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On certain non-hypoelliptic vector fields with finite-codimensional range on the three-torus

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Abstract We study the finiteness of range's codimension for a class of non-globally hypoelliptic vector fields on a torus of dimension three. The linear dependence of certain interactions of the coefficients is crucial. This condition is close to condition (P) of Nirenberg and Treves. Certain obstructions of number-theoretical nature involving Liouville numbers also appear in the results.

Keywords Global solvability · Complex vector fields · Condition (P) · Periodic solutions · Fourier series

Mathematics Subject Classification Primary 35A05 · 35F05; Secondary 35B10 · 35C10

1 Introduction and main results

Our purpose is to present non-globally hypoelliptic vector fields with finite-codimensional range on a torus of dimension three. We recall that a vector field L on $\mathbb{T}^n \simeq \mathbb{R}^n/2\pi\mathbb{Z}^n$ is a linear operator on $\mathscr{C}^\infty(\mathbb{T}^n)$ which satisfies the Leibniz rule. As in [2], we refer to the fact that the codimension of $L\mathscr{C}^\infty(\mathbb{T}^n)$ is finite by saying that L is *strongly solvable* in $\mathbb{T}^n$, while we say that L is *globally hypoelliptic* if the conditions $\mu \in \mathscr{D}'(\mathbb{T}^n)$ and $L\mu \in \mathscr{C}^\infty(\mathbb{T}^n)$ imply that $\mu \in \mathscr{C}^\infty(\mathbb{T}^n)$.

The existence of non-globally hypoelliptic vector fields with finite-codimensional range on $\mathbb{T}^2$ is already known. In [2–4,8] it has been exhibited vector fields with this property.

The operators in [2] are of the form $\partial_t + ib(x,t)\partial_x$, where (x,t) are the coordinates in $\mathbb{T}^2$ and the function b is real-valued and smooth. Since the Nirenberg–Treves condition (P) is

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necessary for the range to have finite codimension (see Corollary 26.4.8 from [11]), for each $x \in \mathbb{T}^1$ the function $t \mapsto b(x,t)$ cannot change sign on $\mathbb{T}^1$; moreover, when the characteristic set is the union of a finite number of one-dimensional orbits which are diffeomorphic to the unit circle, the approach to study the range of this class of operators involves the order of vanishing of b.

In [3,8] the operators are of the form $\partial_t + (a(x) + ib(x))\partial_x$, while in [4] they are of the form $\partial_t + (a(x,t) + ib(x,t))\partial_x$, where a and b are real-valued and smooth. For these vector fields, in addition to the condition (P) of Nirenberg–Treves, certain relations between the order of vanishing of a and b are connected to the study of range's codimension.

In dimension three, let us start by considering the vector field $\partial_t + ib(x,t)\partial_x + (\alpha + i\beta)\partial_y$ on $\mathbb{T}^3_{(x,y,t)}$, where α and β are real numbers. Notice that we have added a constant part to the vector field treated in [2]. When $\beta \neq 0$, condition (P) of Nirenberg–Treves implies that a necessary condition for the strong solvability is that the function $t \in \mathbb{T}^1 \mapsto \xi b(x,t) + \eta\beta$ does not change sign, for all $x \in \mathbb{T}^1$ and $\xi, \eta \in \mathbb{R}$. Hence, it follows from Lemma 3.1 of [6] that the function b must depend only on the variable x. The operator $\partial_t + ib(x)\partial_x + (\alpha + i\beta)\partial_y$ on $\mathbb{T}^3_{(x,y,t)}$ is a particular form of

$$L = \partial_t + (a_1(x) + ib_1(x))\partial_x + (a_2(x) + ib_2(x))\partial_y, \tag{1}$$

where a_j and b_j are smooth real-valued functions on $\mathbb{T}^1$, $j = 1, 2$.

In this article, we study the strong solvability of (1).

The case in which $a_2 + ib_2 = \lambda(a_1 + ib_1)$ ($\lambda \in \mathbb{R}$) was treated in a recent work.[1] In this case, we have an operator of the form

$$\mathscr{L} \doteq \partial_t + (a(x) + ib(x))(\partial_x + \lambda\partial_y).$$

When $(a + ib)^{-1}(0) = \emptyset$, this operator is strongly solvable if and only if it is globally hypoelliptic. Indeed, this result follows from [5] by considering $\frac{1}{a+ib}\mathscr{L}$. On the other hand, when $a + ib$ is not identically zero, but vanishes at some point, then the operator $\mathscr{L}$ is neither strongly solvable nor globally hypoelliptic, since the distributions $\delta(x - x^*) \otimes \delta(y - y^*) \otimes 1_t$ belong to the kernel of the transpose operator ${}^t\mathscr{L}$, for all $(x^*, y^*) \in (a_1 + ib_1)^{-1}(0) \times \mathbb{T}^1$, and $\mathscr{L}(\delta(x - x^*) \otimes 1_y \otimes 1_t) = 0$. Hence, a weaker notion of solvability was considered, which is the closedness of the range. As in [7], when an operator has closed range we say that it is *globally solvable*. For the global solvability, condition (P) is not necessary in general (for instance, in [5] we have vector fields of tube type which are globally solvable but do not satisfy condition (P); e.g., $\partial_t + i\cos(t)\partial_y + i\sin(t)\partial_x$ on $\mathbb{T}^3$). Without assuming condition (P), b may change sign between two consecutive zeros of $a + ib$. However, for the global solvability of $\mathscr{L}$, b may change sign at most once between consecutive zeros; moreover, this change of sign is connected with relations between the order of vanishing of a and b.

Following the historical aspects, the properties described in the second, third, and last paragraph will also appear in our results. Indeed, the approach used in this article is inspired by the results in dimension two, but mainly it is inspired by the new developments in higher dimension mentioned in the paragraph above (see also [5,6]).

Certain necessary conditions for the solvability of L are derived from the results in dimension two. For instance, results of [3,7] imply that $\partial_t + (a_1(x) + ib_1(x))\partial_x$ is not globally solvable when $a_1 + ib_1$ does not vanish identically and $a_1 + ib_1$ has a zero of infinite order. As a consequence, by using partial Fourier series in the variable y, it follows that our operator L [given by (1)] is not globally solvable in this case; hence, it is not strongly solvable. Since L is trivially non-strongly solvable when $a_1 + ib_1$ vanishes identically, we may assume that either

[1] Recent work (not published yet) from the authors A. P. Bergamasco, P. L. Dattori da Silva and R. B. Gonzalez.

$a_1 + ib_1$ never vanishes or $(a_1 + ib_1)^{-1}(0) \neq \emptyset$ and $a_1 + ib_1$ has only zeros of finite order. Only in this second case, we find non-globally hypoelliptic vector fields which are strongly solvable, which makes this case the most interesting one. In fact, when $a_1 + ib_1$ never vanishes, L is strongly solvable (resp. globally hypoelliptic) if and only if $\tilde{L} \doteq (1/(a + ib))L$ is strongly solvable (resp. globally hypoelliptic). From [5] we see that $\tilde{L}$ is strongly solvable if and only if it is globally hypoelliptic.

Note that the condition $(a_1 + ib_1)(x^*) = 0$ implies that $L(\delta(x - x^*) \otimes 1_y \otimes 1_t) = 0$; hence, L is not globally hypoelliptic on any neighborhood of $\{x^*\} \times \mathbb{T}^1 \times \mathbb{T}^1$.

Throughout this article, we will assume that $(a_1 + ib_1)^{-1}(0) \neq \emptyset$ and that $a_1 + ib_1$ has only zeros of finite order.

We write

$$(a_1 + ib_1)^{-1}(0) = \{x_1 < x_2 < \cdots < x_N\} \quad \text{and} \quad x_{N+1} \doteq x_1 + 2\pi.$$

At each zero $x_\ell \in (a_1 + ib_1)^{-1}(0)$, we denote by n_ℓ and m_ℓ the order of vanishing of a and b, respectively.

As mentioned above, by using partial Fourier series in the variable y, we may verify that L is not strongly solvable if the vector field $\partial_t + (a_1 + ib_1)(x)\partial_x$ is not strongly solvable on $\mathbb{T}^2$. Hence, results from [3,8] imply that L is not strongly solvable if there exists $\ell \in \{1, \ldots, N\}$ such that either $m_\ell = 1$ or $m_\ell \geq 2$ and $m_\ell \geq 2n_\ell - 1$ (see Theorems 1.1, 2.2 and 3.1, and Proposition 4.1 in [8]). Likewise, L is not strongly solvable if b_1 changes sign between two consecutive zeros of $a_1 + ib_1$.

We will verify that the linear dependence of b_1 and $a_2b_1 - a_1b_2$ (as functions belonging to $\mathscr{C}^\infty(\mathbb{T}^1, \mathbb{R})$) is crucial for the strong solvability of our operator L. In contrast with [5], this linear dependence takes into account both the real and the imaginary part of the coefficients of the operator.

Similar to what happens in [1,5–7,9,10,12], and many others, certain obstructions of number-theoretical nature, such as Diophantine conditions, also appear. We recall that an irrational number α is said to be a *Liouville number* if there exists a sequence $(p_n, q_n) \in \mathbb{Z} \times \mathbb{N}$ such that $q_n \to \infty$ and $|p_n - \alpha/q_n| < (q_n)^{-n}$, for all $n \in \mathbb{N}$.

We will prove the following description related to the strong solvability of (1):

Theorem 1 *Let L be the operator given by (1) and assume that $(a_1 + ib_1)^{-1}(0) \neq \emptyset$. If L is strongly solvable, then the following conditions hold:*

(i) *$a_1 + ib_1$ vanishes only of finite order; moreover, b_1 does not change sign between two consecutive zeros of $a_1 + ib_1$, and at each zero $x_\ell \in (a_1 + ib_1)^{-1}(0) = \{x_1 < x_2 < \cdots < x_N\}$ we have $2 \leq m_\ell < 2n_\ell - 1$.*

(ii) *for each ℓ, either $b_2(x_\ell) \neq 0$ or $b_2(x_\ell) = 0$ and $a_2(x_\ell)$ is a non-Liouville irrational number.*

(iii) *b_1 and $a_2b_1 - a_1b_2$ are $\mathbb{R}$-linearly dependent; moreover, $a_2b_1 - a_1b_2 = \lambda b_1$ with λ an irrational number.*

Under conditions above, L is strongly solvable provided λ is a non-Liouville irrational number. In this case, we note that the range's codimension of L is $1 + \sum_{\ell=1}^{N} \min\{m_\ell, n_\ell\}$.

The proof of Theorem 1 is given in Sects. 2 and 3.

Example 1 For $\alpha \in \mathbb{R}$, it follows from Theorem 1 that the operator

$$\partial_t + i(1 - \cos(x))\partial_x + (\alpha + i \sin(x))\partial_y$$

is strongly solvable on $\mathbb{T}^3$ if and only if α is a non-Liouville irrational number.

In contrast, for any $\alpha \in \mathbb{R}$, the operator of tube type

$$\partial_t + i(1-\cos(t))\partial_x + (\alpha + i\sin(t))\partial_y$$

is never even globally solvable on $\mathbb{T}^3$, since the functions $1-\cos(t)$ and $\sin(t)$ are $\mathbb{R}-$linearly independent and the first of them has nonzero mean (see Theorem 1.1 of [5]).

We may also destroy the strong solvability of a vector field of tube type when we exchange the dependence on the variable t by a dependence on the variable x.

Example 2 If α is a non-Liouville irrational number, then Theorem 1.3 of [5] implies that

$$\partial_t + (\cos(t) + i\cos^2(t))\partial_x + \alpha\partial_y \tag{2}$$

is globally hypoelliptic. For this class of operators global hypoellipticity is equivalent to strong solvability. Hence, operator (2) is strongly solvable. On the other hand, by Theorem 1 we see that the operator

$$\partial_t + (\cos(x) + i\cos^2(x))\partial_x + \alpha\partial_y$$

is not strongly solvable, since the order of vanishing at each zero does not satisfy condition (i).

Notice also that, even in the case of constant coefficients, in general the assumption that the constant λ [given in (iii)—Theorem 1] is a non-Liouville irrational number is not necessary for the strong solvability of L. For instance, consider $L = \partial_t + i\partial_x + (\alpha + i\beta)\partial_y$. For this L we have $\lambda = \alpha$. If α and β are Liouville numbers such that the vector (α, β) is not a Liouville vector, then L is strongly solvable (see [1] for a construction of two Liouville numbers such that the pair is not a Liouville vector). On the other hand, if α is a non-Liouville irrational number, then L is strongly solvable.

The next results are other consequences of Theorem 1.

Corollary 1 *Consider the operator*

$$L = \partial_t + (a_1 + ib_1)(x)\partial_x + a_2(x)\partial_y$$

and assume that $(a_1 + ib_1)^{-1}(0) \neq \emptyset$. *Then* L *is strongly solvable if and only if the following conditions hold:*

(i) $a_1 + ib_1$ *vanishes only of finite order and at each zero* $x_\ell \in (a_1 + ib_1)^{-1}(0) = \{x_1 < x_2 < \cdots < x_N\}$ *we have* $2 \leq m_\ell < 2n_\ell - 1$, *where* m_ℓ *and* n_ℓ *are the order of vanishing of* a_1 *and* b_1, *respectively, at* x_ℓ.
(ii) b_1 *does not change sign between two consecutive zeros of* $a_1 + ib_1$, *and* $a_2 \equiv \alpha$, *where* α *is a non-Liouville irrational number.*

For the next result, we consider the operator

$$L = \partial_t + (a_1 + ib_1)(x)\partial_x + (\alpha + i\beta)\partial_y, \quad \alpha \in \mathbb{R}, \;\; \beta \in \mathbb{R}\backslash\{0\}, \tag{3}$$

and we assume that $(a_1 + ib_1)^{-1}(0) \neq \emptyset$ and that $a_1 + ib_1$ vanishes only of finite order.

Corollary 2 *Consider the operator* L *given by* (3). *If* L *is strongly solvable, then* $a_1 = \gamma b_1$, *with* $\gamma \in \mathbb{R}$ *such that* $\alpha - \gamma\beta$ *is an irrational number. Moreover,* b_1 *does not change sign between two consecutive zeros of* $a_1 + ib_1$, *and* b_1 *vanishes only of finite order greater than two. On the other hand, under conditions above,* L *is strongly solvable if* $\alpha - \gamma\beta$ *is a non-Liouville irrational number.*

We point out that even operators with a very similar structure may diverge with respect to the strong solvability.

Example 3 By Corollary 2 we see that the operator

$$\partial_t + (\sqrt{2}+i)\cos^2(x)\partial_x + i\sqrt{2}\partial_y$$

is not strongly solvable, while the operator

$$\partial_t + (\sqrt{2}+i)\cos^2(x)\partial_x + i\sqrt{3}\partial_y$$

is strongly solvable.

2 Necessary conditions

In this section, we establish necessary conditions for the strong solvability of operator (1), which is given by

$$L = \partial_t + (a_1(x) + ib_1(x))\partial_x + (a_2(x) + ib_2(x))\partial_y,$$

under the general assumptions that $(a_1 + ib_1)^{-1}(0) \neq \emptyset$ and that $a_1 + ib_1$ vanishes only of finite order. Recall that, in this case, we are writing $(a_1+ib_1)^{-1}(0) = \{x_1 < x_2 < \cdots < x_N\}$, $x_{N+1} = x_1 + 2\pi$, and we are denoting by n_ℓ (respectively m_ℓ) the order of vanishing of a (respectively b) at $x_\ell \in (a_1 + ib_1)^{-1}(0)$.

As we have already mentioned in the previous section, it follows from the results in dimension two (see [3,8]), that the conditions given in item (i) in Theorem 1 are necessary for the strong solvability of our operator L.

From now on we will deal with the new conditions which appear in (ii) and (iii) in Theorem 1.

Lemma 2, Propositions 1, 2 and item (i) in Proposition 3 show the necessity of condition (iii) in Theorem 1, while the necessity of item (ii) in Theorem 1 follows from Lemmas 1 and 3.

Before we proceed, we recall that L is globally solvable if and only if $L\mathscr{C}^\infty(\mathbb{T}^3) = (\ker {}^tL)^\circ$, where ${}^tL\colon \mathscr{D}'(\mathbb{T}^3) \to \mathscr{D}'(\mathbb{T}^3)$ denotes the transpose operator of L. Moreover, L is strongly solvable if and only if L is globally solvable and $\dim\ker {}^tL < \infty$. Both this two characterizations will be used throughout this article.

Lemma 1 *If there exists ℓ such that $b_2(x_\ell) = 0$ and $a_2(x_\ell)$ is rational, then* $\dim\ker {}^tL = \infty$.

Proof Since $a_2(x_\ell)$ is a rational number, there exists a (infinite) sequence $(j_n, k_n) \in \mathbb{Z}^2\backslash\{(0,0)\}$ such that $k_n + j_n a_2(x_\ell) = 0$, for all n. By using partial Fourier series in the variables y and t, we may verify that the distributions

$$\delta(x - x_\ell) \otimes e^{ij_n y} \otimes e^{ik_n t}$$

belong to $\ker {}^tL$, which implies that $\dim\ker {}^tL = \infty$.

The proof that condition (ii) in Theorem 1 is necessary for the strong solvability of L will be completed in the end of this section.

We now focus on the condition (iii), which is related to the linear dependence of b_1 and $a_2b_1 - a_1b_2$. We claim that L is not strongly solvable if b_1 and $a_2b_1 - a_1b_2$ are $\mathbb{R}$-linearly independent functions.

Lemma 2 *Suppose that b_1 and $a_2b_1 - a_1b_2$ are $\mathbb{R}$-linearly independent. There exist an interval $(x_{\ell_0}, x_{\ell_0+1})$ and nonzero integers p and q such that $qb_1(x) + p(a_2b_1 - a_1b_2)(x)$ changes sign on $(x_{\ell_0}, x_{\ell_0+1})$. Moreover, we may choose p and q so that $a_2(x_\kappa)p/q \neq -1$, for $\kappa = \ell_0, \ell_0 + 1$.*

Proof If b_1 and $a_2b_1 - a_1b_2$ are linearly dependent on $(x_\ell, x_{\ell+1})$, then there exists $\lambda_\ell \in \mathbb{R}$ such that $(a_2b_1 - a_1b_2)(x) = \lambda_\ell b_1(x)$, for all $x \in (x_\ell, x_{\ell+1})$. Since b_1 vanishes of finite order at each x_ℓ, it follows that $\lambda_\ell = \lambda_j$, for every $j, \ell = 1, \dots, N$. This implies that b_1 and $a_2b_1 - a_1b_2$ are linearly dependent on $\mathbb{T}^1$. Hence, if they are linearly independent, then there exists $\ell_0 \in \{1, \dots, N\}$ such that they are linearly independent on $(x_{\ell_0}, x_{\ell_0+1})$. By applying the proof of Lemma 3.1 of [6] on the interval $(x_{\ell_0}, x_{\ell_0+1})$, we obtain infinitely many nonzero integers $p \neq q$, which produce infinitely many rationals p/q, so that $\theta(x) \doteq qb_1(x) + p(a_2b_1 - a_1b_2)(x)$ changes sign on $(x_{\ell_0}, x_{\ell_0+1})$.

Proposition 1 *Let p and q be nonzero integers such that $\theta(x) \doteq qb_1(x)+p(a_2b_1-a_1b_2)(x)$ changes sign only once on $(x_{\ell_0}, x_{\ell_0+1})$. In this case we have* $\dim\ker {}^tL = \infty$.

Proof We will exhibit infinitely many linearly independent distributions in $\ker {}^tL$.

Assume that there is $\eta \in (x_{\ell_0}, x_{\ell_0+1})$ such that $\theta \geq 0$ on (x_{ℓ_0}, η) and $\theta \leq 0$ on (η, x_{ℓ_0+1}) (the other case is analogous).

For $x \in (x_{\ell_0}, x_{\ell_0+1})$, set

$$\Theta(x) = \int_\eta^x \left(\frac{\theta(s)}{a_1^2(s) + b_1^2(s)} + i\,\frac{(q + pa_2(s))a_1(s) + pb_1(s)b_2(s)}{a_1^2(s) + b_1^2(s)} \right) \mathrm{d}s.$$

For each positive integer n, the function

$$\psi_n(x) = \begin{cases} \exp\{n\Theta(x)\}, & x \in (x_{\ell_0}, x_{\ell_0+1}) \\ 0, & x \in \mathbb{T}^1 \backslash (x_{\ell_0}, x_{\ell_0+1}) \end{cases}$$

belongs to $\mathsf{L}^\infty(\mathbb{T}^1) \subset \mathscr{D}'(\mathbb{T}^1)$. We will find distributions μ_n such that

$$\left(\frac{\psi_n}{a_1 + ib_1} - \mu_n \right)(x) \otimes e^{-in(py+qt)} \in \ker {}^tL \backslash \{0\} \subset \mathscr{D}'(\mathbb{T}^3). \tag{4}$$

Note that since a_1+ib_1 vanishes only of finite order, we may take $\psi_n/(a_1+ib_1) \in \mathscr{D}'(\mathbb{T}^1)$ so that $(a_1 + ib_1)\psi_n/(a_1 + ib_1) = 1$ and satisfying $\operatorname{supp}(\psi_n/(a_1 + ib_1)) \subset \operatorname{supp}(\psi_n)$. By defining $\omega_n = \psi_n' - inq\psi_n/(a_1+ib_1) - inp(a_2+ib_2)\psi_n/(a_1+ib_1)$ we obtain $\operatorname{supp}(\omega_n) \subset \{x_{\ell_0}, x_{\ell_0+1}\}$. Hence, there exist positive integers r_{ℓ_0} and r_{ℓ_0+1}, and constants $\alpha_{\ell_0}^m$ and $\alpha_{\ell_0+1}^m$, such that

$$\omega_n = \sum_{m=0}^{r_{\ell_0}} \alpha_{\ell_0}^m \delta^{(m)}(x - x_{\ell_0}) + \sum_{m=0}^{r_{\ell_0+1}} \alpha_{\ell_0+1}^m \delta^{(m)}(x - x_{\ell_0+1}).$$

Finally, we claim that we may take constants $\beta_{\ell_0}^m$ and $\beta_{\ell_0+1}^m$ such that the distribution

$$\mu_n = \sum_{m=0}^{r_{\ell_0}} \beta_{\ell_0}^m \delta^{(m)}(x - x_{\ell_0}) + \sum_{m=0}^{r_{\ell_0+1}} \beta_{\ell_0+1}^m \delta^{(m)}(x - x_{\ell_0+1})$$

satisfies $((a_1 + ib_1)\mu_n)' - inq\mu_n - inp(a_2 + ib_2)\mu_n = \omega_n$ and, therefore, it satisfies (4). Indeed, the constants must satisfy the following: for $\kappa = \ell_0, \ell_0 + 1$, $0 < m < r_\kappa$, and with $C_n^m \doteq (-1)^n \binom{m}{n}$,

$$\begin{aligned}\alpha_\kappa^{r_\kappa} &= \beta_\kappa^{r_\kappa}(C_1^{r_\kappa}(a_1+ib_1)'(x_\kappa) - inq - inpC_0^{r_\kappa}(a_2+ib_2)(x_\kappa))\\ &= \beta_\kappa^{r_\kappa}\left[C_1^{r_\kappa}a_1'(x_\kappa)+npb_2(x_\kappa)\right] + i\beta_\kappa^{r_\kappa}\left[C_1^{r_\kappa}b_1'(x_\kappa) - n(q+pa_2(x_\kappa))\right], \end{aligned} \tag{5}$$

$$\begin{aligned}\alpha_\kappa^{m} &= \beta_\kappa^{m}(C_1^{m}(a_1+ib_1)'(x_\kappa) - inq - inpC_0^{m}(a_2+ib_2)(x_\kappa))\\ &\quad+\sum_{j=m+1}^{r_\kappa}\beta_\kappa^j\left[C_{j-m+1}^j(a_1+ib_1)^{(j-m+1)}(x_\kappa) - inpC_{j-m}^j(a_2+ib_2)^{(j-m)}(x_\kappa)\right]\\ &= \beta_\kappa^{m}\left[C_1^{m}a_1'(x_\kappa)+npb_2(x_\kappa)\right] + i\beta_\kappa^{m}\left[C_1^{m}b_1'(x_\kappa) - n(q+pa_2(x_\kappa))\right]\\ &\quad+\sum_{j=m+1}^{r_\kappa}\beta_\kappa^j\left[C_{j-m+1}^j(a_1+ib_1)^{(j-m+1)}(x_\kappa) - inpC_{j-m}^j(a_2+ib_2)^{(j-m)}(x_\kappa)\right], \end{aligned} \tag{6}$$

and

$$\alpha_\kappa^0 = -\beta_\kappa^0 in(q+p(a_2+ib_2)(x_\kappa)) - inp\sum_{j=1}^{r_\kappa}\beta_\kappa^j C_j^j(a_2+ib_2)^{(j)}(x_\kappa). \tag{7}$$

By Lemma 1, we may assume that either $b_2(x_\kappa) \neq 0$ or $b_2(x_\kappa) = 0$ and $a_2(x_\kappa) \notin \mathbb{Q}$. Hence, for n large enough, we may solve Eqs. (5), (6), and (7) above.

Since the distributions given in (4) are linearly independent, it follows that $\dim\ker {}^tL = \infty$.

The existence of the integers p and q such that $qb_1(x)+p(a_2b_1-a_1b_2)(x)$ changes sign between consecutive zeros of a_1+ib_1 was crucial to show that the dimension of the kernel of ${}^tL$ is not finite. Although $\dim\ker {}^tL = \infty$, the range of L might be closed. Indeed, there exist situations where L is globally solvable (closed range) and, for all $(j,k) \in \mathbb{Z}^2\backslash\{(0,0)\}$, the function $kb_1(x) + j(a_2b_1-a_1b_2)(x)$ changes sign (but at most once) between consecutive zeros of a_1+ib_1 (see Remark 1 in the end of Sect. 3).

The next result shows that the range of L may not be closed when there exist integers p and q such that $qb_1(x) + p(a_2b_1-a_1b_2)(x)$ changes sign twice or more.

Proposition 2 *Let p and q be nonzero integers such that $\theta(x) \doteq qb_1(x)+p(a_2b_1-a_1b_2)(x)$ changes sign twice (or more) on $(x_{\ell_0}, x_{\ell_0+1})$. Suppose also that $a_2(x_\kappa)p/q \neq -1$ and that $2 \le m_\kappa < 2n_\kappa - 1$, for $\kappa = \ell_0, \ell_0+1$. In this case, L is not globally solvable.*

Proof With the information that θ changes sign at least twice on $(x_{\ell_0}, x_{\ell_0+1})$, we will construct a function $f \in (\ker {}^tL)^\circ\backslash L\mathscr{C}^\infty(\mathbb{T}^3)$, which will be given in the form

$$f(x,y,t) = \sum_{n=1}^{\infty}\hat{f}(x,np,nq)e^{in(py+qt)}. \tag{8}$$

Each Fourier coefficient $\hat{f}(\cdot,np,nq)$ will be given in $\mathscr{C}_c^\infty((x_{\ell_0}, x_{\ell_0+1}))$.

Notice that $\theta = qb_1 + p(a_2b_1 - a_1b_2)$ vanishes at x_κ, $\kappa = \ell_0, \ell_0+1$. Moreover, since $q + pa_2(x_\kappa) \neq 0$ (by hypothesis), the order of vanishing of θ at x_κ, o_κ, satisfies $o_\kappa \le m_\kappa$.

Since $o_{\ell_0} < \infty$, we have $\theta > 0$ or $\theta < 0$ on $(x_{\ell_0}, x_{\ell_0}+\delta)$, with $\delta > 0$ sufficiently small. Without loss of generality, we may assume that $\theta > 0$ on $(x_{\ell_0}, x_{\ell_0}+\delta)$.

The assumption that θ changes sign at least twice implies that there exist ξ_1 and ξ_2 such that $x_{\ell_0}+\delta < \xi_1 < \xi_2 < x_{\ell_0+1}$, $\theta(\xi_1) < 0$ and $\theta(\xi_2) > 0$.

Set $\sigma = \inf\{x \in (\xi_1,\xi_2);\ \theta(s) \ge 0,\ s \in (x,\xi_2)\}$ and take $\psi_1 \in \mathscr{C}_c^\infty((\sigma,\xi_2))$ so that $\theta > 0$ on $\operatorname{supp}\psi_1$, $\psi_1 \ge 0$ and $\int_{x_{\ell_0}}^{x_{\ell_0+1}}\psi_1(x)\mathrm{d}x = 1$.

We now choose $\eta < \sigma$ so that $\theta(\eta) < 0$ and $\int_\eta^x \theta(s)/(a_1^2 + b_1^2)(s)\mathrm{d}s > 0$, for all $x \in \operatorname{supp}\psi_1$. For $\delta > 0$ sufficiently small we also have $\theta < 0$ on $(\eta, \eta + \delta)$. Take $\psi_2 \in \mathscr{C}_c^\infty((\eta, \eta + \delta))$ so that $\psi_2 \geq 0$ and $\int_{x_{\ell_0}}^{x_{\ell_0+1}} \psi_2(x)\mathrm{d}x = 1$.

Set $\xi_0 \doteq \sup \operatorname{supp}\psi_2$, $M \doteq -\int_{\xi_0}^{\eta+\delta} \theta(s)/(a_1^2 + b_1^2)(s)\mathrm{d}s > 0$, and

$$d_n \doteq -\int_{x_{\ell_0}}^{x_{\ell_0+1}} \psi_2(x) \exp\left\{n \int_\eta^x \frac{\theta(s)}{a_1^2(s) + b_1^2(s)}\mathrm{d}s\right\} \mathrm{d}x,$$

and define

$$\begin{aligned}
&\hat{f}(x, np, nq) \\
&= (a_1 + ib_1)(x) \exp\left\{-nM - in \int_\eta^x \frac{qa_1(s) + p(a_1a_2 + b_1b_2)(s)}{a_1^2(s) + b_1^2(s)}\mathrm{d}s\right\} \\
&\quad \times \left[\psi_2(x) + d_n\psi_1(x) \exp\left\{-n \int_\eta^x \frac{\theta(s)}{a_1^2(s) + b_1^2(s)}\mathrm{d}s\right\}\right],
\end{aligned}$$

for all $n \in \mathbb{N}$. Note that $\operatorname{supp}\hat{f}(\cdot, np, nq) \subset (x_{\ell_0}, x_{\ell_0+1})$.

Since

$$|d_n| \leq \exp\left\{n \int_\eta^{\inf \operatorname{supp}\psi_2} \frac{\theta(s)}{a_1^2(s) + b_1^2(s)}\mathrm{d}s\right\} < 1$$

and, for all $x \in \operatorname{supp}\psi_1$,

$$\exp\left\{-n \int_\eta^x \frac{\theta(s)}{a_1^2(s) + b_1^2(s)}\mathrm{d}s\right\} < 1,$$

it follows that $\hat{f}(\cdot, np, nq)$ decays rapidly, thanks to e^{-nM}. Thus, f given by (8) belongs to $\mathscr{C}^\infty(\mathbb{T}^3)$. Moreover, for all $\mu \in \ker {}^tL$ we have

$$\begin{aligned}
&\langle\hat{\mu}(\cdot, -np, -nq), \hat{f}(\cdot, np, nq)\rangle \\
&= C_n e^{-nM} \int_{x_{\ell_0}}^{x_{\ell_0+1}} \left(\psi_2(x) \exp\left\{n \int_\eta^x \frac{\theta(s)}{a_1^2(s) + b_1^2(s)}\mathrm{d}s\right\} + d_n\psi_1(x)\right) \mathrm{d}x = 0,
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $f \in (\ker {}^tL)^\circ$.

Finally, if there exists $u \in \mathscr{C}^\infty(\mathbb{T}^3)$ such that $Lu = f$, then

$$\hat{f}(x, np, nq) = (a_1 + ib_1)(x)\partial_x\hat{u}(x, j, k) + in\,[kq + p(a_2 + ib_2)(x)]\,\hat{u}(x, j, k), \tag{9}$$

for all $x \in \mathbb{T}^1$, which implies that

$$\frac{\hat{f}(x, np, nq)}{(a_1 + ib_1)(x)} = \partial_x\hat{u}(x, np, nq) + \frac{n\,[\theta(x) + i(qa_1(x) + p(a_1a_2 + b_1b_2)(x))]}{a_1^2(x) + b_1^2(x)}\hat{u}(x, np, nq),$$

for all $x \in (x_{\ell_0}, x_{\ell_0+1})$ and $n \in \mathbb{N}$.

For $x \in (x_{\ell_0}, x_{\ell_0+1})$, set

$$E_n(x) = \exp\left\{n \int_\eta^x \frac{\theta(s) + i[qa_1(s) + p(a_1a_2 + b_1b_2)(s)]}{a_1^2(s) + b_1^2(s)}\mathrm{d}s\right\}.$$

We may write $\hat{u}(x, np, nq) = w_n(x) + v_n(x)$, where

$$w_n(x) = C_n E_{-n}(x), \;\; C_n \in \mathbb{C},$$

and

$$v_n(x) = E_{-n}(x) \int_{x_{\ell_0}}^{x} E_n(x') \hat{f}(x', np, nq)/(a_1 + ib_1)(x')\mathrm{d}x'.$$

Since $q + pa_2(x_{\ell_0}) \neq 0$, it follows from (9) that $\hat{u}(x_{\ell_0}, np, nq) = 0$. We also have $v_n(x_{\ell_0}) = 0$, since $\operatorname{supp} \hat{f}(\cdot, np, nq) \subset (x_{\ell_0}, x_{\ell_0+1})$. On the other hand,

$$|w_n(x)| = |C_n| \exp\left\{ -n \int_{\eta}^{x} \frac{\theta(s)}{a_1^2(s) + b_1^2(s)} \mathrm{d}s \right\}$$

and the conditions $\theta > 0$ on a neighborhood of x_{ℓ_0}, $o_{\ell_0} \leq m_{\ell_0}$ and $2 \leq m_{\ell_0} < 2n_{\ell_0} - 1$ imply that $\lim_{x \to x_{\ell_0}^+} w_n(x) = \infty$, provided $C_n \neq 0$. Hence, we must have $C_n = 0$.

Finally, with

$$K_n = \exp\left\{ -nM - n \int_{\eta}^{\eta+\delta} \frac{\theta(s)}{a_1^2(s) + b_1^2(s)} \mathrm{d}s \right\}$$

we have

$$\begin{aligned} |\hat{u}(\eta + \delta, np, nq)| &= K_n \int_{x_{\ell_0}}^{\eta+\delta} \psi_2(x') \exp\left\{ n \int_{\eta}^{x'} \frac{\theta(s)}{a_1^2(s) + b_1^2(s)} \mathrm{d}s \right\} \mathrm{d}x' \\ &\geq \exp\left\{ n \left(-\int_{\eta}^{\eta+\delta} + \int_{\xi_0}^{\eta+\delta} + \int_{\eta}^{\xi_0} \right) \frac{\theta(s)}{a_1^2(s) + b_1^2(s)} \mathrm{d}s \right\} = 1, \end{aligned}$$

where we recall that $\xi_0 = \sup \operatorname{supp} \psi_2$.

We have a contradiction, since $\hat{u}(\cdot, np, nq)$ must decay rapidly.

Lemma 2 and Propositions 1 and 2 imply that L is not strongly solvable when b_1 and $a_2b_1 - a_1b_2$ are $\mathbb{R}$-linearly independent.

Next proposition implies that L is not strongly solvable if there exists a rational number λ such that $a_2b_1 - a_1b_2 = \lambda b_1$. This completes the proof that condition (iii) in Theorem 1 is necessary for the strong solvability of L.

Proposition 3 *Let L be given by (1). Suppose that $(a_1 + ib_1)^{-1}(0) \neq \emptyset$ and that $a_1 + ib_1$ vanishes only of finite order. Write $(a_1 + ib_1)^{-1}(0) = \{x_1 < \cdots < x_N\}$ and let n_ℓ (respectively m_ℓ) be the order of vanishing of a_1 (respectively b_1) at x_ℓ, $\ell = 1, \ldots, N$. Assume that either $b_2(x_\ell) \neq 0$ or $b_2(x_\ell) = 0$ and $a_2(x_\ell)$ is an irrational number, for each $\ell = 1, \ldots, N$.*

(i) *If $a_2b_1 - a_1b_2 = \lambda b_1$, with $\lambda \in \mathbb{Q}$, then* $\dim \ker {}^tL = \infty$.
(ii) *Suppose that $a_2b_1 - a_1b_2 = \lambda b_1$, with λ an irrational number. In addition, assume $2 \leq m_\ell < 2n_\ell - 1$, for each $\ell = 1, \ldots, N$ and suppose that b_1 does not change sign between two consecutive zeros of $a_1 + ib_1$. Under these conditions,* $\dim \ker {}^tL < \infty$; *moreover, a distribution μ belongs to* $\ker {}^tL$ *if and only if*

$$\mu = (K_0/(a_1 + ib_1)) \otimes 1_y \otimes 1_t + \sum_{\ell=1}^{N} \sum_{m=0}^{r_\ell - 1} K_{\ell m} \delta^{(m)}(x - x_\ell) \otimes 1_y \otimes 1_t,$$

where $r_\ell = \min\{m_\ell, n_\ell\}$ and K_0, $K_{\ell m}$, $m = 0, \ldots, r_\ell - 1$ are constants.

Proof (i) If $a_2b_1 - a_1b_2 = \lambda b_1$, with $\lambda = p/q$ ($p \in \mathbb{Z}$, $q \in \mathbb{N}$), then for ξ_ℓ be fixed in $(x_\ell, x_{\ell+1})$ ($\ell \in \{1, \dots, N\}$) and for $n \in \mathbb{N}$, set

$$\psi_n(x) = \begin{cases} \exp\left\{ in \int_{\xi_\ell}^{x} \frac{a_1(s)(qa_2(s)-p)+qb_1(s)b_2(s)}{a_1^2(s)+b_1^2(s)} \mathrm{d}s \right\}, & x \in (x_\ell, x_{\ell+1}) \\ 0, & x \in \mathbb{T}^1 \backslash (x_\ell, x_{\ell+1}). \end{cases}$$

Since either $b_2(x_\ell) \neq 0$ or $a_2(x_\ell)$ is an irrational number, we may proceed as before [see arguments between Eqs. (4) and (7)] to verify that there exists a unique $\mu_n \in \mathscr{D}'(\mathbb{T}^1)$, such that

$$\left(\frac{\psi_n}{a_1 + ib_1} - \mu_n \right)(x) \otimes e^{in(qy-pt)}$$

belongs to $\ker {}^tL \backslash \{0\}$. Therefore, $\dim \ker {}^tL = \infty$.

(ii) Assume now that $a_2b_1 - a_1b_2 = \lambda b_1$, with λ an irrational number.

Given $\mu \in \ker {}^tL$, we use partial Fourier series in the variables y and t to write

$$\mu = \sum_{(j,k)\in\mathbb{Z}^2} \hat{\mu}(x, j, k) \otimes e^{i(jy+kt)}.$$

It follows that ${}^tL\mu = 0$ if and only if

$$\partial_x((a_1 + ib_1)(x)\hat{\mu}(x, j, k)) + i(k + j(a_2 + ib_2)(x))\hat{\mu}(x, j, k) = 0, \tag{10}$$

for all $(j, k) \in \mathbb{Z}^2$. Define $\nu_{jk} = (a_1 + ib_1)\hat{\mu}(\cdot, j, k)$. On each $(x_\ell, x_{\ell+1})$, $\ell = 1, \dots, N$, we have $\frac{\nu_{jk}}{a_1+ib_1} = \hat{\mu}(\cdot, j, k)$. Hence

$$\partial_x \nu_{jk} + i(k + j(a_2 + ib_2))\frac{\nu_{jk}}{a_1 + ib_1} = 0,$$

which implies that

$$\exp\left\{ \int_{\xi_\ell}^{x} \frac{i(k + j(a_2 + ib_2))}{(a_1 + ib_1)(s)} \mathrm{d}s \right\} \nu_{jk} = C_{jk}^{\ell} \text{ on } (x_\ell, x_{\ell+1}),$$

where $\xi_\ell \in (x_\ell, x_{\ell+1})$ is fixed and C_{jk}^ℓ is a constant.

Since b_1 does not change sign, $2 \leq m_\ell < 2n_\ell - 1$ and $2 \leq m_{\ell+1} < 2n_{\ell+1} - 1$, the function

$$\begin{aligned} &\exp\left\{ \int_{\xi_\ell}^{x} \frac{i(k + j(a_2 + ib_2))}{(a_1 + ib_1)(s)} \mathrm{d}s \right\} \\ &\quad = \exp\left\{ \int_{\xi_\ell}^{x} \frac{(k + \lambda j)b_1(s) + i[a_1(s)(k + ja_2(s)) + jb_1(s)b_2(s)]}{(a_1^2 + b_1^2)(s)} \mathrm{d}s \right\} \end{aligned}$$

is flat at either x_ℓ or $x_{\ell+1}$ provided $(j, k) \neq (0, 0)$ (recall that λ is an irrational number). Hence, for $(j, k) \neq (0, 0)$ we have $C_{jk}^\ell = 0$ and $(a_1 + ib_1)\hat{\mu}(\cdot, j, k) = 0$ on each $(x_\ell, x_{\ell+1})$, which implies that $\operatorname{supp} \hat{\mu}(\cdot, j, k) \subset (a_1 + ib_1)^{-1}(0)$.

Thus, for $(j,k)\neq(0,0)$ we have

$$\hat{\mu}(x,j,k)=\sum_{\ell=1}^{N}\sum_{m=0}^{\tilde{r}_\ell}\beta_{jk}^{\ell m}\delta^{(m)}(x-x_\ell),\quad \beta_{jk}^{\ell m}\in\mathbb{C},$$

where $\tilde{r}_\ell$ is a positive integer.

By using (10) we will show that all the constants $\beta_{jk}^{\ell m}$ are zero. Indeed, as above [see Eqs. (5)–(7)], (10) implies that, for each ℓ be fixed, and for $0<m<\tilde{r}_\ell$, we have

$$\begin{aligned}
0&=\beta_{jk}^{\ell\tilde{r}_\ell}\left(C_1^{\tilde{r}_\ell}(a_1+ib_1)'(x_\ell)+ik+ijC_0^{\tilde{r}_\ell}(a_2+ib_2)(x_\ell)\right)\\
&=\beta_{jk}^{\ell\tilde{r}_\ell}i(k+j(a_2+ib_2)(x_\ell)),\\
0&=\beta_{jk}^{\ell m}\left(C_1^{m}(a_1+ib_1)'(x_\ell)+i(k+jC_0^{m}(a_2+ib_2)(x_\ell))\right)\\
&\quad+\sum_{n=m+1}^{\tilde{r}_\ell}\beta_{jk}^{\ell n}\left[C_{n-m+1}^{n}(a_1+ib_1)^{(n-m+1)}(x_\ell)+ijC_{n-m}^{n}(a_2+ib_2)^{(n-m)}(x_\ell)\right],
\end{aligned}$$

and

$$0=\beta_{jk}^{\ell 0}i(k+j(a_2+ib_2)(x_\ell))+ij\sum_{n=1}^{\tilde{r}_\ell}\beta_{jk}^{\ell n}C_n^{n}(a_2+ib_2)^{(n)}(x_\ell).$$

Equations above imply that all the constants $\beta_{jk}^{\ell m}$ are zero, since either $b_2(x_\ell)\neq 0$ or $a_2(x_\ell)$ is an irrational number, for each $\ell\in\{1,\dots,N\}$.

Hence, $\hat{\mu}(\cdot,j,k)=0$, for all $(j,k)\in\mathbb{Z}^2\backslash\{(0,0)\}$.

Finally, since $\hat{\mu}(\cdot,0,0)$ satisfies $\partial_x((a_1+ib_1)(x)\hat{\mu}(x,0,0))=0$, it follows that $(a_1+ib_1)(x)\hat{\mu}(x,0,0)=K_0$. In particular,

$$\mu=(K_0/(a_1+ib_1))\otimes 1_y\otimes 1_t+\sum_{\ell=1}^{N}\sum_{m=0}^{r_\ell-1}K_{\ell m}\delta^{(m)}(x-x_\ell)\otimes 1_y\otimes 1_t,$$

with $r_\ell=\min\{m_\ell,n_\ell\}$.

Therefore, $\ker {}^tL=\operatorname{span}\{(1/(a_1+ib_1))\otimes 1_y\otimes 1_t,\ \delta^{(m)}(x-x_\ell)\otimes 1_y\otimes 1_t,\ 0\leq m<r_\ell,\ \ell=1,\dots,N\}$ and $\dim\ker {}^tL=1+\sum_{\ell=1}^{N}r_\ell<\infty$.

Our next result completes the proof that condition (ii) in Theorem 1 is necessary for the strong solvability of L.

Lemma 3 *Under the assumptions in item (ii)-Proposition 3, the operator L is not globally solvable if there exists $\ell\in\{1,\dots,N\}$ such that both $b_2(x_\ell)=0$ and $a_2(x_\ell)$ is a Liouville irrational number.*

Proof If $a_2(x_\ell)$ is a Liouville irrational number, then there exists $(j_n,k_n)\in\mathbb{Z}^2\backslash\{(0,0)\}$ such that $|j_n|+|k_n|\geq n$ and $|k_n+j_na_2(x_\ell)|<(|j_n|+|k_n|)^{-n}$, for all $n\in\mathbb{N}$. Since the sequence $(|j_n|+|k_n|)^{-n/2}$ decays rapidly,

$$f(x,y,t)=\sum_{n=1}^{\infty}(|j_n|+|k_n|)^{-n/2}e^{i(j_ny+k_nt)}$$

belongs to $\mathscr{C}^\infty(\mathbb{T}^3)$. Moreover, Proposition 3 implies that $f\in(\ker {}^tL)^\circ$.

If $u \in \mathscr{C}^\infty(\mathbb{T}^3)$ and $Lu = f$, then

$$\begin{aligned}&(|j_n| + |k_n|)^{-n/2}\\ &\quad = (a_1 + ib_1)(x)\partial_x(\hat{u}(x, j_n, k_n)) + i(k_n + j_n(a_2 + ib_2)(x))\hat{u}(x, j_n, k_n),\end{aligned}$$

for all $n \in \mathbb{N}$ and for all $x \in \mathbb{T}^1$. In particular,

$$i(k_n + j_n a_2(x_\ell))\hat{u}(x_\ell, j_n, k_n) = (|j_n| + |k_n|)^{-n/2}.$$

Hence

$$|\hat{u}(x_\ell, j_n, k_n)| \geq (|j_n| + |k_n|)^{n/2}.$$

Since the sequence $(|j_n|+|k_n|)^{n/2}$ does not decay rapidly, there is no $u \in \mathscr{C}^\infty(\mathbb{T}^3)$ such that $Lu = f$. Therefore, L is not globally solvable (in particular, L is not strongly solvable).

The proof that conditions (i)–(iii) in Theorem 1 are necessary for the strong solvability of L is complete.

3 Sufficient conditions

This section is devoted to prove that conditions given in Theorem 1 imply that L, given by (1), is strongly solvable.

Under the assumptions (i)–(iii) in Theorem 1, Proposition 3 implies that $\dim \ker {}^tL < \infty$. To complete the proof of Theorem 1 we must show that L is globally solvable, that is, $L\mathscr{C}^\infty(\mathbb{T}^3) = (\ker {}^tL)^\circ$.

We begin with a result about solvability modulo functions which are flat along $(a_1 + ib_1)^{-1}(0) \times \mathbb{T}^2$. By following the same line as Lemma 2.1 of [7], we obtain:

Lemma 4 *Consider the operator L given by (1). Suppose that $a_1 + ib_1$ vanishes only of finite order and $(a_1 + ib_1)^{-1}(0) = \{x_1 < \cdots < x_N\} \neq \emptyset$. If $2 \leq m_\ell < 2n_\ell - 1$ and either $b_2(x_\ell) \neq 0$ or $a_2(x_\ell)$ is a non-Liouville irrational number, then given $f \in (\ker {}^tL)^\circ$ there exists $u \in \mathscr{C}^\infty(\mathbb{T}^3)$ such that $Lu - f$ is flat along $(a_1 + ib_1)^{-1}(0) \times \mathbb{T}^2$.*

Proof As in [7], it is enough to find u smooth in a neighborhood of $\{x_0\} \times \mathbb{T}^2$ such that $Lu - f$ is flat along $\{x_0\} \times \mathbb{T}^2$, for each $x_0 \in (a_1 + ib_1)^{-1}(0)$.

Denote by n and m the order of vanishing of a_1 and b_1, respectively, at x_0. Set $r = \min\{m, n\} \geq 2$. By using formal Taylor series in a neighborhood $\Omega = (x_0 - \epsilon, x_0 + \epsilon) \times \mathbb{T}^2$, we may write

$$\begin{aligned} a_1 + ib_1 &\simeq (a_{1r} + ib_{1r})(x - x_0)^r + (a_{1r+1} + ib_{1r+1})(x - x_0)^{r+1} + \cdots \\ a_2 + ib_2 &\simeq (a_{20} + ib_{20}) + (a_{21} + ib_{21})(x - x_0) + \cdots \\ u &\simeq u_0(y, t) + u_1(y, t)(x - x_0) + u_2(y, t)(x - x_0)^2 + \cdots \\ f &\simeq f_0(y, t) + f_1(y, t)(x - x_0) + f_2(y, t)(x - x_0)^2 + \cdots . \end{aligned}$$

It follows that $Lu - f$ is flat along $\{x_0\} \times \mathbb{T}^2$ if and only if

$$\partial_t u_j(y, t) + \sum_{k=0}^{j} (a_{2k} + ib_{2k})\partial_y u_{j-k}(y, t) = f_j(y, t), \quad j = 0, \ldots, r - 1, \tag{11}$$

and, for $j \geq r$, $\quad f_j(y,t) =$

$$\partial_t u_j(y,t) + \sum_{k=0}^{j}(a_{2k} + ib_{2k})\partial_y u_{j-k}(y,t) + \sum_{k=r}^{j}(a_{1k} + ib_{1k})(j-k+1)u_{j-k+1}(y,t) \tag{12}$$

For each $j = 0, \dots, r-1$, the distribution $\delta^{(j)}(x - x_0) \otimes 1_y \otimes 1_t$ belongs to $\ker {}^t L$. Thus,

$$\begin{aligned}\hat{f}_j(0,0) &= (2\pi)^{-2} \int_0^{2\pi}\int_0^{2\pi} (j!)^{-1}\partial_x^j f(x_0, y, t)\mathrm{d}y\mathrm{d}t \\ &= (2\pi)^{-2}(j!)^{-1}(-1)^j \left\langle \delta^{(j)}(x - x_0) \otimes 1_y \otimes 1_t, f \right\rangle = 0.\end{aligned}$$

Moreover, since either $b_{20} \neq 0$ or a_{20} is a non-Liouville irrational number, we may find smooth functions $u_j(y,t)$ satisfying Eq. (11) in $\mathbb{T}^2$, for $j = 0, \dots, r-1$.

Note that, for any constant $C_j \in \mathbb{C}$, $u_j + C_j$ is still a solution.

The next equation is

$$\begin{aligned}&\partial_t u_r(y,t) + (a_{20} + ib_{20})\partial_y u_r(y,t) \\ &\quad = f_r(y,t) - \sum_{k=1}^{r}(a_{2k} + ib_{2k})\partial_y u_{r-k}(y,t) - (a_{1r} + ib_{1r})(u_1(y,t) + C_1).\end{aligned}$$

In order to obtain a smooth solution u_r in $\mathbb{T}^2$, it is enough to take $C_1 = \hat{f}_r(0,0)/(a_{1r} + ib_{1r}) - \hat{u_1}(0,0)$.

Proceeding in a similar way we may solve all the Eq. (12).

Proposition 4 *Suppose that $a_1 + ib_1$ vanishes only of finite order and $(a_1 + ib_1)^{-1}(0) = \{x_1 < \cdots < x_N\} \neq \emptyset$. Assume that $2 \leq m_\ell < 2n_\ell - 1$ and either $b_2(x_\ell) \neq 0$ or $a_2(x_\ell)$ is a non-Liouville irrational number, for each $\ell = 1, \dots, N$. Moreover, assume that b_1 do not change sign and $a_2 b_1 - a_1 b_2 = \lambda b_1$, with λ a non-Liouville irrational number. In this case, if L is given by (1), then for each $f \in (\ker {}^t L)^\circ$ which is flat along $(a_1 + ib_1)^{-1}(0) \times \mathbb{T}^2$ there exists $u \in \mathscr{C}^\infty(\mathbb{T}^3)$ such that $Lu = f$.*

Proof For each f belonging to $(\ker {}^t L)^\circ$ and flat along $(a_1 + ib_1)^{-1}(0) \times \mathbb{T}^2$, in order to find u belonging to $\mathscr{C}^\infty(\mathbb{T}^3)$ such that $Lu = f$, we use partial Fourier series in the variables y and t so that it is enough to find a rapid decreasing sequence of smooth functions $\hat{u}(\cdot, j, k)$ which satisfy the equations

$$(a_1 + ib_1)(x)\partial_x \hat{u}(x, j, k) + i(k + j(a_2 + ib_2)(x))\hat{u}(x, j, k) = \hat{f}(x, j, k), \tag{13}$$

for $(j,k) \in \mathbb{Z}^2$ and $x \in \mathbb{T}^1$.

When $(j,k) = (0,0)$, we have the equation

$$\partial_x \hat{u}(x, 0, 0) = \hat{f}(x, 0, 0)/(a_1 + ib_1)(x).$$

Since $1/(a_1+ib_1)(x)\otimes 1_y \otimes 1_t$ belongs to $\ker {}^t L$, we have $\int_0^{2\pi} \hat{f}(x,0,0)/(a_1+ib_1)(x)\mathrm{d}x = 0$. Hence

$$\hat{u}(x, 0, 0) = \int_0^{2\pi} \hat{f}(x', 0, 0)/(a_1 + ib_1)(x')\mathrm{d}x'$$

is a smooth solution.

For $(j,k)\in\mathbb{Z}^2\backslash\{(0,0)\}$, it is enough to find a sequence of smooth solutions $\hat{u}(\cdot,j,k)$ on each interval $(x_\ell,x_{\ell+1})$, such that each $\hat{u}(\cdot,j,k)$ is flat at $\{x_\ell,x_{\ell+1}\}$ and such that the sequence $\hat{u}(\cdot,j,k)$ decays rapidly on $(x_\ell,x_{\ell+1})$.

On each $(x_\ell,x_{\ell+1})$, Eq. (13) becomes

$$\begin{aligned}&\hat{f}(x,j,k)/(a_1+ib_1)(x)\\&=\partial_x\hat{u}(x,j,k)+\frac{(k+\lambda j)b_1(x)+i(ka_1(x)+j(a_1a_2+b_1b_2)(x))}{a_1^2(x)+b_1^2(x)}\hat{u}(x,j,k),\end{aligned}\tag{14}$$

where, by hypothesis, λ is a non-Liouville irrational number.

Without loss of generality, we may assume that $b_1\geq 0$ on $(x_\ell,x_{\ell+1})$.

For $x\in(x_\ell,x_{\ell+1})$ set

$$\theta_{jk}(x)=\frac{(k+\lambda j)b_1(x)+i(ka_1(x)+j(a_1a_2+b_1b_2)(x))}{a_1^2(x)+b_1^2(x)}.$$

If (j,k) is such that $k+\lambda j>0$, we choose

$$\hat{u}(x,j,k)=\int_{x_\ell}^{x}\exp\left\{-\int_{x'}^{x}\theta_{jk}(s)\mathrm{d}s\right\}\frac{\hat{f}(x',j,k)}{(a_1+ib_1)(x')}\mathrm{d}x'\tag{15}$$

as a smooth solution of (14) on $(x_\ell,x_{\ell+1})$. Since f is flat along $\{x_\ell\}\times\mathbb{T}^2$, for each $n\in\mathbb{N}$ there exists $C_n>0$, which does not depend on (j,k), such that

$$|\hat{u}(x,j,k)|\leq C_n(x-x_\ell)^n.$$

In particular, $\hat{u}(\cdot,j,k)$ is flat at x_ℓ. We will see that it is also flat at $x_{\ell+1}$. For $h>0$ sufficiently small we have

$$\begin{aligned}\hat{u}(x_{\ell+1}-h,j,k)&=\int_{x_\ell}^{x_{\ell+1}-2h}\exp\left\{-\int_{x'}^{x_{\ell+1}-h}\theta_{jk}(s)\mathrm{d}s\right\}\frac{\hat{f}(x',j,k)}{(a_1+ib_1)(x')}\mathrm{d}x'\\&\quad+\int_{x_{\ell+1}-2h}^{x_{\ell+1}-h}\exp\left\{-\int_{x'}^{x_{\ell+1}-h}\theta_{jk}(s)\mathrm{d}s\right\}\frac{\hat{f}(x',j,k)}{(a_1+ib_1)(x')}\mathrm{d}x'.\end{aligned}\tag{16}$$

The first integral satisfies

$$\begin{aligned}&\left|\int_{x_\ell}^{x_{\ell+1}-2h}\exp\left\{-\int_{x'}^{x_{\ell+1}-h}\theta_{jk}(s)\mathrm{d}s\right\}\frac{\hat{f}(x',j,k)}{(a_1+ib_1)(x')}\mathrm{d}x'\right|\\&\leq\|\hat{f}(\cdot,j,k)/(a_1+ib_1)\|_\infty\int_{x_\ell}^{x_{\ell+1}-2h}\exp\left\{-\int_{x_{\ell+1}-2h}^{x_{\ell+1}-h}\frac{(k+\lambda j)b_1(s)}{a_1^2(s)+b_1^2(s)}\mathrm{d}s\right\}\mathrm{d}x'.\end{aligned}$$

Moreover, since $2\leq m_{\ell+1}<2n_{\ell+1}-1$, for s near $x_{\ell+1}$ we may write $b_1/(a_1^2+b_1^2)(s)=(x_{\ell+1}-s)^{-\rho}\beta(s)$, where $\rho\geq 2$ and $0<r\leq\beta(s)$. Hence

$$\begin{aligned}&\int_{x_\ell}^{x_{\ell+1}-2h}\exp\left\{-\int_{x_{\ell+1}-2h}^{x_{\ell+1}-h}\frac{(k+\lambda j)b_1(s)}{a_1^2(s)+b_1^2(s)}\mathrm{d}s\right\}\mathrm{d}x'\\&\leq|x_{\ell+1}-x_\ell|\exp\left\{\frac{-(k+\lambda j)r}{\rho-1}\left(\frac{2^{\rho-1}-1}{2^{\rho-1}h^{\rho-1}}\right)\right\}\\&\leq|x_{\ell+1}-x_\ell|\frac{(\rho-1)2^{\rho-1}h^{\rho-1}}{(k+\lambda j)r(2^{\rho-1}-1)}\end{aligned}\tag{17}$$

and, since λ is a non-Liouville irrational number, there exist constants $C > 0$ and $\gamma > 0$, which do not depend on (j, k), such that

$$\|\hat{f}(\cdot, j, k)/(a_1 + ib_1)\|_\infty \int_{x_\ell}^{x_{\ell+1}-2h} \exp\left\{-\int_{x_{\ell+1}-2h}^{x_{\ell+1}-h} \frac{(k+\lambda j)b_1(s)}{a_1^2(s)+b_1^2(s)} \mathrm{d}s\right\} \mathrm{d}x'$$
$$\leq C\|\hat{f}(\cdot, j, k)/(a_1 + ib_1)\|_\infty (|j| + |k|)^\gamma h^{\rho-1}. \tag{18}$$

For the second integral in (16), since f is flat along $\{x_{\ell+1}\} \times \mathbb{T}^2$, for each $n \in \mathbb{N}$ there exists $C_n > 0$ (which does not depend on (j, k)) such that

$$\left|\int_{x_{\ell+1}-2h}^{x_{\ell+1}-h} \exp\left\{-\int_{x'}^{x_{\ell+1}-h} \theta_{jk}(s)\mathrm{d}s\right\} \frac{\hat{f}(x', j, k)}{(a_1 + ib_1)(x')} \mathrm{d}x'\right| \tag{19}$$
$$\leq \int_{x_{\ell+1}-2h}^{x_{\ell+1}-h} \left|\frac{\hat{f}(x', j, k)}{(a_1 + ib_1)(x')}\right| \mathrm{d}x' \leq C_n h^n. \tag{20}$$

It follows from (17) and (20) that each $\hat{u}(\cdot, j, k)$ is flat at $x_{\ell+1}$. Moreover, the rapid decaying of $\hat{f}(\cdot, j, k)/(a_1 + ib_1)$ and since λ is a non-Liouville irrational number, estimates (18) and (19) imply that, for each nonnegative integer m, we may find a constant $C > 0$ such that

$$(|j| + |k|)^m |\hat{u}(x, j, k)| \leq C,$$

for all $x \in (x_\ell, x_{\ell+1})$ and for all $(j, k) \in \mathbb{Z}^2$ such that $k + \lambda j > 0$.

Proceeding in a similar way, we verify that each derivative $\partial_x^n \hat{u}(\cdot, j, k)$ satisfies a similar estimate. Hence, the sequence $\hat{u}(\cdot, j, k)$ given by (15) decays rapidly on $(x_\ell, x_{\ell+1})$.

Finally, for (j, k) such that $k + \lambda j < 0$, we choose

$$\hat{u}(x, j, k) = -\int_x^{x_{\ell+1}} \exp\left\{-\int_{x'}^{x} \theta_{jk}(s)\mathrm{d}s\right\} \frac{\hat{f}(x', j, k)}{(a_1 + ib_1)(x')} \mathrm{d}x'$$

as a smooth solution of (14) on $(x_\ell, x_{\ell+1})$.

Proceeding as above, we verify that each $\hat{u}(\cdot, j, k)$ is flat at $\{x_\ell, x_{\ell+1}\}$ and the sequence $\hat{u}(\cdot, j, k)$ decays rapidly on $(x_\ell, x_{\ell+1})$.

Since $\ell \in \{1, \ldots, N\}$ was arbitrary fixed, the proof is complete.

Summarizing, Proposition 3, Lemma 4 and Proposition 4 imply that L is strongly solvable under the conditions given in Theorem 1. Moreover, Proposition 3 implies that the codimension of the range is $1 + \sum_{\ell=1}^{N} \min\{m_\ell, n_\ell\}$. The proof of Theorem 1 is then complete.

Remark 1 Proposition 4 still hods true if we allow b_1 changing sign at most once between two consecutive zeros of $a_1 + ib_1$. Indeed, without loss of generality, assume that there exists $\eta \in (x_\ell, x_{\ell+1})$ such that $b_1 \geq 0$ on (x_ℓ, η) and $b_1 \leq 0$ on $(\eta, x_{\ell+1})$.

For the indexes (j, k) such that $k + \lambda j > 0$, we define the solution $\hat{u}(x, j, k)$ as (15), and for the indexes (j, k) such that $k + \lambda j < 0$, we define the solution $\hat{u}(x, j, k)$ as

$$\hat{u}(x, j, k) = \int_\eta^x \exp\left\{-\int_{x'}^{x} \theta_{jk}(s)\mathrm{d}s\right\} \frac{\hat{f}(x', j, k)}{(a_1 + ib_1)(x')} \mathrm{d}x', \tag{21}$$

for $x \in (x_\ell, x_{\ell+1})$, where we recall that

$$\theta_{jk}(x) = \frac{(k + \lambda j)b_1(x) + i(ka_1(x) + j(a_1a_2 + b_1b_2)(x))}{a_1^2(x) + b_1^2(x)}.$$

Proceeding as in the proof of Proposition 4 we see that the solutions given by (21) are flat at $\{x_\ell, x_{\ell+1}\}$ and we also see that this sequence decays rapidly.

These arguments will also imply that the solutions given by (15) are flat at x_ℓ. To see that these solutions are flat at $x_{\ell+1}$ and to see that this sequence of solutions also decays rapidly, the following will be useful: since $b_1 \geq 0$ on (x_ℓ, η) and $b_1 \leq 0$ on $(\eta, x_{\ell+1})$, the identity

$$\left|\exp\left\{\int_\eta^x \theta_{jk}(s)\mathrm{d}s\right\}\right| = \exp\left\{(k+\lambda j)\int_\eta^x \frac{b_1}{a_1^2+b_1^2}(s)\mathrm{d}s\right\} \tag{22}$$

implies that, for all $k+\lambda j > 0$, function (22) is bounded in $(x_\ell, x_{\ell+1})$. Furthermore, since $2 \leq m_\sigma < 2n_\sigma - 1$, for $\sigma = \ell, \ell+1$, we have that the function

$$\psi_{jk}(x) \doteq \begin{cases} \exp\left\{\int_\eta^x \theta_{jk}(s)\mathrm{d}s\right\}, & \text{if} \quad x \in (x_\ell, x_{\ell+1}) \\ 0, & \text{if} \quad x \in \mathbb{T}^1 \backslash (x_\ell, x_{\ell+1}) \end{cases}$$

is flat at both x_ℓ and $x_{\ell+1}$. Hence, ψ_{jk} and also $\psi_{jk}/(a_1+ib_1)$ belong to $C^\infty(\mathbb{T}^1)$. Simple calculations show that $(\psi_{jk}/(a_1+ib_1))(x)e^{-i(jy+kt)}$ belongs to $\ker {}^t\mathsf{L}$; hence,

$$\int_{x_\ell}^{x_{\ell+1}} \exp\left\{\int_\eta^x \theta_{jk}(s)\mathrm{d}s\right\} \frac{\hat{f}(x,j,k)}{(a_1+ib_1)(x)}\mathrm{d}x = 0,$$

for all $k+\lambda j > 0$. In particular, we can rewrite each solution $\hat{u}(\cdot, j, k)$ in the following form

$$\hat{u}(x,j,k) = -\int_x^{x_{\ell+1}} \exp\left\{-\int_{x'}^x \theta_{jk}(s)\mathrm{d}s\right\} \frac{\hat{f}(x',j,k)}{(a_1+ib_1)(x')}\mathrm{d}x',$$

for $x \in (x_\ell, x_{\ell+1})$. By using this expression and proceeding as in the proof of Proposition 4, we can show that, for $k+\lambda j > 0$, $\hat{u}(\cdot, j, k)$ is flat at $x_{\ell+1}$ and that this sequence of solutions decays rapidly.

Together with Lemma 4 this result gives globally solvable vector fields with infinite codimensional range. For instance, $L = \partial_t + (\sin^3(x) + i\sin^3(2x))\partial_x + \sqrt{2}\partial_y$.

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