

Optimal partition problems for the fractional Laplacian

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Abstract In this work, we prove an existence result for an optimal partition problem of the form

$$\min\{F_s(A_1, \dots, A_m) : A_i \in \mathcal{A}_s, A_i \cap A_j = \emptyset \text{ for } i \neq j\},$$

where F_s is a cost functional with suitable assumptions of monotonicity and lower semicontinuity, \mathcal{A}_s is the class of admissible domains and the condition $A_i \cap A_j = \emptyset$ is understood in the sense of Gagliardo s -capacity, where $0 < s < 1$. Examples of this type of problem are related to fractional eigenvalues. As the main outcome of this article, we prove some type of convergence of the s -minimizers to the minimizer of the problem with $s = 1$, studied in [5].

Keywords Fractional partial equations · Fractional capacities · Optimal partition

Mathematics Subject Classification 35R11 · 49Q10

1 Introduction

Throughout this article, we consider $\Omega \subset \mathbb{R}^n$ to be a fixed Lipschitz domain, that is an open bounded subset of \mathbb{R}^n with Lipschitz boundary. Fix $0 < s < 1$ and $m \in \mathbb{N}$. We consider optimal partition problems of the form

$$\min \{F_s(A_1, \dots, A_m) : A_i \in \mathcal{A}_s(\Omega), A_i \cap A_j = \emptyset \text{ for } i \neq j\}, \quad (1.1)$$

where F_s is a cost functional which satisfies some lower semicontinuity and monotonicity assumptions and $\mathcal{A}_s(\Omega)$ denotes the class of admissible domains.

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Optimal partition problems were studied by several authors: Bucur, Buttazzo and Henrot [5], Bucur and Velichkov [6], Caffarelli and Lin [8], Conti, Terracini and Verzini [9, 10], Helffer, Hoffmann-Ostenhof and Terracini [19], among others.

In [8], Caffarelli and Lin established the existence of classical solutions to an optimal partition problem for the Dirichlet eigenvalue, as well as the regularity of free interfaces. One more recent work about regularity of solutions to optimal partition problems involving eigenvalues of the Laplacian is [23], where Ramos, Tavares and Terracini used the existence result of [5] and proved that the free boundary of the optimal partition is locally a $C^{1,\alpha}$ -hypersurface up to a residual set.

Conti, Terracini and Verzini proved in [9] the existence of the minimal partition for a problem in N -dimensional domains related to the method of nonlinear eigenvalues introduced by Nehari in [21]. Moreover, they showed some connections between the variational problem and the behavior of competing species systems with large interaction.

Tavares and Terracini proved in [26] the existence of infinitely many sign-changing solutions for the system of m -Schrödinger equations with competition interactions and the relation between the energies associated and an optimal partition problem which involves m -eigenvalues of the Laplacian operator.

In a recent work [16], we studied a general shape optimization problem where $m = 1$.

To mention some references which have to do with optimal partition problems involving fractional operators, we suggest to look through [27], [29], and references therein too.

A class of optimal partition problems involving the half-Laplacian operator and a subcritical cost functional was considered by Zilio in [29]. That work encompasses findings about optimal regularity of the density-functions which characterize the partitions, for the entire set of minimizers. Besides, a numerical-related scheme and its consequences are shown.

In [27], Terracini-Verzini-Zilio consider a class of competition-diffusion nonlinear systems involving the half-Laplacian, including the fractional Gross-Pitaevskii system.

For more references related to optimal partition problems see, for instance, [1, 2, 4, 7, 10, 18, 22, 25]

The goal of this article is to prove the existence of an optimal partition for the problem (1.1), where F_s is decreasing in each coordinate and lower semicontinuous for a suitable notion of convergence in $\mathcal{A}_s(\Omega)$, which is the set of admissible domains. This existence result is carried out in Sect. 3. The dependence on s is related to the Gagliardo s -capacity measure and the fractional Laplacian operator $(-\Delta)^s$, and we will detail that and other preliminaries in Sect. 2.

We follow the ideas given by Bucur, Buttazzo and Henrot in [5], where the existence of solution to (1.1) in the case $s = 1$ was proved.

Furthermore, we prove convergence of minimums and optimal partition shapes to those of the case $s = 1$, studied in [5]. This last aim is accomplished in Sect. 4, and we consider it the most interesting contribution of this work.

At the end, we include “Appendix” with useful properties of s -capacity. Most of those results we suppose are well known. Despite that, we decided to incorporate them for completeness.

2 Preliminaries and statements

2.1 Notations and preliminaries

Given $s \in (0, 1)$ we consider the fractional Laplacian, that for smooth functions u is defined as

$$\begin{aligned}
 (-\Delta)^s u(x) &:= c(n, s) \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \\
 &= -\frac{c(n, s)}{2} \int_{\mathbb{R}^n} \frac{u(x+z) - 2u(x) + u(x-z)}{|z|^{n+2s}} dz.
 \end{aligned}$$

where $c(n, s) := (\int_{\mathbb{R}^n} \frac{1 - \cos \xi_1}{|\xi|^{n+2s}} d\xi)^{-1}$ is a normalization constant.

The constant $c(n, s)$ is chosen in such a way that the following identity holds,

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)),$$

for u in the Schwarz class of rapidly decreasing and infinitely differentiable functions, where \mathcal{F} denotes the Fourier transform. See [14, Proposition 3.3].

The natural functional setting for this operator is the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined as

$$\begin{aligned}
 H^s(\mathbb{R}^n) &:= \left\{ u \in L^2(\mathbb{R}^n) : \frac{u(x) - u(y)}{|x - y|^{\frac{n}{2} + s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\} \\
 &= \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathcal{F}(u)(\xi)|^2 d\xi < \infty \right\}
 \end{aligned}$$

which is a Banach space endowed with the norm $\|u\|_s^2 := \|u\|_2^2 + [u]_s^2$, where the term

$$[u]_s^2 := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy$$

is the so-called Gagliardo semi-norm of u .

To contemplate the *boundary* condition, we work in $H_0^s(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ in the norm $\|\cdot\|_s$. As we are dealing with a Lipschitz domain Ω , $H_0^s(\Omega)$ coincides with the space of functions vanishing outside Ω , i.e.,

$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\},$$

See [17, Corollary 1.4.4.5] for a proof of the identity above.

Definition 2.1 Given $A \subset \Omega$, for any $0 < s < 1$, we define the Gagliardo s -capacity of A relative to Ω as

$$\text{cap}_s(A, \Omega) = \inf \{ [u]_s^2 : u \in C_c^\infty(\Omega), u \geq 1 \text{ in a neighborhood of } A \}.$$

We say that a subset A of Ω is an s -quasi-open subset of Ω if there exists a decreasing sequence $\{G_k\}_{k \in \mathbb{N}}$ of open sets such that $\lim_{k \rightarrow \infty} \text{cap}_s(G_k, \Omega) = 0$ and $A \cup G_k$ is an open set.

We denote by $\mathcal{A}_s(\Omega)$ the class of all s -quasi-open subsets of Ω .

In the case $s = 1$ the definitions are completely analogous with $\|\nabla u\|_2^2$ instead of $[u]_s^2$.

We say that a property $P(x)$ holds s -quasi everywhere on $E \subset \Omega$ (s -q.e. on E), if $\text{cap}_s(\{x \in E : P(x) \text{ does not hold}\}, \Omega) = 0$.

A function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is said s -quasi-continuous if there exists a decreasing sequence $\{G_k\}_{k \in \mathbb{N}}$ of open sets such that $\lim_{k \rightarrow \infty} \text{cap}_s(G_k, \Omega) = 0$ and $u|_{\mathbb{R}^n \setminus G_k}$ is continuous.

The following theorem allows us to work with s -quasi-continuous functions instead of the classical fractional Sobolev ones.

Theorem 2.2 (Theorem 3.7, [28]) *For every function $u \in H_0^s(\Omega)$ there exist a unique $\tilde{u} : \mathbb{R}^n \rightarrow \mathbb{R}$ s -quasi-continuous function such that $u = \tilde{u}$ a.e. in \mathbb{R}^n .*

From this point, we identify a function $u \in H_0^s(\Omega)$ with its s -quasi-continuous representative.

For $A \in \mathcal{A}_s(\Omega)$, we consider the fractional Sobolev space

$$H_0^s(A) := \{u \in H_0^s(\Omega) : u = 0 \text{ s-q.e. in } \mathbb{R}^n \setminus A\}.$$

To go into detail about s -capacity we refer the reader, for instance, to [24,28].

2.2 Statements

Given $A \in \mathcal{A}_s(\Omega)$, we denote by $u_A^s \in H_0^s(A)$ the unique weak solution to

$$(-\Delta)^s u_A^s = 1 \quad \text{in } A, \quad u_A^s = 0 \quad \text{in } \mathbb{R}^n \setminus A. \tag{2.1}$$

With this notation, we define the following notion of set convergence.

Definition 2.3 (Strong γ_s -convergence) Let $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ and $A \in \mathcal{A}_s(\Omega)$. We say that $A_k \xrightarrow{\gamma_s} A$ if $u_{A_k}^s \rightarrow u_A^s$ strongly in $L^2(\Omega)$.

Let $m \in \mathbb{N}$, $\{(A_1^k, \dots, A_m^k)\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ and $(A_1, \dots, A_m) \in \mathcal{A}_s(\Omega)^m$. We say $(A_1^k, \dots, A_m^k) \xrightarrow{\gamma_s} (A_1, \dots, A_m)$ if $A_i^k \xrightarrow{\gamma_s} A_i$ for every $i = 1, \dots, m$.

Definition 2.4 (Weak γ_s -convergence) Let $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ and $A \in \mathcal{A}_s(\Omega)$. We say that $A_k \xrightarrow{\gamma_s} A$ if there exists a function $u \in L^2(\Omega)$ such that $u_{A_k}^s \rightarrow u$ strongly in $L^2(\Omega)$ and $A = \{u > 0\} \in \mathcal{A}_s(\Omega)$.

Let $m \in \mathbb{N}$ and $\{(A_1^k, \dots, A_m^k)\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ and $(A_1, \dots, A_m) \in \mathcal{A}_s(\Omega)^m$. We say $(A_1^k, \dots, A_m^k) \xrightarrow{\gamma_s} (A_1, \dots, A_m)$ if $A_i^k \xrightarrow{\gamma_s} A_i$ for every $i = 1, \dots, m$.

Remark 2.5 We want to emphasize the difference between strong and weak γ_s -convergence. In the weak γ_s -convergence, the $L^2(\Omega)$ -limit function u of the sequence $\{u_{A_k}^s\}_{k \in \mathbb{N}}$ is not required to be a solution of (2.1) in A (the weak γ_s -limit), i.e., it is not required that $u \neq u_A^s$. That is the main hassle we should get through to arrive at the compactness result on $\mathcal{A}_s(\Omega)$, in Sect. 3.1.

Let $m \in \mathbb{N}$ be fixed and $0 < s \leq 1$. Let $F_s : \mathcal{A}_s(\Omega)^m \rightarrow [0, \infty]$ be such that

- F_s is weak γ_s -lower semicontinuous, that is,

$$F_s(A_1, \dots, A_m) \leq \liminf_{k \rightarrow \infty} F_s(A_1^k, \dots, A_m^k),$$

for every $\{(A_1^k, \dots, A_m^k)\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ and $(A_1, \dots, A_m) \in \mathcal{A}_s(\Omega)^m$ such that $(A_1^k, \dots, A_m^k) \xrightarrow{\gamma_s} (A_1, \dots, A_m)$.

- F_s is decreasing, that is, for every $(A_1, \dots, A_m), (B_1, \dots, B_m) \in \mathcal{A}_s(\Omega)^m$ such that $A_i \subset B_i$ for $i = 1, \dots, m$, we have

$$F_s(A_1, \dots, A_m) \geq F_s(B_1, \dots, B_m).$$

Under these assumptions, we are able to recover the existence result of [5], for the fractional case. Rigorously speaking, we have the following theorem.

Theorem 2.6 Let $F_s : \mathcal{A}_s(\Omega)^m \rightarrow [0, \infty]$ be a decreasing and weak γ_s -lower semicontinuous functional. Then, there exists a solution to

$$\min \{F_s(A_1, \dots, A_m) : A_i \in \mathcal{A}_s(\Omega), \text{ cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j\}. \tag{2.2}$$

The proof of Theorem 2.6 is carried out in Sect. 3 and we use ideas from [5] and [16].

Now, we present the main point of this article, that is the convergence of minimums and optimal partition shapes to those of the case $s = 1$.

Once we know the existence of an optimal partition shape for each $0 < s < 1$, we want to analyze the limit of these minimizers and its minimum values when $s \uparrow 1$. To this aim, we need a suitable relationship between the cost functionals F_s , $0 < s \leq 1$ and a notion of set convergence.

Let us start with the notion of set convergence. For $A \in \mathcal{A}_1(\Omega)$, we introduce the analogous notation $u_A^1 \in H_0^1(A)$ for the unique weak solution to

$$-\Delta u_A^1 = 1 \text{ in } A, \quad u_A^1 = 0 \text{ in } \mathbb{R}^n \setminus A.$$

Definition 2.7 (γ -convergence) Let $0 < s_k \uparrow 1$, $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_{s_k}(\Omega)$ and $A \in \mathcal{A}_1(\Omega)$. We say that $A_k \xrightarrow{\gamma} A$ if $u_{A_k}^{s_k} \rightarrow u_A^1$ strongly in $L^2(\Omega)$.

Let $m \in \mathbb{N}$, $(A_1^k, \dots, A_m^k) \in \mathcal{A}_{s_k}(\Omega)^m$ and $(A_1, \dots, A_m) \in \mathcal{A}_1(\Omega)^m$. We say that $(A_1^k, \dots, A_m^k) \xrightarrow{\gamma} (A_1, \dots, A_m)$ if $u_{A_i^k}^{s_k} \rightarrow u_{A_i}^1$ strongly in $L^2(\Omega)$, for every $i = 1, \dots, m$.

Let $m \in \mathbb{N}$ and $0 < s \leq 1$. Let $F_s : \mathcal{A}_s(\Omega)^m \rightarrow [0, \infty]$ be decreasing and weak γ_s -lower semicontinuous functionals. Then, there exists (A_1^s, \dots, A_m^s) solution to

$$m_s := \min \{ F_s(B_1, \dots, B_m) : B_i \in \mathcal{A}_s(\Omega), \text{cap}_s(B_i \cap B_j, \Omega) = 0 \text{ for } i \neq j \}. \quad (2.3)$$

The case $s = 1$ was solved in [5]. For $0 < s < 1$, apply Theorem 2.6.

Assume the following hypotheses over the cost functionals:

(H1) Continuity. For every $(A_1, \dots, A_m) \in \mathcal{A}_1(\Omega)^m$,

$$F_1(A_1, \dots, A_m) = \lim_{s \uparrow 1} F_s(A_1, \dots, A_m).$$

(H2) Liminf inequality. For every $0 < s_k \uparrow 1$, $(A_1^k, \dots, A_m^k) \in \mathcal{A}_{s_k}(\Omega)^m$ and $(A_1, \dots, A_m) \in \mathcal{A}_1(\Omega)^m$ such that $(A_1^k, \dots, A_m^k) \xrightarrow{\gamma} (A_1, \dots, A_m)$,

$$F_1(A_1, \dots, A_m) \leq \liminf_{k \rightarrow \infty} F_{s_k}(A_1^k, \dots, A_m^k).$$

These conditions (H1)-(H2) are natural and analogous to those consider in [16], where a similar shape optimization problem was studied with $m = 1$.

Now, we are able to establish the main result.

Theorem 2.8 Let $m \in \mathbb{N}$ be fixed and $0 < s \leq 1$. Let $F_s : \mathcal{A}_s(\Omega)^m \rightarrow [0, \infty]$ be a decreasing and weak γ_s -lower semicontinuous functional, and such that (H1)-(H2) are verified. Then,

$$m_1 = \lim_{s \uparrow 1} m_s, \quad (2.4)$$

where m_s is defined in (2.3).

Moreover, if (A_1^s, \dots, A_m^s) is a minimizer of (2.3), then, there exist a subsequence $0 < s_k \uparrow 1$, $(\tilde{A}_1^{s_k}, \dots, \tilde{A}_m^{s_k}) \in \mathcal{A}_{s_k}(\Omega)^m$ and $(A_1^1, \dots, A_m^1) \in \mathcal{A}_1(\Omega)^m$ such that $\tilde{A}_i^{s_k} \supset A_i^{s_k}$ and

$$(\tilde{A}_1^{s_k}, \dots, \tilde{A}_m^{s_k}) \xrightarrow{\gamma} (A_1^1, \dots, A_m^1),$$

where (A_1^1, \dots, A_m^1) is a minimizer of (2.3) with $s = 1$.

The proof of Theorem 2.8 is carried out in Sect. 4, and we use again ideas from [16].

2.3 Examples

Given $A \in \mathcal{A}_s(\Omega)$, consider the problem

$$(-\Delta)^s u = \lambda^s u \quad \text{in } A, \quad u \in H_0^s(A) \tag{2.5}$$

where $\lambda^s \in \mathbb{R}$ is the eigenvalue parameter. It is well known that there exists a discrete sequence $\{\lambda_k^s(A)\}_{k \in \mathbb{N}}$ of positive eigenvalues of (2.5) approaching $+\infty$ whose corresponding eigenfunctions $\{u_k^s\}_{k \in \mathbb{N}}$ form an orthogonal basis in $L^2(A)$. Moreover, the following variational characterization holds for the eigenvalues

$$\lambda_k^s(A) = \min_{u \perp W_{k-1}} \frac{c(n, s)}{2} \frac{[u]_s^2}{\|u\|_2^2}, \tag{2.6}$$

where W_k is the space spanned by the first k eigenfunctions u_1^s, \dots, u_k^s .

Due to (2.6) and the stability result proved in [3, Theorem 1.2], we know that $\lambda_k^s(A) \rightarrow \lambda_k^1(A)$, when $s \uparrow 1$, for every $k \in \mathbb{N}$.

Consider functionals $F_s(A_1, \dots, A_m) = \Phi_s(\lambda_{k_1}^s(A_1), \dots, \lambda_{k_m}^s(A_m))$. Theorem 2.6 claims that for every $(k_1, \dots, k_m) \in \mathbb{N}^m$, the minimum

$$\min\{\Phi_s(\lambda_{k_1}^s(A_1), \dots, \lambda_{k_m}^s(A_m)) : A_i \in \mathcal{A}_s(\Omega), \text{cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j\}$$

is achieved, where $\Phi_s : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$, is increasing in each coordinate and lower semicontinuous.

Moreover, if $\Phi_s(t_1, \dots, t_m) \rightarrow \Phi_1(t_1, \dots, t_m)$ for every $(t_1, \dots, t_m) \in \mathbb{R}^m$ and

$$\Phi_1(t_1, \dots, t_m) \leq \liminf_{k \rightarrow \infty} \Phi_{s_k}(t_1^k, \dots, t_m^k),$$

for every $(t_1^k, \dots, t_m^k) \rightarrow (t_1, \dots, t_m)$, then Theorem 2.8 together with the existence result of [5] imply that

$$\begin{aligned} & \min\{\Phi_1(\lambda_{k_1}(A_1), \dots, \lambda_{k_m}(A_m)) : A_i \in \mathcal{A}_1(\Omega), \text{cap}_1(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j\} \\ & = \lim_{s \uparrow 1} \min\{\Phi_s(\lambda_{k_1}^s(A_1), \dots, \lambda_{k_m}^s(A_m)) : A_i \in \mathcal{A}_s(\Omega), \text{cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j\}. \end{aligned}$$

3 Proof of Theorem 2.6

In this section, we adapted the ideas from [5], where the authors consider the Laplacian operator, to recover their results for the fractional case. Despite the similarity of the proofs, we include them for the reader’s convenience and recalling that in the context of this article we need the nonlocal tools proved in [16].

3.1 Certain compactness on $\mathcal{A}_s(\Omega)$

Consider \mathcal{K}_s given by

$$\mathcal{K}_s := \{w \in H_0^s(\Omega) : w \geq 0, (-\Delta)^s w \leq 1 \text{ in } \Omega\}. \tag{3.1}$$

Proposition 3.1 (Proposition 3.3 and Lemma 3.5, [16]) \mathcal{K}_s is convex, closed and bounded in $H_0^s(\Omega)$. Moreover, if $u, v \in \mathcal{K}_s$, then, $\max\{u, v\} \in \mathcal{K}_s$.

Proposition 3.2 (Lemma 3.2, [16]) Given $A \in \mathcal{A}_s(\Omega)$, u_A^s is the solution to

$$\max \{w \in H_0^s(\Omega) : w \leq 0 \text{ in } \mathbb{R}^n \setminus A, (-\Delta)^s w \leq 1 \text{ in } \Omega\}.$$

Moreover, $u_A^s \in \mathcal{K}_s$, for every $A \in \mathcal{A}_s(\Omega)$.

From now on, we understand the identity $A = \{u_A^s > 0\}$ in the sense of the Gagliardo s -capacity, thanks to Proposition A.5.

Remark 3.3 The class $\mathcal{A}_s(\Omega)$ is sequentially pre-compact with respect to the weak γ_s -convergence. Indeed, given a sequence $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$, we know that $\{u_{A_k}^s\}_{k \in \mathbb{N}} \subset \mathcal{K}_s$. By Proposition 3.1, there exists a subsequence $\{u_{A_{k_j}}^s\}_{j \in \mathbb{N}} \subset \{u_{A_k}^s\}_{k \in \mathbb{N}}$ and a function $u \in \mathcal{K}_s$ such that $u_{A_{k_j}}^s \rightarrow u$ strongly in $L^2(\Omega)$. Denote by $A := \{u > 0\}$. Then, $A_{k_j} \xrightarrow{\gamma_s} A$.

Next proposition allows us to pass from the weak γ_s -convergence to the strong one, if we are willing to enlarge the sequence involved.

Proposition 3.4 *Let $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ and $A, B \in \mathcal{A}_s(\Omega)$ be such that $A_k \xrightarrow{\gamma_s} A \subset B$.*

Then, there exists a subsequence $\{A_{k_j}\}_{j \in \mathbb{N}} \subset \{A_k\}_{k \in \mathbb{N}}$ and a sequence $\{B_{k_j}\}_{j \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ such that $A_{k_j} \subset B_{k_j}$ and $B_{k_j} \xrightarrow{\gamma_s} B$.

Proof Since $A_k \xrightarrow{\gamma_s} A \subset B$, we know that $u_{A_k}^s \rightarrow u$ strongly in $L^2(\Omega)$, where $\{u > 0\} = A$. As a consequence of Proposition 3.1, $u \in \mathcal{K}_s$. Moreover, by Proposition 3.2, $u \leq u_A^s$. Since $A \subset B$, $u_A^s \leq u_B^s$. Then, $u \leq u_B^s$.

Denote by $B^\varepsilon = \{u_B^s > \varepsilon\}$ and consider $\{u_{A_k \cup B^\varepsilon}^s\}_{k \in \mathbb{N}} \subset \mathcal{K}_s$. Again by Proposition 3.1, there exists a subsequence $\{A_{k_j}\}_{j \in \mathbb{N}} \subset \{A_k\}_{k \in \mathbb{N}}$ such that $u_{A_{k_j} \cup B^\varepsilon}^s \rightarrow u^\varepsilon$ strongly in $L^2(\Omega)$.

Due to the convergence $u_{A_{k_j}}^s \rightarrow u$ strongly in $L^2(\Omega)$ and $u \leq u_B^s$, we conclude from [16, Lemma 3.6], $u^\varepsilon \leq u_B^s$.

Inside the proof of [16, Lemma 3.7], it was shown that $(u_B^s - \varepsilon)^+ \leq u_{B^\varepsilon}^s$. Since $B^\varepsilon \subset A_{k_j} \cup B^\varepsilon$, it follows that $u_{B^\varepsilon}^s \leq u_{A_{k_j} \cup B^\varepsilon}^s$. So, taking the limit $j \rightarrow \infty$, we obtain

$$(u_B^s - \varepsilon)^+ \leq u_{B^\varepsilon}^s \leq u^\varepsilon \leq u_B^s.$$

The sequence $\{u^\varepsilon\}_{\varepsilon > 0}$ is contained in \mathcal{K}_s . So, by Proposition 3.1, up to a subsequence, we know it has a weak limit in $H_0^s(\Omega)$. But, the previous inequality tells that this weak limit should be u_B^s . In addition, $u^\varepsilon \rightarrow u_B^s$ strongly in $L^2(\Omega)$.

Thus, there exists a sequence $\varepsilon_j \downarrow 0$ such that $u_{A_{k_j} \cup B^{\varepsilon_j}}^s \rightarrow u_B^s$ strongly in $L^2(\Omega)$. That is, $A_{k_j} \cup B^{\varepsilon_j} =: B_{k_j} \xrightarrow{\gamma_s} B$, where $\{B_{k_j}\}_{j \in \mathbb{N}}$ is the enlarged sequence. □

3.2 An auxiliary functional

Fix $m \in \mathbb{N}$ and $0 < s < 1$. Let $F_s : \mathcal{A}_s(\Omega)^m \rightarrow [0, \infty]$ be a decreasing and strong γ_s -lower semicontinuous functional.

We define a functional $G_s : \mathcal{K}_s^m \rightarrow [0, \infty]$

$$G_s(w_1, \dots, w_m) := \inf \left\{ \liminf_{k \rightarrow \infty} J_s(w_1^k, \dots, w_m^k) : w_i^k \rightarrow w_i \text{ strongly in } L^2(\Omega) \right\}, \quad (3.2)$$

where $J_s : \mathcal{K}_s^m \rightarrow [0, \infty]$ is defined as

$$J_s(w_1, \dots, w_m) := \inf \left\{ F_s(A_1, \dots, A_m) : A_i \in \mathcal{A}_s(\Omega), u_{A_i}^s \leq w_i \text{ for } i = 1, \dots, m \right\}$$

and \mathcal{K}_s was given in (3.1).

We will show that G_s satisfies the following properties:

- (G₁) G_s is decreasing on \mathcal{K}_s^m , that is $G_s(u_1, \dots, u_m) \geq G_s(v_1, \dots, v_m)$, if $u_i \leq v_i$ for every $i = 1, \dots, m$.
- (G₂) G_s is lower semicontinuous on \mathcal{K}_s with respect to the strong topology on $L^2(\Omega)$,
- (G₃) $G_s(u_{A_1}^s, \dots, u_{A_m}^s) = F_s(A_1, \dots, A_m)$ for every $(A_1, \dots, A_m) \in \mathcal{A}_s(\Omega)^m$.

The conditions (G₁) and (G₂) are easy to check, and it is the content of next proposition.

Proposition 3.5 *With the notation above, G_s satisfies (G₁) and (G₂).*

Proof By construction, it is clear that G_s verifies (G₂).

To prove (G₁), let $(u_1, \dots, u_m), (v_1, \dots, v_m) \in \mathcal{K}_s^m$ such that $u_i \leq v_i$ for every $i = 1, \dots, m$.

Take $\{u_i^k\}_{k \in \mathbb{N}} \subset \mathcal{K}_s$ such that $u_i^k \rightarrow u_i$ strongly in $L^2(\Omega)$ for every $i = 1, \dots, m$ and

$$G_s(u_1, \dots, u_m) = \lim_{k \rightarrow \infty} J_s(u_1^k, \dots, u_m^k).$$

Consider $v_i^k := \max\{v_i, u_i^k\}$ for every $i = 1, \dots, m$ and $k \in \mathbb{N}$. By Proposition 3.1, we obtain that $v_i^k \in \mathcal{K}_s$. In addition, $v_i^k \rightarrow \max\{v_i, u_i\} = v_i$ strongly in $L^2(\Omega)$, for every $i = 1, \dots, m$. Thus, noticing that J_s is decreasing, we have

$$G_s(v_1, \dots, v_m) \leq \liminf_{k \rightarrow \infty} J_s(v_1^k, \dots, v_m^k) \leq \lim_{k \rightarrow \infty} J_s(u_1^k, \dots, u_m^k) = G_s(u_1, \dots, u_m).$$

□

Now, we prove the most important property of G_s , which is the connection with the cost functional F_s .

Proposition 3.6 *The functional G_s satisfies (G₃).*

Proof By definition of G_s (3.2), it is clear that $G_s(u_{A_1}^s, \dots, u_{A_m}^s) \leq F_s(A_1, \dots, A_m)$, for every $(A_1, \dots, A_m) \in \mathcal{A}_s(\Omega)^m$.

To obtain the other inequality, it is enough to prove that for every sequence $\{u_i^k\}_{k \in \mathbb{N}} \subset \mathcal{K}_s(\Omega)$ such that $u_i^k \rightarrow u_{A_i}^s$ strongly in $L^2(\Omega)$ for $i = 1, \dots, m$, we have

$$F_s(A_1, \dots, A_m) \leq \liminf_{k \rightarrow \infty} J_s(u_1^k, \dots, u_m^k).$$

By definition of J_s , there exists $\{(A_1^k, \dots, A_m^k)\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ such that

$$u_{A_i^k}^s \leq u_i^k \text{ for } i = 1, \dots, m, \text{ and } F_s(A_1^k, \dots, A_m^k) \leq J_s(u_1^k, \dots, u_m^k) + \frac{1}{k}. \tag{3.3}$$

By Remark 3.3, there exists $v_i \in \mathcal{K}_s$ such that $u_{A_i^k}^s \rightarrow v_i$ strongly in $L^2(\Omega)$, up to a subsequence. That is, $A_i^k \xrightarrow{\gamma_s} B_i := \{v_i > 0\} \in \mathcal{A}_s(\Omega)$, for every $i = 1, \dots, m$.

Moreover, taking the limit in $u_{A_i^k}^s \leq u_i^k$, we obtain that $v_i \leq u_{A_i}^s$ for every $i = 1, \dots, m$. In addition, we have $B_i \subset A_i = \{u_{A_i}^s > 0\}$. We are able to apply Proposition 3.4, to obtain the existence of subsequences $\{A_i^{k_j}\}_{j \in \mathbb{N}}, \{B_i^{k_j}\}_{j \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ such that $A_i^{k_j} \subset B_i^{k_j}$ and $B_i^{k_j} \xrightarrow{\gamma_s} A_i$.

Now, by using the strong γ_s -lower semicontinuity and decreasing property of F_s and (3.3), we conclude

$$\begin{aligned} F_s(A_1, \dots, A_m) &\leq \liminf_{j \rightarrow \infty} F_s(B_1^{k_j}, \dots, B_m^{k_j}) \\ &\leq \liminf_{j \rightarrow \infty} F_s(A_1^{k_j}, \dots, A_m^{k_j}) \\ &\leq \liminf_{j \rightarrow \infty} J_s(u_1^{k_j}, \dots, u_m^{k_j}), \end{aligned}$$

which implies the remaining inequality $F_s(A_1, \dots, A_m) \leq G_s(u_{A_1}^s, \dots, u_{A_m}^s)$. \square

The decreasing property of a functional F_s makes equivalent its weak and strong γ_s -lower semicontinuity, which is the content of next theorem.

Theorem 3.7 *Let $m \in \mathbb{N}$ and $0 < s < 1$. Let $F_s : \mathcal{A}_s(\Omega)^m \rightarrow [0, \infty]$ be a decreasing functional. Then, the following assertions are equivalent*

- 1 F_s is weakly γ_s -lower semicontinuous.
- 2 F_s is strong γ_s -lower semicontinuous.

Proof Since every strongly γ_s -convergent sequence $\{A_k\}_{k \in \mathbb{N}}$ is, in addition, weakly γ_s -convergent, (1) \Rightarrow (2) is clear. (See definitions and Proposition A.5). Let us see the converse.

Now, suppose F_s is strongly γ_s -lower semicontinuous. To arrive at the weakly γ_s -lower semicontinuity of F_s from the strong one, the strategy is to take into account the auxiliary functional G_s defined in (3.2) and its properties.

Fix $\{(A_1^k, \dots, A_m^k)\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ and $(A_1, \dots, A_m) \in \mathcal{A}_s(\Omega)^m$ such that

$$(A_1^k, \dots, A_m^k) \xrightarrow{\gamma_s} (A_1, \dots, A_m).$$

That means $u_{A_i^k}^s \rightarrow u_i$ strongly in $L^2(\Omega)$ and $A_i = \{u_i > 0\}$, for $i = 1, \dots, m$.

Since for every $i = 1, \dots, m$, $\{u_{A_i^k}^s\}_{k \in \mathbb{N}} \subset \mathcal{K}_s$, by Proposition 3.1, $u_i \in \mathcal{K}_s$. Moreover, by Proposition 3.2, $u_i \leq u_{A_i}^s$. Then, we can use (G_3) , the decreasing property of G_s , so that we obtain

$$G_s(u_{A_1}^s, \dots, u_{A_m}^s) \leq G_s(u_1, \dots, u_m). \tag{3.4}$$

On the other hand, by recalling (G_1) , the relationship between F_s and G_s , we get the following identities

$$F_s(A_1, \dots, A_m) = G_s(u_{A_1}^s, \dots, u_{A_m}^s) \quad \text{and} \quad F_s(A_1^k, \dots, A_m^k) = G_s(u_{A_1^k}^s, \dots, u_{A_m^k}^s), \tag{3.5}$$

for every $k \in \mathbb{N}$.

Now, due to (G_2) (the $L^2(\Omega)$ -lower semicontinuity of G_s) in addition to $u_{A_i^k}^s \rightarrow u_i$ strongly in $L^2(\Omega)$ for every $i = 1, \dots, m$, we connect (3.4) and (3.5) to conclude that

$$\begin{aligned} F_s(A_1, \dots, A_m) &= G_s(u_{A_1}^s, \dots, u_{A_m}^s) \leq G_s(u_1, \dots, u_m) \\ &\leq \liminf_{k \rightarrow \infty} G_s(u_{A_1^k}^s, \dots, u_{A_m^k}^s) \\ &= \liminf_{k \rightarrow \infty} F_s(A_1^k, \dots, A_m^k). \end{aligned}$$

Since $\{(A_1^k, \dots, A_m^k)\}_{k \in \mathbb{N}}$ is an arbitrary weak γ_s -convergent sequence, we get that F_s is weak γ_s -lower semicontinuous, as we desired. \square

3.3 Existence of an optimal partition

With the help of the previous outcomes of this section, we are able to prove existence of a minimal partition shape for (2.2).

Proof of Theorem 2.6 Denote by

$$\alpha := \inf \{ F_s(A_1, \dots, A_m) : A_i \in \mathcal{A}_s(\Omega), \text{cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j \}.$$

Let $\{(A_1^k, \dots, A_m^k)\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ be such that

$$\text{cap}_s(A_i^k \cap A_j^k, \Omega) = 0 \text{ for } i \neq j, \text{ and } \lim_{k \rightarrow \infty} F_s(A_1^k, \dots, A_m^k) = \alpha.$$

By Remark 3.3, there exist $A_1 \in \mathcal{A}_s(\Omega)$ and a subsequence $\{A_1^{k_j}\}_{j \in \mathbb{N}} \subset \{A_1^k\}_{k \in \mathbb{N}}$ such that $A_1^{k_j} \xrightarrow{\gamma_s} A_1$. Now, consider $\{A_2^{k_j}\}_{j \in \mathbb{N}}$ and apply again Remark 3.3. Thus, there exist $A_2 \in \mathcal{A}_s(\Omega)$ and a subsequence $\{A_2^{k_{j_l}}\}_{l \in \mathbb{N}} \subset \{A_2^{k_j}\}_{j \in \mathbb{N}}$ such that $A_2^{k_{j_l}} \xrightarrow{\gamma_s} A_2$ for $l = 1, 2$. Repeating this argument, we find a sequence $\{(A_1^k, \dots, A_m^k)\}_{k \in \mathbb{N}}$ and $(A_1, \dots, A_m) \in \mathcal{A}_s(\Omega)$ such that $A_i^k \xrightarrow{\gamma_s} A_i$ for every $i = 1, \dots, m$.

Since F_s is weak γ_s -lower semicontinuous, we obtain

$$F_s(A_1, \dots, A_m) \leq \liminf_{k \rightarrow \infty} F_s(A_1^k, \dots, A_m^k) = \alpha. \tag{3.6}$$

To finish the proof, let us see $\text{cap}_s(A_i \cap A_j, \Omega) = 0$ for $i \neq j$ be satisfied.

Let $i, j \in \{1, \dots, m\}$ be such that $i \neq j$. Notice that this product $u_{A_i^k}^s \cdot u_{A_j^k}^s$ is an s -continuous function too, by Lemma A.1, and $u_{A_i^k}^s \cdot u_{A_j^k}^s = 0$ s -q.e. in $\mathbb{R}^n \setminus (A_i^k \cap A_j^k)$. Moreover, since $\text{cap}_s(A_i^k \cap A_j^k, \Omega) = 0$, we have $u_{A_i^k}^s \cdot u_{A_j^k}^s = 0$ s -q.e. in \mathbb{R}^n .

By [28, Lemma 3.8], there exist subsequences $\{u_{A_i^k}^s\}_{k \in \mathbb{N}}$ and $\{u_{A_j^k}^s\}_{k \in \mathbb{N}}$, denoted with the same index, which converge s -q.e. to u_i and u_j , respectively. Then, passing to the limit, we obtain $u_i \cdot u_j = 0$ s -q.e. in \mathbb{R}^n . That is $\text{cap}_s(\{u_i \cdot u_j \neq 0\}, \Omega) = 0$. But, $\{u_i \cdot u_j \neq 0\} = A_i \cap A_j$.

We have shown that (A_1, \dots, A_m) is admissible for the minimization problem (2.2) and recalling (3.6) the result is proved. □

Due to Theorems 3.7 and 2.6, we can establish the next immediate corollary.

Corollary 3.8 *Let $F_s : \mathcal{A}_s(\Omega)^m \rightarrow [0, \infty]$ be a decreasing and strong γ_s -lower semicontinuous functional. Then, there exists a solution to (2.2).*

4 Proof of Theorem 2.8

This is the main part of the article, where we study the behavior of optimal partition shapes obtained in Sect. 3 and their minimum values. Again, we use some results from [16].

Lemma 4.1 (Lemma 4.1, [16]) *Let $0 < s_k \uparrow 1$ and let $u_k \in \mathcal{K}_{s_k}$. Then, there exists $u \in H_0^1(\Omega)$ and a subsequence $\{u_{k_j}\}_{j \in \mathbb{N}} \subset \{u_k\}_{k \in \mathbb{N}}$ such that $u_{k_j} \rightarrow u$ strongly in $L^2(\Omega)$.*

Moreover, if $u_k \in \mathcal{K}_{s_k}$ is such that $u_k \rightarrow u$ strongly in $L^2(\Omega)$, then $u \in \mathcal{K}_1$.

Next proposition gives an idea of the limit behavior of u_A^s when the domains also are varying with s .

Proposition 4.2 (Proposition 4.5, [16]) *Let $0 < s_k \uparrow 1$, $A^k \in \mathcal{A}_{s_k}(\Omega)$ be such that $u_{A^k}^{s_k} \rightarrow u$ strongly in $L^2(\Omega)$. Then, there exist $\tilde{A}^k \in \mathcal{A}_{s_k}(\Omega)$ such that $A^k \subset \tilde{A}^k$ and \tilde{A}^k γ -converges to $A := \{u > 0\}$.*

Now we are ready to prove the main result of this article.

Proof of Theorem 2.8 First, notice that m_1 is achieved by [5, Theorem 3.2].

Let $0 < s_k \uparrow 1$. By Theorem 2.6, there exists $(A_1^k, \dots, A_m^k) \in \mathcal{A}_{s_k}(\Omega)^m$ such that

$$\text{cap}_{s_k}(A_i^k \cap A_j^k, \Omega) = 0 \text{ for } i \neq j \text{ and } F_{s_k}(A_1^k, \dots, A_m^k) = m_k, \tag{4.1}$$

where $m_k = m_{s_k}$ defined in (2.2).

Let $(A_1, \dots, A_m) \in \mathcal{A}_1(\Omega)^m$ be such that $\text{cap}_1(A_i \cap A_j, \Omega) = 0$ for $i \neq j$. Since $0 < s_k \uparrow 1$, we can assume $0 < \varepsilon_0 < s_k \uparrow 1$, for some fixed ε_0 .

Now, recalling Corollary A.7 and Remark A.8, we know that (A_1, \dots, A_m) belongs to

$$\{(B_1, \dots, B_m) : B_i \in \mathcal{A}_{s_k}(\Omega), \text{cap}_{s_k}(B_i \cap B_j, \Omega) = 0 \text{ for } i \neq j\},$$

for every $k \in \mathbb{N}$. This fact and condition (H_1) imply that

$$\limsup_{k \rightarrow \infty} F_{s_k}(A_1^k, \dots, A_m^k) \leq \lim_{k \rightarrow \infty} F_{s_k}(A_1, \dots, A_m) = F_1(A_1, \dots, A_m).$$

It follows that

$$\limsup_{k \rightarrow \infty} m_k \leq m_1. \tag{4.2}$$

To see the remaining inequality, let us denote $u_i^k := u_{A_i^k}^{s_k} \in \mathcal{K}_{s_k}$. By Lemma 4.1, there is $u_i \in \mathcal{K}_1$ such that, up to a subsequence, $u_i^k \rightarrow u_i$ strongly in $L^2(\Omega)$ and a.e. in Ω .

Denote by $A_i := \{u_i > 0\} \in \mathcal{A}_1(\Omega)$ for every $i = 1, \dots, m$. We claim that $\text{cap}_1(A_i \cap A_j, \Omega) = 0$ for $i \neq j$.

Indeed, let $i \neq j$ be fixed. For each $k \in \mathbb{N}$, due to Lemma A.2 and (4.1), we know that

$$|\{u_i^k \cdot u_j^k \neq 0\}| = |A_i^k \cap A_j^k| \leq C(n, s_k) \text{cap}_{s_k}(A_i^k \cap A_j^k, \Omega) = 0.$$

Then, $u_i^k \cdot u_j^k = 0$ a.e. in \mathbb{R}^n . Since $u_i^k \rightarrow u_i$ a.e. in Ω for $l = 1, 2$, we conclude $u_i \cdot u_j = 0$ a.e. in Ω , it is still true in $\mathbb{R}^n \setminus \Omega$ considering that they belong to $H_0^s(\Omega)$. So, $u_i \cdot u_j = 0$ a.e. in \mathbb{R}^n .

Reminding that we are working with 1-quasi-continuous representative functions in $H_0^1(\Omega)$, the previous identity $u_i \cdot u_j = 0$ a.e. in \mathbb{R}^n and [20, Lemma 3.3.30] tells that $u_i \cdot u_j = 0$ 1-q.e. in \mathbb{R}^n . That means $\text{cap}_1(A_i \cap A_j, \Omega) = 0$.

Consequently, (A_1, \dots, A_m) is admissible to the problem 2.2 with $s = 1$ and we obtain $m_1 \leq F_1(A_1, \dots, A_m)$.

Moreover, by Proposition 4.2, there exists $\tilde{A}_i^k \in \mathcal{A}_{s_k}(\Omega)$ such that $A_i^k \subset \tilde{A}_i^k$ and $(\tilde{A}_1^k, \dots, \tilde{A}_m^k)$ γ -converges to (A_1, \dots, A_m) .

Finally, from condition (H_2) and the decreasing property of F_{s_k} , we conclude that

$$\begin{aligned} m_1 &\leq F_1(A_1, \dots, A_m) \leq \liminf_{k \rightarrow \infty} F_{s_k}(\tilde{A}_1^k, \dots, \tilde{A}_m^k) \\ &\leq \liminf_{k \rightarrow \infty} F_{s_k}(A_1^k, \dots, A_m^k) = \liminf_{k \rightarrow \infty} m_k. \end{aligned}$$

Therefore, from previous conclusion and (4.2) we have the identity (2.4), so that the results follow. □

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Appendix A. Some useful properties of s -capacity

The following lemmas address some basic properties of s -capacity and s -quasi-continuous functions. We suppose those results are well known, and we include them for completeness.

Lemma A.1 *Let $u, v : \mathbb{R}^n \rightarrow \mathbb{R}$ be s -quasi-continuous functions. Then, the product $u \cdot v$ is also an s -quasi-continuous function.*

Proof By definition, there exist decreasing sequences $\{A_k\}_{k \in \mathbb{N}}$ and $\{B_k\}_{k \in \mathbb{N}}$ of open sets such that $\lim_{k \rightarrow \infty} \text{cap}_s(A_k, \Omega) = \lim_{k \rightarrow \infty} \text{cap}_s(B_k, \Omega) = 0$ and $u|_{\mathbb{R}^n \setminus A_k}, v|_{\mathbb{R}^n \setminus B_k}$ are continuous.

Consider $C_k := A_k \cup B_k$. Then, $\{C_k\}_{k \in \mathbb{N}}$ is a decreasing sequence of open sets such that $\lim_{k \rightarrow \infty} \text{cap}_s(C_k, \Omega) = 0$, since $\text{cap}_s(C_k, \Omega) \leq \text{cap}_s(A_k, \Omega) + \text{cap}_s(B_k, \Omega)$ by [28, Proposition 3.6]. Moreover, $(u \cdot v)|_{\mathbb{R}^n \setminus C_k}$ is continuous. □

Next lemma gives a relation between the Lebesgue measure and the s -capacity of a subset $A \subset \Omega$. The proof is easy and follows [15, Section 4.7, Theorem 2 VI], where it was shown with the classical capacity measure ($s = 1$).

Lemma A.2 *For every $A \subset \Omega$, $|A| \leq C(\Omega, s) \text{cap}_s(A, \Omega)$, where $C(\Omega, s)$ is the Poincaré’s constant in $H_0^s(\Omega)$.*

Proof For every $\varepsilon > 0$, there exists a function $u_\varepsilon \in H_0^s(\Omega)$ such that $u_\varepsilon \geq 1$ a.e. in a neighborhood of A and

$$[u_\varepsilon]_s^2 \leq \text{cap}_s(A, \Omega) + \varepsilon.$$

On the other hand, by Poincaré’s inequality,

$$|A| = \int_A 1 \, dx \leq \int_{\mathbb{R}^n} u_\varepsilon^2 \, dx \leq C(\Omega, s) [u_\varepsilon]_s^2 \leq C(\Omega, s) (\text{cap}_s(A, \Omega) + \varepsilon).$$

Take the limit $\varepsilon \downarrow 0$ to obtain the result. □

For every $A \in \mathcal{A}_s(\Omega)$, we will show that $A = \{u_A^s > 0\}$ in the sense of $\text{cap}_s(\cdot, \Omega)$. To prove this aim, we need some previous results which are modifications from [11, Lemma 2.1] and [12, Proposition 5.5].

Lemma A.3 *Let $A \in \mathcal{A}_s(\Omega)$. Then, there exists an increasing sequence $\{v_k\}_{k \in \mathbb{N}} \subset H_0^s(\Omega)$ of nonnegative functions, such that $\sup_{k \in \mathbb{N}} v_k = 1_A$ s -q.e. on Ω .*

We omit the proof since it is completely analogous to that of [11, Lemma 2.1].

We prove a density result in $H_0^s(A)$, for $A \in \mathcal{A}_s(\Omega)$, which is similar to [12, Proposition 5.5].

Lemma A.4 *Let $A \in \mathcal{A}_s(\Omega)$. Then, $\{\varphi u_A^s : \varphi \in C_c^\infty(\Omega)\}$ is dense in $H_0^s(A)$.*

Proof In order to prove the lemma, it is sufficient to see that we can approximate any nonnegative function $w \in H_0^s(A)$ with $(-\Delta)^s w \in L^\infty(\Omega)$, since $L^\infty(\Omega)$ is dense in $H^{-s}(\Omega)$ and $w = w^+ - w^-$. Indeed, for an arbitrary function $w \in H_0^s(\Omega)$, we know that $(-\Delta)^s w =: f \in H^{-s}(\Omega)$.

Denote by $f := (-\Delta)^s w$. Then,

$$(-\Delta)^s w \leq \|f\|_{L^\infty(\Omega)} = \|f\|_{L^\infty(\Omega)} (-\Delta)^s u_A^s \quad \text{in } A.$$

By comparison, we obtain $0 \leq w \leq cu_A^s$, where $c := \|f\|_{L^\infty(\Omega)}$.

For every $\varepsilon > 0$, consider $(w - c\varepsilon)^+ \in H_0^s(\Omega)$. Thus,

$$\{(w - c\varepsilon)^+ > 0\} \subset \{u_A^s > \varepsilon\}. \tag{A.1}$$

Notice that $u_A^s \in L^\infty(\Omega)$ by [13, Theorem 4.1]. Observe that, using (A.1), $\varepsilon < u_A^s \leq \|u_A^s\|_{L^\infty(\Omega)}$ in $\{(w - c\varepsilon)^+ > 0\}$. Then, the function $\frac{(w - c\varepsilon)^+}{u_A^s}$ belongs to $H_0^s(\Omega)$. So, there exists a sequence $\{\varphi_k^\varepsilon\}_{k \in \mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_k^\varepsilon \rightarrow \frac{(w - c\varepsilon)^+}{u_A^s}$ strongly in $H_0^s(\Omega)$, when $k \rightarrow \infty$. Therefore, $\varphi_k^\varepsilon u_A^s \rightarrow (w - c\varepsilon)^+$ strongly in $H_0^s(\Omega)$, when $k \rightarrow \infty$.

On the other hand, $(w - c\varepsilon)^+ \rightarrow w$ strongly in $H_0^s(\Omega)$, when $\varepsilon \downarrow 0$.

Consequently, by a diagonal argument, there exist subsequences $\varepsilon_j \downarrow 0$ and $\{\varphi_{k_j}^{\varepsilon_j}\}_{j \in \mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_{k_j}^{\varepsilon_j} u_A^s \rightarrow w$ strongly in $H_0^s(\Omega)$. □

The following proposition is an essential component to relate domains and functions. It also contributes to the proofs of principal results Theorems 2.6 and 2.8.

Proposition A.5 *Let $A \in \mathcal{A}_s(\Omega)$. Then, $A = \{u_A^s > 0\}$ in sense of $\text{cap}_s(\cdot, \Omega)$. That is, $\text{cap}_s(A \Delta \{u_A^s > 0\}, \Omega) = 0$.*

Proof It is clear that $u_A^s = 0$ s -q.e. on $\mathbb{R}^n \setminus A$. So, $\{u_A^s > 0\} \subset A$.

To see $A \subset \{u_A^s > 0\}$, we use the previous lemmas.

By Lemma A.3, there exists an increasing sequence $\{v_k\}_{k \in \mathbb{N}} \subset H_0^s(\Omega)$ of nonnegative functions, such that $\sup_{k \in \mathbb{N}} v_k = 1_A$ s -q.e. on Ω .

For every v_k , by Lemma A.4, there exists a sequence $\{\varphi_j^k\}_{j \in \mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_j^k u_A^s \rightarrow v_k$ strongly in $H_0^s(\Omega)$ and s -q.e., when $j \rightarrow \infty$. Since $\varphi_j^k u_A^s = 0$ s -q.e. in $\{u_A^s = 0\}$, then $v_k = 0$ s -q.e. in $\{u_A^s = 0\}$. Therefore, $1_A = 0$ s -q.e. in $\{u_A^s = 0\}$, which implies $A \subset \{u_A^s > 0\}$. □

Now, we prove a key estimate used in Sect. 4, which is a simply remark following the proof of [14, Proposition 2.2]. Notice that we are interested in finding a positive constant connecting in some sense $\text{cap}_s(\cdot, \Omega)$ and $\text{cap}_1(\cdot, \Omega)$. But, we also want that this constant does not depend on s . As our goal in Sect. 4 is related to the limit case $s \uparrow 1$, we can assume $0 < \varepsilon_0 < s < 1$ for some ε_0 and that will be enough to obtain this desired and *independent* constant.

As we said before, the proof of next lemma follows [14, Proposition 2.2] and, despite the similarity, it is included since we want to analyze how the constant depends on s .

Lemma A.6 *Let $\varepsilon_0 > 0$ and $\varepsilon_0 < s < 1$. Then, there exists a constant $C > 0$ such that for every $u \in H_0^1(\Omega)$*

$$(1 - s)[u]_s^2 \leq C \|\nabla u\|_{L^2(\Omega)}^2.$$

and $C = C(\Omega, n, \varepsilon_0)$ does not depend on s .

Proof Let $u \in C_c^\infty(\Omega)$, we split $[u]_s^2$ into two pieces.

For the first part, use the change of variable $z = y - x$ and observe that for $z \in B_1(0) \setminus \{0\}$ and $\varphi(t) := u(x + tz)$ for $t \in [0, 1]$ we estimate

$$\frac{|u(x + z) - u(x)|}{|z|} = \frac{\left| \int_0^1 \varphi'(t) dt \right|}{|z|} = \frac{\left| \int_0^1 \nabla u(x + tz) \cdot z dt \right|}{|z|} \leq \int_0^1 |\nabla u(x + tz)| dt.$$

Now, use the previous remark and Jensen’s inequality to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|y-x| < 1\}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy &= \int_{\mathbb{R}^n} \int_{B_1(0)} \frac{|u(x) - u(z+x)|^2}{|z|^{n+2s}} dz dx \\ &= \int_{\mathbb{R}^n} \int_{B_1(0)} \frac{|u(x) - u(z+x)|^2}{|z|^2 |z|^{n+2(s-1)}} dz dx \\ &\leq \int_{\mathbb{R}^n} \int_{B_1(0)} \left(\int_0^1 \frac{|\nabla u(x + tz)|}{|z|^{\frac{n}{2}+s-1}} dt \right)^2 dz dx \\ &\leq \int_{B_1(0)} \frac{1}{|z|^{n+2(s-1)}} \int_0^1 \|\nabla u\|_{L^2(\Omega)}^2 dt dz \\ &\leq \|\nabla u\|_{L^2(\Omega)}^2 \int_{B_1(0)} \frac{1}{|z|^{n+2(s-1)}} dz \\ &= \frac{|B_1(0)|}{2(1-s)} \|\nabla u\|_{L^2(\Omega)}. \end{aligned}$$

For the remaining part, use $|a - b|^2 \leq 2(a^2 + b^2)$ and easily follows

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|y-x| \geq 1\}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy &\leq 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|y-x| \geq 1\}} \frac{|u(x)|^2 + |u(y)|^2}{|x - y|^{n+2s}} dx dy \\ &\leq 4 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|y-x| \geq 1\}} \frac{|u(x)|^2}{|x - y|^{n+2s}} dx dy \\ &\leq \int_{\mathbb{R}^n} |u(x)|^2 \left(\int_{\{|z| \geq 1\}} \frac{1}{|z|^{n+2s}} dz \right) dx \\ &= \frac{|B_1(0)|}{2s} \|u\|_{L^2(\Omega)} \\ &\leq \frac{|B_1(0)|}{2\varepsilon_0} C_1(\Omega, n) \|\nabla u\|_{L^2(\Omega)}, \end{aligned}$$

where $C_1(\Omega, n)$ is the constant of classical Poincaré’s inequality in $H_0^1(\Omega)$.

Then, put together the two estimates to conclude

$$\begin{aligned} (1-s)[u]_s^2 &\leq (1-s) \left(\frac{C_1(\Omega, n)}{2\varepsilon_0} + \frac{1}{2(1-s)} \right) |B_1(0)| \|\nabla u\|_{L^2(\Omega)} \\ &\leq \left(\frac{C_1(\Omega, n)}{2\varepsilon_0} + \frac{1}{2} \right) |B_1(0)| \|\nabla u\|_{L^2(\Omega)} \\ &= C(\Omega, n, \varepsilon_0) \|\nabla u\|_{L^2(\Omega)}. \end{aligned}$$

□

Automatically, we obtain an estimate relating the s -capacity and the 1-capacity.

Corollary A.7 *Let $\varepsilon_0 > 0$ and $\varepsilon_0 < s < 1$. Then, there exists a constant $C > 0$ such that for every $A \subset \Omega$*

$$(1 - s)\text{cap}_s(A, \Omega) \leq C\text{cap}_1(A, \Omega),$$

and $C = C(\Omega, n, \varepsilon_0)$ does not depend on s .

We deduce other useful remark from Lemma A.6: Every 1-quasi open set is also an s -quasi-open, for $0 < s < 1$.

Remark A.8 For every $0 < s < 1$, $\mathcal{A}_1(\Omega) \subset \mathcal{A}_s(\Omega)$. Moreover, if $0 < s < t \leq 1$, then $\mathcal{A}_t(\Omega) \subset \mathcal{A}_s(\Omega)$.

Proof Let $A \in \mathcal{A}_1(\Omega)$. There exists a decreasing sequence of open sets $\{G_k\}_{k \in \mathbb{N}}$ such that $A \cup G_k$ is open and $\text{cap}_1(G_k, \Omega) \rightarrow 0$.

Let $0 < s < 1$. By Corollary A.7, $\text{cap}_s(G_k, \Omega) \rightarrow 0$. Then, $A \in \mathcal{A}_s(\Omega)$.

To prove $\mathcal{A}_t(\Omega) \subset \mathcal{A}_s(\Omega)$ for $0 < s < t \leq 1$, use definitions of capacity and [14, Proposition 2.1]. \square

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