

Well-posedness and Gevrey analyticity of the generalized Keller–Segel system in critical Besov spaces

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Abstract In this paper, we study the Cauchy problem for the generalized Keller–Segel system with the cell diffusion being ruled by fractional diffusion:

$$\begin{cases} \partial_t u + \Lambda^\alpha u + \nabla \cdot (u \nabla \psi) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ -\Delta \psi = u & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases}$$

In the case $1 < \alpha \leq 2$, we prove local well-posedness for any initial data and global well-posedness for small initial data in critical Besov spaces $\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ with $1 \leq p < \infty$, $1 \leq q \leq \infty$, and analyticity of solutions for initial data $u_0 \in \dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ with $1 < p < \infty$, $1 \leq q \leq \infty$. Moreover the global existence and analyticity of solutions with small initial data in critical Besov spaces $\dot{B}_{\infty,1}^{-\alpha}(\mathbb{R}^n)$ is also established. In the limit case $\alpha = 1$, we prove global well-posedness for small initial data in critical Besov spaces $\dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ with $1 \leq p < \infty$ and $\dot{B}_{\infty,1}^{-1}(\mathbb{R}^n)$ and show analyticity of solutions for small initial data in $\dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ with $1 < p < \infty$ and $\dot{B}_{\infty,1}^{-1}(\mathbb{R}^n)$, respectively.

Keywords Generalized Keller–Segel system · Chemotaxis model · Well-posedness · Gevrey analyticity · Decay

Mathematics Subject Classification 35B65 · 35K15 · 35M11 · 92C17

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1 Introduction

In this paper, we are concerned with the nonlinear nonlocal evolution equations generalizing the well-known Keller–Segel model of chemotaxis:

$$\begin{cases} \partial_t u + \Lambda^\alpha u + \nabla \cdot (u \nabla \psi) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ -\Delta \psi = u & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$, u and ψ are two unknown functions which stand for the cell density and the concentration of the chemical attractant, respectively, and the anomalous (normal) diffusion is modeled by a fractional power of the Laplacian with $1 \leq \alpha \leq 2$. The positive operator $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ is defined by

$$\Lambda^\alpha f(x) := c(\alpha, n) P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} dy$$

and $c(\alpha, n)$ is a normalization constant. A simple alternative representation is given through the Fourier transform as $\Lambda^\alpha f = \mathcal{F}^{-1}[|\xi|^\alpha \mathcal{F} f(\xi)]$, where \mathcal{F} and \mathcal{F}^{-1} are the Fourier transform and the inverse Fourier transform, respectively.

Obviously, the choice $\alpha = 2$ in the system (1.1) corresponds to a simplified system of

$$\begin{cases} \partial_t u - \Delta u = -\nabla \cdot (u \nabla \psi) & \text{in } \mathbb{R}^n \times (0, \infty), \\ \partial_t \psi - \Delta \psi = u - \psi & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x), \quad \psi(x, 0) = \psi_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

The system (1.2) is a mathematical model of chemotaxis, which is formulated by Keller and Segel [35] in 1970, while it is also connected with astrophysical models of gravitational self-interaction of massive particles in a cloud or a nebula, see Biler et al. [6].

In biology, chemotaxis is the directed movement of an organism in response to ambient chemical gradients that are often segregated by the cells themselves. The system (1.2) describes the manner in which cellular slime molds aggregate owing to the motion of the cells, which move toward higher concentration of a chemical substance which they produce themselves. In those cases where the chemical products are attractive (and they are called chemoattractants), they lead to the phenomenon known as chemotactic collapse: the cells accumulate in small regions of space giving rise to high density configurations. This phenomenon exhibits that the system (1.2) admits finite time blowup solutions for large-enough initial data. It was actually conjectured by Childress and Percus [18] that in a two-dimensional domain $\Omega \subset \mathbb{R}^2$, there exists a threshold c_0 such that if the initial mass $m = \int_\Omega u_0(x) dx < c_0$, then the solution exists globally in time, while if $m = \int_\Omega u_0(x) dx > c_0$, then the solution blows up in finite time. For various simplified versions of the Keller–Segel system (1.2), the conjecture has been essentially verified, see [27, 28] for a comprehensive review of these aspects. Jager and Luckhaus [33] considered the system (1.2) with Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^2$ and showed that for sufficiently small $\frac{1}{|\Omega|} \int_\Omega u_0(x) dx$, there exists a unique smooth global positive solution, while for large $\frac{1}{|\Omega|} \int_\Omega u_0(x) dx$, there exists radial solutions which explode in finite time. Herrero and Velázquez [25, 26] studied the system (1.2) with no-flux boundary conditions on a disk and showed by the method of matched asymptotic expansion that there exists a nonnegative radial initial data (u_0, ψ_0) with $\int_\Omega u_0(x) dx > 8\pi$ such that the solution (u, ψ) corresponding to the initial data (u_0, ψ_0) blows up only at the origin in finite time and u has a Dirac delta-type singularity at the origin.

Biler [4], Gajewski and Zacharias [24], Nagai et al. [45] subsequently proved global existence of nonnegative solution under the condition $\int_{\Omega} u_0(x)dx < 4\pi$ and existence of radial solutions on a disk under the condition $\int_{\Omega} u_0(x)dx < 8\pi$. Moreover there exists a detailed description of the asymptotic behavior of solutions of (1.2) in the case $\int_{\Omega} u_0(x)dx < 8\pi$ to [15], in the limit case $\int_{\Omega} u_0(x)dx = 8\pi$ to [14] and in the radially symmetric case to [8, 9]. For more results related to this topic, we refer the reader to see [10, 21, 40, 41, 44, 53].

Since the chemical concentration ψ is determined by the Poisson equation, the second equation of (1.1) gives rise to the coefficient $\nabla\psi$ in the first equation of (1.1), when ψ is represented as the volume potential of u :

$$\psi(x, t) = (-\Delta)^{-1}u(x, t) = \begin{cases} \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{u(y, t)}{|x-y|^{n-2}} dy, & n \geq 3, \\ -\frac{1}{2\pi} \int_{\mathbb{R}^2} u(y, t) \log|x-y| dy, & n = 2, \end{cases}$$

where ω_n denotes the surface area of the unit sphere in \mathbb{R}^n , the system (1.1) amounts essentially to the following differential–integral Fokker–Planck system:

$$u = e^{-t\Lambda^\alpha} u_0 - \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla \cdot [u \nabla (-\Delta)^{-1}u] d\tau, \tag{1.3}$$

where $e^{-t\Lambda^\alpha} := \mathcal{F}^{-1}[e^{-t|\xi|^\alpha} \mathcal{F}]$. We may find the solution of (1.3) by using the contraction mapping argument for the mapping $u \mapsto \mathbb{F}(u)$ with

$$\mathbb{F}(u) := e^{-t\Lambda^\alpha} u_0 - \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla \cdot [u \nabla (-\Delta)^{-1}u] d\tau.$$

The invariant space for solving the integral equation (1.3) requires us to analyze the scaling invariance property of the system (1.1). Set

$$u_\lambda(x, t) := \lambda^\alpha u(\lambda x, \lambda^\alpha t), \quad \psi_\lambda(x, t) := \lambda^{\alpha-2} \psi(\lambda x, \lambda^\alpha t).$$

Then if u solves (1.1) with initial data u_0 (ψ can be determined by u), so does u_λ with initial data $u_{0\lambda}$ (ψ_λ can be determined by u_λ), where $u_{0\lambda}(x) := \lambda^\alpha u_0(\lambda x)$. In particular, the norm of $u_0 \in \dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ ($1 \leq p, q \leq \infty$) is scaling invariant under the above change of scale.

Notice that in the case of classical Brownian diffusion $\alpha = 2$, the solvability of the system (1.1) has been relatively well developed in various classes of functions and distributions, such as the Lebesgue space $L^1(\mathbb{R}^n) \cap L^{\frac{n}{2}}(\mathbb{R}^n)$ by Corrias et al. [19], the Sobolev space $L^1(\mathbb{R}^n) \cap W^{2,2}(\mathbb{R}^n)$ by Kozono and Sugiyama [36], the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ by Ogawa and Shimizu [46], the Besov space $\dot{B}_{1,2}^0(\mathbb{R}^2)$ by Ogawa and Shimizu [47], the Besov space $\dot{B}_{p,\infty}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ and Fourier–Herz space $\dot{B}_2^{-2}(\mathbb{R}^n)$ by Iwabuchi [30] and the pseudomeasure space $\mathcal{PM}^{n-2}(\mathbb{R}^n)$ by Biler et al. [5], for more results, see Lemarié-Rieusset [39] and the references therein. We mention here that, for the drift–diffusion system (1.2) of bipolar type, recently, the author of this paper, Liu and Cui [57], proved that small data global existence and large data local existence of solutions in critical Besov space $\dot{B}_{p,q}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $1 < p < 2n$ and $1 \leq q \leq \infty$. Subsequently, Deng and Li [20] established a dichotomy for well-posedness and ill-posedness issues in two dimensions; more precisely, they proved that the bipolar type drift–diffusion system is well-posed in $\dot{B}_{4,2}^{-\frac{3}{2}}(\mathbb{R}^2)$, and ill-posed in $\dot{B}_{4,q}^{-\frac{3}{2}}(\mathbb{R}^2)$ for $2 < q \leq \infty$. Very recently, Iwabuchi and Ogawa [32] finally proved that the bipolar type drift–diffusion system is ill-posed in $\dot{B}_{p,q}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $2n < p \leq \infty$ and $1 \leq q \leq \infty$, or $p = 2n$ and $2 < q \leq \infty$.

For general fractional diffusion case $1 < \alpha < 2$, the system (1.1) was first studied by Escudero [22], where it was used to describe the spatiotemporal distribution of a population density of random walkers undergoing Lévy flights. Moreover the author proved that the one-dimensional system (1.1) possesses global in time solutions not only in the case of $\alpha = 2$ but also in the case $1 < \alpha < 2$. Biler and Karch [7] proved local and global solutions with small initial data of the system (1.1) in critical Lebesgue space $L^{\frac{n}{\alpha}}(\mathbb{R}^n)$ for $1 < \alpha < 2$; they also proved the finite time blowup of nonnegative solutions with some initial data imposed on the large-mass or high-concentration conditions. Biler and Wu [11] established global well-posedness of the system (1.1) with small initial data in the critical Besov spaces $\dot{B}_{2,q}^{1-\alpha}(\mathbb{R}^2)$ for $1 < \alpha < 2$. Wu and Zheng [51] proved a local well-posedness with any initial data and global well-posedness with small initial data in critical Fourier–Herz space $\dot{B}_q^{2-2\alpha}(\mathbb{R}^n)$ for $1 < \alpha \leq 2$ and $2 \leq q \leq \infty$ and proved ill-posedness in $\dot{B}_q^{-2}(\mathbb{R}^n)$ and $\dot{B}_{\infty,q}^{-2}(\mathbb{R}^n)$ with $\alpha = 2$ and $2 < q \leq \infty$. Zhai [55] proved the global existence, uniqueness and stability of solutions with small initial data in critical Besov spaces with general potential type nonlinear term. Parts of these results were also generalized for the fractional power drift–diffusion system of bipolar type, please refer to [11, 43, 48, 50] and the references therein.

In this paper, we aim at studying well-posedness and Gevrey analyticity of the generalized Keller–Segel system (1.1) with initial data in critical Besov spaces $\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ for $1 \leq \alpha \leq 2$ and $1 \leq p, q \leq \infty$. The first novelty of this paper is that we resort to the Fourier localization technique and the Bony’s paraproduct theory to address well-posedness issues of the system (1.1) in critical Besov spaces either $\dot{B}_{\infty,1}^{-\alpha}(\mathbb{R}^n)$ with $1 < \alpha < 2$ or $\dot{B}_{\infty,1}^{-1}(\mathbb{R}^n)$ with $\alpha = 1$. These critical spaces are marginal cases adapted to the system (1.1). The second novelty of this paper is that we employ the Gevrey class regularity to prove analyticity of solutions. The choice of this argument is motivated by the work of Foias and Temam [23] for estimating space analyticity radius of the Navier–Stokes equations (similar results were extended by many authors to various equations, see [1, 2, 12, 13, 29, 37] for more details). The result characterizes space analyticity radius of solutions and has an important physical interpretation: at this length scale, the viscous effects and the nonlinear inertial effects are roughly comparable; below this length scale the Fourier spectrum decays exponentially. As a consequence of analyticity result, we obtain temporal decay rates of higher-order Besov norms of solutions.

Now we state main results of this paper. Let us denote by Λ_1 the Fourier multiplier whose symbol is given by $|\xi|_1 = |\xi_1| + \dots + |\xi_n|$, and we refer the reader to see Sect. 2 for the definitions of the stationary/time dependent homogeneous Besov spaces.

Theorem 1.1 *Let $n \geq 2, 1 < \alpha \leq 2$. Assume that $u_0 \in \dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ with $1 \leq p, q \leq \infty$. Then we have the following results:*

- (i) *(Well-posedness for $1 \leq p < \infty$) Let $1 \leq p < \infty$. Then there exists a $T^* = T^*(u_0) > 0$ such that the system (1.1) has a unique solution $u \in \mathcal{X}_{T^*}$, where*

$$\mathcal{X}_{T^*} := \tilde{L}^\infty \left(0, T^*; \dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n) \right) \cap \tilde{L}^{\rho_1} \left(0, T^*; \dot{B}_{p,q}^{s_1}(\mathbb{R}^n) \right) \cap \tilde{L}^{\rho_2} \left(0, T^*; \dot{B}_{p,q}^{s_2}(\mathbb{R}^n) \right) \tag{1.4}$$

with

$$s_1 = -1 + \frac{n}{p} + \varepsilon, \quad s_2 = -1 + \frac{n}{p} - \varepsilon, \quad \rho_1 = \frac{\alpha}{\alpha - 1 + \varepsilon}, \quad \rho_2 = \frac{\alpha}{\alpha - 1 - \varepsilon}, \quad 0 < \varepsilon < \alpha - 1.$$

If $T^* < \infty$, then

$$\|u\|_{\tilde{L}_{T^*}^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_{T^*}^{\rho_2}(\dot{B}_{p,q}^{s_2})} = \infty.$$

Moreover if $u_0 \in \dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ is sufficiently small, then $T^* = \infty$.

(ii) (Analyticity for $1 < p < \infty$) Let $1 < p < \infty$. Then the solution obtained in (i) satisfies

$$e^{t^{\frac{1}{\alpha}} \Lambda_1} u \in \mathcal{X}_{T^*}. \tag{1.5}$$

Moreover if $u_0 \in \dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ is sufficiently small, then $T^* = \infty$.

(iii) (Well-posedness for $p = \infty$) Let $1 < \alpha < 2$ and $p = \infty$. Suppose that $\|u_0\|_{\dot{B}_{\infty,1}^{-\alpha}}$ is sufficiently small. Then the system (1.1) has a unique solution u satisfying

$$u \in \tilde{L}^\infty(0, \infty; \dot{B}_{\infty,1}^{-\alpha}(\mathbb{R}^n)) \cap \tilde{L}^1(0, \infty; \dot{B}_{\infty,1}^0(\mathbb{R}^n)). \tag{1.6}$$

(iv) (Analyticity for $p = \infty$) Let $1 < \alpha < 2$ and $p = \infty$. Then the solution obtained in (iii) satisfies

$$e^{t^{\frac{1}{\alpha}} \Lambda_1} u \in \tilde{L}^\infty(0, \infty; \dot{B}_{\infty,1}^{-\alpha}(\mathbb{R}^n)) \cap \tilde{L}^1(0, \infty; \dot{B}_{\infty,1}^0(\mathbb{R}^n)). \tag{1.7}$$

(v) (Decay rate for $1 < p \leq \infty$) For any $\sigma \geq 0$, $1 < p < \infty$ or $p = \infty$ and $q = 1$, the global solution obtained in (i) and (iii) satisfies

$$\|\Lambda^\sigma u(t)\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}} \leq C_\sigma t^{-\frac{\sigma}{\alpha}} \|u_0\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}}, \tag{1.8}$$

where $C_\sigma := \|\Lambda^\sigma e^{-\Lambda_1}\|_{L^1}$.

Remark 1.1 We mention here that Bourgain and Pavlović [17] proved ill-posedness for the 3D Navier–Stokes equations in $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$. Subsequently, Yoneda [54] proved ill-posedness in some function spaces, which are larger than $\dot{B}_{\infty,2}^{-1}(\mathbb{R}^3)$ but smaller than $\dot{B}_{\infty,q}^{-1}(\mathbb{R}^3)$ with $2 < q \leq \infty$; Wang [49] finally proved ill-posedness for the 3D Navier–Stokes equations in $\dot{B}_{\infty,q}^{-1}(\mathbb{R}^3)$ for all $1 \leq q \leq 2$. Note that when $\alpha = 2$, $\dot{B}_{\infty,q}^{-1}(\mathbb{R}^n)$ for the Navier–Stokes equations corresponds to $\dot{B}_{\infty,q}^{-2}(\mathbb{R}^n)$ for the system (1.1); therefore, we cannot expect the well-posedness of the system (1.1) in $\dot{B}_{\infty,q}^{-2}(\mathbb{R}^n)$ for $1 \leq q \leq \infty$. However, when $1 < \alpha < 2$, Theorem 1.1 shows that the system (1.1) is well-posedness in $\dot{B}_{\infty,1}^{-\alpha}(\mathbb{R}^n)$.

Remark 1.2 We emphasize here that the exponential operator $e^{t^{\frac{1}{\alpha}} \Lambda_1}$ is quantified by the operator Λ_1 , whose symbol is given by the l^1 norm $|\xi|_1 = \sum_{i=1}^n |\xi_i|$, rather than the usual operator $\Lambda = \sqrt{-\Delta}$, whose symbol is given by the l^2 norm $|\xi| = (\sum_{i=1}^n |\xi_i|^2)^{\frac{1}{2}}$. This approach enables us to avoid cumbersome recursive estimation of higher-order derivatives and intricate combinatorial arguments to get the desired decay estimates of solutions, see [52, 56].

Remark 1.3 The method we shall use to prove well-posedness of (1.1) in critical Besov spaces $\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ is the Chemin mono-norm method, which is different from the methods used in [30, 55].

Remark 1.4 In [34], Kato obtained the first and second asymptotic profiles of solutions to the parabolic system (1.2) in the Lebesgue framework; particularly, the optimal asymptotic rate of the first asymptotic profiles is established. Similar results have been extended to the fractional power drift–diffusion system of bipolar type, see Ogawa and Yamamoto [48]. Compared with the decay result in Theorem 1.1, (1.8) gives us the decay rates of solutions in critical Besov spaces and is compatible with the decay rates of solutions to the linear fractional power dissipative equation, and the result is an immediate by-product of Gevrey regularity of solutions.

Corresponding to Theorem 1.1, in the case $\alpha = 1$, we obtain the following results.

Theorem 1.2 *Let $n \geq 2, \alpha = 1$. Assume that $u_0 \in \dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Then we have the following results:*

(i) *(Well-posedness for $1 \leq p < \infty$) Let $1 \leq p < \infty$. Suppose that $\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}$ is sufficiently small. Then the system (1.1) has a unique solution u satisfying*

$$u \in \tilde{L}^\infty \left(0, \infty; \dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n) \right). \tag{1.9}$$

(ii) *(Analyticity for $1 < p < \infty$) Let $1 < p < \infty$. Then the solution obtained in (i) satisfies*

$$e^{t^{\frac{1}{2n}} \Lambda_1} u \in \tilde{L}^\infty \left(0, \infty; \dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n) \right). \tag{1.10}$$

(iii) *(Well-posedness for $p = \infty$) Let $p = \infty$. Suppose that $\|u_0\|_{\dot{B}_{\infty,1}^{-1}}$ is sufficiently small. Then the system (1.1) has a unique solution u satisfying*

$$u \in \tilde{L}^\infty \left(0, \infty; \dot{B}_{\infty,1}^{-1}(\mathbb{R}^n) \right) \cap \tilde{L}^1 \left(0, \infty; \dot{B}_{\infty,1}^0(\mathbb{R}^n) \right). \tag{1.11}$$

(iv) *(Analyticity for $p = \infty$) Let $p = \infty$. Then the solution obtained in (iii) satisfies*

$$e^{t^{\frac{1}{2n}} \Lambda_1} u \in \tilde{L}^\infty \left(0, \infty; \dot{B}_{\infty,1}^{-1}(\mathbb{R}^n) \right) \cap \tilde{L}^1 \left(0, \infty; \dot{B}_{\infty,1}^0(\mathbb{R}^n) \right). \tag{1.12}$$

(v) *(Decay rate for $1 < p \leq \infty$) For any $\sigma \geq 0$ and $1 < p \leq \infty$, the global solution obtained in (i) and (iii) satisfies*

$$\|\Lambda^\sigma u(t)\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}} \leq \tilde{C}_\sigma t^{-\sigma} \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}, \tag{1.13}$$

where $\tilde{C}_\sigma := \|\Lambda^\sigma e^{-\frac{1}{2n} \Lambda_1}\|_{L^1}$.

Remark 1.5 In the case $\alpha = 1$, since the dissipative operator $e^{-\frac{1}{2}t\Lambda}$ is not strong enough to dominate the operator $e^{t\Lambda_1}$, we need to define Gevrey operator more carefully. By noticing a simple fact that $\frac{1}{2n}|\xi|_1 < \frac{1}{2}|\xi|$ for all $\xi \in \mathbb{R}^n$, the Gevrey operator can be defined by $e^{\frac{1}{2n}t\Lambda_1}u$.

Remark 1.6 In Miao and Wu [42], by making use of modulus of continuity and Fourier localization technique, the authors proved that the critical Burgers equations $\partial_t u + u\partial_x u + \Lambda u = 0$ is well-posed in critical Besov space $\dot{B}_{p,1}^{\frac{1}{p}}$ with $p \in [1, \infty)$, while the well-posedness in the limit case $p = \infty$ is successfully addressed by Iwabuchi [31]. The main crux in their proof is an optimal a priori estimates for the transport–diffusion equation, which is distinct from the method used in our paper; our proof is based on the fundamental estimates of the linear fractional power dissipative equation in the Chemin–Lerner mixed type space, see Sect. 4.

Before ending this section, let us sketch, for example, the proof of analyticity part in Theorem 1.1. Setting $U(t) = e^{t^{\frac{1}{2n}} \Lambda_1} u(t)$. Then we see that $U(t)$ satisfies the following integral equation:

$$\begin{aligned} U(t) &= e^{t^{\frac{1}{2n}} \Lambda_1 - t\Lambda^\alpha} u_0 - \int_0^t e^{\left[\left(t^{\frac{1}{2n}} - \tau^{\frac{1}{2n}} \right) \Lambda_1 - (t-\tau)\Lambda^\alpha \right]} \nabla \\ &\quad \times e^{\tau^{\frac{1}{2n}} \Lambda_1} \left(e^{-\tau^{\frac{1}{2n}} \Lambda_1} U(\tau) e^{-\tau^{\frac{1}{2n}} \Lambda_1} \nabla (-\Delta)^{-1} U(\tau) \right) d\tau. \end{aligned}$$

Note that since $e^{t^{\frac{1}{\alpha}}|\xi|_1}$ can be dominated by $e^{-t|\xi|^\alpha}$ provided that $|\xi|$ is sufficiently large, the behavior of the linear term $e^{t^{\frac{1}{\alpha}}\Lambda_1 - t\Lambda^\alpha} u_0$ closely resembles that of $e^{-t\Lambda^\alpha} u_0$. In order to tackle with the nonlinear term, we resort to [38] and [2] to find out the nice boundedness property of the following bilinear operator:

$$\begin{aligned} \mathcal{B}_t(f, g) &:= e^{t^{\frac{1}{\alpha}}\Lambda_1} \left(e^{-t^{\frac{1}{\alpha}}\Lambda_1} f e^{-t^{\frac{1}{\alpha}}\Lambda_1} g \right) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix \cdot (\xi + \eta)} e^{t^{\frac{1}{\alpha}}(|\xi + \eta|_1 - |\xi|_1 - |\eta|_1)} \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta. \end{aligned}$$

Based on the desired properties of $\mathcal{B}_t(f, g)$, we can modify the argument of the proof of well-posedness results in Theorem 1.1 to obtain Gevrey regularity.

This paper is organized as follows: We shall collect some basic facts on the Littlewood–Paley dyadic decomposition theory and the various product laws in Besov spaces in Sect. 2 and then prove Theorem 1.1 in Sect. 3 and Theorem 1.2 in Sect. 4.

2 Preliminaries

2.1 Notations

In this paper, we shall use the following notations. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $|x|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$ and $|x| = |x|_2$, and thus, we denote by Λ_1 the Fourier multiplier whose symbol is given by $|\xi|_1 = \sum_{i=1}^n |\xi_i|$ and by $\Lambda = \sqrt{-\Delta}$ the Fourier multiplier whose symbol is given by $|\xi| = \left(\sum_{i=1}^n |\xi_i|^2\right)^{\frac{1}{2}}$. For any function space X and the operator $\mathcal{T} : X \rightarrow X$, we denote

$$\mathcal{T}X := \{Tf : f \in X\} \text{ and } \|f\|_{\mathcal{T}X} := \|\mathcal{T}f\|_X.$$

The linear space of all multipliers on L^p is denoted by \mathcal{M}_p and the norm on which is defined by

$$\|f\|_{\mathcal{M}_p} := \sup \{ \|\mathcal{F}^{-1}[f\mathcal{F}g]\|_{L^p} : \forall g \in \mathcal{S}(\mathbb{R}^n), \|g\|_{L^p} = 1 \}.$$

For two constants A and B , the notation $A \lesssim B$ means that there is a uniform constant C (always independent of x, t), which may vary from line to line, such that $A \leq CB$.

2.2 Littlewood–Paley theory and Besov spaces

The proofs of Theorems 1.1 and 1.2 are formulated by the dyadic decomposition in the Littlewood–Paley theory. Let us briefly explain how it may be built in \mathbb{R}^n . Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing function and $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions be the dual set of $\mathcal{S}(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a smooth radial function valued in $[0, 1]$ such that φ is supported in the shell $\mathcal{C} = \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, and

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then for any $f \in \mathcal{S}'(\mathbb{R}^n)$, we set for all $j \in \mathbb{Z}$,

$$\Delta_j f := \varphi(2^{-j}D)f \text{ and } S_j f := \sum_{k \leq j-1} \Delta_k f. \tag{2.1}$$

By telescoping the series, we have the following homogeneous Littlewood–Paley decomposition:

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ for } f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n),$$

where $\mathcal{P}(\mathbb{R}^n)$ is the set of polynomials (see [3]). We remark here that the Littlewood–Paley decomposition satisfies the property of almost orthogonality, that is to say, for any $f, g \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, the following properties hold:

$$\Delta_i \Delta_j f \equiv 0 \text{ if } |i - j| \geq 2 \text{ and } \Delta_i(S_{j-1}f \Delta_j g) \equiv 0 \text{ if } |i - j| \geq 5. \tag{2.2}$$

Using the above decomposition, the stationary/time-dependent homogeneous Besov space can be defined as follows:

Definition 2.1 Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, we set

$$\|f\|_{\dot{B}_{p,q}^s} := \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{\frac{1}{q}} & \text{for } 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_{L^p} & \text{for } q = \infty. \end{cases}$$

Then the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^n)$ is defined by

- For $s < \frac{n}{p}$ (or $s = \frac{n}{p}$ if $q = 1$), we define

$$\dot{B}_{p,q}^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) : \|f\|_{\dot{B}_{p,q}^s} < \infty \right\}.$$

- If $k \in \mathbb{N}$ and $\frac{n}{p} + k \leq s < \frac{n}{p} + k + 1$ (or $s = \frac{n}{p} + k + 1$ if $q = 1$), then $\dot{B}_{p,q}^s(\mathbb{R}^n)$ is defined as the subset of distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\partial^\beta f \in \dot{B}_{p,r}^{s-k}(\mathbb{R}^n)$ whenever $|\beta| = k$.

Definition 2.2 For $0 < T \leq \infty$, $s \leq \frac{n}{p}$ (resp. $s \in \mathbb{R}$), $1 \leq p, q, \rho \leq \infty$. We define the mixed time-space $\tilde{L}^\rho(0, T; \dot{B}_{p,q}^s(\mathbb{R}^n))$ as the completion of $\mathcal{C}([0, T]; \mathcal{S}(\mathbb{R}^n))$ by the norm

$$\|f\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^s)} := \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \left(\int_0^T \|\Delta_j f(\cdot, t)\|_{L^p}^\rho dt \right)^{\frac{q}{\rho}} \right)^{\frac{1}{q}} < \infty$$

with the usual change if $\rho = \infty$ or $q = \infty$. For simplicity, we use $\|f\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^s)}$ instead of $\|f\|_{\tilde{L}_\infty^\rho(\dot{B}_{p,q}^s)}$.

In what follows, we shall frequently use the following Bony’s homogeneous paraproduct decomposition, which is a mathematical tool to define a generalized product between two temperate distributions (see [16]). Let f and g be two temperate distributions, the paraproduct between f and g is defined by

$$T_f g := \sum_{j \in \mathbb{Z}} S_{j-1} f \Delta_j g = \sum_{j \in \mathbb{Z}} \sum_{k \leq j-2} \Delta_k f \Delta_j g.$$

Formally, we have the following Bony’s decomposition:

$$fg = T_f g + T_g f + R(f, g),$$

where

$$R(f, g) := \sum_{j \in \mathbb{Z}} \sum_{|j-j'| \leq 1} \Delta_j f \Delta_{j'} g.$$

2.3 Essential lemmas

For the convenience of the readers, we list some basic facts of the Littlewood–Paley theory, one may refer to [3, 38] for more details.

Lemma 2.3 [3, 38] *Let \mathcal{B} be a ball, and \mathcal{C} a ring in \mathbb{R}^n . There exists a constant C such that for any positive real number λ , any nonnegative integer k and any couple of real numbers (p, r) with $1 \leq p \leq r \leq \infty$, we have*

$$\text{supp } \mathcal{F}(f) \subset \lambda \mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^r} \leq C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{r})} \|f\|_{L^p}, \tag{2.3}$$

$$\text{supp } \mathcal{F}(f) \subset \lambda \mathcal{C} \Rightarrow C^{-1-k} \lambda^k \|f\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \leq C^{1+k} \lambda^k \|f\|_{L^p}. \tag{2.4}$$

Lemma 2.4 [3, 38] *Let f be a smooth function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree m . Then for any $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, and*

$$s - m < \frac{n}{p}, \text{ or } s - m = \frac{n}{p} \text{ and } q = 1,$$

the operator $f(D)$ is continuous from $\dot{B}_{p,q}^s(\mathbb{R}^n)$ to $\dot{B}_{p,q}^{s-m}(\mathbb{R}^n)$.

Lemma 2.5 [50] *Let \mathcal{C} be a ring in \mathbb{R}^n . There exist two positive constants κ and \mathcal{K} such that for any $p \in [1, \infty]$ and any couple (t, λ) of positive real numbers, we have*

$$\text{supp } \mathcal{F}(f) \subset \lambda \mathcal{C} \Rightarrow \|e^{t\Delta^\alpha} f\|_{L^p} \leq \mathcal{K} e^{-\kappa \lambda^\alpha t} \|f\|_{L^p}. \tag{2.5}$$

3 The case $1 < \alpha \leq 2$: Proof of Theorem 1.1

In this section, we prove Gevrey analyticity of the system (1.1) in critical Besov spaces $\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ with $1 < \alpha \leq 2$, $1 < p \leq \infty$ and $1 \leq q \leq \infty$. The proof is based on an adequate modification of the proof of local in time existence with any initial data and global in time existence with small initial data to the system (1.1), and thus, we begin with the detailed proof of the first part of Theorem 1.1.

3.1 The case $1 \leq p < \infty$: well-posedness

In this subsection, we intend to establish local well-posedness with any initial data and global well-posedness with small initial data to the system (1.1) in critical Besov spaces $\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ for $1 \leq p < \infty$. Firstly, we are concerned with the Cauchy problem of the fractional power dissipative equation:

$$\begin{cases} \partial_t u + \Lambda^\alpha u = f, & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

Proposition 3.1 ([11]) *Let $s \in \mathbb{R}$, $1 \leq p, q, \rho_1 \leq \infty$ and $0 < T \leq \infty$. Assume that $u_0 \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ and $f \in \tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\frac{\alpha}{\rho_1}-\alpha}(\mathbb{R}^n))$. Then (3.1) has a unique solution $u \in \bigcap_{\rho_1 \leq \rho \leq \infty} \tilde{L}_T^\rho(\dot{B}_{p,q}^{s+\frac{\alpha}{\rho}}(\mathbb{R}^n))$. In addition, there exists a constant $C > 0$ depending only on α and n such that for any $\rho_1 \leq \rho \leq \infty$, we have*

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^{s+\frac{\alpha}{\rho}})} \leq C \left(\|u_0\|_{\dot{B}_{p,q}^s} + \|f\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\frac{\alpha}{\rho_1}-\alpha})} \right). \tag{3.2}$$

In particular, if $f \in \tilde{L}_T^1(\dot{B}_{p,q}^s(\mathbb{R}^n))$, then we have

$$\|u\|_{\tilde{L}_T^\infty(\dot{B}_{p,q}^s) \cap \tilde{L}_T^1(\dot{B}_{p,q}^{s+\alpha})} \leq C \left(\|u_0\|_{\dot{B}_{p,q}^s} + \|f\|_{\tilde{L}_T^1(\dot{B}_{p,q}^s)} \right). \tag{3.3}$$

Next, by using in a fundamental way the algebraical structure of the system (1.1), we aim at establishing the following crucial bilinear estimates in time-dependent Besov spaces.

Lemma 3.2 *Let $s > 0$, $1 \leq p < \infty$, $1 \leq q, \rho, \rho_1, \rho_2 \leq \infty$ with $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$. Then for any $\varepsilon > 0$, $0 < T \leq \infty$, we have*

$$\begin{aligned} \|u \nabla(-\Delta)^{-1} v + v \nabla(-\Delta)^{-1} u\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^s)} &\lesssim \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} \\ &\quad + \|u\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\varepsilon})}. \end{aligned} \tag{3.4}$$

Moreover if we choose $\varepsilon = 0$, then (3.4) also holds for $q = 1$.

Proof Thanks to Bony’s paraproduct decomposition, we have

$$u \nabla(-\Delta)^{-1} v + v \nabla(-\Delta)^{-1} u := I_1 + I_2 + I_3, \tag{3.5}$$

where

$$\begin{aligned} I_1 &:= \sum_{j' \in \mathbb{Z}} \Delta_{j'} u \nabla(-\Delta)^{-1} S_{j'-1} v + \Delta_{j'} v \nabla(-\Delta)^{-1} S_{j'-1} u, \\ I_2 &:= \sum_{j' \in \mathbb{Z}} S_{j'-1} u \nabla(-\Delta)^{-1} \Delta_{j'} v + S_{j'-1} v \nabla(-\Delta)^{-1} \Delta_{j'} u, \\ I_3 &:= \sum_{j' \in \mathbb{Z}} \sum_{|j'-j''| \leq 1} \Delta_{j'} u \nabla(-\Delta)^{-1} \Delta_{j''} v + \Delta_{j'} v \nabla(-\Delta)^{-1} \Delta_{j''} u. \end{aligned}$$

In the sequel, we estimate I_i ($i = 1, 2, 3$) one by one. For I_1 , we need only to deal with the first term $\sum_{j' \in \mathbb{Z}} \Delta_{j'} u \nabla(-\Delta)^{-1} S_{j'-1} v$, while the second one can be done analogously, thus using the facts (2.1) and (2.2), and applying Hölder’s inequality and Lemmas 2.3 and 2.4, one has

$$\begin{aligned} \left\| \Delta_j \sum_{j' \in \mathbb{Z}} \Delta_{j'} u \nabla(-\Delta)^{-1} S_{j'-1} v \right\|_{L_T^\rho(L^p)} &\lesssim \sum_{|j'-j| \leq 4} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} \|\nabla(-\Delta)^{-1} S_{j'-1} v\|_{L_T^{\rho_2}(L^\infty)} \\ &\lesssim \sum_{|j'-j| \leq 4} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} \sum_{k \leq j'-2} 2^{(-1+\frac{n}{p})k} \|\Delta_k v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim \sum_{|j'-j| \leq 4} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} \sum_{k \leq j'-2} 2^{\varepsilon k} 2^{(-1+\frac{n}{p}-\varepsilon)k} \|\Delta_k v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim \sum_{|j'-j| \leq 4} 2^{-sj'} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} \|v\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})}. \end{aligned} \tag{3.6}$$

Multiplying (3.6) by 2^{sj} , then taking l^q norm to the resulting inequality, we obtain

$$\left\| \sum_{j' \in \mathbb{Z}} \Delta_{j'} u \nabla(-\Delta)^{-1} S_{j'-1} v \right\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^s)} \lesssim \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})},$$

which implies that

$$\|I_1\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^s)} \lesssim \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} + \|u\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\varepsilon})}. \tag{3.7}$$

Similarly, for the first term of I_2 , applying Hölder’s inequality and Lemmas 2.3 and 2.4 again, we see that

$$\begin{aligned} & \left\| \Delta_j \sum_{j' \in \mathbb{Z}} S_{j'-1} u \nabla (-\Delta)^{-1} \Delta_{j'} v \right\|_{L_T^\rho(L^p)} \\ & \lesssim \sum_{|j'-j| \leq 4} \sum_{k \leq j'-2} 2^{\frac{n}{p}k} \|\Delta_k u\|_{L_T^{\rho_2}(L^p)} 2^{-j'} \|\Delta_{j'} v\|_{L_T^{\rho_1}(L^p)} \\ & \lesssim \sum_{|j'-j| \leq 4} \sum_{k \leq j'-2} 2^{(1+\varepsilon)k} 2^{(-1+\frac{n}{p}-\varepsilon)k} \|\Delta_k u\|_{L_T^{\rho_2}(L^p)} 2^{-j'} \|\Delta_{j'} v\|_{L_T^{\rho_1}(L^p)} \\ & \lesssim \sum_{|j'-j| \leq 4} 2^{-sj'} 2^{(s+\varepsilon)j'} \|\Delta_{j'} v\|_{L_T^{\rho_1}(L^p)} \|u\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})}, \end{aligned} \tag{3.8}$$

which yields directly that

$$\left\| \sum_{j' \in \mathbb{Z}} S_{j'-1} u \nabla (-\Delta)^{-1} \Delta_{j'} v \right\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^s)} \lesssim \|u\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\varepsilon})}.$$

Thus, we obtain

$$\|I_2\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^s)} \lesssim \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} + \|u\|_{\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s+\varepsilon})}. \tag{3.9}$$

Now we tackle with the most tricky term I_3 . Based on careful analysis of the algebraical structure of the system (1.1), we can split I_3 into the following three terms for $m = 1, 2, \dots, n$:

$$I_3 := K_1 + K_2 + K_3, \tag{3.10}$$

where

$$\begin{aligned} K_1 &:= \sum_{j' \in \mathbb{Z}} \sum_{|j'-j''| \leq 1} (-\Delta) \left\{ ((-\Delta)^{-1} \Delta_{j'} u) (\partial_m (-\Delta)^{-1} \Delta_{j''} v) \right\}, \\ K_2 &:= \sum_{j' \in \mathbb{Z}} \sum_{|j'-j''| \leq 1} 2 \nabla \cdot \left\{ ((-\Delta)^{-1} \Delta_{j'} u) (\partial_m \nabla (-\Delta)^{-1} \Delta_{j''} v) \right\}, \\ K_3 &:= \sum_{j' \in \mathbb{Z}} \sum_{|j'-j''| \leq 1} \partial_m \left\{ ((-\Delta)^{-1} \Delta_{j'} u) \Delta_{j''} v \right\}. \end{aligned}$$

Moreover since K_2 can be treated similarly to K_3 , we treat K_1 and K_3 only. We first consider the case $2 \leq p < \infty$, and by using Hölder’s inequality and Lemmas 2.3 and 2.4, it follows from (2.2) that there exists $N_0 \in \mathbb{N}$ such that

$$\begin{aligned} \|\Delta_j K_1\|_{L_T^\rho(L^p)} &\lesssim 2^{\left(2+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} \|\partial_m (-\Delta)^{-1} \Delta_{j''} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{\left(2+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} 2^{\left(-2-s-\frac{n}{p}\right)j'} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} 2^{\left(-1+\frac{n}{p}-\varepsilon\right)j''} \|\Delta_{j''} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{-sj} \sum_{j' \geq j-N_0} 2^{-\left(2+s+\frac{n}{p}\right)(j'-j)} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} \|v\|_{L_T^{\rho_2}\left(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon}\right)}. \end{aligned} \tag{3.11}$$

$$\begin{aligned} \|\Delta_j K_3\|_{L_T^\rho(L^p)} &\lesssim 2^{\left(1+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} \|\Delta_{j''} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{\left(1+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} 2^{\left(-1-s-\frac{n}{p}\right)j'} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} 2^{\left(-1+\frac{n}{p}-\varepsilon\right)j''} \|\Delta_{j''} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{-sj} \sum_{j' \geq j-N_0} 2^{-\left(1+s+\frac{n}{p}\right)(j'-j)} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^p)} \|v\|_{L_T^{\rho_2}\left(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon}\right)}. \end{aligned} \tag{3.12}$$

On the other hand, in the case that $1 \leq p < 2$, we choose $2 < p' \leq \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, it follows that

$$\begin{aligned} \|\Delta_j K_1\|_{L_T^\rho(L^p)} &\lesssim 2^{\left(2+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_T^{\rho_1}(L^{p'})} \|\partial_m (-\Delta)^{-1} \Delta_{j''} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{\left(2+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} 2^{\left(-2+n\left(\frac{1}{p}-\frac{1}{p'}\right)\right)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^{p'})} 2^{-j''} \|\Delta_{j''} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{\left(2+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-\left(2+n+s-\frac{n}{p}\right)j'} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^{p'})} 2^{\left(-1+\frac{n}{p}-\varepsilon\right)j'} \|\Delta_{j'} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{-sj} \sum_{j' \geq j-N_0} 2^{-\left(2+n+s-\frac{n}{p}\right)(j'-j)} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^{p'})} \|v\|_{L_T^{\rho_2}\left(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon}\right)}. \end{aligned} \tag{3.13}$$

$$\begin{aligned} \|\Delta_j K_3\|_{L_T^\rho(L^p)} &\lesssim 2^{\left(1+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_T^{\rho_1}(L^{p'})} \|\Delta_{j''} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{\left(1+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} 2^{\left(-2+n\left(\frac{1}{p}-\frac{1}{p'}\right)\right)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^{p'})} \|\Delta_{j''} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{\left(1+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-\left(1+n+s-\frac{n}{p}\right)j'} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^{p'})} 2^{\left(-1+\frac{n}{p}-\varepsilon\right)j'} \|\Delta_{j'} v\|_{L_T^{\rho_2}(L^p)} \\ &\lesssim 2^{-sj} \sum_{j' \geq j-N_0} 2^{-\left(1+n+s-\frac{n}{p}\right)(j'-j)} 2^{(s+\varepsilon)j'} \|\Delta_{j'} u\|_{L_T^{\rho_1}(L^{p'})} \|v\|_{L_T^{\rho_2}\left(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon}\right)}. \end{aligned} \tag{3.14}$$

Under the hypotheses of Lemma 3.2, we have

$$2 + s + \frac{n}{p} > 0, \quad 1 + s + \frac{n}{p} > 0, \quad 2 + n + s - \frac{n}{p} > 0, \quad 1 + n + s - \frac{n}{p} > 0.$$

Then we infer from the estimates (3.11)–(3.14) that for all $1 \leq p < \infty$,

$$\|I_3\|_{\tilde{L}_T^\rho\left(\dot{B}_{p,q}^s\right)} \lesssim \|u\|_{\tilde{L}_T^{\rho_1}\left(\dot{B}_{p,q}^{s+\varepsilon}\right)} \|v\|_{\tilde{L}_T^{\rho_2}\left(\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon}\right)}. \tag{3.15}$$

Hence plugging (3.7), (3.9) and (3.15) into (3.5), we get (3.4). The proof of Lemma 3.2 is complete. \square

Now we are in a position to prove well-posedness of the system (1.1) in the case that $1 < \alpha \leq 2$ and $1 \leq p < \infty$. Define the map

$$\mathbb{F} : u(t) \rightarrow e^{-t\Lambda^\alpha} u_0 - \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla \cdot (u \nabla (-\Delta)^{-1} u) d\tau \tag{3.16}$$

in the metric space ($I = [0, T]$):

$$\mathcal{D}_T := \left\{ u : \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})} \leq \eta, \quad d(u, v) := \|u - v\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})} \right\}$$

with

$$s_1 = -1 + \frac{n}{p} + \varepsilon, \quad s_2 = -1 + \frac{n}{p} - \varepsilon, \quad \rho_1 = \frac{\alpha}{\alpha - 1 + \varepsilon}, \quad \rho_2 = \frac{\alpha}{\alpha - 1 - \varepsilon}, \quad 0 < \varepsilon < \alpha - 1.$$

Applying Proposition 3.1 and Lemma 3.2 by choosing $\rho = \frac{\alpha}{2\alpha - 2}$, for any $u, v \in \mathcal{D}_T$, we see that

$$\begin{aligned} \|\mathbb{F}(u)\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})} &\lesssim \|e^{-t\Lambda^\alpha} u_0\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})} + \|u \nabla (-\Delta)^{-1} u\|_{\tilde{L}_T^{\frac{\alpha}{2\alpha-2}}(\dot{B}_{p,q}^{-1+\frac{n}{p}})} \\ &\lesssim \|e^{-t\Lambda^\alpha} u_0\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})} + \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})}^2, \end{aligned} \tag{3.17}$$

and

$$d(\mathbb{F}(u), \mathbb{F}(v)) \lesssim \eta d(u, v). \tag{3.18}$$

Based on these two estimates (3.17) and (3.18), applying the standard contraction mapping argument (cf. [38]), we can show that if we choose T is properly small, then \mathbb{F} is a contraction mapping from (\mathcal{D}_T, d) into itself, and we omit the details here. Therefore there exists $u \in \mathcal{D}_T$ such that $\mathbb{F}(u) = u$, which is a unique solution of the system (1.1). Moreover by Proposition 3.1, we have

$$\|u\|_{\tilde{L}_T^\infty(\dot{B}_{p,q}^{-\alpha+\frac{n}{p}})} \lesssim \|u_0\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}} + \|u\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})}^2 \lesssim \|u_0\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}} + \eta^2.$$

Thus the solution u can be extended step by step, and finally there is a maximal time T^* such that

$$u \in \tilde{L}^\infty\left(0, T^*; \dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)\right) \cap \tilde{L}^{\rho_1}\left(0, T^*; \dot{B}_{p,q}^{s_1}(\mathbb{R}^n)\right) \cap \tilde{L}^{\rho_2}\left(0, T^*; \dot{B}_{p,q}^{s_2}(\mathbb{R}^n)\right).$$

If $T^* < \infty$ and $\|u\|_{\tilde{L}_{T^*}^{\rho_1}(\dot{B}_{p,q}^{s_1}) \cap \tilde{L}_{T^*}^{\rho_2}(\dot{B}_{p,q}^{s_2})} < \infty$, we claim that the solution can be extended beyond the maximal time T^* . Indeed, let us consider the integral equation

$$u(t) = e^{-(t-T)\Lambda^\alpha} u(T) - \int_T^t e^{-(t-\tau)\Lambda^\alpha} \nabla \cdot (u \nabla (-\Delta)^{-1} u) d\tau. \tag{3.19}$$

As we have proved before, we can show that if we choose T sufficiently close to T^* , then

$$\begin{aligned} \|u(t)\|_{\tilde{L}^{\rho_1}(T, T^*; \dot{B}_{p,q}^{s_1}) \cap \tilde{L}^{\rho_2}(T, T^*; \dot{B}_{p,q}^{s_2})} &\leq \|u(T)\|_{\tilde{L}^{\rho_1}(T, T^*; \dot{B}_{p,q}^{s_1}) \cap \tilde{L}^{\rho_2}(T, T^*; \dot{B}_{p,q}^{s_2})} \\ &\quad + \|u\|_{\tilde{L}^{\rho_1}(T, T^*; \dot{B}_{p,q}^{s_1}) \cap \tilde{L}^{\rho_2}(T, T^*; \dot{B}_{p,q}^{s_2})}^2. \end{aligned} \tag{3.20}$$

Note that (3.20) is analogous to (3.17), which yields immediately that the solution exists on $[T, T^*]$. This is a contradiction to the fact that T^* is maximal. Moreover observe that if $\|u_0\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}}$ is sufficiently small, we can directly choose $T = \infty$ in (3.17) and (3.18), which yields global well-posedness of (1.1) with small initial data. We conclude the proof of the first part of Theorem 1.1.

3.2 The case $1 < p < \infty$: Gevrey analyticity

In this subsection, we prove analyticity of the system (1.1) with initial data in $\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ with $1 < \alpha \leq 2$ and $1 < p < \infty$. We first recall the following two elementary results.

Lemma 3.3 (Lemma 3.2 in [2]) *Consider the operator $E_\alpha := e^{-[(t-s)\frac{1}{\alpha}+s\frac{1}{\alpha}-t\frac{1}{\alpha}]\Lambda_1}$ for $0 \leq s \leq t$. Then E_α is either the identity operator or is the Fourier multiplier with L^1 kernel whose L^1 -norm is bounded independent of s and t .*

Lemma 3.4 (Lemma 3.3 in [2]) *Assume that the operator $F_\alpha := e^{t\frac{1}{\alpha}\Lambda_1-\frac{1}{2}t\Lambda^\alpha}$ for $t \geq 0$. Then F_α is the Fourier multiplier which maps boundedly $L^p \rightarrow L^p$ for $1 < p < \infty$, and its operator norm is uniformly bounded with respect to $t \geq 0$.*

Proposition 3.5 *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q, \rho_1 \leq \infty$ and $0 < T \leq \infty$. Assume that $u_0 \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ and $f \in \tilde{L}_T^{\rho_1}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s+\frac{\alpha}{\rho_1}-\alpha}(\mathbb{R}^n))$. Then (3.1) has a unique solution $u \in \bigcap_{\rho_1 \leq \rho \leq \infty} \tilde{L}_T^\rho(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s+\frac{\alpha}{\rho}}(\mathbb{R}^n))$. In addition, there exists a constant $C > 0$ depending only on α and n such that for any $\rho_1 \leq \rho \leq \infty$, we have*

$$\|u\|_{\tilde{L}_T^\rho(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s+\frac{\alpha}{\rho}})} \leq C \left(\|u_0\|_{\dot{B}_{p,q}^s} + \|f\|_{\tilde{L}_T^{\rho_1}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s+\frac{\alpha}{\rho_1}-\alpha})} \right). \tag{3.21}$$

Proof Since Proposition 3.1 has already ensured that (3.1) has a unique solution u , it suffices to prove that the inequality (3.21) holds. For this purpose, setting $U(t) = e^{t\frac{1}{\alpha}\Lambda_1}u(t)$, then applying $\Delta_j e^{t\frac{1}{\alpha}\Lambda_1}$ to (3.1) and taking L^p norm to the resulting equality imply that

$$\|\Delta_j U(t)\|_{L^p} \leq \left\| e^{t\frac{1}{\alpha}\Lambda_1-t\Lambda^\alpha} \Delta_j u_0 \right\|_{L^p} + \left\| \int_0^t e^{t\frac{1}{\alpha}\Lambda_1-(t-\tau)\Lambda^\alpha} \Delta_j f(\tau) d\tau \right\|_{L^p}. \tag{3.22}$$

It follows from Lemmas 3.4 and 2.5 that there exists $\kappa > 0$ such that

$$\begin{aligned} \left\| e^{t\frac{1}{\alpha}\Lambda_1-t\Lambda^\alpha} \Delta_j u_0 \right\|_{L^p} &= \left\| e^{t\frac{1}{\alpha}\Lambda_1-\frac{t}{2}\Lambda^\alpha} e^{-\frac{t}{2}\Lambda^\alpha} \Delta_j u_0 \right\|_{L^p} \\ &\lesssim \left\| e^{-\frac{t}{2}\Lambda^\alpha} \Delta_j u_0 \right\|_{L^p} \lesssim e^{-\kappa 2^{\alpha j} t} \|\Delta_j u_0\|_{L^p}. \end{aligned} \tag{3.23}$$

Notice the fact that we can rewrite

$$e^{t\frac{1}{\alpha}\Lambda_1-(t-\tau)\Lambda^\alpha} = e^{-\left[(t-\tau)\frac{1}{\alpha}+\tau\frac{1}{\alpha}-t\frac{1}{\alpha}\right]\Lambda_1+\left[(t-\tau)\frac{1}{\alpha}\Lambda_1-\frac{t-\tau}{2}\Lambda^\alpha\right]-\frac{t-\tau}{2}\Lambda^\alpha} e^{\tau\frac{1}{\alpha}\Lambda_1}.$$

It follows from Lemmas 3.3 and 3.4 that

$$\begin{aligned} \left\| \int_0^t e^{t\frac{1}{\alpha}\Lambda_1-(t-\tau)\Lambda^\alpha} \Delta_j f(\tau) d\tau \right\|_{L^p} &\lesssim \int_0^t \left\| e^{-\frac{t-\tau}{2}\Lambda^\alpha} \Delta_j e^{\tau\frac{1}{\alpha}\Lambda_1} f(\tau) \right\|_{L^p} d\tau \\ &\lesssim \int_0^t e^{-\kappa(t-\tau)2^{\alpha j}} \left\| \Delta_j e^{\tau\frac{1}{\alpha}\Lambda_1} f(\tau) \right\|_{L^p} d\tau. \end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24), we see that

$$\|\Delta_j U(t)\|_{L^p} \lesssim e^{-\kappa 2^{\alpha j} t} \|\Delta_j u_0\|_{L^p} + \int_0^t e^{-\kappa(t-\tau)2^{\alpha j}} \left\| \Delta_j e^{t\frac{1}{\alpha}\Lambda_1} f(\tau) \right\|_{L^p} d\tau. \tag{3.25}$$

Taking $L^\rho([0, T])$ norm to (3.25) and using Young’s inequality,

$$\begin{aligned} \|\Delta_j U(t)\|_{L_T^\rho(L^p)} &\lesssim \left(\frac{1 - e^{-\kappa\rho 2^{\alpha j} T}}{\kappa\rho 2^{\alpha j}} \right)^{\frac{1}{\rho}} \|\Delta_j u_0\|_{L^p} \\ &\quad + \left(\frac{1 - e^{-\kappa\rho 2^{\alpha j} T}}{\kappa\rho 2^{\alpha j}} \right)^{\frac{1}{\rho_2}} \left\| \Delta_j e^{t\frac{1}{\alpha}\Lambda_1} f(\tau) \right\|_{L_T^{\rho_1}(L^p)}, \end{aligned} \tag{3.26}$$

where $\frac{1}{\rho} + 1 = \frac{1}{\rho_2} + \frac{1}{\rho_1}$. Finally, multiplying $2^{(s+\frac{\alpha}{\rho})j}$ and taking the l^q norm to (3.26), we conclude that

$$\begin{aligned} \|U\|_{\tilde{L}_T^\rho(\dot{B}_{p,q}^{s+\frac{\alpha}{\rho}})} &\lesssim \left[\sum_{j \in \mathbb{Z}} \left(\frac{1 - e^{-\kappa\rho 2^{\alpha j} T}}{\kappa\rho} \right)^{\frac{q}{\rho}} (2^{sj} \|\Delta_j u_0\|_{L^p})^q \right]^{\frac{1}{q}} \\ &\quad + \left[\sum_{j \in \mathbb{Z}} \left(\frac{1 - e^{-\kappa\rho 2^{\alpha j} T}}{\kappa\rho 2^{\alpha j}} \right)^{\frac{q}{\rho_2}} \left(2^{(s+\frac{\alpha}{\rho_1}-\alpha)j} \left\| \Delta_j e^{t\frac{1}{\alpha}\Lambda_1} f \right\|_{L_T^{\rho_1}(L^p)} \right)^q \right]^{\frac{1}{q}} \\ &\lesssim \|u_0\|_{\dot{B}_{p,q}^s} + \|f\|_{\tilde{L}^{\rho_1}(0,T;e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s+\frac{\alpha}{\rho_1}-\alpha})}, \end{aligned}$$

which leads to (3.21). □

We also need to establish the corresponding result as Lemma 3.2 in terms of the operator $e^{t\frac{1}{\alpha}\Lambda_1}$.

Lemma 3.6 *Let $s > 0, 1 < p < \infty, 1 \leq q, \rho, \rho_1, \rho_2 \leq \infty$ with $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$. Then for any $\varepsilon > 0, 0 < T \leq \infty$, we have*

$$\begin{aligned} &\|u \nabla(-\Delta)^{-1} v + v \nabla(-\Delta)^{-1} u\|_{\tilde{L}_T^\rho(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^s)} \\ &\lesssim \|u\|_{\tilde{L}_T^{\rho_1}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s+\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_2}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} \\ &\quad + \|u\|_{\tilde{L}_T^{\rho_2}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{-1+\frac{n}{p}-\varepsilon})} \|v\|_{\tilde{L}_T^{\rho_1}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s+\varepsilon})}. \end{aligned} \tag{3.27}$$

Moreover if we choose $\varepsilon = 0$, then (3.27) also holds for $q = 1$.

Proof Set $U(t) = e^{t\frac{1}{\alpha}\Lambda_1} u(t), V(t) = e^{t\frac{1}{\alpha}\Lambda_1} v(t)$. Then as Lemma 3.2, we use Bony’s paraproduct decomposition to get

$$\begin{aligned} e^{t\frac{1}{\alpha}\Lambda_1} (u \nabla(-\Delta)^{-1} v + v \nabla(-\Delta)^{-1} u) &= e^{t\frac{1}{\alpha}\Lambda_1} (e^{-t\frac{1}{\alpha}\Lambda_1} U e^{-t\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} V \\ &\quad + e^{-t\frac{1}{\alpha}\Lambda_1} V e^{-t\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} U) \\ &:= J_1 + J_2 + J_3, \end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
 J_1 &:= e^{t\frac{1}{\alpha}\Lambda_1} \sum_{j' \in \mathbb{Z}} e^{-t\frac{1}{\alpha}\Lambda_1} \Delta_{j'} U e^{-t\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} S_{j'-1} V \\
 &\quad + e^{-t\frac{1}{\alpha}\Lambda_1} \Delta_{j'} V e^{-t\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} S_{j'-1} U, \\
 J_2 &:= e^{t\frac{1}{\alpha}\Lambda_1} \sum_{j' \in \mathbb{Z}} e^{-t\frac{1}{\alpha}\Lambda_1} S_{j'-1} U e^{-t\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} \Delta_{j'} V \\
 &\quad + e^{-t\frac{1}{\alpha}\Lambda_1} S_{j'-1} V e^{-t\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} \Delta_{j'} U, \\
 J_3 &:= e^{t\frac{1}{\alpha}\Lambda_1} \sum_{j' \in \mathbb{Z}} \sum_{|j'-j''| \leq 1} e^{-t\frac{1}{\alpha}\Lambda_1} \Delta_{j'} U e^{-t\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} \Delta_{j''} V \\
 &\quad + e^{-t\frac{1}{\alpha}\Lambda_1} \Delta_{j'} V e^{-t\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} \Delta_{j''} U.
 \end{aligned}$$

To estimate the terms J_i ($i = 1, 2, 3$), we use an idea as in [37] and [1] and consider the following bilinear operator $\mathcal{B}_t(f, g)$ of the form

$$\begin{aligned}
 \mathcal{B}_t(f, g) &:= e^{t\frac{1}{\alpha}\Lambda_1} \left(e^{-t\frac{1}{\alpha}\Lambda_1} f e^{-t\frac{1}{\alpha}\Lambda_1} g \right) \\
 &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix \cdot (\xi + \eta)} e^{t\frac{1}{\alpha}(|\xi + \eta| - |\xi| - |\eta|)} \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta. \tag{3.29}
 \end{aligned}$$

Note that we can split the domain of integration into sub-domains, depending on the sign of ξ_j , of η_j and of $\xi_j + \eta_j$. Indeed, for $\varsigma = (\varsigma_1, \dots, \varsigma_n), \mu = (\mu_1, \dots, \mu_n), \nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$ such that $\varsigma_i, \mu_i, \nu_i \in \{-1, 1\}$, we denote

$$\begin{aligned}
 D_\varsigma &:= \{ \eta : \varsigma_i \eta_i \geq 0, \quad i = 1, 2, \dots, n \}, \\
 D_\mu &:= \{ \xi : \mu_i \xi_i \geq 0, \quad i = 1, 2, \dots, n \}, \\
 D_\nu &:= \{ \xi + \eta : \nu_i (\xi_i + \eta_i) \geq 0, \quad i = 1, 2, \dots, n \}.
 \end{aligned}$$

Let χ_D be the characteristic function on the domain D . Then we can rewrite $\mathcal{B}_t(f, g)$ as

$$\mathcal{B}_t(f, g) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix \cdot (\xi + \eta)} \chi_{D_\nu} e^{t\frac{1}{\alpha}(|\xi + \eta| - |\xi| - |\eta|)} \chi_{D_\mu} \hat{f}(\xi) \chi_{D_\varsigma} \hat{g}(\eta) d\xi d\eta.$$

By this observation, we introduce the monodimensional operators:

$$K_1 f := \frac{1}{2\pi} \int_0^{+\infty} e^{ix\xi} \hat{f}(\xi) d\xi, \quad K_{-1} f := \frac{1}{2\pi} \int_{-\infty}^0 e^{ix\xi} \hat{f}(\xi) d\xi,$$

and

$$L_{t, \varepsilon_1, \varepsilon_2} f := f \text{ if } \varepsilon_1 \varepsilon_2 = 1, \quad L_{t, \varepsilon_1, \varepsilon_2} f := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} e^{-2t\frac{1}{\alpha}|\xi|} \hat{f}(\xi) d\xi \text{ if } \varepsilon_1 \varepsilon_2 = -1.$$

Moreover for $t > 0$, we define the operator

$$Z_{t, \varsigma, \mu} := K_{\mu_1} L_{t, \varsigma_1, \mu_1} \otimes \dots \otimes K_{\mu_n} L_{t, \varsigma_n, \mu_n}. \tag{3.30}$$

We mention here that the above tensor product (3.30) means that the j -th operator in the tensor product acts on the j -th variable of the function $f(x_1, \dots, x_n)$. Then an elementary calculation yields the following identity:

$$\mathcal{B}_t(f, g) = \sum_{\varsigma, \mu, \nu \in \{-1, 1\}^{n \times 3}} K_{\varsigma_1} \otimes \dots \otimes K_{\varsigma_n}(Z_{t, \varsigma, \mu} f Z_{t, \varsigma, \nu} g). \tag{3.31}$$

Noticing that for $\xi + \eta \in D_\nu$, $\xi \in D_\mu$ and $\eta \in D_\varsigma$, $e^{t \frac{1}{\alpha} (|\xi + \eta|_1 - |\xi|_1 - |\eta|_1)}$ must belong to the following set:

$$\mathbb{E} := \left\{ 1, e^{-2t \frac{1}{\alpha} |\xi_i + \eta_i|_1}, e^{-2t \frac{1}{\alpha} |\xi_i|_1}, e^{-2t \frac{1}{\alpha} |\eta_i|_1}, i = 1, 2, \dots, n \right\}.$$

Moreover it is clear that $\chi_{D_\varsigma}, \chi_{D_\mu}, \chi_{D_\nu} \in \mathcal{M}_p$, and every element in \mathbb{E} are the Fourier multipliers on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, which yield that the operators K_ς and $Z_{t, \varsigma, \mu}$ defined above are combinations of the identity operator and of the Fourier multipliers on $L^p(\mathbb{R}^n)$ (including Hilbert transform). Hence the operators K_ς and $Z_{t, \varsigma, \mu}$ are bounded linear operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and the corresponding operator norm of $Z_{t, \varsigma, \mu}$ is bounded independent of $t \geq 0$. Moreover for $1 < p, p_1, p_2 < \infty$,

$$\|\mathcal{B}_t(f, g)\|_{L^p} \lesssim \|Z_{t, \varsigma, \mu} f Z_{t, \varsigma, \nu} g\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \quad \text{with} \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}.$$

Since the nice boundedness property of the bilinear operator $\mathcal{B}_t(f, g)$, we can follow the proof of Lemma 3.2 to complete the proof of Lemma 3.4. Indeed, we take the first term of J_1 as an example:

$$\begin{aligned} & \left\| \Delta_j e^{t \frac{1}{\alpha} \Lambda_1} \sum_{j' \in \mathbb{Z}} e^{-t \frac{1}{\alpha} \Lambda_1} \Delta_{j'} U e^{-t \frac{1}{\alpha} \Lambda_1} \nabla(-\Delta)^{-1} S_{j'-1} V \right\|_{L_T^p(L^p)} \\ &= \left\| \Delta_j \sum_{j' \in \mathbb{Z}} \mathcal{B}_t(\Delta_{j'} U, \nabla(-\Delta)^{-1} S_{j'-1} V) \right\|_{L_T^p(L^p)} \\ &\lesssim \sum_{|j'-j| \leq 4} \left\| K_{\varsigma_1} \otimes \dots \otimes K_{\varsigma_n}(Z_{t, \varsigma, \mu} \Delta_{j'} U Z_{t, \varsigma, \nu} \nabla(-\Delta)^{-1} S_{j'-1} V) \right\|_{L_T^p(L^p)} \\ &\lesssim \sum_{|j'-j| \leq 4} \|Z_{t, \varsigma, \mu} \Delta_{j'} U\|_{L_T^{p_1}(L^p)} \|Z_{t, \varsigma, \nu} \nabla(-\Delta)^{-1} S_{j'-1} V\|_{L_T^{p_2}(L^\infty)} \\ &\lesssim \sum_{|j'-j| \leq 4} \|Z_{t, \varsigma, \mu} \Delta_{j'} U\|_{L_T^{p_1}(L^p)} \sum_{k \leq j'-2} 2^{(-1 + \frac{n}{p})k} \|Z_{t, \varsigma, \nu} \nabla(-\Delta)^{-1} \Delta_k V\|_{L_T^{p_2}(L^p)} \\ &\lesssim \sum_{|j'-j| \leq 4} \|\Delta_{j'} U\|_{L_T^{p_1}(L^p)} \sum_{k \leq j'-2} 2^{\varepsilon k} 2^{(-1 + \frac{n}{p} - \varepsilon)k} \|\Delta_k V\|_{L_T^{p_2}(L^p)} \\ &\lesssim \sum_{|j'-j| \leq 4} 2^{-sj'} 2^{(s+\varepsilon)j'} \|\Delta_{j'} U\|_{L_T^{p_1}(L^p)} \|V\|_{\tilde{L}_T^{p_2}(\dot{B}_{p, q}^{-1 + \frac{n}{p} - \varepsilon})}. \end{aligned}$$

The other terms can be established analogously, and thus, we get the desired estimate (3.27). □

Combining Proposition 3.5 and Lemma 3.6, returning to the mapping (3.16), we obtain

$$\begin{aligned} \|\mathbb{F}(u)\|_{\tilde{L}_T^{\rho_1}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s_1})\cap\tilde{L}_T^{\rho_2}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s_2})} &\lesssim \left\| e^{t\frac{1}{\alpha}\Lambda_1-t\Lambda^\alpha}u_0 \right\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1})\cap\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})} \\ &\quad + \|u\nabla(-\Delta)^{-1}u\|_{\tilde{L}_T^{\rho_1}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s_1})\cap\tilde{L}_T^{\rho_2}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s_2})} \\ &\lesssim \left\| e^{-\frac{t}{2}\Lambda^\alpha}u_0 \right\|_{\tilde{L}_T^{\rho_1}(\dot{B}_{p,q}^{s_1})\cap\tilde{L}_T^{\rho_2}(\dot{B}_{p,q}^{s_2})} \\ &\quad + \|u\|_{\tilde{L}_T^{\rho_1}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s_1})\cap\tilde{L}_T^{\rho_2}(e^{t\frac{1}{\alpha}\Lambda_1}\dot{B}_{p,q}^{s_2})}^2. \end{aligned} \tag{3.32}$$

Based on the above estimate (3.32), by applying the standard contraction mapping argument, we complete the proof, as desired.

3.3 The case $1 < \alpha < 2$ and $p = \infty$: well-posedness

In this subsection, we focus on the limit case $p = \infty$. We first intend to prove the following result.

Lemma 3.7 *For $1 \leq \alpha < 2$, we have*

$$\begin{aligned} \|u\nabla(-\Delta)^{-1}v + v\nabla(-\Delta)^{-1}u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^{1-\alpha})} &\lesssim \|u\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} \|v\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \\ &\quad + \|u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})}. \end{aligned} \tag{3.33}$$

Proof Following from Lemma 3.2, by applying Hölder’s inequality, Lemmas 2.3 and 2.4, we estimate the terms I_i ($i = 1, 2, 3$) as follows:

$$\begin{aligned} &\|\Delta_j I_1\|_{L_t^1(L^\infty)} \\ &\lesssim \sum_{|j'-j|\leq 4} \left(\|\Delta_{j'}u\|_{L_t^1(L^\infty)} \|\nabla(-\Delta)^{-1}S_{j'-1}v\|_{L_t^\infty(L^\infty)} \right. \\ &\quad \left. + \|\Delta_{j'}v\|_{L_t^1(L^\infty)} \|\nabla(-\Delta)^{-1}S_{j'-1}u\|_{L_t^\infty(L^\infty)} \right) \\ &\lesssim \sum_{|j'-j|\leq 4} \left(\|\Delta_{j'}u\|_{L_t^1(L^\infty)} \sum_{k\leq j'-2} 2^{(\alpha-1)k} 2^{-\alpha k} \|\Delta_k v\|_{L_t^\infty(L^\infty)} \right. \\ &\quad \left. + \|\Delta_{j'}v\|_{L_t^1(L^\infty)} \sum_{k\leq j'-2} 2^{(\alpha-1)k} 2^{-\alpha k} \|\Delta_k u\|_{L_t^\infty(L^\infty)} \right) \\ &\lesssim 2^{(\alpha-1)j} \sum_{|j'-j|\leq 4} \left(\|\Delta_{j'}u\|_{L_t^1(L^\infty)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} + \|\Delta_{j'}v\|_{L_t^1(L^\infty)} \|u\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} \right). \end{aligned}$$

This along with Definition 2.2 leads to

$$\|I_1\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^{1-\alpha})} \lesssim \|u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} + \|u\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} \|v\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)}. \tag{3.34}$$

Similarly, for I_2 , we obtain

$$\begin{aligned} & \|\Delta_j I_2\|_{L_t^1(L^\infty)} \\ & \lesssim \sum_{|j'-j|\leq 4} 2^{-j'} \sum_{k\leq j'-2} \left(\|\Delta_k u\|_{L_t^1(L^\infty)} \|\Delta_{j'} v\|_{L_t^\infty(L^\infty)} + \|\Delta_k v\|_{L_t^1(L^\infty)} \|\Delta_{j'} u\|_{L_t^\infty(L^\infty)} \right) \\ & \lesssim 2^{(\alpha-1)j} \sum_{|j'-j|\leq 4} 2^{-\alpha j'} \left(\|\Delta_{j'} v\|_{L_t^\infty(L^\infty)} \|u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} + \|\Delta_{j'} u\|_{L_t^\infty(L^\infty)} \|v\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \right), \end{aligned}$$

which yields directly to

$$\|I_2\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^{1-\alpha})} \lesssim \|u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} + \|u\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} \|v\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)}. \tag{3.35}$$

To treat with the remainder term I_3 , as Lemma 3.2, we split $I_3 = K_1 + K_2 + K_3$ for $m = 1, 2, \dots, n$ and consider K_1 and K_3 only. It follows from Hölder’s inequality, Lemmas 2.3 and 2.4 that

$$\begin{aligned} \|\Delta_j K_1\|_{L_t^1(L^\infty)} & \lesssim 2^{2j} \sum_{j'\geq j-N_0} \sum_{|j'-j''|\leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_t^1(L^\infty)} \|\partial_m (-\Delta)^{-1} \Delta_{j''} v\|_{L_t^\infty(L^\infty)} \\ & \lesssim 2^{2j} \sum_{j'\geq j-N_0} \sum_{|j'-j''|\leq 1} 2^{-2j'} \|\Delta_{j'} u\|_{L_t^1(L^\infty)} 2^{-j''} \|\Delta_{j''} v\|_{L_t^\infty(L^\infty)} \\ & \lesssim 2^{2j} \sum_{j'\geq j-N_0} 2^{(\alpha-3)j'} \|\Delta_{j'} u\|_{L_t^1(L^\infty)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} \\ & \lesssim 2^{(\alpha-1)j} \sum_{j'\geq j-N_0} 2^{(\alpha-3)(j'-j)} \|\Delta_{j'} u\|_{L_t^1(L^\infty)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} \|\Delta_j K_3\|_{L_t^1(L^\infty)} & \lesssim 2^j \sum_{j'\geq j-N_0} \sum_{|j'-j''|\leq 1} 2^{-2j'} \|\Delta_{j'} u\|_{L_t^1(L^\infty)} \|\Delta_{j''} v\|_{L_t^\infty(L^\infty)} \\ & \lesssim 2^j \sum_{j'\geq j-N_0} 2^{(\alpha-2)j'} \|\Delta_{j'} u\|_{L_t^1(L^\infty)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})} \\ & \lesssim 2^{(\alpha-1)j} \sum_{j'\geq j-N_0} 2^{(\alpha-2)(j'-j)} \|\Delta_{j'} u\|_{L_t^1(L^\infty)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})}. \end{aligned} \tag{3.37}$$

Under the assumption $1 \leq \alpha < 2$, we have $\alpha - 3 < 0$ and $\alpha - 2 < 0$. Hence putting the above estimates (3.36) and (3.37) together, and multiplying $2^{(1-\alpha)j}$ to the resulting inequality, then taking l^1 norm yields that

$$\|I_3\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^{1-\alpha})} \lesssim \|u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha})}. \tag{3.38}$$

Thanks to (3.34), (3.35) and (3.38), we get (3.33). The proof of Lemma 3.7 is complete. \square

In order to prove the third part of Theorem 1.1, we consider the resolution space $\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha}(\mathbb{R}^n)) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0(\mathbb{R}^n))$. Then for the mapping (3.16), we infer from Proposition 3.1 and Lemma 3.7 that

$$\begin{aligned} \|\mathbb{F}(u)\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} & \lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-\alpha}} + \|u \nabla (-\Delta)^{-1} u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^{1-\alpha})} \\ & \lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-\alpha}} + \|u\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)}^2. \end{aligned} \tag{3.39}$$

As before, applying the standard contraction mapping argument, we can show that if $\|u_0\|_{\dot{B}_{\infty,1}^{-\alpha}}$ is sufficiently small, then \mathbb{F} is a contraction mapping from some suitable metric space into itself, and this leads to that the system (1.1) admits a unique solution in $u \in \tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha}(\mathbb{R}^n)) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0(\mathbb{R}^n))$. We complete the proof, as desired.

3.4 The case $1 < \alpha < 2$ and $p = \infty$: Gevrey analyticity

Set $U(t) := e^{t\frac{1}{\alpha}\Lambda_1}u(t)$. Then $U(t)$ satisfies the following integral equation

$$U(t) = e^{t\frac{1}{\alpha}\Lambda_1 - t\Lambda^\alpha} u_0 - \int_0^t \left[e^{t\frac{1}{\alpha}\Lambda_1 - (t-\tau)\Lambda^\alpha} \nabla \cdot \left(e^{-\tau\frac{1}{\alpha}\Lambda_1} U \cdot e^{-\tau\frac{1}{\alpha}\Lambda_1} \nabla(-\Delta)^{-1} U \right) \right] d\tau. \tag{3.40}$$

Consider the linear part, since the symbol $e^{t\frac{1}{\alpha}|\xi|^{1-\frac{1}{2}|\xi|^\alpha}$ is uniformly bounded for all ξ and decays exponentially for $|\xi| \gg 1$, when localized in dyadic blocks in the Fourier spaces, the Fourier multiplier $F_\alpha := e^{t\frac{1}{\alpha}\Lambda_1 - \frac{1}{2}t\Lambda^\alpha}$ maps uniformly bounded from L^∞ to L^∞ for all $t \geq 0$. Then by Young’s inequality, we have

$$\left\| e^{t\frac{1}{\alpha}\Lambda_1 - t\Lambda^\alpha} u_0 \right\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \lesssim \left\| e^{-\frac{1}{2}t\Lambda^\alpha} u_0 \right\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-\alpha}}.$$

For the nonlinear part, by proceeding the same line as the proof of Lemma 3.6, and observe that in general, the operators K_ζ and $Z_{t,\zeta,\mu}$ defined in Lemma 3.6 do not map L^∞ to L^∞ boundedly. However, when localized in dyadic blocks in the Fourier spaces, these operators are bounded in L^∞ . Therefore we can follow the calculations line by line from (3.34) to (3.38) in the proof of Lemma 3.7 to deal with the nonlinear term, and finally together with the estimate of the linear term ensure that

$$\|u(t)\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-\alpha}} + \|u(t)\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-\alpha}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)}^2.$$

This completes the proof, as desired.

3.5 Decay rate of solution

In this subsection, we focus on the decay rate estimates of solutions obtained in Theorem 1.1. The proof is based on the following result.

Lemma 3.8 *For all $\sigma \geq 0$ and $1 < \alpha \leq 2$, the operator $\Lambda^\sigma e^{-t\frac{1}{\alpha}\Lambda_1}$ is the convolution operator with a kernel $K_\sigma(t) \in L^1(\mathbb{R}^n)$ for all $t > 0$. Moreover*

$$\|K_\sigma(t)\|_{L^1} \leq C_\sigma t^{-\frac{\sigma}{\alpha}}, \tag{3.41}$$

where $C_\sigma := \|\Lambda^\sigma e^{-\Lambda_1}\|_{L^1}$.

Proof It suffices to consider the operator $\Lambda^\sigma e^{-\Lambda_1}$ and its kernel $\hat{k}_\sigma(\xi) = |\xi|^\sigma e^{-|\xi|_1}$ due to the general case can be obtained by using the scaling: $\xi \mapsto t^{\frac{1}{\alpha}}\xi$. It is clear that $\hat{k}_\sigma(\xi) = |\xi|^\sigma e^{-|\xi|_1} \in L^1$. Thus k_σ is a continuous bounded function. Moreover if $\sigma > 0$, we introduce a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ so that $0 \notin \text{Supp } \phi$ and $\sum_{j \in \mathbb{Z}} \phi(2^j \xi) = 1$. Then $|\xi|^\sigma \phi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, and if we write $|\xi|^\sigma \phi(\xi) = \hat{\Phi}_\sigma(\xi)$ and $\theta = 1 - \sum_{j \geq 0} \phi(2^j \xi)$, then we have

$$\hat{k}_\sigma(\xi) = \sum_{j \geq 0} 2^{-j\sigma} \hat{\Phi}_\sigma(2^j \xi) e^{-|\xi|_1} + \theta(\xi) |\xi|^\sigma e^{-|\xi|_1}.$$

Hence

$$\|k_\sigma\|_{L^1} \leq \sum_{j \geq 0} 2^{-j\sigma} \|\Phi_\sigma\|_{L^1} \|\mathcal{F}^{-1}(e^{-|\xi|1})\|_{L^1} + \|\mathcal{F}^{-1}(\theta(\xi)|\xi|^\sigma e^{-|\xi|1})\|_{L^1} < \infty.$$

We complete the proof of Lemma 3.8. □

Now the existence parts of Theorem 1.1 tell us that if the initial data u_0 is sufficiently small in critical Besov spaces $\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}(\mathbb{R}^n)$ for either $1 < \alpha \leq 2, 1 < p < \infty$ and $1 \leq q \leq \infty$ or $1 < \alpha < 2, p = \infty$ and $q = 1$, then the solution is in the Gevrey class. Consequently, for all $\sigma \geq 0$, applying Lemma 3.8, we get the following time decay of mild solution in terms of the homogeneous Besov norm:

$$\begin{aligned} \|\Lambda^\sigma u(t)\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}} &= \left\| \Lambda^\sigma e^{-t\frac{1}{\alpha}\Lambda_1} e^{t\frac{1}{\alpha}\Lambda_1} u(t) \right\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}} \\ &\leq C_\sigma t^{-\frac{\sigma}{\alpha}} \left\| e^{t\frac{1}{\alpha}\Lambda_1} u(t) \right\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}} \\ &\leq C_\sigma t^{-\frac{\sigma}{\alpha}} \|u_0\|_{\dot{B}_{p,q}^{-\alpha+\frac{n}{p}}}. \end{aligned} \tag{3.42}$$

This completes the proof, as desired.

4 The case $\alpha = 1$: the Proof of Theorem 1.2

In this section, we consider the case $\alpha = 1$ for the system (1.1) with initial data in critical spaces $\dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ ($1 \leq p \leq \infty$). The global well-posedness with small initial data and Gevrey analyticity will be established in the case that $1 \leq p < \infty$ and $p = \infty$, respectively.

4.1 The case $1 \leq p < \infty$: well-posedness

We first recall some time-space estimates for solutions of the linear evolution equation:

$$\begin{cases} \partial_t u + \Lambda u = f(x, t), & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.1}$$

Proposition 4.1 ([29]) *Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$ and $0 < T \leq \infty$. Assume that $u_0 \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ and $f \in \tilde{L}_T^1(\dot{B}_{p,q}^s(\mathbb{R}^n))$. Then (4.1) has a unique solution $u \in \tilde{L}_T^\infty(\dot{B}_{p,q}^s(\mathbb{R}^n)) \cap \tilde{L}_T^1(\dot{B}_{p,q}^{s+1}(\mathbb{R}^n))$. In addition, there exists a constant $C > 0$ depending only on n such that*

$$\|u\|_{\tilde{L}_T^\infty(\dot{B}_{p,q}^s) \cap \tilde{L}_T^1(\dot{B}_{p,q}^{s+1})} \leq C \left(\|u_0\|_{\dot{B}_{p,q}^s} + \|f\|_{\tilde{L}_T^1(\dot{B}_{p,q}^s)} \right). \tag{4.2}$$

Now for any initial data $u_0 \in \dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)$, we consider the resolution space $\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n))$. Slightly modifying the proof of Lemma 3.7, we get the following result.

Lemma 4.2 *For any $u, v \in \tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})$, we have*

$$\|u \nabla(-\Delta)^{-1} v + v \nabla(-\Delta)^{-1} u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \lesssim \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}. \tag{4.3}$$

Proof We calculate the estimation of I_1 as follows:

$$\begin{aligned}
 & \|\Delta_j I_1\|_{L_t^\infty(L^p)} \\
 & \lesssim \sum_{|j'-j|\leq 4} \left(\|\Delta_{j'} u\|_{L_t^\infty(L^p)} \|\nabla(-\Delta)^{-1} S_{j'-1} v\|_{L_t^\infty(L^\infty)} \right. \\
 & \quad \left. + \|\Delta_{j'} v\|_{L_t^\infty(L^p)} \|\nabla(-\Delta)^{-1} S_{j'-1} u\|_{L_t^\infty(L^\infty)} \right) \\
 & \lesssim \sum_{|j'-j|\leq 4} \left(\|\Delta_{j'} u\|_{L_t^\infty(L^p)} \sum_{k\leq j'-2} 2^{(-1+\frac{n}{p})k} \|\Delta_k v\|_{L_t^\infty(L^p)} \right. \\
 & \quad \left. + \|\Delta_{j'} v\|_{L_t^\infty(L^p)} \sum_{k\leq j'-2} 2^{(-1+\frac{n}{p})k} \|\Delta_k u\|_{L_t^\infty(L^p)} \right) \\
 & \lesssim \sum_{|j'-j|\leq 4} \left(\|\Delta_{j'} u\|_{L_t^\infty(L^p)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} + \|\Delta_{j'} v\|_{L_t^\infty(L^p)} \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \right). \tag{4.4}
 \end{aligned}$$

Multiplying $2^{(-1+\frac{n}{p})j}$ to (4.4), then taking l^1 norm to the resulting inequality, we get

$$\|I_1\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \lesssim \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}. \tag{4.5}$$

Similarly, for I_2 ,

$$\begin{aligned}
 & \|\Delta_j I_2\|_{L_t^\infty(L^p)} \\
 & \lesssim \sum_{|j'-j|\leq 4} \left(\sum_{k\leq j'-2} 2^k 2^{(-1+\frac{n}{p})k} \|\Delta_k u\|_{L_t^\infty(L^p)} 2^{-j'} \|\Delta_{j'} v\|_{L_t^\infty(L^p)} \right. \\
 & \quad \left. + \sum_{k\leq j'-2} 2^k 2^{(-1+\frac{n}{p})k} \|\Delta_k v\|_{L_t^\infty(L^p)} 2^{-j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} \right) \\
 & \lesssim \sum_{|j'-j|\leq 4} \left(\|\Delta_{j'} v\|_{L_t^\infty(L^p)} \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} + \|\Delta_{j'} u\|_{L_t^\infty(L^p)} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \right), \tag{4.6}
 \end{aligned}$$

which leads directly to

$$\|I_2\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \lesssim \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}. \tag{4.7}$$

Moreover for the remainder term $I_3 = K_1 + K_2 + K_3$. In the case that $2 \leq p < \infty$, K_1 and K_3 can be estimated as follows (K_2 can be done analogously):

$$\begin{aligned}
 & \|\Delta_j K_1\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(2+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_t^\infty(L^p)} \|\partial_m (-\Delta)^{-1} \Delta_{j''} v\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(2+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} 2^{-2j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} 2^{-j''} \|\Delta_{j''} v\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(2+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-(1+\frac{2n}{p})j'} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} v\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(1-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-(1+\frac{2n}{p})(j'-j)} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} \|v\|_{L_t^\infty\left(\dot{B}_{p,1}^{-1+\frac{n}{p}}\right)}, \tag{4.8}
 \end{aligned}$$

$$\begin{aligned}
 & \|\Delta_j K_3\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(1+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_t^\infty(L^p)} \|\Delta_{j''} v\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(1+\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-\frac{2n}{p}j'} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} v\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(1-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{\left(-\frac{2n}{p}\right)(j'-j)} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} \|v\|_{L_t^\infty\left(\dot{B}_{p,1}^{-1+\frac{n}{p}}\right)}. \tag{4.9}
 \end{aligned}$$

In the case that $1 \leq p < 2$, there exists $2 \leq p' \leq \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ such that

$$\begin{aligned}
 & \|\Delta_j K_1\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(2+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_t^\infty(L^{p'})} \|\partial_m (-\Delta)^{-1} \Delta_{j''} v\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(2+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} 2^{\left(-2+n\left(\frac{1}{p}-\frac{1}{p'}\right)\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} 2^{-j''} \|\Delta_{j''} v\|_{L_t^{\rho_1}(L^p)} \\
 & \lesssim 2^{\left(2+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-(n+1)j'} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} v\|_{L_t^{\rho_1}(L^p)} \\
 & \lesssim 2^{\left(1-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-(n+1)(j'-j)} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} \|v\|_{L_t^\infty\left(\dot{B}_{p,1}^{-1+\frac{n}{p}}\right)}, \tag{4.10}
 \end{aligned}$$

$$\begin{aligned}
 & \|\Delta_j K_3\|_{L_t^\infty(L^p)} \lesssim 2^{\left(1+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} \sum_{|j'-j''| \leq 1} \|(-\Delta)^{-1} \Delta_{j'} u\|_{L_t^\infty(L^{p'})} \|\Delta_{j''} v\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(1+n-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-nj'} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} v\|_{L_t^\infty(L^p)} \\
 & \lesssim 2^{\left(1-\frac{n}{p}\right)j} \sum_{j' \geq j-N_0} 2^{-n(j'-j)} 2^{\left(-1+\frac{n}{p}\right)j'} \|\Delta_{j'} u\|_{L_t^\infty(L^p)} \|v\|_{L_t^\infty\left(\dot{B}_{p,1}^{-1+\frac{n}{p}}\right)}. \tag{4.11}
 \end{aligned}$$

Thus putting the above estimates (4.8)–(4.11) together, we obtain for all $1 \leq p < \infty$,

$$\|I_3\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \lesssim \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}. \tag{4.12}$$

Combining (4.5), (4.7) and (4.12), we conclude that (4.3) holds. The proof of Lemma 4.2 is complete. \square

Based on Proposition 4.1 and Lemma 4.2, consider the mapping (3.16), we obtain

$$\begin{aligned} \|\mathbb{F}(u)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} &\lesssim \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}} + \|u \nabla(-\Delta)^{-1} u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \\ &\lesssim \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}} + \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}^2. \end{aligned} \tag{4.13}$$

Thus, if $\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}$ is sufficiently small, we can prove that \mathbb{F} is a contraction mapping from some suitable metric space into itself, which implies that the system (1.1) admits a unique solution in $\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n))$. The proof is complete, as desired.

4.2 The case $1 < p < \infty$: Gevrey analyticity

Notice that when $\alpha = 1$, the dissipation term $e^{-t\Lambda}$ is not strong enough to overcome the operator $e^{t\Lambda_1}$. Therefore we need to define the Gevrey operator more carefully. Since $\frac{1}{2n}|\xi|_1 < \frac{1}{2}|\xi|$ for all $\xi \in \mathbb{R}^n$, we define

$$U(t) := e^{\frac{1}{2n}t\Lambda_1} u(t).$$

Then $U(t)$ satisfies the following integral equation:

$$U(t) = e^{\frac{1}{2n}t\Lambda_1 - t\Lambda} u_0 - \int_0^t \left[e^{\frac{1}{2n}t\Lambda_1 - (t-\tau)\Lambda} \nabla \cdot (e^{-\frac{1}{2n}\tau\Lambda_1} U \cdot e^{-\frac{1}{2n}\tau\Lambda_1} \nabla(-\Delta)^{-1} U) \right] d\tau. \tag{4.14}$$

Notice that the operator $e^{\frac{1}{2n}t\Lambda_1 - \frac{1}{2}t\Lambda}$ is a Fourier multiplier which maps uniformly bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and moreover, its operator norm is uniformly bounded with respect to any $t \geq 0$ because the symbol $e^{\frac{1}{2n}t|\xi|_1 - \frac{1}{2}t|\xi|}$ is uniformly bounded and decays exponentially for all $|\xi| \geq 1$. Therefore by Proposition 4.1, the linear term can be treated with

$$\left\| e^{\frac{1}{2n}t\Lambda_1 - t\Lambda} u_0 \right\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \lesssim \left\| e^{-\frac{1}{2}t\Lambda} u_0 \right\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})} \lesssim \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}. \tag{4.15}$$

For the nonlinear term, we rewrite

$$e^{\frac{1}{2n}t\Lambda_1 - (t-\tau)\Lambda} = e^{\frac{1}{2n}(t-\tau)\Lambda_1 - (t-\tau)\Lambda} e^{\frac{1}{2n}\tau\Lambda_1}.$$

Thus based on the nice boundedness properties of the operator $e^{\frac{1}{2n}t\Lambda_1 - \frac{1}{2}t\Lambda}$ and the bilinear operator $\tilde{\mathcal{B}}_t(f, g)$ of the form

$$\tilde{\mathcal{B}}_t(f, g) := e^{\frac{1}{2n}t\Lambda_1} \left(e^{-\frac{1}{2n}t\Lambda_1} f e^{-\frac{1}{2n}t\Lambda_1} g \right),$$

we can proceed along the lines of the proof of Lemma 4.2 to obtain the Gevrey analyticity of the solution. Indeed, the bilinear operator $\tilde{\mathcal{B}}_t(f, g)$ has a similar expression as (3.31), and moreover, the corresponding operators $\tilde{\mathcal{K}}_\zeta$ and $\tilde{\mathcal{Z}}_{t,\zeta,\mu}$ are bounded linear operators on $L^p(\mathbb{R}^n)$

for $1 < p < \infty$, and the corresponding operator norm of $\tilde{Z}_{t,\varsigma,\mu}$ is bounded independent of $t \geq 0$, thus, for $1 < p, p_1, p_2 < \infty$, we still have

$$\|\tilde{\mathcal{B}}_t(f, g)\|_{L^p} \lesssim \|\tilde{Z}_{t,\varsigma,\mu} f \tilde{Z}_{t,\varsigma,\nu} g\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \quad \text{with} \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}.$$

This completes the proof, as desired.

4.3 The case $p = \infty$: well-posedness

In the case $p = \infty$, the resolution space $\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}(\mathbb{R}^n))$ can not be adapted to the system (1.1), and therefore, we turn to consider the resolution space $\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}(\mathbb{R}^n)) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0(\mathbb{R}^n))$. Firstly, from Proposition 4.1, we see that

$$\|e^{-t\Delta} u_0\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-1}}. \tag{4.16}$$

Secondly, from Lemma 3.7, we get

$$\|u \nabla(-\Delta)^{-1} u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \lesssim \|u\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)}^2. \tag{4.17}$$

Hence consider the mapping (3.16), we deduce from Proposition 4.1, (4.16) and (4.17) that

$$\begin{aligned} \|\mathbb{F}(u)\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} &\lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-1}} + \|u \nabla(-\Delta)^{-1} u\|_{\tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \\ &\lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-1}} + \|u\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)}^2. \end{aligned} \tag{4.18}$$

This reveals that, through the standard contraction mapping argument, if $\|u_0\|_{\dot{B}_{\infty,1}^{-1}}$ is sufficiently small, then \mathbb{F} is a contraction mapping from some suitable metric space into itself, which means that the system (1.1) admits a unique solution in $\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}(\mathbb{R}^n)) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0(\mathbb{R}^n))$. The proof is complete, as desired.

4.4 The case $p = \infty$: Gevrey analyticity

To treat the Gevrey analyticity of solution in the case $p = \infty$, it suffices to prove that the following a priori estimate holds:

$$\left\| e^{\frac{1}{2n}t\Lambda_1} u(t) \right\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-1}} + \left\| e^{\frac{1}{2n}t\Lambda_1} u(t) \right\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)}^2. \tag{4.19}$$

Since the symbol $e^{\frac{1}{2n}t|\xi| - \frac{1}{2}t|\xi|^2}$ is uniformly bounded in \mathbb{R}^n and decays exponentially for sufficiently large $|\xi| \gg 1$ with respect to all $t \geq 0$, the estimation of linear part is straightforward due to the fact that when localized in dyadic blocks in the Fourier spaces, the operator $e^{\frac{1}{2n}t\Lambda_1 - \frac{1}{2}t\Lambda}$ maps uniformly bounded from L^∞ to L^∞ with respect to $t \geq 0$. Thus

$$\left\| e^{\frac{1}{2n}t\Lambda_1 - t\Lambda} u_0 \right\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \lesssim \left\| e^{-\frac{1}{2}t\Lambda} u_0 \right\|_{\tilde{L}_t^\infty(\dot{B}_{\infty,1}^{-1}) \cap \tilde{L}_t^1(\dot{B}_{\infty,1}^0)} \lesssim \|u_0\|_{\dot{B}_{\infty,1}^{-1}}. \tag{4.20}$$

For the nonlinear part, following the proofs of Lemmas 3.6 and 3.7, the only difficulty arises from the following bilinear operator $\tilde{\mathcal{B}}_t(f, g)$ of the form

$$\tilde{\mathcal{B}}_t(f, g) = e^{\frac{1}{2n}t\Lambda_1} \left(e^{-\frac{1}{2n}t\Lambda_1} f e^{-\frac{1}{2n}t\Lambda_1} g \right)$$

is not bounded from $L^\infty \times L^\infty$ to L^∞ , more precisely, the corresponding operators \tilde{K}_ζ and $\tilde{Z}_{t,\zeta,\mu}$ in (3.31) do not map L^∞ to L^∞ uniformly bounded. However when localized in dyadic blocks in the Fourier spaces, these operators are bounded in L^∞ . Therefore we can follow the calculations line by line from (3.34) to (3.38) with $\alpha = 1$ in the proof of Lemma 3.7 to complete the estimation of the nonlinear term, which along with (4.20), we arrive at (4.19). The proof is complete, as desired.

4.5 Decay rate of solution

In this subsection, we show the decay rate estimates of solutions obtained in Theorem 1.2. Based on Lemma 3.8, we can show that for all $\sigma \geq 0$, the operator $\Lambda^\sigma e^{-\frac{1}{2n}t\Lambda_1}$ is the convolution operator with a kernel $K_\sigma(t) \in L^1(\mathbb{R}^n)$ for all $t > 0$. Moreover

$$\|K_\sigma(t)\|_{L^1} \leq \tilde{C}_\sigma t^{-\sigma}, \tag{4.21}$$

where $\tilde{C}_\sigma = \|\Lambda^\sigma e^{-\frac{1}{2n}t\Lambda_1}\|_{L^1}$. Now we know that the existence parts of Theorem 1.2 imply that if $u_0 \in \dot{B}_{p,1}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ ($1 < p \leq \infty$) is sufficiently small, then the solution is in the Gevrey class. Consequently for all $\sigma \geq 0$, applying (4.21), we get

$$\begin{aligned} \|\Lambda^\sigma u(t)\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}} &= \left\| \Lambda^\sigma e^{-\frac{1}{2n}t\Lambda_1} e^{\frac{1}{2n}t\Lambda_1} u(t) \right\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}} \\ &\leq \tilde{C}_\sigma t^{-\sigma} \left\| e^{\frac{1}{2n}t\Lambda_1} u(t) \right\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}} \\ &\leq \tilde{C}_\sigma t^{-\sigma} \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}. \end{aligned} \tag{4.22}$$

We complete the proof of Theorem 1.2, as desired.

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