Annali di Matematica (2018) 197:843–868
https://doi.org/10.1007/s10231-017-0706-8

Unique strong and $\mathbb{V}$-attractor of a three-dimensional globally modified two-phase flow model

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Received: 7 June 2017 / Accepted: 9 October 2017 / Published online: 23 October 2017

Abstract In this article, we study a globally modified Allen–Cahn–Navier–Stokes system in a three-dimensional domain. The model consists of the globally modified Navier–Stokes equations proposed in Caraballo et al. (Adv Nonlinear Stud 6(3):411–436, 2006) for the velocity, coupled with an Allen–Cahn model for the order (phase) parameter. We prove the existence and uniqueness of strong solutions. Using the flattening property, we also prove the existence of global $\mathbb{V}$-attractors for the model. Using a limiting argument, we derive the existence of bounded entire weak solutions for the three-dimensional coupled Allen–Cahn–Navier–Stokes system with time-independent forcing.

Keywords Allen–Cahn–Navier–Stokes · Globally modified · Strong solutions · Global attractor

Mathematics Subject Classification 35Q30 · 35Q35 · 35Q72

1 Introduction

It is well accepted that the incompressible Navier–Stokes (NS) equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids [17]. For instance, this approach is used in [2] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system [3,16–18]. In the isothermal compressible case, the existence of a global weak solution is proved in [15]. In the incompressible isothermal case, neglecting chemical reactions and

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other forces, the model reduces to an evolution system which governs the fluid velocity v and the order parameter ϕ. This system can be written as a NS equation coupled with a convective Allen–Cahn equation [17]. The associated initial and boundary value problem was studied in [17] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor.

The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved [34]. In the case of binary fluids, the analysis is even more complicate and the mathematical studied is still at it infancy as noted in [17]. As noted in [16], the mathematical analysis of binary fluid flows is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic patterns appear and then a shear stage in which these patters organize themselves into parallel layers (see, e.g., [32] for experimental snapshots). This model has to take into account the chemical interactions between the two phases at the interface, achieved using a Cahn–Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which a Navier–Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but nonzero thickness, a well-known model is the so-called Model H (cf. [19]). This is a system of equations where an incompressible Navier–Stokes equation for the (mean) velocity v is coupled with a convective Cahn–Hilliard equation for the order parameter ϕ, which represents the relative concentration of one of the fluids.

Many challenges in the mathematical and numerical analysis of the AC–NS equations are related to the fact that the full mathematical theory for the 3D Navier–Stokes equation (NSE) in three dimensions is still lacking at present. Since the uniqueness theorem for the global weak solutions (or the global existence of strong solutions) of the initial-value problem of the 3D Navier–Stokes system is not proved yet, the known theory of global attractors of infinite-dimensional dynamical systems is not applicable to the 3D Navier–Stokes system. This situation is the same for the 3D coupled Allen–Cahn–Navier–Stokes systems. Using regular approximation equations to study the classical 3D Navier–Stokes systems has become an effective tool both from the numerical and the theoretical point of views. As noted in [36], it was demonstrated analytically and numerically in many works that the LANS-α model gives a good approximation in the study of many problems related to turbulence flows. In particular, it was found that the explicit steady analytical solution of the LANS-α model compares successfully with empirical and numerical experiment data for a wide range of Reynolds numbers in turbulent channel and pipe flows [36]. Let us recall that the inviscid 3D LANS-α equations was first proposed in [20,21]. As described in [31], the 3D LANS-α equations are a systems of partial differential equations for the mean velocity in which a nonlinear dispersive mechanism filters the small scales. As such, the 3D LANS-α equations serve as an appropriate model for turbulent flows and a suitable approximation of the 3D NS as documented in [9–12].

In [7], the authors proposed a three-dimensional system of a globally modified Navier–Stokes equations (GMNSE). They studied the existence and uniqueness of strong solutions and established the existence of global V-attractors. As noted in [7], the GMNSE prevents large gradients dominating the dynamic and leading to explosion. Let us recall that some useful results about the three-dimensional NSE are obtained from the GMNSE. In particular, using the GMNSE model, the authors of [7] established the existence of bounded entire solutions of the 3D Navier–Stokes equations. In [30], the authors used the GMNSE to prove that the attainability set of weak solutions of the 3D NS satisfying the energy inequality is weakly compact and weakly connected. Several articles are devoted to the mathematical analysis of the GMNSE, see for instance [5,8,14,24,25,29,30,33], as well as the review paper [4] in which the authors present some recent developments on the GMNSE.

Motivated by the above work, we propose in this article a three-dimensional system of a globally modified AC–NS equations (GMACNSE). We prove the existence and uniqueness of strong solutions as well as the existence of $\mathbb{V}$-attractors, i.e., attractors in the space $\mathbb{V}$ generated by strong solutions [see the definition 2.20 below]. Let us note that the coupling between the Navier–Stokes and the Allen–Cahn equations introduces in the coupled model a highly nonlinear term that makes the analysis more involved.

The article is divided as follows. In the next section, we introduce the GMACNSE and its mathematical setting. The third section studies the existence and uniqueness of strong solutions. In the fourth section, we study the asymptotic behavior of the strong solutions when the forcing term is time independent and we prove the existence of global attractors in $\mathbb{V}$. In the fifth section, we prove that solutions to the GMACNSE converge to weak solution of the AC–NS system. For a time-independent forcing, we also prove the existence of bounded entire weak solutions of the 3D AC–NS equations.

2 A globally modified AC–NS model and its mathematical setting

2.1 Governing equations

In this article, we consider a globally modified version of a model of homogeneous incompressible two-phase flow. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\Re^3$. We consider a globally modified version of the following AC–NS system

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla p - \mathcal{K}\mu\nabla\phi = g, \\ \text{div } v = 0, \\ \dfrac{\partial \phi}{\partial t} + v\cdot\nabla\phi + \mu = 0,\ \mu = -\epsilon\Delta\phi + \alpha f(\phi), \end{cases} \tag{2.1}$$

in $\mathcal{M}\times(0,+\infty)$.

In (2.1), the unknown functions are the velocity $v=(v_1,v_2,v_3)$ of the fluid, the pressure p, the order (phase) parameter ϕ and the (given) external force field g. The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_{\mathcal{M}} \left(\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi)\right) \mathrm{d}s, \tag{2.2}$$

where, e.g., $F(r) = \displaystyle\int_0^r f(\zeta)\mathrm{d}\zeta$. Here, the constants $\nu > 0$ and $\mathcal{K} > 0$ correspond to the kinematic viscosity of the fluid and the capillarity (stress) coefficient, respectively, $\epsilon,\ \alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ϵ is related with the thickness of the interface separating the two fluids. Hereafter, as in [17] we assume that $\epsilon \le \alpha$.

We endow (2.1) with the boundary condition

$$v = 0,\ \frac{\partial\phi}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}\times(0,+\infty), \tag{2.3}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and η is its outward normal.

The initial condition is given by

$$(v, \phi)(0) = (v^0, \phi^0). \tag{2.4}$$

Now, we define the function $F_N : \Re^+ \to \Re^+$ by

$$F_N(r) = \min\{1, N/r\}, \quad r \in \Re^+, \tag{2.5}$$

for some (fixed) $N \in \Re^+$ and we consider the following globally modified AC–NS equations (GMACNSE)

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu \Delta v + F_N(\|v\|)\,[(v \cdot \nabla)v] + \nabla p - F_N(\|(v, \phi)\|_{\mathbb{V}})\,[\mathcal{K}\mu \nabla \phi] = g, \\ \text{div } v = 0, \\ \dfrac{\partial \phi}{\partial t} + F_N(\|(v, \phi)\|_{\mathbb{V}})\,[v \cdot \nabla \phi] + \mu = 0, \ \mu = -\epsilon \Delta \phi + \alpha f(\phi), \end{cases} \tag{2.6}$$

in $\mathcal{M} \times (0, +\infty)$, where $\|v\|$ and $\|(v, \phi)\|_{\mathbb{V}}$ are some norms defined by (2.9) and (2.21) below.

The GMACNSE (2.6) is inspired from the globally modified Navier–Stokes equations (GMNSE) proposed in [7]. As noted in [7] in the case of the GMNSE, the GMACNSE are indeed globally modified. The factors $F_N(\|v\|)$ and $F_N(\|(v, \phi)\|_{\mathbb{V}})$ depend, respectively, on the norms $\|v\|$ and $\|(v, \phi)\|_{\mathbb{V}}$. They prevent large values of $\|v\|$ and $\|(v, \phi)\|_{\mathbb{V}}$ dominating the dynamics. Just like the GMNSE, the GMACNSE violate the basic laws of mechanics, but mathematically the model is well defined. See also [13] for other modifications of the nonlinear term in the NSE.

2.2 Mathematical setting

Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class $\mathcal{C}^2$). We also assume that $f \in \mathcal{C}^1(\Re)$ satisfies

$$\begin{cases} \lim\limits_{|r| \to +\infty} f'(r) > 0, \\ |f'(r)| \le c_f(1 + |r|^k), \quad \forall r \in \Re, \end{cases} \tag{2.7}$$

where c_f is some positive constant and $k \in [1, 2]$ is fixed. It follows from (2.7) that

$$|f(r)| \le c_f \left(1 + |r|^{k+1}\right), \quad \forall r \in \Re. \tag{2.8}$$

If X is a real Hilbert space with inner product $(\cdot, \cdot)_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. We set

$$\mathcal{V}_1 = \left\{ v \in \mathcal{C}_c^\infty(\mathcal{M}) : \ \text{div } v = 0 \text{ in } \mathcal{M} \right\}.$$

We denote by H_1 and V_1 the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^3$ and $(H_0^1(\mathcal{M}))^3$, respectively. The scalar product in H_1 is denoted by $(\cdot, \cdot)$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((u, v)) = \sum_{i=1}^{3} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \|u\| = ((u, u))^{1/2}. \tag{2.9}$$

We now define the operator A_0 by

$$A_0 u = \mathcal{P}\Delta u, \quad \forall u \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}_1$ is the Leray–Helmholtz projector in $L^2(\mathcal{M})$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|A_0\cdot|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the H^2-norm.

Note that from (2.7), we can find $\gamma > 0$ such that

$$\lim_{|r|\to+\infty} f'(r) > 2\gamma > 0. \tag{2.10}$$

We define the linear positive unbounded operator A_γ on $L^2(\mathcal{M})$ by:

$$A_\gamma\phi = -\Delta\phi + \gamma\phi, \quad \forall\phi \in D(A_\gamma), \tag{2.11}$$

where

$$D(A_\gamma) = \left\{\rho \in H^2(\mathcal{M});\ \frac{\partial\rho}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}\right\}.$$

Note that A_γ^{-1} is a compact linear operator on $L^2(\mathcal{M})$ and $|A_\gamma\cdot|_{L^2}$ is a norm on $D(A_\gamma)$ that is equivalent to the H^2-norm.

We introduce the bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0, which are defined from $D(A_0)\times D(A_0)$ into H_1, $D(A_0)\times D(A_\gamma)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M})\times D(A_\gamma^{3/2})$ into H_1, respectively. More precisely, we set

$$\begin{aligned}
&(B_0(u,v),w) = \int_{\mathcal{M}} [(u\cdot\nabla)v]\cdot w\,\mathrm{d}x = b_0(u,v,w), \quad \forall u,v,w\in D(A_0),\\
&(B_1(u,\phi),\rho) = \int_{\mathcal{M}} [(u\cdot\nabla)\phi]\,\rho\,\mathrm{d}x = b_1(u,\phi,\rho), \quad \forall u\in D(A_0),\ \phi,\rho\in D(A_\gamma),\\
&(R_0(\mu,\phi),w) = \int_{\mathcal{M}} \mu\,[\nabla\phi\cdot w]\,\mathrm{d}x = b_1(w,\phi,\mu), \quad \forall w\in D(A_0),\ (\mu,\phi)\in L^2(\mathcal{M})\times D(A_\gamma^{3/2}).
\end{aligned} \tag{2.12}$$

Note that

$$R_0(\mu,\phi) = \mathcal{P}_1\mu\nabla\phi.$$

We recall that B_0, B_1 and R_0 satisfy the following estimates (see for instance [16,17,34,35])

$$\begin{aligned}
&|b_0(u,v,w)| \le c|u|_{L^2}^{1/2}\|u\|^{1/2}|A_0v|_{L^2}|w|_{L^2}, \quad \forall u\in V_1, v\in D(A_0)), w\in H_1,\\
&|B_0(u,v)|_{V_1^*} \le c|u|_{L^2}^{1/4}\|u\|^{3/4}|v|_{L^2}^{1/4}\|v\|^{3/4}, \quad \forall u,v\in V_1,\\
&|B_0(u,v)|_{L^2} \le c\|u\|\|v\|^{1/2}|A_0v|_{L^2}^{1/2}, \quad \forall u\in V_1, v\in D(A_0),
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&|b_1(u,\phi,\psi)| \le c|u|_{L^2}^{1/2}\|u\|^{1/2}|A_\gamma\phi|_{L^2}|\psi|_{L^2}, \quad \forall u\in V_1, \phi\in D(A_\gamma)), \psi\in H_2,\\
&|B_1(u,\phi)|_{V_2^*} \le c|u|_{L^2}^{1/4}\|u\|^{3/4}|\phi|_{L^2}^{1/4}\|\phi\|^{3/4}, \quad \forall u\in V_1, \phi\in V_2,\\
&|B_1(v,\phi)|_{L^2} \le c\|v\|\|\phi\|^{1/2}|A_\gamma\phi|_{L^2}^{1/2}, \quad \forall v\in V_1, \phi\in D(A_\gamma),
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
&|R_0(A_\gamma\phi,\rho)|_{V_1^*} \le c\|\rho\|^{1/2}|A_\gamma\rho|_{L^2}^{1/2}|A_\gamma\phi|_{L^2}, \quad \forall\phi,\rho\in D(A_\gamma),\\
&|R_0(A_\gamma\phi,\rho)|_{L^2} \le c|A_\gamma\rho|_{L^2}|A_\gamma\phi|_{L^2}^{1/2}|A_\gamma^{3/3}\phi|_{L^2}^{1/2}, \quad \forall\phi\in D(A_\gamma), \rho\in D(A_\gamma^{3/2}).
\end{aligned} \tag{2.15}$$

For instance to derive (2.15)$_2$, we note that

$$|\langle R_0(A_\gamma\phi,\rho),w\rangle| = |b_1(w,\rho,A_\gamma\phi)| \le c|w|_{L^2}|\nabla\rho|_{L^6}|A_\gamma\phi|_{L^3} \\ \le c|w|_{L^2}|A_\gamma\rho|_{L^2}|A_\gamma\phi|_{L^2}^{1/2}|A_\gamma^{3/2}\phi|_{L^2}^{1/2}, \tag{2.16}$$

which gives (2.15)$_2$.

Hereafter we set

$$b_0^N(u,v,w) = F_N(\|v\|)b_0(u,v,w),\ \ \langle B_0^N(u,v),w\rangle = b_0^N(u,v,w),\quad \forall u,v,w\in V_1,$$
$$b_1^N(v,\phi,\psi) = F_N(\|(v,\phi)\|_{\mathbb{V}})b_1(v,\phi,\psi),\ \ \langle B_1^N(v,\phi),\psi\rangle = b_0^N(v,\phi,\psi),\quad \forall v\in V_1,\ \phi,\psi\in V_2,$$
$$\langle R_0^N(A_\gamma\phi,\phi),w\rangle = F_N(\|(v,\phi)\|_{\mathbb{V}})\langle R_0(A_\gamma\phi,\phi),w\rangle,\quad \forall(v,\phi)\in V_1\times D(A_\gamma),w\in V_1. \tag{2.17}$$

It follows from (2.13–2.15) and (2.5) that

$$|b_0^N(u,v,w)| \le cN\|u\|\|w\|,\quad \forall u,v,\in V_1,$$
$$\|B_0^N(u,v)\|_{V_1^*} \le c|u|_{L^2}^{1/4}\|u\|^{3/4}|v|_{L^2}^{1/4}\|v\|^{3/4},\quad \forall u,v\in V_1,$$
$$\|B_0^N(u,v)\|_{V_1^*} \le cN\|u\|,\quad \forall u,v\in V_1. \tag{2.18}$$

We also note that

$$b_0^N(u,v,v) = 0,\quad \forall u,v\in V_1,$$
$$b_1^N(v,\phi,\phi) = 0,\quad \forall v\in V_1,\phi\in V_2,$$
$$b_1^N(v,\phi,A_\gamma\phi) = \left\langle R_0^N(A_\gamma\phi,\phi),v\right\rangle,\quad \forall(v,\phi)\in V_1\times D(A_\gamma). \tag{2.19}$$

Now we define the Hilbert spaces $\mathbb{Y}$ and $\mathbb{V}$ by

$$\mathbb{Y} = H_1\times H^1(\mathcal{M}),\ \ \mathbb{V} = V_1\times D(A_\gamma) \tag{2.20}$$

endowed with the scalar products whose associated norms are

$$|(v,\phi)|_{\mathbb{Y}}^2 = \mathcal{K}^{-1}|v|_{L^2}^2 + \epsilon\left(|\nabla\phi|_{L^2}^2+\gamma|\phi|_{L^2}^2\right) = \mathcal{K}^{-1}|v|_{L^2}^2 + \epsilon|A_\gamma^{1/2}\phi|_{L^2}^2,$$
$$\|(v,\phi)\|_{\mathbb{V}}^2 = \|v\|^2 + |A_\gamma\phi|_{L^2}^2. \tag{2.21}$$

We also set

$$f_\gamma(r) = f(r) - \alpha^{-1}\epsilon\gamma r$$

and observe that f_γ still satisfies (2.10) with γ in place of 2γ since $\epsilon\le\alpha$. Also its primitive $F_\gamma(r) = \int_0^r f_\gamma(\zeta)\zeta$ is bounded from below.

Hereafter, we will denote by $\lambda>0$ a constant such that

$$\lambda|v|_{L^2}^2 \le \|v\|^2,\quad \lambda|A_\gamma^{1/2}\phi|_{L^2}^2 \le |A_\gamma\phi|_{L^2}^2,\quad \forall(v,\phi)\in\mathbb{V}. \tag{2.22}$$

Using the notations above, we rewrite (2.6), (2.3), (2.4) in the form

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} + \nu A_0 v + B_0^N(v,v) = \mathcal{K}R_0^N(\epsilon A_\gamma\phi,\phi) + g, \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}t} + \mu + B_1^N(v,\phi) = 0,\ \ \mu = \epsilon A_\gamma\phi + \alpha f_\gamma(\phi), \\ (v,\phi)(0) = (v^0,\phi^0). \end{cases} \tag{2.23}$$

Remark 2.1 In the formulation (2.23), the term $\mu\nabla\phi$ is replaced by $\epsilon A_\gamma \nabla\phi$. This is justified since $f_\gamma'(\phi)\nabla\phi$ is the gradient $F_\gamma(\phi)$ and can be incorporated into the pressure gradient, see [17] for details.

Definition 2.1 Suppose that $(v^0, \phi^0) \in \mathbb{Y}$ and $g \in L^2(0, T; H_1)$ for all $T > 0$. A weak solution to (2.23) is any pair $(v, \phi) \in L^2(0, T; \mathbb{V})$ such that

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} + \nu A_0 v + B_0^N(v, v) = \mathcal{K} R_0^N(\epsilon A_\gamma \phi, \phi) + g \ \text{ in } \mathcal{D}'\left(0, \infty; V_1'\right), \\ \frac{\mathrm{d}\phi}{\mathrm{d}t} + \mu + B_1^N(v, \phi) = 0, \ \mu = \epsilon A_\gamma \phi + \alpha f_\gamma(\phi) \ \text{ in } \mathcal{D}'\left(0, \infty; V_2'\right), \\ (v, \phi)(0) = \left(v^0, \phi^0\right). \end{cases} \tag{2.24}$$

Remark 2.2 Note that if $(v, \phi) \in L^2(0, T; \mathbb{V})$ satisfies (2.23), it follows from (2.18) that $\frac{\mathrm{d}}{\mathrm{d}t}(v, \phi) \in L^2([0, T); \mathbb{V}^*)$ and consequently, $(v, \phi) \in \mathcal{C}([0, T); \mathbb{Y})$.

Hereafter, for any $(w, \psi) \in \mathbb{Y}$, we set

$$\mathcal{E}(w, \psi) = |(w, \psi)|_{\mathbb{Y}}^2 + 2\langle F_\gamma(\psi), 1\rangle + \alpha_0, \tag{2.25}$$

where $\alpha_0 > 0$ is a constant large enough and independent of (w, ψ) such that $\mathcal{E}(w, \psi)$ is nonnegative (note that F_γ is bounded from below).

We can check that (see [16] for details) there exists a monotone non-decreasing function Q_0 (independent of time and the initial condition) such that

$$|(w, \psi)|_{\mathbb{Y}}^2 \le \mathcal{E}(w, \psi) \le Q_0\left(|(w, \psi)|_{\mathbb{Y}}^2\right) \equiv C_f\left(1 + |(w, \psi)|_{\mathbb{Y}}^2 + |\nabla\phi|_{L^2}^{k+2}\right), \quad \forall (w, \psi) \in \mathbb{Y}, \tag{2.26}$$

where k is the integer that appears in (2.7)–(2.8) and $C_f > 0$ is a constant.

By taking the scalar product in H_1 of $(2.23)_1$ with v, then taking the scalar product in $L^2(\mathcal{M})$ of $(2.23)_3$ with μ, we derive that (v, ϕ) satisfies the energy equality

$$\mathcal{E}(t) - \mathcal{E}(s) + \int_s^t \left(2\nu\|v\|^2 + 2|\mu|_{L^2}^2\right) \mathrm{d}\zeta = 2\int_s^t \langle g, v\rangle \mathrm{d}\zeta \quad \text{for all } 0 \le s \le t, \tag{2.27}$$

where $\mathcal{E}(t) = \mathcal{E}(v(t), \phi(t))$.

The weak formulation of (2.23) with F_N replaced by 1 is studied in [16,17], where the existence and uniqueness of solution was proved in the two-dimensional case. See also [1]. Hereafter, to simplify the notation, we set $\mathcal{K} = 1$.

We recall from [7] the following properties of F_N.

Lemma 1

$$\begin{aligned} |F_N(p) - F_N(r)| &\le \frac{|p - r|}{r}, \quad \forall p, r \in \Re^+, \ r \ne 0, \\ |F_N(\|v_1\|) - F_N(\|v_1\|)| &\le \frac{\|v_1 - v_2\|}{\|v_2\|}, \quad \forall v_1, v_2 \in V_1, \ v_1 \ne 0, \\ |F_M(p) - F_N(r)| &\le \frac{|M - N|}{r} + \frac{|p - r|}{r}, \quad \forall p, r, M, N \in \Re^+, \ r \ne 0. \end{aligned} \tag{2.28}$$

Proof See [7]. □

3 Existence and uniqueness of strong solution

Theorem 1 *There exists at most one weak solution* (v, ϕ) *of* (2.23) *such that* $(v, \phi) \in L^2(0, T; D(A_0) \times D(A_\gamma^{3/2}))$.

Proof Let $(v_i, \phi_i), i = 1, 2$ be weak solutions to (2.23) that belong to $L^2(0, T; D(A_0) \times D(A_\gamma^{3/2}))$. Let us set $(w, \psi) = (v_1, \phi_1) - (v_2, \phi_2)$. Then (w, ψ) satisfies

$$\begin{cases} \frac{\mathrm{d}w}{\mathrm{d}t} + \nu A_0 w + B_0^N(v_1, v_1) - B_0^N(v_2, v_2) = R_0^N(\epsilon A_\gamma \phi_1, \phi_1) - R_0^N(\epsilon A_\gamma \phi_2, \phi_2), \\ \frac{\mathrm{d}\psi}{\mathrm{d}t} + \epsilon A_\gamma \psi + B_1^N(v_1, \phi_1) - B_1^N(v_2, \phi_2) + \alpha f_\gamma(\phi_1) - \alpha f_\gamma(\phi_2) = 0, \\ (w, \psi)(0) = (0, 0). \end{cases} \tag{3.1}$$

From [7], we have

$$\begin{aligned} B_0^N(v_1, v_1) - B_0^N(v_2, v_2) &= F_N(\|v_1\|) B_0(w, v_1) + F_N(\|v_2\|) B_0(v_2, w) \\ &\quad + (F_N(\|v_1\|) - F_N(\|v_2\|)) B_0(v_2, v_1), \end{aligned} \tag{3.2}$$

$$|\langle B_0^N(v_1, v_1) - B_0^N(v_2, v_2), w\rangle| \le \frac{\nu}{8}\|w\|^2 + c|A_0 v_2|_{L^2}^2 |w|_{L^2}^2 + c\|v_2\| |A_0 v_2|_{L^2} |w|_{L^2}^2. \tag{3.3}$$

We can also check that

$$\begin{aligned} \Big\langle R_0^N(\epsilon A_\gamma \phi_1, \phi_1) - R_0^N(\epsilon A_\gamma \phi_2, \phi_2), w\Big\rangle &= \langle F_N(\|(v_1, \phi_1)\|_{\mathbb{V}}) R_0(\epsilon A_\gamma \phi_1, \phi_1) \\ &\quad - F_N(\|(v_2, \phi_2)\|_{\mathbb{V}}) R_0(\epsilon A_\gamma \phi_2, \phi_2), w\rangle \\ &= F_N(\|(v_1, \phi_1)\|_{\mathbb{V}})\, b_1(w, \phi_1, \epsilon A_\gamma \psi) \\ &\quad + F_N(\|(v_2, \phi_2)\|_{\mathbb{V}})\, b_1(w, \psi, \epsilon A_\gamma \phi_2) \\ &\quad + F_N(\|(v_1, \phi_1)\|_{\mathbb{V}}) \\ &\quad - F_N(\|(v_2, \phi_2)\|_{\mathbb{V}})\, b_1\big(w, \phi_1, \epsilon A_\gamma \phi_2\big) \\ &\equiv K_2^1 + K_2^2 + K_2^3, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \Big\langle B_1^N(v_1, \phi_1) - B_1^N(v_2, \phi_2), A_\gamma \psi\Big\rangle &= \langle F_N(\|(v_1, \phi_1)\|_{\mathbb{V}}) B_1(v_1, \phi_1) \\ &\quad - F_N(\|(v_2, \phi_2)\|_{\mathbb{V}})\, B_1(v_2, \phi_2), w\rangle \\ &= F_N(\|(v_1, \phi_1)\|_{\mathbb{V}})\, b_1(w, \phi_1, A_\gamma \psi) \\ &\quad + F_N(\|(v_2, \phi_2)\|_{\mathbb{V}})\, b_1(v_2, \psi, A_\gamma \psi) \\ &\quad + F_N(\|(v_1, \phi_1)\|_{\mathbb{V}}) \\ &\quad - F_N(\|(v_2, \phi_2)\|_{\mathbb{V}})\, b_1(v_2, \phi_1, A_\gamma \psi) \\ &\equiv K_3^1 + K_3^2 + K_3^3. \end{aligned} \tag{3.5}$$

From (2.13–2.15), we have

$$\begin{aligned} |b_1(w, \phi_1, \epsilon A_\gamma \psi)| &\le c|w|_{L^2}^{1/2} \|w\|^{1/2} |A_\gamma \phi_1|_{L^2} |A_\gamma \psi|_{L^2}, \\ |b_1(w, \psi, \epsilon A_\gamma \phi_2)| &\le c|w|_{L^2}^{1/2} \|w\|^{1/2} |A_\gamma \psi|_{L^2} |A_\gamma \phi_2|_{L^2}, \\ |b_1(w, \phi_1, \epsilon A_\gamma \phi_2)| &\le c|w|_{L^2}^{1/2} \|w\|^{1/2} |A_\gamma \phi_1|_{L^2} |A_\gamma \phi_2|_{L^2}. \end{aligned} \tag{3.6}$$

It follows from (2.5), (2.28) and (3.4)–(3.6) that

$$\begin{aligned}|K_2^1| &\equiv F_N(\|(v_1,\phi_1)\|_{\mathbb{V}})|b_1(w,\phi_1,\epsilon A_\gamma\psi)| \le cN|w|_{L^2}^{1/2}\|w\|^{1/2}|A_\gamma\psi|_{L^2}\\ &\le \tfrac{\nu}{8}\|w\|^2+\tfrac{\epsilon}{8}|A_\gamma\psi|_{L^2}^2+cN^4|w|_{L^2}^2,\end{aligned} \tag{3.7}$$

$$\begin{aligned}|K_2^2| &\equiv F_N(\|(v_2,\phi_2)\|_{\mathbb{V}})|b_1(w,\psi,\epsilon A_\gamma\phi_2)| \le cN|w|_{L^2}^{1/2}\|w\|^{1/2}|A_\gamma\psi|_{L^2}\\ &\le \tfrac{\nu}{8}\|w\|^2+\tfrac{\epsilon}{8}|A_\gamma\psi|_{L^2}^2+cN^4|w|_{L^2}^2,\end{aligned} \tag{3.8}$$

$$\begin{aligned}|K_2^3| &\equiv |F_N(\|(v_1,\phi_1)\|_{\mathbb{V}})-F_N(\|(v_2,\phi_2)\|_{\mathbb{V}}||b_1(w,\phi_1,\epsilon A_\gamma\phi_2)|\\ &\le c\frac{\|(w,\psi)\|_{\mathbb{V}}}{\|(v_2,\phi_2)\|_{\mathbb{V}}}|w|_{L^2}^{1/2}\|w\|^{1/2}|A_\gamma\phi_1|_{L^2}|A_\gamma\phi_2|_{L^2}\\ &\le \|(w,\psi)\|_{\mathbb{V}}^{3/2}|w|_{L^2}^{1/2}|A_\gamma\phi_1|_{L^2}\\ &\le \tfrac{\nu}{8}\|w\|^2+\tfrac{\epsilon}{8}|A_\gamma\psi|_{L^2}^2+c|w|_{L^2}^2\|\phi_1\|^2|A_\gamma^{3/2}\phi_1|_{L^2}^2.\end{aligned} \tag{3.9}$$

Similarly, from (2.13)–(2.15) we have

$$\begin{aligned}&|b_1(v_2,\psi,\epsilon A_\gamma\psi)| \le c\|v_2\|\|\psi\|^{1/2}|A_\gamma\psi|_{L^2}^{3/2},\\ &|b_1(v_2,\phi_1,\epsilon A_\gamma\psi)| = \langle A_\gamma^{1/2}B_1(v_2,\phi_1),A_\gamma^{1/2}\psi\rangle \le c\|v_2\||A_\gamma\phi_1|_{L^2}\|\psi\|^{1/2}|A_\gamma\psi|_{L^2}^{1/2}.\end{aligned} \tag{3.10}$$

It follows from (2.5), (2.28), (3.5) and (3.10) that

$$\begin{aligned}|K_3^1| &\equiv F_N(\|(v_1,\phi_1)\|_{\mathbb{V}})|b_1\left(w,\phi_1,\epsilon A_\gamma\psi\right)| \le cN|w|_{L^2}^{1/2}\|w\|^{1/2}|A_\gamma\psi|_{L^2}\\ &\le \frac{\nu}{8}\|w\|^2+\frac{\epsilon}{8}|A_\gamma\psi|_{L^2}^2+cN^4|w|_{L^2}^2,\end{aligned} \tag{3.11}$$

$$\begin{aligned}|K_3^2| &\equiv F_N(\|(v_2,\phi_2)\|_{\mathbb{V}})|b_1\left(v_2,\psi,\epsilon A_\gamma\psi\right)| \le cN\|\psi\|^{1/2}|A_\gamma\psi|_{L^2}^{3/2}\\ &\le \frac{\epsilon}{8}|A_\gamma\psi|_{L^2}^2+cN^4\|\psi\|^2,\end{aligned} \tag{3.12}$$

$$\begin{aligned}|K_3^3| &\equiv |F_N(\|(v_1,\phi_1)\|_{\mathbb{V}})-F_N(\|(v_2,\phi_2)\|_{\mathbb{V}}||b_1(v_2,\phi_1,\epsilon A_\gamma\psi)|\\ &= |F_N(\|(v_1,\phi_1)\|_{\mathbb{V}})-F_N(\|(v_2,\phi_2)\|_{\mathbb{V}}|\langle A_\gamma^{1/2}B_1(v_2,\phi_1),A_\gamma^{1/2}\psi\rangle|\\ &\le c\frac{\|(w,\psi)\|_{\mathbb{V}}}{\|(v_2,\phi_2)\|_{\mathbb{V}}}\|v_2\||A_\gamma\phi_1|_{L^2}\|\psi\|^{1/2}|A_\gamma\psi|_{L^2}^{1/2}\\ &\le \frac{\nu}{8}\|w\|^2+\frac{\epsilon}{8}|A_\gamma\psi|_{L^2}^2+c\|\phi_1\|^2|A_\gamma^{3/2}\phi_1|_{L^2}^2\|\psi\|^2.\end{aligned} \tag{3.13}$$

From (2.7)–(2.8), we can check that

$$\alpha|\langle f_\gamma(\phi_1)-f_\gamma(\phi_2),A_\gamma\psi\rangle| \le \frac{\epsilon}{8}|A_\gamma\psi|_{L^2}^2+Q_1\left(\|\phi_1\|,\|\phi_2\|\right)\|\psi\|^2, \tag{3.14}$$

where $Q_1=Q_1(x_1,x_2)$ is a monotone non-decreasing function of x_1 and x_2.

Let us set

$$\mathcal{Y}=|(w,\psi)|_{\mathbb{Y}}^2=|w|_{L^2}^2+\|\psi\|^2,$$

$$\mathcal{Y}_1=c|A_0v_2|_{L^2}^2+c\|v_2\||A_0v_2|_{L^2}+c\|\phi_1\|^2|A_\gamma^{3/2}\phi_1|_{L^2}^2+Q_1\left(\|\phi_1\|,\|\phi_2\|\right)+CN^4.$$

Multiplying $(3.1)_1$ by w and $(3.1)_2$ by $A_\gamma\psi$ and using (3.4), (3.7)–(3.9) and (3.11)–(3.14), we derive that

$$\frac{\mathrm{d}\mathcal{Y}}{\mathrm{d}t}+\nu\|w\|^2+\epsilon|A_\gamma\psi|_{L^2}^2 \le \mathcal{Y}_1\mathcal{Y}, \tag{3.15}$$

and the Gronwall lemma yields $\mathcal{Y} = |(w, \psi)|^2_{\mathbb{Y}} = 0$, i.e., $(v_1, \phi_1) = (v_2, \phi_2)$. □

Theorem 2 *Suppose that $g \in L^2(0, T; H_1)$ for all $T > 0$ and $(v^0, \phi^0) \in \mathbb{V}$ be given. Then there exists a unique weak solution (v, ϕ) of (2.23), which is in fact a strong solution in the sense that*

$$(v, \phi) \in \mathcal{C}(0, T; \mathbb{V}) \cap L^2(0, T; D(A_0) \times D(A_\gamma^{3/2})). \tag{3.16}$$

If the initial condition $(v^0, \phi^0) \in \mathbb{Y}\backslash\mathbb{V}$ and $g \in L^\infty(0, \infty; H_1)$, then every weak solution (v, ϕ) of (2.23) is a strong solution, in the sense that

$$(v, \phi) \in \mathcal{C}(\tau, T; \mathbb{V}) \cap L^2(\tau, T; D(A_0) \times D(A_\gamma^{3/2})) \text{ for all } T > \tau > 0. \tag{3.17}$$

Proof Since the injection of $\mathbb{Y} \subset \mathbb{V}$ is compact, let $\{(w_i, \psi_i), i = 1, 2, 3, \dots\} \subset \mathbb{V}$ be an orthonormal basis of $\mathbb{Y}$, where $\{w_i,\ i = 1, 2 \dots\}$, $\{\psi_i,\ i = 1, 2 \dots\}$ are eigenvectors of A_0 and A_γ, respectively. We set $\mathbb{V}_m = \mathbb{Y}_m = \text{span}\{(w_1, \psi_1), \dots (w_m, \psi_m)\}$.

We look for $(v_m, \phi_m) \in \mathbb{Y}_m$ solution to the ordinary differential equations

$$\begin{cases} \dfrac{dv_m}{dt} + \nu\mathcal{P}^1_m A_0 v_m + B^N_0(v_m, v_m) = \mathcal{P}^1_m \left(R^N_0\left(\epsilon A_\gamma \phi_m, \phi_m\right) + g\right), \\ \dfrac{d\phi_m}{dt} + \mathcal{P}^2_m \left(\mu_m + B^N_1(v_m, \phi_m)\right) = 0,\ \ \mu_m = \epsilon A_\gamma \phi_m + f_\gamma(\phi_m), \\ (v_m, \phi_m)(0) = \mathcal{P}_m(v^0, \phi^0), \end{cases} \tag{3.18}$$

where $\mathcal{P}_m = (\mathcal{P}^1_m, \mathcal{P}^2_m) : H_1 \times L^2(\mathcal{M}) \to \mathbb{V}_m$ is the orthogonal projection. Since $\mathcal{P}_m(0, g)$ is a local Lipschitz function in (v, ϕ), it follows from the theory of ordinary differential equation that this equation has a solution (v_m, ϕ_m), (see also Theorem A1 of [6]). Hereafter C denotes a constant independent of m and depending only on data such as $\mathcal{M}$ and whose value may be different in each inequality.

By taking the scalar product in H_1 of $(3.18)_1$ with v_m, then taking the scalar product in $L^2(\mathcal{M})$ of $(3.18)_3$ with μ_m, we derive that (see [17] for the details)

$$\frac{d\mathcal{E}}{dt} + 2\nu\|v_m\|^2 + 2|\mu_m|^2_{L^2} = 2\langle g, v_m\rangle, \tag{3.19}$$

where $\mathcal{E} = \mathcal{E}(t) = \mathcal{E}(v_m(t), \phi_m(t))$.

From (3.19), it follows that [see (2.26)]

$$\begin{aligned} \mathcal{E}(t) + \int_0^t \left(\nu\|v_m\|^2 + 2|\mu_m|^2_{L^2}\right) ds &\le \mathcal{E}(0) + c\int_0^t \|g\|^2_{V_1^*} ds \\ &\le Q_0\left(|(v^0, \phi^0)|^2_{\mathbb{Y}}\right) + c\int_0^t \|g\|^2_{V_1^*} ds. \end{aligned} \tag{3.20}$$

This proves that (v_m, ϕ_m) is uniformly bounded in $L^\infty(0, T; \mathbb{Y}) \cap L^2(0, T; \mathbb{V})$.

Note that from

$$\mu_m = \epsilon A_\gamma \phi_m + \alpha f_\gamma(\phi_m),$$

we derive that

$$|A_\gamma \phi_m|^2_{L^2} \le c|\mu_m|^2_{L^2} + Q_2\left(\|\phi_m\|^2\right), \tag{3.21}$$

where Q_2 is a monotone non-decreasing function independent of time, the initial condition and m. It follows from (3.20)–(3.21) that $A_\gamma \phi_m$ is bounded in $L^2(0, T; H_2)$.

We conclude that there exists a subsequence of (v_m, ϕ_m) (still) denoted (v_m, ϕ_m) such that

$$(v_m, \phi_m) \to (v, \phi) \ \text{ weak-star in } L^\infty(0, T; \mathbb{Y}),$$

$$(v_m, \phi_m) \to (v, \phi) \text{ weakly in } L^2(0, T; \mathbb{V}), \tag{3.22}$$

where

$$(v, \phi) \in L^\infty(0, T; \mathbb{Y}) \cap L^2(0, T; \mathbb{V}).$$

By a well-known compactness result (see Theorem 5.1 in Chapter 1 of [26] or [34]), we can assume that

$$\begin{aligned}(v_m, \phi_m) &\to (v, \phi) \text{ strongly in } L^2(0, T; \mathbb{Y}),\\ (v_m, \phi_m) &\to (v, \phi) \text{ a.e., in } (0, T) \times \mathcal{M}.\end{aligned} \tag{3.23}$$

The weak convergence in $L^2(0, T; \mathbb{V})$ is not enough to ensure that

$$\begin{aligned}F_N(\|v_m\|) &\to F_N(\|v\|) \text{ as } m \to \infty,\\ F_N(\|(v_m, \phi_m)\|_{\mathbb{V}}) &\to F_N(\|(v, \phi)\|_{\mathbb{V}}) \text{ as } m \to \infty.\end{aligned} \tag{3.24}$$

Therefore, we need to derive stronger a priori estimates. Now taking the inner product in H_1 of $(3.18)_1$ with $2A_0 v_m$, the inner product in $L^2(\mathcal{M})$ of $(3.18)_2$ and $(3.18)_3$ with $2A_\gamma^2 \phi_m$ and adding the resulting equalities gives

$$\begin{aligned}\frac{\mathrm{d}\mathcal{Y}}{\mathrm{d}t} + 2\nu|A_0 v_m|_{L^2}^2 + 2\epsilon|A_\gamma^{3/2}\phi_m|_{L^2}^2 = {}& 2b_1^N(A_0 v_m, \phi_m, \epsilon A_\gamma \phi_m)\\ & - 2b_0^N(v_m, v_m, A_0 v_m) + 2\langle g, A_0 v_m\rangle\\ & - 2\alpha(A_\gamma^{1/2} f_\gamma(\phi_m), A_\gamma^{3/2}\phi_m)_{L^2}\\ & - 2b_1^N(v_m, \phi_m, A_\gamma^2 \phi_m),\end{aligned} \tag{3.25}$$

where

$$\mathcal{Y}(t) = \|(v_m, \phi_m)\|_{\mathbb{V}}^2 = \|v_m(t)\|^2 + |A_\gamma \phi_m(t)|_{L^2}^2.$$

As noted in [7], we have

$$\left|b_0^N(v_m, v_m, A_0 v_m)\right| \le \frac{\nu}{8}|A_0 v_m|_{L^2}^2 + cN^4\|v_m\|^2. \tag{3.26}$$

We can also check that

$$\left|b_1(A_0 v_m, \phi_m, A_\gamma \phi_m)\right| \le c|A_0 v_m|_{L^2}|A_\gamma \phi_m|_{L^2}^{3/2}|A_\gamma^{3/2}\phi_m|_{L^2}^{1/2}, \tag{3.27}$$

$$\begin{aligned}\left|b_1(v_m, \phi_m, A_\gamma^2 \phi_m)\right| &= \left|\langle A_\gamma^{1/2} B_1(v_m, \phi_m), A_\gamma^{3/2}\phi_m\rangle\right|\\ &\le c\|v_m\|^{1/2}|A_0 v_m|_{L^2}^{1/2}|A_\gamma \phi_m|_{L^2}|A_\gamma^{3/2}\phi_m|_{L^2}\\ &\quad + c\|v_m\||A_\gamma \phi_m|_{L^2}^{1/2}|A_\gamma^{3/2}\phi_m|_{L^2}^{3/2}.\end{aligned} \tag{3.28}$$

It follows from (3.27)–(3.28) that

$$\begin{aligned}\left|b_1^N(A_0 v_m, \phi_m, \epsilon A_\gamma \phi_m)\right| &= F_N(\|(v_m, \phi_m)\|_{\mathbb{V}})\left|b_1\left(A_0 v_m, \phi_m, A_\gamma \phi_m\right)\right|\\ &\le cN|A_0 v_m|_{L^2}|A_\gamma \phi_m|_{L^2}^{1/2}|A_\gamma^{3/2}\phi_m|_{L^2}^{1/2}\\ &\le \frac{\nu}{8}|A_0 v_m|_{L^2}^2 + \frac{\epsilon}{8}|A_\gamma^{3/2}\phi_m|_{L^2}^2 + cN^4|A_\gamma \phi_m|_{L^2}^2,\end{aligned} \tag{3.29}$$

$$\begin{aligned}\left|b_1^N(v_m,\phi_m,A_\gamma^2\phi_m)\right| &= F_N\left(\|(v_m,\phi_m)\|_{\mathbb{V}}\right)|\langle B_1(v_m,\phi_m),A_\gamma^2\phi_m\rangle| \\ &\le cN\|v_m\|^{1/2}|A_0v_m|_{L^2}^{1/2}|A_\gamma^{3/2}\phi_m|_{L^2}+cN|A_\gamma\phi_m|_{L^2}^{1/2}|A_\gamma^{3/2}\phi_m|_{L^2}^{3/2} \\ &\le \frac{\nu}{8}|A_0v_m|_{L^2}^2+\frac{\epsilon}{8}|A_\gamma^{3/2}\phi_m|_{L^2}^2+cN^4\left(|A_\gamma\phi_m|_{L^2}^2+\|v_m\|^2\right).\end{aligned} \tag{3.30}$$

We also have

$$\begin{aligned}\alpha\left|\left\langle f_\gamma(\phi_m),A_\gamma^2\phi_m\right\rangle\right| &= \alpha|\langle A_\gamma^{1/2}f_\gamma(\phi_m),A_\gamma^{3/2}\phi_m\rangle| \\ &\le \frac{\epsilon}{8}|A_\gamma^{3/2}\phi_m|_{L^2}^2+Q_2\left(\|\phi_m\|^2\right)|A_\gamma\phi_m|_{L^2}^2.\end{aligned} \tag{3.31}$$

Let us set

$$\mathcal{Y}_1=CN^4+Q_2(\|\phi_m\|^2).$$

It follows from (3.25)–(3.26) and (3.29)–(3.31) that $\mathcal{Y}$ satisfies

$$\frac{\mathrm{d}\mathcal{Y}}{\mathrm{d}t}+\nu|A_0v_m|_{L^2}^2+\epsilon|A_\gamma^{3/2}\phi_m|_{L^2}^2\le\mathcal{Y}_1\mathcal{Y}+c|g|_{L^2}^2. \tag{3.32}$$

Case 1: $(v^0,\phi^0)\in\mathbb{V}$.

We recall that

$$\|(v_m,\phi_m)(0)\|_{\mathbb{V}}=\|\mathcal{P}_m(v^0,\phi^0)\|_{\mathbb{V}}\le\|(v^0,\phi^0)\|_{\mathbb{V}}.$$

Using the Gronwall lemma, we derive from (3.20) and (3.32) that (v_m,ϕ_m) satisfies

$$\|(v_m,\phi_m)(t)\|_{\mathbb{V}}\le C,\quad \int_0^T\left(\nu|A_0v_m|_{L^2}^2+\epsilon|A_\gamma^{3/2}\phi_m|_{L^2}^2\right)\mathrm{d}s\le C, \tag{3.33}$$

which proves that (v_m,ϕ_m) is bounded in $L^\infty(0,T;\mathbb{V})\cap L^2(0,T;D(A_0)\times D(A_\gamma^{3/2}))$.

Using (3.33) and (2.13)–(2.18), we can check that

$$\frac{\mathrm{d}}{\mathrm{d}t}(v_m,\phi_m)\text{ is bounded in }L^2(0,T,\mathbb{Y}). \tag{3.34}$$

Since $D(A_0)\times D(A_\gamma^{3/2})\subset\mathbb{V}\subset\mathbb{Y}$ with compact injections, it follows that there exists $(v,\phi)\in L^\infty(0,T;\mathbb{V})\cap L^2(0,T;D(A_0)\times D(A_\gamma^{3/2}))$ and a subsequence of (v_m,ϕ_m) (still) denoted (v_m,ϕ_m) such that for all $T>0$, we have

$$\begin{aligned}(v_m,\phi_m)&\to(v,\phi)\ \text{ strongly in }\ L^2(0,T;\mathbb{V}),\\ (v_m,\phi_m)&\to(v,\phi)\ \text{ a.e., in }\ (0,T)\times\mathcal{M},\\ (v_m,\phi_m)&\to(v,\phi)\ \text{ weakly-star in }\ L^\infty(0,T;\mathbb{V}),\\ (v_m,\phi_m)&\to(v,\phi)\ \text{ weakly in }\ L^2\left(0,T;D(A_0)\times D\left(A_\gamma^{3/2}\right)\right),\\ \frac{\mathrm{d}}{\mathrm{d}t}(v_m,\phi_m)&\to\frac{\mathrm{d}}{\mathrm{d}t}(v,\phi)\ \text{ weakly in }L^2(0,T,\mathbb{Y}).\end{aligned} \tag{3.35}$$

Since $(v_m,\phi_m)\to(v,\phi)$ in $L^2(0,T;\mathbb{V})$ for all $T>0$, there exists a subsequence (still) denoted (v_m,ϕ_m) such that

$$\|(v_m,\phi_m)\|_{\mathbb{V}}\to\|(v,\phi)\|_{\mathbb{V}}\ \text{ a.e., in }\ (0,\infty),$$

and therefore

$$F_N(\|(v_m,\phi_m)\|_{\mathbb{V}}) \to F_N(\|(v,\phi)\|_{\mathbb{V}}) \text{ a.e., in } (0,\infty),$$
$$F_N(\|v_m\|) \to F_N(\|v\|) \text{ a.e., in } (0,\infty).$$

Therefore as in [7], we can take the limit in (3.18) to derive that (v,ϕ) is a weak solution to (2.23) satisfying (2.27). In fact, let us set

$$Z = R_0^N\left(\epsilon A_\gamma \phi_m, \phi_m\right) - R_0^N\left(\epsilon A_\gamma \phi, \phi\right), \quad (w,\psi) = (v_m,\phi_m) - (v,\phi).$$

Then for any $\theta \in D(A_0)$, we have

$$\begin{aligned}\langle Z,\theta\rangle &= F_N\left(\|(v_m,\phi_m)\|_{\mathbb{V}}\right) b_1(\theta,\phi_m,\epsilon A_\gamma\psi) F_N\left(\|(v,\phi)\|_{\mathbb{V}}\right) b_1(\theta,\psi,\epsilon A_\gamma\phi)\\ &\quad + \left(F_N\left(\|(v_m,\phi_m)\|_{\mathbb{V}}\right) - F_N\left(\|(v,\phi)\|_{\mathbb{V}}\right)\right) b_1(\theta,\phi_m,\epsilon A_\gamma\phi)\\ &= Z^1 + Z^2 + Z^3. \end{aligned} \tag{3.36}$$

From (3.34), (3.35), we can check that as $m \to +\infty$, we have (see some details in Sect. 5)

$$\begin{aligned}\int_0^T Z^1 \mathrm{d}s &= \int_0^T F_N\left(\|(v_m,\phi_m)\|_{\mathbb{V}}\right) b_1(\theta,\phi_m,\epsilon A_\gamma\psi)\mathrm{d}s\\ &= \int_0^T F_N\left(\|(v_m,\phi_m)\|_{\mathbb{V}}\right)\left\langle A_\gamma^{1/2} B_1\left(\theta,\phi_m,\epsilon A_\gamma^{1/2}\psi\right)\right\rangle \mathrm{d}s \longrightarrow 0,\\ \int_0^T Z^2 \mathrm{d}s &= \int_0^T F_N\left(\|(v,\phi)\|_{\mathbb{V}}\right) b_1(\theta,\psi,A_\gamma\phi) \longrightarrow 0,\\ \int_0^T Z^3 \mathrm{d}s &= \int_0^T \left(F_N\left(\|(v_m,\phi_m)\|_{\mathbb{V}}\right) - F_N\left(\|(v,\phi)\|_{\mathbb{V}}\right)\right) b_1(\theta,\phi_m,\epsilon A_\gamma\psi)\mathrm{d}s\\ &= \int_0^T \left(F_N(\|(v_m,\phi_m)\|_{\mathbb{V}}) - F_N(\|(v,\phi)\|_{\mathbb{V}})\right)\left\langle A_\gamma^{1/2} B_1(\theta,\phi_m),\epsilon A_\gamma^{1/2}\psi\right\rangle \mathrm{d}s \longrightarrow 0. \end{aligned} \tag{3.37}$$

The convergence of the other nonlinear terms in (3.18) is proved similarly.

Case 2: $(v^0,\phi^0) \in \mathbb{Y}\backslash\mathbb{V}$, $g \in L^\infty(0,\infty;H_1)$. We proceed as in [7,13,35].

Hereafter we set $|g|_\infty = \|g\|_{L^\infty(0,\infty;H_1)}$.

From (3.32), we derive that

$$\mathcal{Y}(t) \le \mathcal{Y}(t_0)\exp\left(\int_{t_0}^t \mathcal{Y}_1(s)\mathrm{d}s\right) + c|g|_\infty^2 (t-t_0)\exp\left(\int_{t_0}^t \mathcal{Y}_1(s)\mathrm{d}s\right) \text{ for any } 0 \le t_0 \le t. \tag{3.38}$$

From (3.20)–(3.21), we also have

$$\int_t^{t+\tau} \mathcal{Y}(s)\mathrm{d}s \le Q_0\left(|(v_m,\phi_m)(t)|_{\mathbb{Y}}^2\right) + c\tau|g|_\infty^2 + c_1\tau + \int_t^{t+\tau} Q_2(\|\phi_m\|^2)\mathrm{d}s. \tag{3.39}$$

We can also check that (v_m,ϕ_m) satisfy the following estimate [see (4.10) in Sect. 4 for the details]

$$|(v_m,\phi_m)|_{\mathbb{Y}}^2 \le Q_0\left(\left|\left(v^0,\phi^0\right)\right|_{\mathbb{Y}}^2\right)\mathrm{e}^{-\kappa t} + c\int_0^t \mathrm{e}^{-\kappa(t-s)}\left(\|g\|_{V_1^*}^2 + c_1\right)\mathrm{d}s, \tag{3.40}$$

where $\kappa > 0$ is given by (4.8) below. It follows that

$$|(v_m,\phi_m)|_{\mathbb{Y}}^2 \le Q_0\left(|(v^0,\phi^0)|_{\mathbb{Y}}^2\right) + c\left(|g|_\infty^2 + c_1\right) \equiv K_1. \tag{3.41}$$

From (3.41), we have also have

$$\int_{t_0}^{t} \mathcal{Y}_1(s)\mathrm{d}s = \int_{t_0}^{t} \left(CN^4 + Q_2(\|\phi_m\|^2)\right) \mathrm{d}s \leq \left(CN^4 + Q_2(K_1)\right)(t-t_0) \equiv K_3(t-t_0). \tag{3.42}$$

From (3.20) and (3.41), we also derive that

$$\begin{aligned}\int_{t}^{t+\tau} \mathcal{Y}(s)\mathrm{d}s = \int_{t}^{t+\tau} & \|(v_m, \phi_m)\|_{\mathbb{V}}^2 \mathrm{d}s \leq Q_0\left(|(v,\phi)(t)|_{\mathbb{Y}}^2\right) \\ &+ \int_{t}^{t+\tau} Q_2\left(\|\phi_m\|^2\right) \mathrm{d}s + c\tau|g|_{\infty}^2 \\ &\leq Q_0(K_1) + \tau Q_2(K_1) + c\tau|g|_{\infty}^2 \equiv K_2. \end{aligned} \tag{3.43}$$

Let $\rho > 0$ defined by

$$\rho^2 = \frac{2K_2}{\tau} \tag{3.44}$$

and consider the sets

$$D_m = \{s \in [t, t+\tau] : \mathcal{Y}(s) \geq \rho^2\}$$

and let us denote $|D_m|$ the Lebesgue measure of D_m. From (3.43), we have

$$\rho^2|D_m| \leq \int_{D_m} \mathcal{Y}(s)\mathrm{d}s \leq \int_{t}^{t+\tau} \mathcal{Y}(s) \leq \frac{\tau\rho^2}{2}, \tag{3.45}$$

which gives $|D_m| \leq \frac{\tau}{2}$.

From this property, we have that for any given $\tau > 0$ and any $t \geq \tau$, there exists a $t_0 \in (t-\tau, t)$ such that

$$\mathcal{Y}(t_0) \leq 2K_2. \tag{3.46}$$

From (3.38) and (3.52), we deduce that

$$\begin{aligned}\mathcal{Y}(t) &\leq 2K_2 \exp\left(K_3(t-t_0)\right) + c|g|_{\infty}^2(t-t_0)\exp\left((K_3(t-t_0)\right) \\ &\leq 2K_2 \exp(\tau K_3) + c|g|_{\infty}^2 \tau \exp(\tau K_3) \text{ for all } t \geq \tau.\end{aligned} \tag{3.47}$$

From (3.20), (3.53) and (3.32), we derive that the sequence (v_m, ϕ_m) is bounded in $L^{\infty}(0, T; \mathbb{Y}) \cap L^{\infty}(\tau; T; \mathbb{V}) \cap L^2(\tau, T; D(A_0) \times D(A_{\gamma}^{3/2}))$, for all $T > \tau > 0$. Reasoning as in case 1, we can check that the sequence $\frac{\mathrm{d}}{\mathrm{d}t}(v_m, \phi_m)$ is also bounded in $L^2(\tau, T; \mathbb{Y})$ for all $T > \tau > 0$. Hence, there exists an element

$$(v, \phi) \in L^{\infty}(0, T; \mathbb{Y}) \cap L^{\infty}(\tau; T; \mathbb{V}) \cap L^2(\tau, T; D(A_0) \times D(A_{\gamma}^{3/2}))$$

for all $T > \tau > 0$, and a subsequence (still) denoted (v_m, ϕ_m), such that

$$\begin{aligned}
(v_m, \phi_m) &\to (v, \phi) \text{ weakly in } L^2(0, T; \mathbb{V}), \\
(v_m, \phi_m) &\to (v, \phi) \text{ a.e., in } (0, T) \times \mathcal{M}, \\
(v_m, \phi_m) &\to (v, \phi) \text{ strongly in } L^2(0, T; \mathbb{Y}), \\
(v_m, \phi_m) &\to (v, \phi) \text{ strongly in } L^2(\tau, T; \mathbb{V}), \\
(v_m, \phi_m) &\to (v, \phi) \text{ weakly-star in } L^{\infty}(0, T; \mathbb{Y}), \\
(v_m, \phi_m) &\to (v, \phi) \text{ weakly in } L^2(\tau, T; D(A_0) \times D(A_{\gamma}^{3/2})),
\end{aligned}$$

$$
\begin{aligned}
(v_m, \phi_m) &\to (v, \phi) \ \text{ weakly-star in } \ L^\infty(\tau, T; \mathbb{V}),\\
\frac{d}{dt}(v_m, \phi_m) &\to \frac{d}{dt}(v, \phi) \ \text{ weakly in } L^2(\tau, T; \mathbb{Y}).
\end{aligned} \tag{3.48}
$$

As in Case 1, we can take the limit in (3.18) and prove that (v, ϕ) is a solution to (2.23) satisfying (2.27). □

3.1 Continuous dependence on initial values and N

In this part, we prove that the semiflows generated by the solutions $(v_N, \phi_N)(t, (v^0, \phi^0))$ of the GMACNES (2.23) with the parameter N depend continuously on the parameter N and the initial value (v^0, ϕ^0). More precisely, we have the following result.

Theorem 3 *Suppose that $g \in L^2(0, T; H_1)$ for all $T > 0$ and let $N_i > 0$, $(v_i^0, \phi_i^0) \in \mathbb{V}, i = 1, 2$ be given. Let (v_i, ϕ_i) be the solution to (2.23) corresponding to the parameter N_i and the initial value (v_i^0, ϕ_i^0), $i = 1, 2$. Then, there exists an constant C independent of $N_i, (v_i^0, \phi_i^0)$ such that*

$$
\begin{aligned}
&\|(v_1, \phi_1)(t) - (v_2, \phi_2)(t)\|_{\mathbb{V}}^2\\
&\le \left\{\|(v_1^0, \phi_1^0) - (v_2^0, \phi_2^0)\|_{\mathbb{V}}^2 + C(N_1 - N_2)^2 \int_0^t \mathcal{Y}_3(s)ds\right\} \times \exp\left(\int_0^t \mathcal{Y}_1(s)ds\right),\\
&\int_0^t \left(\nu|A_0(v_1 - v_2)|_{L^2}^2 + \epsilon|A_\gamma^{3/2}(\phi_1 - \phi_2)|_{L^2}^2\right) ds\\
&\le \left\{\| \left(v_1^0, \phi_1^0\right) - \left(v_2^0, \phi_2^0\right) \|_{\mathbb{V}}^2 + C(N_1 - N_2)^2 \int_0^t \mathcal{Y}_3(s)ds\right\} \times \exp\left(\int_0^t \mathcal{Y}_1(s)ds\right).
\end{aligned} \tag{3.49}
$$

where

$$
\begin{aligned}
\mathcal{Y}_1 &= c\left(N_1^4 + N_2^4 + |A_\gamma\phi_1|_{L^2}|A_\gamma^{3/2}\phi_1|_{L^2}\right) + \mathcal{Q}_1\left(|A_\gamma\phi_1|_{L^2}, |A_\gamma\phi_2|_{L^2}\right),\\
\mathcal{Y}_3 &= c|A_\gamma\phi_1|_{L^2}|A_\gamma^{3/2}\phi_1|_{L^2}.
\end{aligned} \tag{3.50}
$$

Proof Let us set $(w, \psi) = (v_1, \phi_1) - (v_2, \phi_2)$. Then (w, ψ) satisfies

$$
\begin{cases}
\frac{dw}{dt} + \nu A_0 w + B_0^{N_1}(v_1, v_1) - B_0^{N_2}(v_2, v_2) = R_0^{N_1}(\epsilon A_\gamma\phi_1, \phi_1) - R_0^{N_2}(\epsilon A_\gamma\phi_2, \phi_2),\\
\frac{d\psi}{dt} + \epsilon A_\gamma\psi + B_1^{N_1}(v_1, \phi_1) - B_1^{N_2}(v_2, \phi_2) + \alpha f_\gamma(\phi_1) - \alpha f_\gamma(\phi_2) = 0,\\
(w, \psi)(0) = (v_1^0, \phi_1^0) - (v_2^0, \phi_2^0).
\end{cases} \tag{3.51}
$$

Let us set

$$
\begin{aligned}
&K_1 = B_0^{N_1}(v_1, v_1) - B_0^{N_2}(v_2, v_2),\ K_2 = R_0^{N_1}(A_\gamma\phi_1, \phi_1) - R_0^{N_2}(A_\gamma\phi_2, \phi_2),\\
&K_3 = B_1^{N_1}(v_1, \phi_1) - B_1^{N_2}(v_2, \phi_2).
\end{aligned} \tag{3.52}
$$

We can easily check that (see [7])

$$
\begin{aligned}
|\langle K_1, A_0 w\rangle| \le \frac{\nu}{8}|A_0 w|_{L^2}^2 + CN_1^4\|w\|^2 + c|A_0 v_2|_{L^2}^2\|w\|^2\\
+ c|A_0 v_2|_{L^2}^2(\|w\|^2 + (N_1 - N_2)^2).
\end{aligned} \tag{3.53}
$$

We also note that

$$\begin{aligned}\langle K_2, A_0 w\rangle &= F_{N_1}\left(\|(v_1,\phi_1)\|_{\mathbb{V}}\right) b_1\left(A_0 w, \phi_1, \epsilon A_\gamma \psi\right)\\ &\quad + F_{N_2}\left(\|(v_2,\phi_2)\|_{\mathbb{V}}\right) b_1\left(A_0 w, \psi, \epsilon A_\gamma \phi_2\right)\\ &\quad + \left(F_{N_1}\left(\|(v_1,\phi_1)\|_{\mathbb{V}}\right) - F_{N_2}\left(\|(v_2,\phi_2)\|_{\mathbb{V}}\right)\right) b_1\left(A_0 w, \phi_1, \epsilon A_\gamma \phi_2\right)\\ &= K_2^1 + K_2^2 + K_2^3, \end{aligned} \tag{3.54}$$

$$\begin{aligned}\langle K_3, A_\gamma^2 \psi\rangle &= F_{N_1}\left(\|(v_1,\phi_1)\|_{\mathbb{V}}\right) b_1\left(w, \phi_1, \epsilon A_\gamma^2 \psi\right) + F_{N_2}\left(\|(v_2,\phi_2)\|_{\mathbb{V}}\right) b_1\left(v_2, \psi, \epsilon A_\gamma^2 \psi\right)\\ &\quad + \left(F_{N_1}\left(\|(v_1,\phi_1)\|_{\mathbb{V}}\right) - F_{N_2}\left(\|(v_2,\phi_2)\|_{\mathbb{V}}\right)\right) b_1\left(v_2, \phi_1, \epsilon A_\gamma^2 \psi\right)\\ &= K_3^1 + K_3^2 + K_3^3. \end{aligned} \tag{3.55}$$

From (2.13) to (2.15), we have

$$\begin{aligned} |b_1(A_0 w, \phi_1, \epsilon A_\gamma \psi)| &\le c|A_0 w|_{L^2}|A_\gamma \phi_1|_{L^2}|A_\gamma \psi|_{L^2}^{1/2}|A_\gamma^{3/2}\psi|_{L^2}^{1/2},\\ |b_1(A_0 w, \psi, \epsilon A_\gamma \phi_2)| &\le c|A_0 w|_{L^2}|A_\gamma \phi_2|_{L^2}|A_\gamma^{3/2}\psi|_{L^2}^{1/2}|A_\gamma \psi|_{L^2}^{1/2},\\ |b_1(A_0 w, \phi_1, \epsilon A_\gamma \phi_2)| &\le c|A_0 w|_{L^2}|A_\gamma \phi_2|_{L^2}|A_\gamma \phi_1|_{L^2}^{1/2}|A_\gamma^{3/2}\phi_1|_{L^2}^{1/2}. \end{aligned} \tag{3.56}$$

It follows from (2.28) and (3.56) that

$$\begin{aligned} |K_2^1| &= F_{N_1}(\|(v_1,\phi_1)\|_{\mathbb{V}})|b_1(A_0 w, \phi_1, \epsilon A_\gamma \psi)| \le cN_1|A_0 w|_{L^2}|A_\gamma \psi|_{L^2}^{1/2}|A_\gamma^{3/2}\psi|_{L^2}^{1/2}\\ &\le \frac{\nu}{8}|A_0 w|_{L^2}^2 + \frac{\epsilon}{8}|A_\gamma^{3/2}\psi|_{L^2}^2 + cN_1^4|A_\gamma \psi|_{L^2}^2, \end{aligned} \tag{3.57}$$

$$\begin{aligned} |K_2^2| &= F_{N_2}(\|(v_2,\phi_2)\|_{\mathbb{V}})|b_1(A_0 w, \psi, \epsilon A_\gamma \phi_2)| \le cN_2|A_0 w|_{L^2}|A_\gamma \psi|_{L^2}^{1/2}|A_\gamma^{3/2}\psi|_{L^2}^{1/2}\\ &\le \frac{\nu}{8}|A_0 w|_{L^2}^2 + \frac{\epsilon}{8}|A_\gamma^{3/2}\psi|_{L^2}^2 + cN_2^4|A_\gamma \psi|_{L^2}^2, \end{aligned} \tag{3.58}$$

$$\begin{aligned} |K_2^3| &= |(F_{N_1}(\|(v_1,\phi_1)\|_{\mathbb{V}}) - F_{N_2}(\|(v_2,\phi_2)\|_{\mathbb{V}}))||b_1(A_0 w, \phi_1, \epsilon A_\gamma \phi_2)|\\ &\le c|A_0 w|_{L^2}|A_\gamma \phi_2|_{L^2}|A_\gamma \phi_1|_{L^2}^{1/2}|A_\gamma^{3/2}\phi_1|_{L^2}^{1/2}\left(\frac{|N_1 - N_2|}{\|(v_2,\phi_2)\|_{\mathbb{V}}} + \frac{\|(w,\psi)\|_{\mathbb{V}}}{\|(v_2,\phi_2)\|_{\mathbb{V}}}\right)\\ &\le c|A_0 w|_{L^2}|A_\gamma \phi_1|_{L^2}^{1/2}|A_\gamma^{3/2}\phi_1|_{L^2}^{1/2}(|N_1 - N_2| + \|(w,\psi)\|_{\mathbb{V}})\\ &\le \frac{\nu}{8}|A_0 w|_{L^2}^2 + c|A_\gamma \phi_1|_{L^2}|A_\gamma^{3/2}\phi_1|_{L^2}(|N_1 - N_2|^2 + \|(w,\psi)\|_{\mathbb{V}}^2). \end{aligned} \tag{3.59}$$

Similarly, we can check that

$$\begin{aligned} |b_1(w, \phi_1, A_\gamma^2 \psi)| &= |\langle A_\gamma^{1/2} B_1(w,\phi_1), A_\gamma^{3/2}\psi\rangle| \le c\|w\|^{1/2}|A_0 w|_{L^2}^{1/2}||A_\gamma \phi_1|_{L^2}|A_\gamma^{3/2}\psi|_{L^2},\\ |b_1(v_2, \psi, A_\gamma^2 \psi)| &= |\langle A_\gamma^{1/2} B_1(v_2,\psi), A_\gamma^{3/2}\psi\rangle| \le c\|v_2\||A_\gamma \psi|_{L^2}^{1/2}||A_\gamma^{3/2}\psi|_{L^2}^{3/2},\\ |b_1(v_2, \phi_1, A_\gamma^2 \psi)| &= |\langle A_\gamma^{1/2} B_1(v_2,\phi_1), A_\gamma^{3/2}\psi\rangle| \le c\|v_2\||A_\gamma \phi_1|_{L^2}^{1/2}||A_\gamma^{3/2}\phi_1|_{L^2}^{1/2}|A_\gamma^{3/2}\psi|_{L^2}. \end{aligned} \tag{3.60}$$

It follows from (2.28) and (3.60) that

$$\begin{aligned} |K_3^1| &= F_{N_1}(\|(v_1,\phi_1)\|_{\mathbb{V}})|b_1(w, \phi_1, \epsilon A_\gamma^2 \psi)| \le cN_1\|w\|^{1/2}|A_0 w|_{L^2}^{1/2}||A_\gamma^{3/2}\psi|_{L^2}\\ &\le \frac{\nu}{8}|A_0 w|_{L^2}^2 + \frac{\epsilon}{8}|A_\gamma^{3/2}\psi|_{L^2}^2 + cN_1^4\|w\|^2, \end{aligned} \tag{3.61}$$

$$\begin{aligned} |K_3^2| &= F_{N_2}(\|(v_2,\phi_2)\|_{\mathbb{V}})|b_1(v_2, \psi, \epsilon A_\gamma^2 \psi)| \le cN_2|A_\gamma \psi|_{L^2}^{1/2}||A_\gamma^{3/2}\psi|_{L^2}^{3/2}\\ &\le \frac{\epsilon}{8}|A_\gamma^{3/2}\psi|_{L^2}^2 + cN_2^4|A_\gamma \psi|_{L^2}^2, \end{aligned} \tag{3.62}$$

$$\begin{aligned}|K_3^3| &= |(F_{N_1}(\|(v_1,\phi_1)\|_{\mathbb{V}}) - F_{N_2}(\|(v_2,\phi_2)\|_{\mathbb{V}}))||b_1(v_2,\phi_1,\epsilon A_\gamma^2\psi)|\\ &\le c|A_\gamma\phi_1|_{L^2}^{1/2}||A_\gamma^{3/2}\phi_1|_{L^2}^{1/2}|A_\gamma^{3/2}\psi|_{L^2}(|N_1-N_2|+\|(w,\psi)\|_{\mathbb{V}})\\ &\le \frac{\epsilon}{8}|A_\gamma^{3/2}\psi|_{L^2}^2 + c|A_\gamma\phi_1|_{L^2}||A_\gamma^{3/2}\phi_1|_{L^2}(|N_1-N_2|^2+\|(w,\psi)\|_{\mathbb{V}}^2). \end{aligned} \tag{3.63}$$

Finally, we note that

$$\alpha|\langle f_\gamma(\phi_1)-f_\gamma(\phi_2),A_\gamma^2\psi\rangle| \le Q_1(|A_\gamma\phi_1|_{L^2},|A_\gamma\phi_2|_{L^2})|A_\gamma\psi|_{L^2}^2 + \frac{\epsilon}{8}|A_\gamma^{3/2}\psi|_{L^2}^2. \tag{3.64}$$

Let us set

$$\mathcal{Y} = \|(w,\psi)\|_{\mathbb{V}}^2.$$

Multiplying $(3.51)_1$ by A_0w and $(3.51)_2$ by $A_\gamma^2\psi$ and using (3.52)–(3.54), (3.57)–(3.59) and (3.61)–(3.64), we derive that

$$\frac{\mathrm{d}\mathcal{Y}}{\mathrm{d}t} + \nu|A_0w|_{L^2}^2 + \epsilon|A_\gamma^{3/2}\psi|_{L^2}^2 \le \mathcal{Y}_1\mathcal{Y} + \mathcal{Y}_2. \tag{3.65}$$

where

$$\begin{aligned}\mathcal{Y}_1 &= c(N_1^4+N_2^4+|A_\gamma\phi_1|_{L^2}|A_\gamma^{3/2}\phi_1|_{L^2}) + Q_1(|A_\gamma\phi_1|_{L^2},|A_\gamma\phi_2|_{L^2}),\\ \mathcal{Y}_2 &= c(N_1-N_2)^2|A_\gamma\phi_1|_{L^2}|A_\gamma^{3/2}\phi_1|_{L^2} \equiv (N_1-N_2)^2\mathcal{Y}_3.\end{aligned} \tag{3.66}$$

It follows from the Gronwall lemma that

$$\mathcal{Y}(t) \le \left(\mathcal{Y}(0) + (N_1-N_2)^2\int_0^t \mathcal{Y}_3(s)\mathrm{d}s\right)\exp\left(\int_0^t \mathcal{Y}_1(s)\mathrm{d}s\right), \tag{3.67}$$

and (3.49) follows. □

As a consequence of (3.49), we have a continuous dependence on the initial value and N. More precisely, if we denote by $(v_N,\phi_N)(\cdot,(v^0,\phi^0))$ the solution to (2.23) corresponding to the parameter N and the initial value (v^0,ϕ^0), then the following result holds true.

Corollary 3.1 *We assume that* $T>0$ *and* $g\in L^2(0,T;H_1)$. *Then for any* $(v^0,\phi^0)\in\mathbb{V}$ *and* $N>0$, *we have*

$$(v_M,\phi_M)(\cdot,(w^0,\psi^0)) \longrightarrow (v_N,\phi_N)(\cdot,(v^0,\phi^0)) \text{ in } \mathcal{C}(0,T;\mathbb{V})\cap L^2(0,T;D(A_0)\times D(A_\gamma^{3/2})) \tag{3.68}$$

as $(M,(w^0,\psi^0)) \longrightarrow (N,(v^0,\phi^0))$ *in* $\Re^+\times\mathbb{V}$.

Proof It follows from (3.49). □

4 Existence of global attractor in $\mathbb{V}$ of the GMACNSE

In this section, we assume that $N>0$ and $g\in H_1$ are fixed and we denote by $(v,\phi)\in\mathbb{V}$ the unique strong solution to (2.23). If we set $S_N(t)(v^0,\phi^0)=(v,\phi)(t)$, then it follows from Theorems 1, 2 and 3 that $\{S_N(t)\}_{t\ge0}$ is a $\mathcal{C}^0$ semigroup in $\mathbb{V}$.

Below we recall from [27] a lemma belonging to the family of Gronwall's type lemmas which we shall use in the sequel.

Lemma 2 *We assume that for some $k > 0$, $\tau \in \Re$, we have*

$$y'(s) + ky(s) \leq h(s) \text{ for all } s > \tau,$$

where the functions y, y', h are assumed to be locally integrable and y, h nonnegative on the interval $t < s < t + r$, for some $t \geq \tau$. Then

$$y(t+r) \leq \frac{2}{r} \mathrm{e}^{-k\frac{r}{2}} \int_t^{t+r} y(s) \mathrm{d}s + \mathrm{e}^{-k(t+r)} \int_t^{t+r} \mathrm{e}^{ks} h(s) \mathrm{d}s. \tag{4.1}$$

Proof See [27]. □

4.1 Absorbing set in $\mathbb{Y}$

As in [17], we can check that

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + \kappa \mathcal{E}(t) = \wedge_1(t), \tag{4.2}$$

where

$$\mathcal{E}(t) = |(v, \phi)(t)|_{\mathbb{Y}}^2 + 2\alpha \left(F_\gamma(\phi(t)), 1\right)_{L^2} + C_e, \tag{4.3}$$

and

$$\begin{aligned} \wedge_1(t) = & -2\nu \|v\|^2 + \kappa |v|_{L^2}^2 - 2|\mu|_{L^2}^2 - (2-\kappa)\epsilon(|\nabla\phi)|_{L^2}^2 + \left(\gamma |\phi(t)|_{L^2}^2\right) \\ & + 2\alpha \left[\kappa (F_\gamma(\phi) - f_\gamma(\phi)\phi, 1)_{L^2} - (1-\kappa)(f_\gamma(\phi)\phi, 1)_{L^2}\right] \\ & + 2\langle v, g\rangle + \kappa |\phi(t)|_{L^2}^2 + 2\kappa\alpha C_{F_\gamma} |\mathcal{M}|. \end{aligned} \tag{4.4}$$

From (2.7), we have

$$\begin{aligned} c_* |f_\gamma(y)|(1+|y|) &\leq 2 f_\gamma(y) y + c_f \left(1 + \alpha^{-1}\epsilon\right), \\ F_\gamma(y) - f_\gamma(y) y &\leq c'_f \left(1 + \alpha^{-1}\epsilon\right) |y|^2 + c''_f, \end{aligned} \tag{4.5}$$

for any $y \in \Re$, where c_f, c_*, c'_f and c''_f are positive, sufficiently large constants that depend only on f.

From [17], we also note that

$$\begin{aligned} \wedge_1(t) \leq & -(\nu - \kappa C_m |\mathcal{M}|) \|v(t)\|^2 - 2|\mu(t)|_{L^2}^2 - (2-\kappa)\epsilon |\nabla\phi(t)|_{L^2}^2 \\ & - \left(2 - \kappa \left(1 + 2c'_f(\alpha + \epsilon)\right)(\epsilon\gamma)^{-1}\right) \epsilon\gamma |\phi(t)|_{L^2}^2 \\ & - c_*\alpha(1-\kappa) \left(|f_\gamma(\phi(t))|, 1 + |\phi(t)|\right)_{L^2} + 2\langle v, g\rangle + c_1, \end{aligned} \tag{4.6}$$

where C_m depends on the shape of $\mathcal{M}$, but not its size and c_1 is given by

$$c_1 = 2\kappa\alpha C_{F_\gamma} |\mathcal{M}| + 2\alpha c''_f |\mathcal{M}| + c_f(\alpha + \epsilon)(1-\kappa)|\mathcal{M}|. \tag{4.7}$$

Let us choose $\kappa \in (0, 1)$ as

$$\kappa = \min \left\{ \nu (2C_m |\mathcal{M}|)^{-1}, \left(1 + 2c'_f(\alpha + \epsilon)(\epsilon\gamma)^{-1}\right)^{-1} \right\}. \tag{4.8}$$

From now on, c_i will denote a positive constant independent of the initial data and on time. Let us set

$$2\alpha_1 = \nu - \kappa C_m |\mathcal{M}|, \;\; 2\alpha_2 = \min\left(2 - \kappa, \left(2 - \kappa\left(1 + 2c'_f(\alpha + \epsilon)\right)(\epsilon\gamma)^{-1}\right)\right). \tag{4.9}$$

It follows from (4.3)–(4.8) that

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + \kappa\mathcal{E}(t) + \alpha_1\|v(t)\|^2 + \alpha_2|\nabla\phi(t)|_{L^2}^2 + \epsilon\gamma|\phi(t)|_{L^2}^2 + 2|\mu(t)|_{L^2}^2 + c_3(|f_\gamma(\phi(t)|, 1 + |\phi(t)|)_{L^2} \le c\|g\|_{V_1^*}^2 + c_1, \tag{4.10}$$

which gives

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + \kappa\mathcal{E}(t) + \alpha_1\|v(t)\|^2 + \alpha_2\|\phi(t)\|^2 + 2|\mu(t)|_{L^2}^2 + c_3(|f_\gamma(\phi(t)|, 1 + |\phi(t)|)_{L^2} \le c\|g\|_{V_1^*}^2 + c_1. \tag{4.11}$$

It follows from (4.11) that

$$\mathcal{E}(t) \le \mathcal{E}(0)\mathrm{e}^{-\kappa t} + c\|g_0\|_{V_1^*}^2 + c_1 \le Q_0(|(v^0, \phi^0)|_{\mathbb{Y}}^2)\mathrm{e}^{-\kappa t} + c\|g_0\|_{V_1^*}^2 + c_1, \tag{4.12}$$

$$\mathcal{E}(t) + \int_t^{t+1} \left(\alpha_1\|v(s)\|^2 + \alpha_2|\nabla\phi(s)|_{L^2}^2\right) \mathrm{d}s + \int_t^{t+1} \left[|\nabla\mu(s)|_{L^2}^2 + c_3(|f(\phi(s))|, 1 + |\phi(s)|)_{L^2}\right] \mathrm{d}s \le Q_0(|(v, \phi)(t)|_{\mathbb{Y}}^2)\mathrm{e}^{-\kappa t} + c_1. \tag{4.13}$$

From (2.26), we derive that

$$|(v, \phi)(t)|_{\mathbb{Y}}^2 \le Q_0\left(|(v^0, \phi^0)|_{\mathbb{Y}}^2\right) \mathrm{e}^{-\kappa t} + c\|g\|_{V_1^*}^2 + c_1. \tag{4.14}$$

We conclude that $S_N(t)$ has an absorbing set $\mathcal{B}_{\mathbb{Y}}$ in $\mathbb{Y}$ given by

$$\mathcal{B}_{\mathbb{Y}} = \left\{(v, \phi) \in \mathbb{Y},\ |(v, \phi)|_{\mathbb{Y}}^2 \le 1 + c\|g\|_{V_1^*}^2 + c_1\right\}. \tag{4.15}$$

4.2 Absorbing set in $\mathbb{V}$

Hereafter, we denote by $C_N \equiv C(N) > 0$ a monotone non-decreasing function of the parameter N. From [7], we have

$$|b_0^N(v, v, A_0 v)| \le \frac{\nu}{8}|A_0 v|_{L^2}^2 + C_N|v|_{L^2}^2. \tag{4.16}$$

We also note that

$$\begin{aligned}|\langle R_0(\epsilon A_\gamma\phi, \phi), A_0 v\rangle| &= |b_1(A_0 v, \phi, \epsilon A_\gamma\phi)| \\ &\le c|A_0 v|_{L^2}\|\phi\|^{1/4}|A_\gamma\phi|_{L^2}^{3/4}|A_\gamma\phi|_{L^2}^{1/4}|A_\gamma^{3/2}\phi|_{L^2}^{3/4},\end{aligned} \tag{4.17}$$

from which we derive that

$$\begin{aligned}|\langle F_N(\|(v, \phi)\|_{\mathbb{V}})R_0(\epsilon A_\gamma\phi, \phi), A_0 v\rangle| &\le cN|A_0 v|_{L^2}|A_\gamma^{3/2}\phi|_{L^2}^{3/4}\|\phi\|^{1/4} \\ &\le \tfrac{\nu}{8}|A_0 v|_{L^2}^2 + \tfrac{\epsilon}{8}|A_\gamma^{3/2}\phi|_{L^2}^2 + C_N\|\phi\|^2.\end{aligned} \tag{4.18}$$

Similarly, we have

$$\begin{aligned}|b_1(v, \phi, A_\gamma^2\phi)| &= |\langle A_\gamma^{1/2}B_1(v, \phi), A_\gamma^{3/2}\phi\rangle| \\ &\le c\|v\|^{1/4}|A_0 v|_{L^2}^{3/4}\|\phi\|^{1/4}|A_\gamma\phi|_{L^2}^{3/4}|A_\gamma^{3/2}\phi|_{L^2} \\ &\quad + c|v|_{L^2}^{1/4}\|v\|^{3/4}|A_\gamma\phi|_{L^2}^{1/4}|A_\gamma^{3/2}\phi|_{L^2}^{3/4}|A_\gamma^{3/2}\phi|_{L^2},\end{aligned} \tag{4.19}$$

which gives

$$
\begin{aligned}
|\langle B_1^N(v,\phi), A_\gamma^2\phi\rangle| &= F_N(\|(v,\phi)\|_{\mathbb{V}})|\langle A_\gamma^{1/2}B_1(v,\phi), A_\gamma^{3/2}\phi\rangle| \\
&\le cN|A_0 v|_{L^2}^{3/4}\|\phi\|^{1/4}|A_\gamma^{3/2}\phi|_{L^2} + cN|v|_{L^2}^{1/4}|A_\gamma^{3/2}\phi|_{L^2}^{7/4} \\
&\le \frac{\nu}{8}|A_0 v|_{L^2}^2 + \frac{\epsilon}{8}|A_\gamma^{3/2}\phi|_{L^2}^2 + C_N\left(\|\phi\|^2 + |v|_{L^2}^2\right).
\end{aligned}
\tag{4.20}
$$

Finally from (2.7)–(2.8), we derive that

$$
\begin{aligned}
|\langle f_\gamma(\phi), A_\gamma^2\phi\rangle| &= |\langle A_\gamma^{1/2} f_\gamma(\phi), A_\gamma^{3/2}\phi\rangle| \le |f'_\gamma(\phi)|_{L^2}|A_\gamma^{1/2}\phi|_{L^\infty}|A_\gamma^{3/2}\phi|_{L^2} \\
&\le c(1+\|\phi\|^k)|A_\gamma\phi|_{L^2}^{1/2}|A_\gamma^{3/2}\phi|_{L^2}^{3/2} \le c(1+\|\phi\|^k)\|\phi\|^{1/4}|A_\gamma^{3/2}\phi|_{L^2}^{7/4} \\
&\le \frac{\epsilon}{8}|A_\gamma^{3/2}\phi|_{L^2}^2 + Q_2(\|\phi\|^2).
\end{aligned}
\tag{4.21}
$$

Let us set

$$
\mathcal{Y} = \|(v,\phi)\|_{\mathbb{V}}^2.
$$

Multiplying $(2.23)_1$ by $A_0 v$ and $(2.23)_2$ by $A_\gamma^2\phi$ and using (4.16), (4.18), (4.20)–(4.21), we derive that

$$
\begin{aligned}
\frac{d\mathcal{Y}}{dt} + \nu|A_0 v|_{L^2}^2 + \epsilon|A_\gamma^{3/2}\phi|_{L^2}^2 &\le C_N(\|\phi\|^2 + |v|_{L^2}^2) + Q_2(\|\phi\|^2) + c|g|_{L^2}^2 \\
&\le Q_2((|(v,\phi)|_{\mathbb{Y}}^2) + c|g|_{L^2}^2,
\end{aligned}
\tag{4.22}
$$

and

$$
\frac{d\mathcal{Y}}{dt} + \zeta\mathcal{Y} \le Q_2((|(v,\phi)|_{\mathbb{Y}}^2) + c|g|_{L^2}^2,
\tag{4.23}
$$

where $\zeta = \min(\lambda\nu, \lambda\epsilon)$.

From Lemma 2, we derive that

$$
\mathcal{Y}(t+1) \le 2e^{-\frac{\zeta}{2}}\int_t^{t+1}\mathcal{Y}(s)ds + e^{-\zeta(t+1)}\int_t^{t+1} e^{\zeta s}(Q_2(|(v,\phi)|_{\mathbb{Y}}^2) + c|g|_{L^2}^2)ds.
\tag{4.24}
$$

But from (3.21) and (4.11)–(4.12), we have

$$
\begin{aligned}
\int_t^{t+1}\mathcal{Y}(s)ds &\le \mathcal{E}(t) + \int_t^{t+1}(c|g|_{V_1^*}^2 + c_1 + Q_2(\|\phi\|^2))ds \\
&\le Q_0(|(v^0,\phi^0)|_{\mathbb{Y}}^2)e^{-\kappa t} + c|g|_{V_1^*}^2 + 2c_1 \\
&\quad + Q_2\left[Q_0(|(v^0,\phi^0)|_{\mathbb{Y}}^2)e^{-\kappa t} + c|g|_{V_1^*}^2 + c_1\right],
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
&e^{-\zeta(t+1)}\int_t^{t+1} e^{\zeta s}(Q_2(|(v,\phi)(s)|_{\mathbb{Y}}^2) + c|g|_{L^2}^2)ds \\
&\quad\le Q_2\left[Q_0(|(v^0,\phi^0)|_{\mathbb{Y}}^2)e^{-\kappa t} + c|g|_{V_1^*}^2 + c_1\right] + c|g|_{L^2}^2.
\end{aligned}
\tag{4.26}
$$

Let

$$
R_1 = 2e^{-\frac{\zeta}{2}}\left(c|g|_{V_1^*}^2 + c_1 + Q_2\left(c|g|_{V_1^*}^2 + c_1\right)\right) + c|g|_{L^2}^2.
$$

From (4.24)–(4.26), we derive that the ball

$$
\mathcal{B}_{\mathbb{V}}^N = \left\{(v,\phi)\in\mathbb{V},\ \|(v,\phi)\|_{\mathbb{V}}^2 \le R_1\right\}
\tag{4.27}
$$

is an absorbing set in $\mathbb{V}$.

4.3 Asymptotic compactness in $\mathbb{V}$

In this part, we prove the asymptotic compactness of the semigroup $S_N(t)$. We will prove the following flattening property (see [22,23,28]) of the semigroup $S_N(t)$.

Proposition 1 (Flattening property). *For any bounded set B of $\mathbb{V}$ and any $\epsilon_1 > 0$, there exists $T_{\epsilon_1}(B) > 0$ and a finite dimensional subspace $\mathbb{V}_{\epsilon_1}$ of $\mathbb{V}$ such that $\{\mathcal{P}_{\epsilon_1} S_N(t)B,\ t \geq T_{\epsilon_1}(B)\}$ is bounded and*

$$\|(I-\mathcal{P}_{\epsilon_1})S_N(t)(v^0,\phi^0)\|_{\mathbb{V}} \leq \epsilon_1, \quad \forall t \geq T_{\epsilon_1}(B),\ (v^0,\phi^0) \in B, \tag{4.28}$$

where $\mathcal{P}_{\epsilon_1} : \mathbb{V} \to \mathbb{V}_{\epsilon_1}$ is the projection operator.

Proof Without loss of generality, we can restrict ourselves to $B = \mathcal{B}^N_{\mathbb{V}}$, the absorbing set of $S_N(t)$ in $\mathbb{V}$ given by (4.27). Let $\epsilon_1 > 0$. We will find an integer $N_{\epsilon_1} > 0$ such that the flattening property holds for the N_{ϵ_1}-dimensional subspace $\mathbb{V}_{\epsilon_1}$ of $\mathbb{V}$ spanned by the first eigenfunctions (e_i,ψ_i), $i = 1, 2, \ldots N_{\epsilon_1}$, where (e_i,ψ_i) are the eigenfunctions used in the proof of Theorem 2. Let us denote by λ_i^1, λ_i^2 the eigenvalues defined by

$$A_0 e_i = \lambda_i^1 e_i,\ A_\gamma \psi_i = \lambda_i^2 \psi_i,\ i = 1, 2, \ldots.$$

Let $\lambda = \min(\nu\lambda^1_{N_{\epsilon_1}}, \epsilon\lambda^2_{N_{\epsilon_1}})$.

Since B is a bounded absorbing set and $\|\mathcal{P}_{\epsilon_1}(w,\psi)\|_{\mathbb{V}} \leq \|(w,\psi)\|_{\mathbb{V}},\ \forall (w,\psi) \in \mathbb{V}$, there exists $T_{\epsilon_1}(B) > 0$ such that the set $\{\mathcal{P}_{\epsilon_1} S_N(t)B,\ t \geq T_{\epsilon_1}(B)\}$ is bounded. Let us now prove (4.28).

Let $(v^0,\phi^0) \in \mathcal{B}^N_{\mathbb{V}}$, $(v,\phi)(t) = S_N(t)(v^0,\phi^0)$ and $\mathcal{Y} = \|(v,\phi)\|^2_{\mathbb{V}}$. For t large enough, we know that $(v,\phi)(t)$ is uniformly bounded in $\mathbb{V}$.

From (4.22) we derive that for any $\alpha_2 > 0$, we have

$$\mathrm{e}^{-\alpha_2 t}\int_0^t \mathrm{e}^{\alpha_2 s}\left(\nu|A_0 v|^2_{L^2} + \epsilon|A_\gamma^{3/2}\phi|^2_{L^2}\right)\mathrm{d}s \leq C < \infty. \tag{4.29}$$

Let $(w,\psi) = (I - P_{\epsilon_1})S_N(t)(v^0,\phi^0)$. Multiplying $(2.23)_1$ by $A_0 w$ and $(2.23)_2$ by $A_\gamma^2\psi$, we can easily check that (w,ψ) satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|w\|^2 + |A_\gamma\psi|^2_{L^2}) + 2\nu|A_0 w|^2_{L^2} + 2\epsilon|A_\gamma^{3/2}\psi|^2_{L^2} + b_0^N(v,v,A_0 w)\\ + b_1^N(v,\phi,A_\gamma^2\psi) + \alpha\langle f_\gamma(\phi), A_\gamma^2\psi\rangle = b_1^N(A_0 w,\phi,\epsilon A_\gamma\phi) + \langle g, \mathcal{A}_0 w\rangle.\end{aligned} \tag{4.30}$$

We note that (for t large enough)

$$|b_0^N(v,v,A_0 w)| \leq \frac{\nu}{8}|A_0 w|^2_{L^2} + C_N|A_0 v|_{L^2}, \tag{4.31}$$

$$|b_1^N(A_0 w,\phi,\epsilon A_\gamma\phi)| \leq \frac{\nu}{8}|A_0 w|^2_{L^2} + C_N|A_\gamma^{3/2}\phi|_{L^2}, \tag{4.32}$$

$$\begin{aligned}|b_1^N(v,\phi,A_\gamma^2\psi)| &\leq cN|A_0 v|^{1/2}_{L^2}|A_\gamma^{3/2}\psi|_{L^2} + cN|A_\gamma^{3/2}\phi|^{1/2}_{L^2}|A_\gamma^{3/2}\psi|_{L^2}\\ &\leq \frac{\epsilon}{8}|A_\gamma^{3/2}\psi|^2_{L^2} + C_N\left(|A_0 v|_{L^2} + |A_\gamma^{3/2}\phi|_{L^2}\right),\end{aligned} \tag{4.33}$$

$$\alpha|\langle f_\gamma(\phi), A_\gamma^2\psi\rangle| \leq \frac{\epsilon}{8}|A_\gamma^{3/2}\psi|^2_{L^2} + C. \tag{4.34}$$

Now let

$$\mathcal{Y}_1 = \|w\|^2 + |A_\gamma\psi|^2_{L^2}.$$

It follows from (4.31)–(4.34) that

$$\frac{\mathrm{d}\mathcal{Y}_1}{\mathrm{d}t} + \nu|A_0 w|_{L^2}^2 + \epsilon|A_\gamma^{3/2}\psi|_{L^2}^2 \leq C_N\left(|A_0 v|_{L^2} + |A_\gamma^{3/2}\phi|_{L^2}\right) + C + C|g|_{L^2}^2, \quad (4.35)$$

and

$$\frac{\mathrm{d}\mathcal{Y}_1}{\mathrm{d}t} + \lambda\mathcal{Y}_1 \leq C_N\left(|A_0 v|_{L^2} + |A_\gamma^{3/2}\phi|_{L^2}\right) + C + C|g|_{L^2}^2, \quad (4.36)$$

which gives

$$\begin{aligned}\mathcal{Y}_1(t) &\leq \mathcal{Y}_1(0)\mathrm{e}^{-\lambda t} + \frac{C}{\lambda}|g|_{L^2}^2 + \frac{C}{\lambda} + c\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\left(\nu|A_0 v|_{L^2} + \epsilon|A_\gamma^{3/2}\phi|_{L^2}\right)\mathrm{d}s \\ &\leq \mathcal{Y}_1(0)\mathrm{e}^{-\lambda t} + \frac{C}{\lambda}|g|_{L^2}^2 + \frac{C}{\lambda} + \frac{C}{\sqrt{\lambda}}.\end{aligned} \quad (4.37)$$

Note that we use the fact

$$\begin{aligned}&\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\left(\nu|A_0 v|_{L^2} + \epsilon|A_\gamma^{3/2}\phi|_{L^2}\right)\mathrm{d}s \\ &\leq \left(\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\mathrm{d}s\right)^{1/2}\left(\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\left(\nu|A_0 v|_{L^2}^2 + \epsilon|A_\gamma^{3/2}\phi|_{L^2}^2\right)\mathrm{d}s\right)^{1/2} \leq \frac{C}{\sqrt{\lambda}},\end{aligned} \quad (4.38)$$

since from (4.29), we also have

$$\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\left(\nu|A_0 v|_{L^2}^2 + \epsilon|A_\gamma^{3/2}\phi|_{L^2}^2\right)\mathrm{d}s \leq C. \quad (4.39)$$

Therefore, for N_{ϵ_1} and t large enough, we derive that $\mathcal{Y}_1(t) \leq \epsilon_1$, which proves the flattening property of $S_N(t)$. □

Theorem 4 *If $g \in H_1$, then the GMACNSE (2.6) has a global attractor $\mathcal{A}_N$ in $\mathbb{V}$ for each $N > 0$. Moreover, the set-valued mapping $N \mapsto \mathcal{A}_N$ is upper semicontinuous, i.e.,*

$$\mathrm{dist}_{\mathbb{V}}(\mathcal{A}_N, \mathcal{A}_M) \to 0 \text{ as } M \to N, \quad (4.40)$$

where $\mathrm{dist}_{\mathbb{V}}$ *is the Hausdorff semidistance on* $\mathbb{V}$.

Proof The existence of the global attractor follows from the existence of the absorbing set in $\mathbb{V}$ as well as the flattening property proved above. The upper semicontinuity (4.40) is proved as in [7]. Note that for each $N > 0$, we have $\mathcal{A}_N \subset \mathcal{B}_{\mathbb{V}}^N$ and from (4.27), we have $\mathcal{B}_{\mathbb{V}}^{N_1} \subset \mathcal{B}_{\mathbb{V}}^{N_2}$ for $N_1 \leq N_2$. □

5 Convergence to weak solution of the AC–NS systems

We suppose that $g \in L^2(0, T; H_1)$ for all $T > 0$. Let $(v_N, \phi_N)(t)$ be a weak solution to (2.23) with initial value $(v_N^0, \phi_N^0) \in \mathbb{Y}$, where $(v_N^0, \phi_N^0) \to (v^0, \phi^0)$ weakly in $\mathbb{Y}$ as $N \to +\infty$.

If we set

$$\mathcal{E} = |(v_N, \phi_N)|_{\mathbb{Y}}^2 + 2\langle F_\gamma(\phi_N), 1\rangle + \alpha_0,$$

where α_0 is given in (2.25). Then

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + \nu\|v_N\|^2 + 2|\mu_N|_{L^2}^2 = 2\langle g, v\rangle, \quad (5.1)$$

which gives

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + \nu\|v_N\|^2 + 2|\mu_N|^2_{L^2} \leq c|g|^2_{V_1^*}. \tag{5.2}$$

It follows from (5.2) that (v_N, ϕ_N) is bounded in $L^\infty(0, T; \mathbb{Y}) \cap L^2(0, T; \mathbb{V})$, $\forall T > 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}(v_N, \phi_N)$ is bounded in $L^{4/3}(0, T; V_1^*) \times L^2(0, T; H_2)$. Therefore by a diagonal argument, there exists a subsequence of (v_N, ϕ_N) still denoted (v_N, ϕ_N) such that

$$\begin{aligned} &(v_N, \phi_N) \to (v, \phi) \text{ weakly-star in } L^\infty(0, T; \mathbb{Y}),\\ &(v_N, \phi_N) \to (v, \phi) \text{ weakly in } L^2(0, T; \mathbb{V}),\\ &(v_N, \phi_N) \to (v, \phi) \text{ strongly in } L^2(0, T; \mathbb{Y}), \end{aligned} \tag{5.3}$$

where $(v, \phi) \in L^\infty(0, T; \mathbb{Y}) \cap L^2(0, T; \mathbb{V})$. As in [7], we will prove that (v, ϕ) is a weak solution to the AC–NS (5.17) below. Note that from $\mu_N = \epsilon A_\gamma \phi_N) + \alpha f_\gamma(\phi_N)$, we derive that

$$\int_0^t |A_\gamma \phi_N|^2_{L^2} \mathrm{d}s \leq C. \tag{5.4}$$

Lemma 3 *We have*

$$\begin{aligned} F_N\left(\|v_N(s)\|\right) &\to 1 \text{ in } L^p(0, T; \Re),\\ F_N\left(\|\left(v_N, \phi_N\right)(s)\|\right) &\to 1 \text{ in } L^p(0, T; \Re), \end{aligned} \tag{5.5}$$

as $N \to +\infty$ *for each* $p > 1$.

Proof The proof of $(5.5)_1$ is given in [7], and that of $(5.5)_2$ is similar. □

Using Lemma 3, it is proved in [7] that as $N \to \infty$, we have

$$\int_0^t F_N(\|v_N(s)\|) b_0(v_N, v_N, w)\mathrm{d}s \to \int_0^t b_0(v, v, w)\mathrm{d}s, \quad \forall t \in [0, T],\ w \in D(A_0). \tag{5.6}$$

Let us now focuss on the convergence of the other nonlinear terms that appear in (3.18). We will restrict our attention to the term $R_0^N(\epsilon A_\gamma \phi_N, \phi_N)$ which is the strongest nonlinearity in (2.23). Our goal is to prove that as $N \to \infty$, we have

$$\begin{aligned} &\int_0^t \langle F_N(\|(v_N, \phi_N)(s)\|_{\mathbb{V}}) R_0(\epsilon A_\gamma \phi_N, \phi_N), w\rangle \mathrm{d}s\\ &\quad = \int_0^t F_N(\|(v_N, \phi_N)(s)\|_{\mathbb{V}}) b_1(w, \phi_N, \epsilon A_\gamma \phi_N)\mathrm{d}s\\ &\quad \to \int_0^t b_1(w, \phi, \epsilon A_\gamma \phi)\mathrm{d}s = \int_0^t \langle R_0(\epsilon A_\gamma \phi, \phi), w\rangle, \quad \forall t \in [0, T],\ w \in D(A_0). \end{aligned} \tag{5.7}$$

We proceed as in [7] and we set

$$F_N(s) = F_N(\|(v_N, \phi_N)(s)\|_{\mathbb{V}}),\ r_N(s) = \langle R_0(\epsilon A_\gamma \phi_N, \phi_N), w\rangle,\ r(s) = \langle R_0(\epsilon A_\gamma \phi, \phi), w\rangle.$$

We want to prove that

$$\int_0^t F_N(s) r_N(s)\mathrm{d}s \to \int_0^T r(s)\mathrm{d}s \text{ as } N \to \infty. \tag{5.8}$$

We note that

$$\int_0^t (F_N(s)r_N(s) - r(s))\mathrm{d}s = \int_0^t (F_N(s) - 1)r_N(s)\mathrm{d}s + \int_0^T (r_N(s) - r(s))\mathrm{d}s. \quad (5.9)$$

Note that

$$\begin{aligned} r_N(s) - r(s) &= b_1(w, \phi_N, \epsilon A_\gamma \phi_N) - b_1(w, \phi, \epsilon A_\gamma \phi) \\ &= b_1(w, \phi_N - \phi, \epsilon A_\gamma \phi_N) + b_1\left(w, \phi, \epsilon A_\gamma(\phi_N - \phi)\right) \equiv I_1 + I_2, \quad (5.10) \\ |I_1| &= |b_1(w, \phi_N - \phi, \epsilon A_\gamma \phi_N)| \le c\|w\|\|\phi_N - \phi\|^{1/2}|A_\gamma \phi_N|_{L^2}^{1/2}, \\ |I_2| &= |b_1\left(w, \phi, \epsilon A_\gamma(\phi^N - \phi)\right)| \le c\|w\||A_\gamma\phi|_{L^2}\|\phi_N - \phi\|^{1/2}|A_\gamma(\phi_N - \phi)|_{L^2}^{1/2}. \end{aligned} \quad (5.11)$$

It follows from (5.3) and (5.11) that

$$\int_0^T (r_N(s) - r(s))\mathrm{d}s \to \text{ as } N \to +\infty. \quad (5.12)$$

We also have

$$\left|\int_0^t (F_N(s) - 1)r_N(s)\mathrm{d}s\right|^2 \le \left(\int_0^T |F_N(s) - 1|^2\mathrm{d}s\right)\int_0^T |r_N(s)|^2\mathrm{d}s. \quad (5.13)$$

We note that

$$|r_N(s)| = |b_1(w, \phi_N, \epsilon A_\gamma \phi_N)| \le c\|w\||A_0 w|_{L^2}\|\phi_N\||A_\gamma \phi_N|_{L^2}, \quad (5.14)$$

which gives (see Lemma 1)

$$\begin{aligned} \left|\int_0^t (F_N(s) - 1)r_N(s)\mathrm{d}s\right|^2 &\le \int_0^T |F_N(s) - 1|^2\mathrm{d}s \int_0^T |r_N(s)|^2\mathrm{d}s \\ &\le c\left(\int_0^T |F_N(s) - 1|^2\mathrm{d}s\right)\int_0^T \|w\|^2|A_0 w|_{L^2}^2\|\phi_N\|^2|A_\gamma\phi_N|_{L^2}^2\mathrm{d}s \\ &\le c\left(\int_0^T |F_N(s) - 1|^2\mathrm{d}s\right)\|w\|^2|A_0 w|_{L^2}^2\int_0^T \|\phi_N\|^2|A_\gamma\phi_N|_{L^2}^2\mathrm{d}s \\ &\le C\|w\|^2|A_0 w|_{L^2}^2\int_0^T |F_N(s) - 1|^2\mathrm{d}s \to 0 \text{ as } N \to \infty. \end{aligned} \quad (5.15)$$

Similarly, we can check that

$$\int_0^t F_N(\|(v_N, \phi_N)(s)\|_{\mathbb{V}})b_1(v_N, \phi_N, \rho)\mathrm{d}s \to \int_0^t b_1(v, \phi, \rho)\mathrm{d}s, \quad \forall t \in [0, T], \ \rho \in D(A_\gamma). \quad (5.16)$$

This proves that the limit (v, ϕ) is a weak solution of the following three-dimensional AC–NS system

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} + \nu A_0 v + B_0(v, v) = R_0(\epsilon A_\gamma\phi, \phi) + g, \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}t} + \mu + B_1(v, \phi) = 0, \ \mu = \epsilon A_\gamma\phi + \alpha f_\gamma(\phi), \\ (v, \phi)(0) = (v^0, \phi^0). \end{cases} \quad (5.17)$$

5.1 Existence of bounded entire weak solutions of the AC–NS equations

Hereafter, we assume that the forcing $g \in H_1$. Following similar steps as in [7], we prove the existence of a bounded entire weak solution of the AC–NS equations.

Theorem 5 *There exists a bounded entire weak solution of the AC–NS equations* (5.17). *More precisely, there exists a bounded entire weak solutions of* (5.17) *with initial value* $(v^0, \phi^0) \in \mathcal{U}_0$, *where* $\mathcal{U}_0$ *is a subset of* $\mathbb{Y}$ *consisting of weak* $\mathbb{Y}$*-cluster points of a sequence in* $\mathcal{A}_N$.

Proof The proof is similar to that of Theorem 14 of [7]. Therefore, we omit the details and only give a sketch. We consider a sequence $(v_N^0, \phi_N^0) \in \mathbb{V}$ with $(v_N^0, \phi_N^0) \in \mathcal{A}_N$ for each N. Then $S_N(t)\mathcal{A}_N = \mathcal{A}_N$ for all $t \geq 0$. It follows that there exists an entire strong solution of the GMACNSE (2.23) $(w_N, \psi_N) : \Re \to \mathbb{V}$ with $(w_N, \psi_N)(0) = (v_N^0, \phi_N^0)$ and $(w_N, \psi_N)(t) \in \mathcal{A}_N$ for all $t \in \Re$ and each N. Note that $\mathcal{A}_N \subset \mathcal{B}_{\mathbb{Y}}$ for each N, where $\mathcal{B}_{\mathbb{Y}}$ is the absorbing set in $\mathbb{Y}$ given by (4.15). Since $\mathcal{B}_{\mathbb{Y}}$ is independent of N, it follows that the sequence (w_N, ψ_N) is bounded in $L^\infty(0, T; \mathbb{Y}) \cap L^2(0, T; \mathbb{V})$. Therefore, there exists a subsequence (still) denoted (w_N, ψ_N) which converges to a function $(w, \psi) \in L^\infty(0, T; \mathbb{Y}) \cap L^2(0, T; \mathbb{V})$ weak-star in $L^\infty(0, T; \mathbb{Y})$, weakly in $L^2(0, T; \mathbb{V})$ and strongly in $L^2(0, T; \mathbb{Y})$ for all $T > 0$. Moreover, $(w, \psi) \in \mathcal{B}_{\mathbb{Y}}$ by the weak-star lower semicontinuity of the norm in $L^\infty(0, T; \mathbb{Y})$. As in [7], we can extend this weak solution backward in time and obtain an entire weak solution (w, ψ) of the AC–NS system (5.17) with values in $\mathcal{B}_{\mathbb{Y}}$. □

Acknowledgements The author would like to thank the anonymous referees whose comments help to improve the contain of this article.

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