| u \|_{\Phi + d^2/C}, \tag{2.7}$$

for suitable positive constants C and C'. Here we denoted by

$$d(x) = \operatorname{dist}(x, \complement\Omega), \tag{2.8}$$

the distance of x to the boundary of Ω, and by

$$\| u \|^2_{\Phi} = \int_{\Omega} e^{-2\lambda\Phi(x)} |u(x)|^2 L(dx). \tag{2.9}$$

2.3 Some pseudodifferential calculus

We start with a proposition on the composition of two pseudodifferential operators.

Proposition 2.1 ([10]) *Let Q_1 and Q_2 be of order zero. Then they can be composed and*

$$Q_1^{\Omega} \circ Q_2^{\Omega} = (Q_1 \circ Q_2)^{\Omega} + R^{\Omega},$$

where R^{Ω} is an error term, i.e., an operator whose norm is $\mathscr{O}(1)$ as an operator from $H_{\Phi+(1/C)d^2}$ to $H_{\Phi-(1/C)d^2}$

We shall need also a lower bound for an elliptic operator of order zero.

Proposition 2.2 ([1]) *Let Q a zero-order pseudodifferential operator defined on Ω as above. Assume further that its principal symbol $q_0(x,\xi,\lambda)$ satisfies*

$$|{q_0}_{|_{\Lambda_\Phi \cap \pi^{-1}(\Omega)}}| \ge c_0 > 0.$$

Here π denotes the projection onto the first factor in $\mathbb{C}^n_x \times \mathbb{C}^n_{\xi}$. Then

$$\| u \|_{\tilde{\Phi}} + \| Q^{\Omega} u \|_{\Phi} \ge C \| u \|_{\Phi}, \tag{2.10}$$

where

$$\tilde{\Phi}(x) = \Phi(x) + \frac{1}{C} d^2(x), \tag{2.11}$$

and d has been defined in (2.8).

Proof We have

$$\begin{aligned} &Q^{\Omega}u(x,\lambda) - {q_0}_{|_{\Lambda_\Phi}}(x,\lambda) I^{\Omega}u(x,\lambda) \\ &= \left(\frac{\lambda}{\pi}\right)^n \int_{\Omega} e^{2\lambda\psi(x,\bar{y})} \left[q(x,\bar{y},\lambda) - {q_0}_{|_{\Lambda_\Phi}}(x,\lambda) i(x,\bar{y},\lambda) \right] \\ &\quad \cdot e^{-2\lambda\Phi(y)} u(y) L(dy). \end{aligned}$$

The absolute value of the term in square brackets may be estimated by $C(|x-y|+\lambda^{-1})$. Then

$$
\begin{aligned}
&\|Q^\Omega u - q_{0|_{\Lambda_\Phi}} I^\Omega u\|_\Phi^2 \le C\lambda^{-2}\|u\|_\Phi^2 \\
&\quad + C\int_\Omega \left| \left(\frac{\lambda}{\pi}\right)^n \int_\Omega e^{-\lambda\Phi(x)+2\lambda\psi(x,\bar{y})-\lambda\Phi(y)} |x-y| e^{-\lambda\Phi(y)} u(y) L(\mathrm{d}y)\right|^2 L(\mathrm{d}x) \\
&\quad \le C\left(\frac{\lambda}{\pi}\right)^{2n} \int_\Omega \left(\int_\Omega e^{-\lambda/C|x-y|^2}|x-y|L(\mathrm{d}y)\right) \\
&\quad\quad \cdot \left(\int_\Omega e^{-\lambda/C|x-y|^2}|x-y|e^{-2\lambda\Phi(y)}|u(y)|^2 L(\mathrm{d}y)\right) L(\mathrm{d}x) + C\lambda^{-2}\|u\|_\Phi^2 \\
&\quad \le C\lambda^{-1}\|u\|_\Phi^2.
\end{aligned}
$$

Using (2.7), we may conclude that

$$
\begin{aligned}
\|Q^\Omega u\|_\Phi &\ge \|q_{0|_{\Lambda_\Phi}} I^\Omega u\|_\Phi - C\lambda^{-1/2}\|u\|_\Phi \\
&\ge \|q_{0|_{\Lambda_\Phi}} u\|_\Phi - \|q_{0|_{\Lambda_\Phi}}(I^\Omega - 1)u\|_\Phi - C\lambda^{-1/2}\|u\|_\Phi \\
&\ge c_0\|u\|_\Phi - C\|u\|_{\tilde{\Phi}} - C\lambda^{-1/2}\|u\|_\Phi.
\end{aligned}
$$

This proves the assertion. □

2.4 An *a priori* estimate for sums of squares

Consider now the vector fields X_j defined in Sect. 1. Following [1], we state the FBI version of the estimate (1.2).

Theorem 2.1 *Let P^Ω be the Ω-realization of P (see Eq. (2.5)). Note that, arguing as in [10] we have that*

$$
P^\Omega = \sum_{j=1}^N (X_j^\Omega)^2 + \mathscr{O}(\lambda^2), \tag{2.12}
$$

where $\mathscr{O}(\lambda^2)$ is continuous from $H_{\tilde{\Phi}}$ to $H_{\Phi-(1/C)d^2}$ with norm bounded by $C'\lambda^2$, $\tilde{\Phi}$ given by (2.11).

Let $\Omega_1 \subset\subset \Omega$. Then

$$
\lambda^{\frac{2}{r}}\|u\|_\Phi^2 + \sum_{j=1}^N \|X_j^\Omega u\|_\Phi^2 \le C\left(\langle P^\Omega u, u\rangle_\Phi + \lambda^\alpha \|u\|^2_{\Phi,\Omega\setminus\Omega_1}\right), \tag{2.13}
$$

where α is a positive integer, $u \in L^2(\Omega, e^{-2\Phi}L(\mathrm{d}x))$ and $r = \nu((x_0,\xi_0))$.

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we construct a deformation of Λ_{Φ_0} following the ideas in [10] (see also [1].)

Let us consider the “sum of squares of vector fields” operator P defined in 1.1. Let (x_0,ξ_0) be a characteristic point of P and let $r = \nu(x_0,\xi_0)$.

We perform an FBI transform of the form

$$Tu(x,\lambda)=\int_{\mathbb{R}^n} e^{i\lambda\varphi(x,y)}u(y)\mathrm{d}y,$$

where u is a compactly supported distribution and $\varphi(x,y)$ is a phase function. Even though it does not really matter which phase function we use, the classical phase function will be employed:

$$\varphi_0(x,y)=\frac{i}{2}(x-y)^2,\qquad x\in\mathbb{C}^n,\ y\in\mathbb{R}^n. \tag{3.1}$$

Let us denote by Ω an open neighborhood of the point $\pi_x\mathscr{H}_T(x_0,\xi_0)$ in $\mathbb{C}^n$. Here π_x denotes the space projection $\pi_x\colon\mathbb{C}^n_x\times\mathbb{C}^n_\xi\to\mathbb{C}^n_x$ and $\mathscr{H}_T$ is the complex canonical transformation associated with T:

$$\mathscr{H}_T\colon\ \left\{\left(y,-\frac{2}{i}\frac{\partial\Phi}{\partial y}\right)\right\}\to\left\{\left(x,\frac{2}{i}\frac{\partial\Phi}{\partial x}\right)\right\},$$

($\Phi(x,y)=-\operatorname{Im}\varphi(x,y)$), i.e., in the classical case, once we restrict to $\mathbb{R}^{2n}$,

$$\mathscr{H}_0(y,\eta)=(y-i\eta,\eta),\quad (y,\eta)\in\mathbb{R}^{2n}.$$

For the sake of simplicity, we denote by $x_0\in\mathbb{C}^n$ the point $\pi_x\mathscr{H}_0(x_0,\xi_0)$.

Let $\Phi_0(x,y)=-\operatorname{Im}\varphi_0(x,y)=-\frac{1}{2}(x'-y)^2+\frac{1}{2}x''^2$, where $y\in\mathbb{R}^n$, $x=x'+ix''\in\mathbb{C}^n$. We write also

$$\Phi_0(x)=\text{c.v.}_{y\in\mathbb{R}^n}\Phi_0(x,y)$$

(the critical value of Φ_0 w.r.t. y.).

For $\lambda\geq 1$, let us consider a real analytic function defined near the point $\mathscr{H}_0(x_0,\xi_0)=(x_0-i\xi_0,\xi_0)\in\Lambda_{\Phi_0}$, say $h(x,\xi,\lambda)$. Solve, for small positive t, the Hamilton-Jacobi problem

$$\begin{cases} 2\dfrac{\partial\Phi}{\partial t}(t,x,\lambda)=h\left(x,\dfrac{2}{i}\dfrac{\partial\Phi}{\partial x}(t,x,\lambda),\lambda\right)\\ \Phi(0,x,\lambda)=\Phi_0(x)\end{cases}. \tag{3.2}$$

This is easy to solve since h is real analytic. Set

$$\Phi_t(x,\lambda)=\Phi(t,x,\lambda).$$

We have

$$\Lambda_{\Phi_t}=\exp\left(itH_h\right)\Lambda_{\Phi_0}$$

We choose the function h as

$$h(x,\xi,\lambda)=\lambda^{-\frac{r-1}{r}}|x-x_0|^2\quad\text{on}\ \ \Lambda_{\Phi_0}. \tag{3.3}$$

Keeping in mind the definition of Λ_{Φ_0}, we have that, as a function in $\mathbb{R}^{2n}$

$$h(x,\xi,\lambda)=\lambda^{-\frac{r-1}{r}}\left[|x-x_0|^2+|\xi-\xi_0|^2\right]. \tag{3.4}$$

The function Φ_t can be expanded as a power series in the variable t using both Eq. (3.2) and the Faà di Bruno formula to obtain

$$\Phi_t(x,\lambda)=\Phi_0(x)+\frac{t}{2}\,h(\cdot,\cdot,\lambda)\Big|_{\Lambda_{\Phi_0}}+\mathscr{O}(\lambda^{-1}), \tag{3.5}$$

where h on Λ_{Φ_0} is given by (3.4).

Our purpose is to use the estimate (2.13) where the weight function Φ has been replaced by the weight Φ_t. This is possible using the phase ψ_t in (2.4) and realizing the operator as in (2.5). Here ψ_t is defined as the holomorphic extension of $\psi_t(x, \bar{x}) = \Phi_t(x)$.

We need to restrict the symbol of both P and $P+Q$ to Λ_{Φ_t}; denote by P^t, Q^t the symbols of P, Q restricted to Λ_{Φ_t}.

Noting that

$$
\begin{aligned}
X_j^2\left(x, \frac{2}{i}\partial_x \Phi_t(x,\lambda), \lambda\right) &= X_j^2\left(x, \frac{2}{i}\partial_x \Phi_0(x), \lambda\right) \\
&+ 2t X_j\left(x, \frac{2}{i}\partial_x \Phi_0(x), \lambda\right)\left\langle \partial_\xi X_j\left(x, \frac{2}{i}\partial_x \Phi_0(x), \lambda\right), \frac{2}{i}\partial_x\partial_t \Phi_t(x,\lambda)\Big|_{t=0}\right\rangle \\
&+ \mathscr{O}(t^2\lambda^{2/r}),
\end{aligned}
$$

We then deduce that

$$
P^t(x,\xi,\lambda) = \lambda^2 P(x,\xi) + tR(x,\xi,\lambda) + \mathscr{O}(t^2\lambda^{\frac{2}{r}}), \tag{3.6}
$$

where

$$
R(x,\xi,\lambda) = \lambda^{\frac{1}{r}} \sum_{j=1}^{N} a_j(x,\xi,\lambda) X_j(x,\xi,\lambda).
$$

The analytic extension of P^t is the symbol appearing in the Ω-realization of P^t, $P^{t\,\Omega}$. We point out that the principal symbol of P^t satisfies the assumptions of Theorem 2.1 and, using the a priori inequality (2.13), we can deduce an estimate of the form (2.13) for P^t in the H_{Φ_t} spaces.

Denote by θ the order of the pseudodifferential operator Q. We have

$$
\begin{aligned}
&\lambda^{\frac{2}{r}}\|u\|^2_{\Phi_t} + \sum_{j=1}^{N}\|X_j^\Omega u\|^2_{\Phi_t} \\
&\quad \le C\left(|\langle (P^{t\,\Omega} - tR^\Omega - \mathscr{O}(t^2\lambda^{\frac{2}{r}}))u, u\rangle_{\Phi_t}| + \lambda^\alpha \|u\|^2_{\Phi_t,\Omega\setminus\Omega_1}\right) \\
&\quad = C\left(|\langle (P^{t\,\Omega} + Q^{t\,\Omega} - tR^\Omega - \mathscr{O}(t^2\lambda^{\frac{2}{r}}) - Q^{t\,\Omega})u, u\rangle_{\Phi_t}| \right.\\
&\qquad \left. + \lambda^\alpha \|u\|^2_{\Phi_t,\Omega\setminus\Omega_1}\right)
\end{aligned}
$$

The fourth term in the left-hand side of the scalar product above is easily absorbed on the left provided t is small enough. The fifth term is also absorbed since, being Q of order θ, $\|Q^{t\,\Omega}u\|_{\Phi_t,\Omega} \le \lambda^\theta C\|u\|_{\Phi_t,\Omega}$.

Let us consider the third term in the scalar product above. By Proposition 2.1, we have

$$
R^\Omega = \sum_{j=1}^{N} a_j^\Omega(x,\tilde{D},\lambda) X_j^\Omega(x,\tilde{D},\lambda) + \mathscr{O}(\lambda),
$$

where $\mathscr{O}(\lambda)$ denotes an operator from $H_{\Phi_t+\frac{1}{C}d^2}$ to $H_{\Phi_t-\frac{1}{C}d^2}$ whose norm is bounded by $C\lambda$. Hence

$$
t|\langle R^\Omega u, u\rangle_{\Phi_t}| \le Ct\Big(\lambda^{\frac{2}{r}}\|u\|^2_{\Phi_t} + \sum_{j=1}^{N}\|X_j^\Omega u\|^2_{\Phi_t} + \lambda^2\|u\|^2_{\tilde{\Phi}_t}\Big).
$$

Hence we deduce that there exist a neighborhood Ω_0 of x_0, a positive number δ and a positive integer α such that, for every $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega \subset \Omega_0$, there exists a constant $C > 0$ such that, for $0 < t < \delta$, we have

$$\lambda^{\frac{2}{r}} \|u\|_{\Phi_t,\Omega_1} \leq C \left(\|(P+Q)^{t\,\Omega} u\|_{\Phi_t,\Omega_2} + \lambda^{\alpha} \|u\|_{\Phi_t,\Omega\setminus\Omega_1} \right). \tag{3.7}$$

In other words, Theorem 2.1 holds for the perturbed operator.

Using (3.7), we may finish the proof of Theorem 1.1.

By assumption $\|(P+Q)^{t\,\Omega} u\|_{\Phi_t,\Omega_2} \leq Ce^{-\lambda/C}$, since Λ_{Φ_t} is a small perturbation of Λ_{Φ_0} when t is small.

By our choice of h [see (3.3)], it is also straightforward that $\|u\|_{\Phi_t,\Omega\setminus\Omega_1} \leq Ce^{-\lambda^{1/r}/C}$. Thus, we obtain that

$$\|u\|_{\Phi_t,\Omega_1} \leq C_1 e^{-\lambda^{1/r}/C_1}.$$

On the other hand, $\Phi_t(x,\lambda) = \Phi_0(x) + \frac{t}{2}\, h(\cdot,\cdot,\lambda)\big|_{\Lambda_{\Phi_0}} + \mathscr{O}(\lambda^{-1})$, so that, if we are close enough to the base point on Λ_{Φ_0}, i.e., for $x \in \Omega_3$, for a fixed small positive value of t, we have

$$\Phi_t(x) - \Phi_0(x) \leq \frac{\lambda^{-1+1/r}}{C_2(t)}.$$

Therefore $\|u\|_{\Phi_0,\Omega_3} \leq ce^{-\lambda^{1/r}/c}$, which proves Theorem 1.1.

4 Proof of Theorem 1.2

We are going to proceed in the same way as in the previous section, but using the (order zero) function h of the assumption. First of all, we deform Λ_{Φ_0} according to (3.2). Next we want to deduce a priori estimates for $P+Q$ where the weight function Φ_0 is replaced by Φ_t. For the sake of simplicity, let us write (1.3) as

$$\{h(x,\xi), X_j(x,\xi)\} = \alpha(x,\xi)X(x,\xi), \tag{4.1}$$

where X denotes a vector whose components are the symbols of the vector fields and α is a $N \times N$ matrix with entries being real analytic symbols. As before we have $\Lambda_{\Phi_t} = \exp(itH_h)\Lambda_{\Phi_0}$.

Denote by Y_j^t, $j = 1, \ldots, N$, the restriction to Λ_{Φ_t} of X_j. We have $Y_j^t = X_j \circ \exp(itH_h)$, so that, by our assumptions,

$$\partial_t Y^t = i\{h, X\} \circ \exp(itH_h).$$

We deduce that

$$\begin{cases} 2\dfrac{\partial Y^t}{\partial t}(x,\xi) = i(\alpha \circ \exp(itH_h))(x,\xi)Y^t(x,\xi) \\ Y^t(x,\xi)\big|_{t=0} = X(x,\xi) \end{cases}.$$

From this relation, we deduce that there is a $N \times N$ matrix, whose entries are real analytic symbols depending real analytically on the real parameter t, $b_t(x,\xi)$, such that

$$Y^t(x,\xi) = b_t(x,\xi)X(x,\xi), \tag{4.2}$$

and that $b_0 = \mathrm{Id}_N$. Hence b_t is non-singular if t is small enough.

Denote by X^t the holomorphic extension of $\mathrm{Re}\, Y^t$; since X is real on Λ_{Φ_0}, using (4.2), we have that

$$X^t(x,\xi) = \beta_t(x,\xi) X(x,\xi), \tag{4.3}$$

where $\beta_{t=0}(x,\xi) = \mathrm{Id}_N$. In particular β_t is non-singular, provided t is small.

Then we have

$$\begin{aligned} P(x,\tilde{D}) = &\sum_{i,j=1}^{N} X_i^t(x,\tilde{D}) a_{ij}^t(x,\tilde{D};\lambda) X_j^t(x,\tilde{D}) \\ &+ \lambda^{-1} \sum_{j=1}^{N} b_j^t(x,\tilde{D};\lambda) X_j^t(x,\tilde{D}) + \lambda^{-2} c^t(x,\tilde{D};\lambda), \end{aligned} \tag{4.4}$$

for suitable analytic pseudodifferential operators a_{ij}^t, b_j^t, c^t of order zero.

We can apply Theorem 2.1 and deduce that

$$\lambda^{\frac{2}{r}} \|u\|_{\Phi_t,\Omega_1} \le C\left(\|Pu\|_{\Phi_t,\Omega} + \lambda^{\alpha} \|u\|_{\Phi_t,\Omega\setminus\Omega_1}\right),$$

where $\Omega_1 \subset\subset \Omega$, α is a fixed positive integer and P denotes the realization on Ω of the given operator P. Let Q the realization on Ω of the real analytic pseudodifferential operator of order $\theta < 2/r$ in the statement of Theorem 1.2. We have

$$\lambda^{\frac{2}{r}} \|u\|_{\Phi_t,\Omega_1} \le C\left(\|(P+Q)u\|_{\Phi_t,\Omega} + \|Qu\|_{\Phi_t,\Omega} + \lambda^{\alpha} \|u\|_{\Phi_t,\Omega\setminus\Omega_1}\right). \tag{4.5}$$

Let us consider the second term in the right-hand side of the above inequality. We have

$$\|Qu\|_{\Phi_t,\Omega} \le C_1 \lambda^{\theta} \|u\|_{\Phi_t,\Omega} \le C_1 \lambda^{\theta} \left(\|u\|_{\Phi_t,\Omega_1} + \|u\|_{\Phi_t,\Omega\setminus\Omega_1}\right)$$

Since $\theta < 2/r$ the first term of above inequality is absorbed on the left-hand side of (4.5) provided λ is large enough. Hence we have

$$\lambda^{\frac{2}{r}} \|u\|_{\Phi_t,\Omega_1} \le C\left(\|(P+Q)u\|_{\Phi_t,\Omega} + \lambda^{\alpha} \|u\|_{\Phi_t,\Omega\setminus\Omega_1}\right), \tag{4.6}$$

for a suitable new positive constant C.

Assume now that $(x_0, \xi_0) \notin WF_a((P+Q)u)$. We may choose Ω in such a way that

$$\|(P+Q)u\|_{\Phi_0,\Omega} \le C e^{-\lambda/C}, \tag{4.7}$$

for a suitable positive constant C. From

$$\Phi_t(x) = \Phi_0(x) + \frac{1}{2}\int_0^t h\left(x, \frac{2}{i}\partial_x \Phi_s(x)\right) \mathrm{d}s, \tag{4.8}$$

using the fact that $h_{|_{\Lambda_{\Phi_0}}} \ge 0$, and recalling that $\Lambda_{\Phi_t} = \exp(itH_h)\Lambda_{\Phi_0}$, we deduce that $h_{|_{\Lambda_{\Phi_t}}} \ge 0$ so that

$$\Phi_t(x) \ge \Phi_0(x), \qquad x \in \Omega. \tag{4.9}$$

Hence, by (4.9) and (4.7),

$$\|(P+Q)u\|_{\Phi_t,\Omega} \le C e^{-\lambda/C}, \tag{4.10}$$

for a suitable positive constant C.

Let us now estimate the second term in the right-hand side of (4.6). We point out that

$$h\big|_{\Lambda_{\Phi_0}\cap\Omega\setminus\Omega_1} \geq a > 0.$$

It follows, because of (4.8), that

$$\Phi_t(x) \geq \Phi_0(x) + c't, \qquad x \in \Omega \setminus \Omega_1. \tag{4.11}$$

Then

$$\begin{aligned}\|u\|^2_{\Phi_t,\Omega\setminus\Omega_1} &= \int_{\Omega\setminus\Omega_1} e^{-2\lambda\Phi_t(x)}|u(x)|^2 L(\mathrm{d}x)\\ &\leq \int_{\Omega\setminus\Omega_1} e^{-2\lambda\Phi_0(x)-2\lambda c't}|u(x)|^2 L(\mathrm{d}x)\\ &\leq Ce^{-2\lambda c't}\lambda^N\\ &\leq Ce^{-\lambda c''t}, \qquad t > 0.\end{aligned}$$

By (4.6), we deduce that $\|u\|_{\Phi_t,\Omega_1} \leq C\exp(-\lambda t/C)$, for a suitable positive constant C. Let now $\Omega_2 \subset\subset \Omega_1$ be a neighborhood of x_0 such that $\Phi_t \leq \Phi_0 + t/(2C)$ in Ω_2. We conclude that

$$\|u\|^2_{\Phi_0,\Omega_2} \leq Ce^{-\lambda t/C}, \qquad t > 0.$$

This proves the theorem.

A Appendix

We collect here a few facts concerning the hypoellipticity of pseudodifferential perturbations of sums of squares.

Let k be an integer, $k \geq 2$, and consider

$$P(x, D) = D_1^2 + x_1^{2(k-1)}D_2^2, \quad x \in \mathbb{R}^2.$$

Let

$$Q(x, D) = \lambda|D_2|^{2/k}.$$

Q is microlocally elliptic near points in $\mathrm{Char}(P) = \{(x,\xi) \in \mathbb{R}^4 \mid x_1 = \xi_1 = 0, \xi_2 \neq 0\}$. Here λ is a constant that we shall choose later.

Performing a Fourier transform w.r.t. x_2, and the dilation (we recall that $\xi_2 \neq 0$)

$$x_1 \to |\xi_2|^{-1/k}x_1,$$

$P + Q$ becomes, modulo a microlocally elliptic factor which we can disregard,

$$D_1^2 + x_1^{2(k-1)} + \lambda.$$

Let $\varphi_\lambda(x_1)$ be such that

$$-\varphi_\lambda'' + x_1^{2(k-1)}\varphi_\lambda + \lambda\varphi_\lambda = 0.$$

This is possible since the above operator, by [3], has a discrete, positive, simple spectrum, so that, if λ is the opposite of an eigenvalue, φ_λ, the associated eigenfunction, satisfies the above equation. It is well known that $\varphi_\lambda \in \mathscr{S}(\mathbb{R})$, i.e., is rapidly decreasing at infinity.

Consider

$$u(x)=\int_0^{+\infty}e^{ix_2\rho}\varphi_\lambda(x_1\rho^{1/k})(1+\rho^4)^{-1}\mathrm{d}\rho. \tag{A.1}$$

We see immediately that $(P+Q)u=0$. Let us show that $u\notin C^\infty$.

Let us assume first that $\varphi_\lambda(0)\neq 0$. Then

$$u(0,x_2)=\varphi_\lambda(0)\int_0^{+\infty}e^{ix_2\rho}(1+\rho^4)^{-1}\mathrm{d}\rho,$$

and it is obvious that it cannot be smooth since we cannot take an arbitrary derivative w.r.t. x_2.

If $\varphi_\lambda(0)=0$, then necessarily $\varphi_\lambda'(0)\neq 0$. It suffices then to consider

$$(\partial_{x_1}u)(0,x_2)=\varphi_\lambda'(0)\int_0^{+\infty}e^{ix_2\rho}(1+\rho^4)^{-1}\rho^{1/k}\mathrm{d}\rho,$$

and argue exactly as in the preceding case.

This shows that a pseudodifferential perturbation of the same order as the subellipticity index does not preserve the C^∞ hypoellipticity. Analogous argument for the analytic hypoellipticity.

We also point out that allowing a general pseudodifferential perturbation of order equal to the subellipticity index may lead to both a hypoelliptic and a non-hypoelliptic operator.

Consider for instance, microlocally near the point $(0,e_2)$, P as above and $Q=\lambda|D_2|^{2/k}+\mu(x_2)|D_2|^\varepsilon$, with $\varepsilon<2/k$. Then $P+Q$ can be analytic hypoelliptic, G^s hypoelliptic for some s, or not even C^∞ hypoelliptic, depending on the analytic function μ. We do not wish to give any detail about this since it goes far beyond the scope of the present note.

References

1. Albano, P., Bove, A., Chinni, G.: Minimal microlocal Gevrey regularity for "sums of squares". Int. Math. Res. Notices **2009**(12), 2275–2302 (2009)
2. Albano, P., Bove, A.: Wave Front Set of Solutions to Sums of Squares of Vector Fields, vol. 221, p. 1039. American Mathematical Society, New York (2013)
3. Berezin, F.A., Shubin, M.A.: The Schrödinger Equation, Mathematics and Its Applications (Soviet Series), vol. 66. Kluwer Academic Publishers Group, Dordrecht (1991)
4. Bolley, P., Camus, J., Nourrigat, J.: La condition de Hörmander-Kohn pour les opérateurs pseudo-différentiels. Commun. Partial Differ. Eq. **7**, 197–221 (1982)
5. Braun Rodrigues, N., Chinni, G., Cordaro, P.D., Jahnke, M.R.: Lower order perturbation and global analytic vectors for a class of globally analytic hypoelliptic operators. Proc. Am. Math. Soc. **144**(12), 5159–5170 (2016)
6. Chinni, G., Cordaro, P.D.: On global analytic and Gevrey hypoellipticity on the torus and the Métivier inequality. Commun. Partial Differ. Eq. **42**(1), 121–141 (2017)
7. Derridj, M.: Une problème aux limitespour une classe d'opérateurs du second ordre hypoelliptiques. Ann. Inst. Fourier Univ. Grenoble **21**(4), 99–148 (1971)
8. Derridj, M., Zuily, C.: Régularité analytique et Gevrey d'opérateurs elliptiques dégénérés. J. Math. Pures Appl. **52**, 309–336 (1973)
9. Gilioli, A., Trèves, F.: An example in the solvability theory of linear PDE's. Am. J. Math. **96**, 367–385 (1974)
10. Grigis, A., Sjöstrand, J.: Front d'onde analytique et somme de carrés de champs de vecteurs. Duke Math. J. **52**, 35–51 (1985)
11. Hörmander, L.: Hypoelliptic second order differential equations. Acta Math. **119**, 147–171 (1967)
12. Hörmander, L.: The Analysis of Partial Differential Operators, I. Springer, Berlin (1985)
13. Hörmander, L.: The Analysis of Partial Differential Operators, III. Springer, Berlin (1985)

14. Métivier, G.: Analytic hypoellipticity for operators with multiple characteristics. Commun. Partial Differ. Eq. **6**(1), 1–90 (1981)
15. Mughetti, M.: Hypoellipticity and higher order Levi conditions. J. Differ. Eq. **257**(4), 1246–1287 (2014)
16. Preiss Rothschild, L., Stein, E.M.: Hypoelliptic differential operators and nilpotent groups. Acta Math. **137**, 247–320 (1976)
17. Sjöstrand, J.: Analytic wavefront set and operators with multiple characteristics. Hokkaido Math. J. **12**, 392–433 (1983)
18. Sjöstrand, J.: Singularités analytiques microlocales. Astérisque **95**, (1982)
19. Sjöstrand, J.: Lectures on resonances. unpublished lecture notes, (2002). http://www.math.polytechnique.fr/~sjoestrand/CoursgbgWeb.pdf
20. Stein, E.M.: An example on the Heisenberg group related to the Lewy operator. Invent. Math. **69**, 209–216 (1982)