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Removable sets for continuous solutions of quasilinear elliptic equations with nonlinear source or absorption terms

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Abstract This paper establishes removable singularity theorems for nonnegative continuous solutions of quasilinear elliptic equations with nonlinear source or absorption terms when an exceptional set is conditioned in terms of the regularity of Hausdorff measure and a uniform Minkowski property. These weaker conditions enable us to consider some fractal sets as an exceptional set.

Keywords Removable singularity · Quasilinear elliptic equation · Nonlinear potential estimate · Iteration method · Fractal set

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1 Introduction

This paper deals with the removability of singularities of nonnegative continuous solutions of quasilinear elliptic equations with a nonlinear source term

$$-\Delta_p u = u^q \tag{1.1}$$

and with a nonlinear absorption term

$$\Delta_p u = u^q, \tag{1.2}$$

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where $\Delta_p u = \mathrm{div}(\|\nabla u\|^{p-2}\nabla u)$ is the p-Laplacian of u on $\mathbb{R}^n$ with $1 < p < n$ and the equations are understood in the weak sense. In particular, we are interested in establishing removability theorems under an appropriate growth condition on solutions and a weak assumption on an exceptional set.

In [14], Serrin has established removability theorems and regularity theorems for solutions of second-order quasilinear elliptic equations. It should be noted that his results are applicable to the above equations in only the case $0 < q \le p-1$. Our study is motivated by the following special one of [14, Theorem 11].

Theorem A *Let Ω be an open set in $\mathbb{R}^n$ and let E be a compact smooth manifold in Ω of dimension $m < n-p$. Assume that $0 < q \le p-1$. If u is a nonnegative continuous solution of (1.1) or (1.2) in $\Omega\setminus E$ satisfying growth condition*

$$u(x) = O(d(x,E)^{(p-n+m)/(p-1)+\delta}) \tag{1.3}$$

for some $\delta > 0$, then u can be extended to the whole of Ω as a continuous solution.

There arise the following natural questions which complement and extend Theorem A:

(i) Can growth condition (1.3) be replaced by the weaker one

$$u(x) = o(d(x,E)^{(p-n+m)/(p-1)}) \ ?$$

(ii) Is Theorem A true when $q > p-1$ and E is a fractal set?

When E is a singleton, it is possible to obtain more detailed results. In fact, we know the classification of a removable isolated singularity and the local behavior near an isolated point of nonnegative continuous solutions of each Eqs. (1.1) and (1.2) with $p-1 < q < n(p-1)/(n-p)$. See Bidaut-Véron [1], Friedman and Véron [4] and Véron [17]. When $q \ge n(p-1)/(n-p)$, it is also known that any isolated point is removable for nonnegative continuous solutions of (1.2). See Vázquez and Véron [16] and Véron [17]. Moreover, as mentioned in [17], the last result can be extended to a singular set lying on a compact smooth manifold of dimension $m < n-p$ when $q \ge (n-m)(p-1)/(n-m-p)$. It seems that there are no removability results for continuous solutions of (1.1) and (1.2) with $p-1 < q < (n-m)(p-1)/(n-m-p)$, although Skrypnik [15] studied question (i) in only the case $q \le p-1$ using an argument not applied to question (ii). Thus the purpose of this paper is to give answers to the above questions in the subcritical case. We prove that if Ω is an open set in $\mathbb{R}^n$; E is a compact subset of Ω of dimension $m < n-p$ satisfying a weak condition and $0 < q < (n-m)(p-1)/(n-m-p)$, then E is removable for nonnegative continuous solutions of each Eqs (1.1) and (1.2) in $\Omega\setminus E$ satisfying the growth condition in the above question. Results are obtained for equations of more general form than (1.1) and (1.2) [see (2.1) and (2.2) below]. One of the difficulties in this problem is to show the local boundedness of extended solutions of (1.1). We give a new iteration technique with the help of the Wolff potential estimate established by Kilpeläinen and Malý [9,10]. Also, we establish a removability theorem for p-subharmonic functions satisfying a growth condition which is applied to show a result for (1.2). Finally, we note that the recent interesting results by Bidaut-Véron [2] and Phuc and Verbitsky [12,13] concerning removable singularities for (1.1) were established in the framework of entropy solutions and p-superharmonic functions, so extended solutions do not necessarily belong to the appropriate Sobolev space. Thus, our results do not follow from their's and are more delicate.

2 Notation and main results

Throughout this paper, we suppose that Ω is an open set in $\mathbb{R}^n$ ($n \geq 2$); E is a compact subset of Ω and $1 < p < n$. The symbol C stands for an absolute positive constant whose value is unimportant and may vary at each occurrence. If necessary, we write $C_1, C_2, \dots$ to specify them.

Let us consider quasilinear elliptic equations of the form

$$- \operatorname{div} \mathcal{A}(x, \nabla u) = \mathcal{B}(x, u, \nabla u) \tag{2.1}$$

and

$$\operatorname{div} \mathcal{A}(x, \nabla u) = \mathcal{B}(x, u, \nabla u), \tag{2.2}$$

where $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy nonlinear structure conditions stated below. The notation $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ stand for the Euclidean norm and inner product on $\mathbb{R}^n$, respectively. Let $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function, that is, the mapping $x \mapsto \mathcal{A}(x, \xi)$ is measurable on Ω for every $\xi \in \mathbb{R}^n$ and the mapping $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous on $\mathbb{R}^n$ for every $x \in \Omega$. We always assume the following (A1)–(A3):

(A1) *Coercivity and growth*: there are positive constants c_1 and c_2 such that for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$\langle \mathcal{A}(x, \xi), \xi \rangle \geq c_1 \|\xi\|^p \quad \text{and} \quad \|\mathcal{A}(x, \xi)\| \leq c_2 \|\xi\|^{p-1}.$$

(A2) *Monotonicity*: for all $x \in \Omega$ and $\xi_1, \xi_2 \in \mathbb{R}^n$ satisfying $\xi_1 \neq \xi_2$,

$$\langle \mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2), \xi_1 - \xi_2 \rangle > 0.$$

(A3) *Homogeneity*: if $\lambda \in \mathbb{R} \setminus \{0\}$, then for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$\mathcal{A}(x, \lambda \xi) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \xi).$$

Let q be a positive constant and let $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Borel function satisfying

(B1) *Sign*: if $t \geq 0$, then $\mathcal{B}(x, t, \xi) \geq 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.
(B2) *Growth for* (2.1): there is a positive constant c_3 such that for all $x \in \Omega$, $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$|\mathcal{B}(x, t, \xi)| \leq c_3 (1 + |t|^q).$$

The constant q in (B2) need not be $p - 1$, and its range is restricted in terms of the fractal dimension of an exceptional set E. When we discuss (2.2), we assume the following growth condition instead of (B2):

(B2′) *Growth for* (2.2): there exists $b \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ such that for all $x \in \Omega$, $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$|\mathcal{B}(x, t, \xi)| \leq b(t).$$

Prototypes of Eqs. (2.1) and (2.2) are quasilinear elliptic equations of Lane–Emden type

$$-\Delta_p u = |u|^{q-1} u \quad \text{and} \quad \Delta_p u = |u|^{q-1} u$$

and $\Delta_p u = e^u$. We say that u is a *(weak) solution* of (2.1) in Ω if it belongs to $W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and satisfies (2.1) in Ω in the weak sense: for all $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \langle \mathcal{A}(x, \nabla u(x)), \nabla \phi(x) \rangle \, \mathrm{d}x = \int_\Omega \mathcal{B}(x, u(x), \nabla u(x)) \phi(x) \, \mathrm{d}x.$$

A (weak) solution of (2.2) in Ω is defined in a similar manner. If u is a nonnegative solution of (2.1) [resp. (2.2)] in $\Omega \setminus E$, then (B2) [resp. (B2′)] implies $\mathcal{B}(\cdot, u, \nabla u) \in L^{\infty}_{\mathrm{loc}}(\Omega \setminus E)$. It follows from [11, Theorem 4.11] that u has a continuous representative on $\Omega \setminus E$. Therefore we only consider continuous solutions of (2.1) and (2.2).

Let us consider the following exceptional set E. The Euclidean distance from a point $x \in \mathbb{R}^n$ to E is denoted by $d(x, E)$. For $r > 0$, we write

$$E(r) := \{x \in \mathbb{R}^n : d(x, E) < r\}.$$

Also, $B(x, r)$ stands for the open ball of center x and radius r. The m-dimensional Hausdorff measure and the n-dimensional Lebesgue measure of a set A are denoted, respectively, by $\mathcal{H}^m(A)$ and $|A|$. When $m = 0$, $\mathcal{H}^0$ is interpreted as the counting measure. Let m be a real number such that $0 \le m < n$. We say that a compact set E is a *Lipschitz set of dimension* m if there exist positive constants r_0 and $C_0 \ge 1$ such that for all $x \in E$, $0 < r < r_0$ and $0 < R < r_0$,

$$\frac{1}{C_0} r^m \le \mathcal{H}^m(E \cap B(x, r)) \le C_0 r^m \tag{2.3}$$

and

$$|E(r) \cap B(x, R)| \le C_0 r^{n-m} R^m. \tag{2.4}$$

If only (2.4) is satisfied, then we call E a *uniform Minkowski set of dimension* m. Examples of Lipschitz sets will be given in Sect. 7.

Our main results are as follows. The first one is for Eq. (2.1).

Theorem 2.1 *Let* $0 \le m < n - p$ *and let* E *be a uniform Minkowski set in* Ω *of dimension* m. *If*

$$0 < q < \frac{(n-m)(p-1)}{n-m-p}, \tag{2.5}$$

then E *is removable for nonnegative continuous solutions of* (2.1) *in* $\Omega \setminus E$ *satisfying the growth condition: at each* $y \in E$,

$$u(x) = o(d(x, E)^{(p-n+m)/(p-1)}) \quad \text{as } x \to y. \tag{2.6}$$

For Eq. (2.2), we have the following.

Theorem 2.2 *Let* $0 \le m < n - p$ *and let* E *be a Lipschitz set in* Ω *of dimension* m. *Then* E *is removable for nonnegative continuous solutions of* (2.2) *in* $\Omega \setminus E$ *satisfying* (2.6) *at each* $y \in E$.

Remark 2.3 Note from Vázquez and Véron [16, Lemma 1.3] that any nonnegative continuous solution of (1.2) satisfies

$$u(x) \le C d(x, E)^{-p/(q-p+1)} \quad \text{near } E,$$

because its proof is independent of the shape of E. Thus if $q > (n-m)(p-1)/(n-m-p)$, then (2.6) is always satisfied for such solutions, and hence the above removability result holds without any growth condition.

The rest of this paper is organized as follows. In Sect. 3, we collect several known results from nonlinear potential theory and basic estimates for the Wolff potential which are used in the subsequent sections. Proofs of Theorems 2.1 and 2.2 are given in Sects. 4 and 5, respectively. In Sect. 7, we show that some fractal sets are Lipschitz sets.

3 Preliminary material

We start with the definition of super/subsolutions of

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = 0. \tag{3.1}$$

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *supersolution* of (3.1) in Ω if the inequality

$$\int_\Omega \langle \mathcal{A}(x, \nabla u(x)), \nabla \phi(x) \rangle \, \mathrm{d}x \ge 0$$

holds for all nonnegative functions $\phi \in C_0^\infty(\Omega)$. If $-u$ is a supersolution of (3.1) in Ω, then we call u a *subsolution* of (3.1) in Ω. A continuous function $h \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be $\mathcal{A}$-*harmonic* on Ω if the equality

$$\int_\Omega \langle \mathcal{A}(x, \nabla h(x)), \nabla \phi(x) \rangle \, \mathrm{d}x = 0$$

holds for all $\phi \in C_0^\infty(\Omega)$. Rather than arguing in the framework of super/subsolutions, it is convenient to introduce the notion of $\mathcal{A}$-super/$\mathcal{A}$-subharmonicity. A function $u : \Omega \to (-\infty, +\infty]$ is called $\mathcal{A}$-*superharmonic* on Ω if $u \not\equiv +\infty$ on each component of Ω, if u is lower semicontinuous on Ω and if for each open set ω with $\overline{\omega} \subset \Omega$ and each continuous function h on $\overline{\omega}$ which is $\mathcal{A}$-harmonic on ω, the inequality $u \ge h$ on $\partial\omega$ implies $u \ge h$ on ω. Also, a function $u : \Omega \to [-\infty, +\infty)$ is called $\mathcal{A}$-*subharmonic* on Ω if $-u$ is $\mathcal{A}$-superharmonic on Ω. Note that the class of $\mathcal{A}$-superharmonic functions is closed under a positive scalar multiplication λu ($\lambda > 0$), a constant addition $u + C$ and the minimum operation $\min\{u, v\}$. A substantial difference between supersolutions and $\mathcal{A}$-superharmonic functions is that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ or not. Indeed, the following facts can be found in [5, Corollaries 7.18, 7.20, 7.21 and Theorem 7.22].

Lemma 3.1 *The following statements hold.*

(i) *If u is a supersolution of* (3.1) *in Ω, then there exists an $\mathcal{A}$-superharmonic function v on Ω such that $v = u$ almost everywhere on Ω.*
(ii) *Let u be an $\mathcal{A}$-superharmonic function on Ω. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, then u is a supersolution of* (3.1) *in Ω.*
(iii) *Let u be an $\mathcal{A}$-superharmonic function on Ω. If u is locally bounded above on Ω, then $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and u is a supersolution of* (3.1) *in Ω.*
(iv) *If u is $\mathcal{A}$-superharmonic on Ω, then*

$$u(x) = \operatorname{ess} \liminf_{y \to x} u(y)$$

for each $x \in \Omega$.

These facts are also noteworthy matters when we define the Riesz measure associated with an $\mathcal{A}$-superharmonic function. Following the book [5, pp. 381–382] and [9], we recall the definition of the Riesz measure. Let u be an $\mathcal{A}$-superharmonic function on Ω and $k \in \mathbb{N}$. Then, the truncated function $u_k := \min\{u, k\}$ belongs to $W^{1,p}_{\mathrm{loc}}(\Omega)$ and $\{\nabla u_k\}$ converges almost everywhere on Ω as $k \to \infty$. The *very weak gradient* of u is defined by

$$Du := \lim_{k \to \infty} \nabla u_k.$$

Then $Du \in L^{p-1}_{\mathrm{loc}}(\Omega)$ by [5, Theorem 7.46]. Moreover, by [5, Theorem 21.2], there exists a unique Radon measure μ on Ω such that $-\operatorname{div}\mathcal{A}(x, Du) = \mu$ in Ω in the sense of distributions, i.e., for all $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \langle \mathcal{A}(x, Du(x)), \nabla\phi(x)\rangle \, \mathrm{d}x = \int_\Omega \phi \, \mathrm{d}\mu.$$

This measure μ is called the *Riesz measure* associated with u.

Also, we need the Wolff potential estimate for $\mathcal{A}$-superharmonic functions. Let μ be a Radon measure on Ω. The *Wolff potential* of μ is defined by

$$\mathbf{W}^\mu_{1,p}(x, r) := \int_0^r \left(\frac{\mu(B(x,t))}{t^{n-p}} \right)^{1/(p-1)} \frac{\mathrm{d}t}{t},$$

whenever $B(x, r) \subset \Omega$. The following important estimate has been established by Kilpeläinen and Malý [10, Theorem 1.6].

Lemma 3.2 *Let u be a nonnegative $\mathcal{A}$-superharmonic function on $B(x_0, 3r)$ and let μ be the Riesz measure associated with u. Then there exists a constant $C > 1$ depending only on p, n and the structural constants c_1 and c_2 such that*

$$\frac{1}{C}\mathbf{W}^\mu_{1,p}(x_0, r) \le u(x_0) \le C\left\{ \inf_{B(x_0,r)} u + \mathbf{W}^\mu_{1,p}(x_0, 2r) \right\}.$$

The following lemma is a consequence of [10, Theorem 4.20].

Lemma 3.3 *Let u be an $\mathcal{A}$-superharmonic function on Ω and let μ be the Riesz measure associated with u. If ω is an open subset of Ω with $\mu(\omega) = 0$, then u is real valued and continuous on ω.*

In order to obtain a growth estimate near E of the Wolff potential, we need the following lemma which was proved in [6, Lemma 2.2].

Lemma 3.4 *Let $0 \le m < n$ and let E be a uniform Minkowski set of dimension m. If $m - n < \lambda < 0$, then there exists a positive constant C depending only on C_0, λ, m and n such that for all $x \in E$, $0 < r < r_0$ and $0 < R < r_0$,*

$$\int_{E(r)\cap B(x,R)} d(y, E)^\lambda \, dy \le C r^{\lambda+n-m} R^m.$$

Lemma 3.2 and the next basic estimate play crucial roles in the proof of Theorem 2.1.

Lemma 3.5 *Assume that $0 \le m < n - p$ and E is a uniform Minkowski set of dimension m. Let $m - n < \lambda < 0$, let $0 < r < r_0/3$ and let $B(x, r) \subset E(r_0/3)$. If μ is a Radon measure on $B(x, r)$ for which there is a constant $C_1 > 0$ such that for all $0 < t < r$,*

$$\mu(B(x,t)) \le C_1 \int_{B(x,t)} d(y, E)^\lambda \, dy,$$

then

$$\mathbf{W}^\mu_{1,p}(x, r) \le \begin{cases} C d(x, E)^{(\lambda+p)/(p-1)} & \text{if } \lambda < -p, \\ C\left(1 + \log^+ \dfrac{1}{d(x,E)}\right) & \text{if } \lambda = -p, \\ C & \text{if } \lambda > -p, \end{cases} \tag{3.2}$$

where C is a constant depending only on C_1, λ, p, r_0, C_0 and n.

Proof If $0 < t \le d(x, E)/2$, then we have for all $y \in B(x, t)$,

$$\frac{1}{2} d(x, E) \le d(y, E) \le 2d(x, E),$$

and so

$$\mu(B(x, t)) \le C d(x, E)^{\lambda} t^n.$$

Therefore, for any pair (x, r) satisfying $d(x, E) \ge 2r$, we get

$$\begin{aligned} \mathbf{W}^{\mu}_{1,p}(x, r) &\le C d(x, E)^{\lambda/(p-1)} \int_0^{d(x,E)/2} t^{p/(p-1)} \frac{\mathrm{d}t}{t} \\ &\le C d(x, E)^{(\lambda+p)/(p-1)}, \end{aligned}$$

and so (3.2) holds for such an (x, r). Let us consider the other (x, r). If $d(x, E)/2 < t < r$, then $B(x, t) \subset B(x^*, 3t)$ for some $x^* \in E$. From Lemma 3.4 with $r = R = 3t$, we get

$$\mu(B(x, t)) \le C \int_{B(x^*,3t)} d(y, E)^{\lambda}\, dy \le C t^{\lambda+n}.$$

Therefore

$$\begin{aligned} \mathbf{W}^{\mu}_{1,p}(x, r) &\le C\left(d(x, E)^{\lambda/(p-1)} \int_0^{d(x,E)/2} t^{p/(p-1)} \frac{\mathrm{d}t}{t} + \int_{d(x,E)/2}^{r} t^{(\lambda+p)/(p-1)} \frac{\mathrm{d}t}{t} \right) \\ &\le C\left(d(x, E)^{(\lambda+p)/(p-1)} + \int_{d(x,E)/2}^{r} t^{(\lambda+p)/(p-1)} \frac{\mathrm{d}t}{t} \right). \end{aligned}$$

Since the last integral is dominated by

$$\begin{cases} C d(x, E)^{(\lambda+p)/(p-1)} & \text{if } (\lambda + p)/(p - 1) < 0, \\ \log \dfrac{r_0}{d(x, E)} & \text{if } (\lambda + p)/(p - 1) = 0, \\ C r_0^{(\lambda+p)/(p-1)} & \text{if } (\lambda + p)/(p - 1) > 0 \end{cases}$$

with some positive constant C depending only on λ and p, we obtain the required estimate. □

Lemma 3.6 *Assume that $0 \le m < n - p$ and E is a Lipschitz set of dimension m. Then there exists a constant $C > 1$ depending only on m, p, n, r_0, C_0 and $\mathcal{H}^m(E)$ such that for any pair $x \in E(r_0/3)\backslash E$ and $r > 3d(x, E)$,*

$$\frac{1}{C} d(x, E)^{(p-n+m)/(p-1)} \le \mathbf{W}^{\mathcal{H}^m|_E}_{1,p}(x, r) \le C d(x, E)^{(p-n+m)/(p-1)}. \tag{3.3}$$

Proof Let $x \in E(r_0/3)\backslash E$ and let $r > 3d(x, E)$. Take $x^* \in E$ so that $\|x - x^*\| = d(x, E)$.

We first show the second inequality in (3.3). Since $B(x,t)\cap E\subset B(x^*,2t)\cap E$, we have by (2.3)

$$
\begin{aligned}
\mathbf{W}_{1,p}^{\mathcal{H}^m|_E}(x,r) &= \int_{d(x,E)}^{r}\left(\frac{\mathcal{H}^m(B(x,t)\cap E)}{t^{n-p}}\right)^{1/(p-1)}\frac{\mathrm{d}t}{t}\\
&\le C\int_{d(x,E)}^{\min\{r,r_0/2\}} t^{(p-n+m)/(p-1)}\frac{\mathrm{d}t}{t}\\
&\quad + \mathcal{H}^m(E)^{1/(p-1)}\int_{\min\{r,r_0/2\}}^{r} t^{(p-n)/(p-1)}\frac{\mathrm{d}t}{t}\\
&\le Cd(x,E)^{(p-n+m)/(p-1)} + C\mathcal{H}^m(E)^{1/(p-1)}r_0^{(p-n)/(p-1)}.
\end{aligned}
$$

Since E is compact, the standard finite covering argument shows that $\mathcal{H}^m(E)$ is finite. Therefore, the last quantity is bounded by a constant multiple of $d(x,E)^{(p-n+m)/(p-1)}$.

Next, we show the first inequality in (3.3). If $t>2d(x,E)$, then $B(x^*,t/2)\subset B(x,t)$. Therefore

$$
\begin{aligned}
\mathbf{W}_{1,p}^{\mathcal{H}^m|_E}(x,r) &\ge \int_{2d(x,E)}^{3d(x,E)}\left(\frac{\mathcal{H}^m(B(x,t)\cap E)}{t^{n-p}}\right)^{1/(p-1)}\frac{\mathrm{d}t}{t}\\
&\ge \frac{1}{C}\int_{2d(x,E)}^{3d(x,E)} t^{(p-n+m)/(p-1)}\frac{\mathrm{d}t}{t}\\
&\ge \frac{1}{C}d(x,E)^{(p-n+m)/(p-1)}.
\end{aligned}
$$

Thus the lemma is proved. □

Lemma 3.7 *Assume that $0\le m<n-p$ and E is a Lipschitz set of dimension m. Let ω be a bounded open set in $\mathbb{R}^n$ containing E. Then there exists a positive $\mathcal{A}$-superharmonic function g on $\mathbb{R}^n$ satisfying*

$$
-\operatorname{div}\mathcal{A}(x,Dg)=\mathcal{H}^m|_E \quad \text{in } \mathbb{R}^n \tag{3.4}
$$

in the sense of distributions and

$$
\frac{1}{C_2}d(x,E)^{(p-n+m)/(p-1)} \le g(x) \le C_2 d(x,E)^{(p-n+m)/(p-1)} \tag{3.5}
$$

for all $x\in\omega\setminus E$ and some constant $C_2>1$.

Proof The existence of positive $\mathcal{A}$-superharmonic functions g satisfying (3.4) was proved by Kilpeläinen [8, Theorem 2.10]. Fix $r>3\operatorname{diam}\omega$ and take an open ball B in $\mathbb{R}^n$ so that $B(x,3r)\subset B$ for all $x\in\omega$. Let $x\in\omega\setminus E$. By Lemma 3.2, we have

$$
\frac{1}{C}\mathbf{W}_{1,p}^{\mathcal{H}^m|_E}(x,r)\le g(x)\le C\left\{\inf_{B(x,r)} g+\mathbf{W}_{1,p}^{\mathcal{H}^m|_E}(x,2r)\right\}.
$$

Since $B(x,r)\setminus\omega\ne\emptyset$ and g is continuous on $\mathbb{R}^n\setminus E$ by Lemma 3.3, we have

$$
\inf_{B(x,r)} g\le \inf_{B(x,r)\setminus\omega} g\le \sup_{B(x,r)\setminus\omega} g\le \sup_{B\setminus\omega} g\le C<\infty.
$$

Hence, if $x\in E(r_0/3)\setminus E$, then (3.5) follows from Lemma 3.6. Also, the minimum principle and the continuity yield

$$
0<\min_{\partial\omega} g=\min_{\overline{\omega}} g\le \max_{\overline{\omega}\setminus E(r_0/3)} g<\infty.
$$

Since $r_0/3 \le d(x, E) \le \operatorname{diam}\omega$ for all $x \in \omega\backslash E(r_0/3)$, we see that (3.5) is true for $x \in \omega\backslash E(r_0/3)$. This completes the proof. □

The following lemma was proved in [11, Lemma 2.117].

Lemma 3.8 *Let u be a nonnegative supersolution of (3.1) in $B(x_0, 4r)$ and let $\eta \in C_0^\infty(B(x_0, r))$ be a nonnegative function satisfying $\|\nabla\eta\| \le C_3/r$ for some constant C_3. Then there exists a positive constant C depending only on C_3, c_1, c_2, p and n such that*

$$\int_{B(x_0,r)} \|\nabla u\|^{p-1}\eta^{p-1}\|\nabla\eta\|\, dx \le Cr^{n-p}\left(\inf_{B(x_0,r/2)} u\right)^{p-1}.$$

Finally, we make some remarks on an exceptional set E. Let E be a uniform Minkowski set of dimension m. From (2.4), we see that the m-dimensional upper Minkowski content of E defined by

$$\mathcal{M}^m(E) := \limsup_{r\to 0+} \frac{|E(r)|}{r^{n-m}}$$

is finite. It is known that the m-dimensional Hausdorff measure of E is not greater than $C\mathcal{M}^m(E)$ for some positive constant C depending only on m and n. Therefore, if $0 \le m \le n-p$, then the p-capacity of E is zero (see [5, Theorem 2.27]). Since [5, Theorem 7.35] states that a compact set of p-capacity zero is removable for nonnegative $\mathcal{A}$-superharmonic functions, we have the following lemma.

Lemma 3.9 *Let $0 \le m \le n-p$. If E is a uniform Minkowski set in Ω of dimension m, then E is removable for nonnegative $\mathcal{A}$-superharmonic functions on $\Omega\backslash E$.*

Moreover, for the Riesz measure associated with an $\mathcal{A}$-super/$\mathcal{A}$-subharmonic function, we get the following lemma.

Lemma 3.10 *Let $0 \le m \le n-p$, let E be a uniform Minkowski set in Ω of dimension m and let u be an $\mathcal{A}$-super/$\mathcal{A}$-subharmonic function on Ω with the Riesz measure μ. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, then $\mu(E) = 0$.*

Proof Let $r > 0$ be small. Take $\eta \in C_0^\infty(E(r))$ so that $0 \le \eta \le 1$ and $\|\nabla\eta\| \le C/r$ on $E(r)$ and $\eta = 1$ on E. By (A1) and the Hölder inequality, we have

$$\begin{aligned}
\mu(E) &\le \int_{E(r)} \eta\, \mathrm{d}\mu = \pm\int_{E(r)} \langle\mathcal{A}(x, \nabla u(x)), \nabla\eta(x)\rangle\, \mathrm{d}x \\
&\le c_2 \int_{E(r)} \|\nabla u\|^{p-1}\|\nabla\eta\|\, \mathrm{d}x \\
&\le c_2\left(\int_{E(r)} \|\nabla u\|^p\, \mathrm{d}x\right)^{(p-1)/p}\left(\int_{E(r)} \|\nabla\eta\|^p\, \mathrm{d}x\right)^{1/p}.
\end{aligned}$$

Since

$$\int_{E(r)} \|\nabla\eta\|^p\, \mathrm{d}x \le Cr^{-p}|E(r)| \le Cr^{-p+n-m} \le C,$$

we obtain from $\|\nabla u\| \in L^p_{\mathrm{loc}}(\Omega)$ that $\mu(E) = 0$ after $r \to 0$ in the above inequality. □

4 Proof of Theorem 2.1

In order to give the proof of Theorem 2.1, we prepare two crucial lemmas.

Lemma 4.1 *Assumptions on m, q and E are the same as in Theorem 2.1. Let u be a nonnegative solution of (2.1) in $\Omega \setminus E$ satisfying growth condition (2.6) at each $y \in E$. Then there exists a nonnegative $\mathcal{A}$-superharmonic function $\overline{u}$ on Ω such that $\overline{u} = u$ almost everywhere on $\Omega \setminus E$ and*

$$\int_{\Omega} \langle \mathcal{A}(x, D\overline{u}(x)), \nabla \phi(x) \rangle \, dx = \int_{\Omega \setminus E} \mathcal{B}(x, u(x), \nabla u(x)) \phi(x) \, dx$$
$$= \int_{\Omega} \mathcal{B}(x, \overline{u}(x), D\overline{u}(x)) \phi(x) \, dx \tag{4.1}$$

for all $\phi \in C_0^\infty(\Omega)$. In particular, the Riesz measure associated with $\overline{u}$ is given by

$$d\mu(x) = \mathcal{B}(x, \overline{u}(x), D\overline{u}(x)) \, dx$$

on Borel subsets of Ω.

Proof The nonnegativity of u and (B1) imply that u is a supersolution of (3.1) in $\Omega \setminus E$. By Lemmas 3.1(i) and 3.9, there is a nonnegative $\mathcal{A}$-superharmonic function $\overline{u}$ on Ω such that $\overline{u} = u$ almost everywhere on $\Omega \setminus E$. Consider the truncated function $\overline{u}_k := \min\{\overline{u}, k\}$ for $k \in \mathbb{N}$, which is locally bounded and $\mathcal{A}$-superharmonic on Ω and so is a supersolution of (3.1) in Ω by Lemma 3.1(iii). Then

$$D\overline{u} = \lim_{k \to \infty} \nabla \overline{u}_k = \nabla u \quad \text{a.e. on } \Omega \setminus E. \tag{4.2}$$

Let μ_k be the Riesz measure associated with $\overline{u}_k$. Note from [9, Remark 2.2] that μ_k converges weakly to the Riesz measure μ associated with $\overline{u}$, i.e., for all $\phi \in C_0^\infty(\Omega)$,

$$\lim_{k \to \infty} \int_{\Omega} \phi \, \mathrm{d}\mu_k = \int_{\Omega} \phi \, \mathrm{d}\mu.$$

We show that $\mu(E) = 0$. Let $\varepsilon > 0$. By the compactness of E and growth condition (2.6), there is a positive constant r such that $E(10r) \subset \Omega$ and

$$u(x) \leq \varepsilon d(x, E)^{(p-n+m)/(p-1)} \quad \text{for all } x \in E(2r) \setminus E. \tag{4.3}$$

Let $z \in E$. We take $\eta \in C_0^\infty(B(z, 2r))$ such that $0 \leq \eta \leq 1$ and $\|\nabla \eta\| \leq C/r$ on $B(z, 2r)$ and $\eta = 1$ on $B(z, r)$. Then, by (A1) and Lemma 3.8,

$$\begin{aligned}
\mu_k(B(z, r)) &\leq \int_{B(z,2r)} \eta^p \, \mathrm{d}\mu_k = \int_{B(z,2r)} \langle \mathcal{A}(x, \nabla \overline{u}_k(x)), \nabla \eta(x)^p \rangle \, \mathrm{d}x \\
&\leq pc_2 \int_{B(z,2r)} \|\nabla \overline{u}_k\|^{p-1} \eta^{p-1} \|\nabla \eta\| \, \mathrm{d}x \\
&\leq Cr^{n-p} \left(\inf_{B(z,r)} \overline{u}_k \right)^{p-1}.
\end{aligned}$$

From (4.3), we get

$$\mu_k(B(z, r)) \leq Cr^{n-p} \cdot (\varepsilon r^{(p-n+m)/(p-1)})^{p-1} \leq C\varepsilon^{p-1} r^m. \tag{4.4}$$

Consider the covering $\{B(z, r/5)\}_{z \in E}$ of $\overline{E(r/10)}$. By the compactness and the basic covering lemma, we find N-points z_j in E such that $\{B(z_j, r/5)\}_{j=1}^N$ are mutually disjoint and $\overline{E(r/10)}$

is covered by $\{B(z_j, r)\}_{j=1}^N$. Since all the $B(z_j, r/5)$ is contained in $E(r)$, we have $N \le Cr^{-m}$ by (2.4). This, together with (4.4), yields

$$\mu_k(E(r/10)) \le \sum_{j=1}^{N} \mu_k(B(z_j, r)) \le C\varepsilon^{p-1}.$$

Now, we take $\psi \in C_0^\infty(E(r/10))$ such that $0 \le \psi \le 1$ on $E(r/10)$ and $\psi = 1$ on E. The weak convergence $\mu_k \to \mu$ implies that

$$\mu(E) \le \int \psi \, \mathrm{d}\mu = \lim_{k\to\infty} \int \psi \, \mathrm{d}\mu_k \le \liminf_{k\to\infty} \mu_k(E(r/10)) \le C\varepsilon^{p-1}.$$

Hence we obtain $\mu(E) = 0$ as required.

Finally, we show (4.1). As mentioned in the first paragraph, $\overline{u}$ is an $\mathcal{A}$-superharmonic function on Ω with the associated Riesz measure μ. Since u is a solution of (2.1) in $\Omega\backslash E$, it follows from (4.2) that for all $\phi \in C_0^\infty(\Omega\backslash E)$,

$$\begin{aligned}\int_{\Omega\backslash E} \langle \mathcal{A}(x, D\overline{u}(x)), \nabla\phi(x)\rangle \, \mathrm{d}x &= \int_{\Omega\backslash E} \langle \mathcal{A}(x, \nabla u(x)), \nabla\phi(x)\rangle \, \mathrm{d}x \\ &= \int_{\Omega\backslash E} \mathcal{B}(x, u(x), \nabla u(x))\phi(x) \, \mathrm{d}x.\end{aligned}$$

The uniqueness of the Riesz measure μ implies that $\mathrm{d}\mu = \mathcal{B}(\cdot, u, \nabla u)\, \mathrm{d}x$ on Borel subsets of $\Omega\backslash E$. This and $\mu(E) = |E| = 0$ yield that for all $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\int_{\Omega} \langle \mathcal{A}(x, D\overline{u}(x)), \nabla\phi(x)\rangle \, \mathrm{d}x &= \int_{\Omega} \phi \, \mathrm{d}\mu = \int_{\Omega\backslash E} \phi \, \mathrm{d}\mu \\ &= \int_{\Omega\backslash E} \mathcal{B}(x, u(x), \nabla u(x))\phi(x) \, \mathrm{d}x \\ &= \int_{\Omega} \mathcal{B}(x, \overline{u}(x), D\overline{u}(x))\phi(x) \, \mathrm{d}x.\end{aligned}$$

This completes the proof. □

To show the local boundedness of $\overline{u}$ obtained in Lemma 4.1, we need an elementary lemma concerning the width of $E(r)$.

Lemma 4.2 *Let E be a compact set in $\mathbb{R}^n$ with no interior and let $\rho > 0$. Then there exists a positive constant ρ_0 depending only on E and ρ such that $B(z, \rho)\backslash E(\rho_0) \neq \emptyset$ for all $z \in E$.*

Proof Suppose to the contrary that for each $k \in \mathbb{N}$ there is $z_k \in E$ such that $B(z_k, \rho) \subset E(1/k)$. Taking a subsequence if necessary, we may assume that $\{z_k\}$ converges to $z_0 \in E$. Then we see that there is $k_\rho \in \mathbb{N}$ such that $B(z_0, \rho/2) \subset B(z_k, \rho)$ for all $k \ge k_\rho$. This implies that $B(z_0, \rho/2) \subset \bigcap_{k\ge k_\rho} E(1/k) = E$, which contradicts that E has no interior. □

Lemma 4.3 *Let E be a compact set in $\mathbb{R}^n$ with no interior and let $\rho > 0$. Then there exists a positive constant ρ_1 depending only on E and ρ such that $B(x, 2\rho)\backslash E(\rho_1) \neq \emptyset$ for all $z \in E$ and $x \in B(z, \rho)$.*

Proof Let $\rho_1 := \min\{\rho_0, \rho\}$, where ρ_0 is the constant in Lemma 4.2. If $d(x, E) \ge \rho$, then $x \in B(x, 2\rho)\backslash E(\rho_1)$. If $d(x, E) < \rho$, then we take $x^* \in E$ with $\|x^* - x\| = d(x, E)$. Then $B(x^*, \rho)\backslash E(\rho_0) \neq \emptyset$ by Lemma 4.2, and so $B(x, 2\rho)\backslash E(\rho_1) \neq \emptyset$. Thus the lemma follows. □

Lemma 4.4 *Assumptions on m, q and E are the same as in Theorem 2.1. Let u be a nonnegative solution of (2.1) in $\Omega \setminus E$ satisfying growth condition (2.6) at each $y \in E$ and let $\overline{u}$ be a nonnegative $\mathcal{A}$-superharmonic function on Ω obtained in Lemma 4.1. Then $\overline{u} \in L^\infty(E(r))$ for some $r > 0$.*

Proof We first consider the case

$$p-1 < q < \frac{(n-m)(p-1)}{n-m-p}.$$

Let N be the smallest natural number satisfying

$$N \geq \frac{\log \dfrac{p(p-1)}{(n-m)(p-1)-q(n-m-p)}}{\log \dfrac{q}{p-1}},$$

which is equivalent to

$$(p-n+m)\left(\frac{q}{p-1}\right)^N + p\left(\frac{q}{p-1}\right)^{N-1} + \cdots + p\left(\frac{q}{p-1}\right) + p \geq 0.$$

Define finitely many numbers by

$$\lambda_1 := (p-n+m)\frac{q}{p-1}$$

and for $k = 2, \ldots, N+1$,

$$\lambda_k := (p-n+m)\left(\frac{q}{p-1}\right)^k + p\left(\frac{q}{p-1}\right)^{k-1} + \cdots + p\left(\frac{q}{p-1}\right).$$

The case $N = 1$ needs only the final step of the following iteration argument, so we consider the case $N > 1$. In below, we note that

$$\frac{\lambda_k + p}{p-1} q = \lambda_{k+1} \quad (k = 1, \ldots, N)$$

and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{N-1} < -p \leq \lambda_N.$$

Let $z \in E$. In view of (2.6), we take $0 < r_1 < \min\{1, r_0/6\}$ such that $B(z, 4r_1) \subset E(r_0/3) \cap \Omega$ and

$$\overline{u}(x) \leq d(x, E)^{(p-n+m)/(p-1)} \quad \text{for all } x \in B(z, 3r_1), \tag{4.5}$$

and define $r_{k+1} := r_k/4$ inductively. Then

$$B(x, 3r_{k+1}) \subset B(z, r_k) \quad \text{for all } x \in B(z, r_{k+1}). \tag{4.6}$$

Let $x \in B(z, r_1)$. Taking Lemma 4.1 into account, we consider the Wolff potential $\mathbf{W}^\mu_{1,p}(x, 2r_1)$ with $d\mu = \mathcal{B}(\cdot, \overline{u}, D\overline{u})\, dx$. By Lemmas 3.2, 4.3 and (4.5), we have

$$\overline{u}(x) \leq C\left\{\inf_{B(x,r_1)} \overline{u} + \mathbf{W}^\mu_{1,p}(x, 2r_1)\right\} \leq C\{1 + \mathbf{W}^\mu_{1,p}(x, 2r_1)\}, \tag{4.7}$$

where the last constant C may depend on r_1. Since

$$\begin{aligned}\mathcal{B}(y, \overline{u}(y), D\overline{u}(y)) &\le c_3\{1 + \overline{u}(y)^q\}\\ &\le Cd(y, E)^{\lambda_1} \quad \text{for all } y \in B(x, 2r_1)\end{aligned} \tag{4.8}$$

by (B2) and (4.5), it follows from Lemma 3.5 that

$$\mathbf{W}^{\mu}_{1,p}(x, 2r_1) \le Cd(x, E)^{(\lambda_1+p)/(p-1)},$$

and so

$$\overline{u}(x) \le Cd(x, E)^{(\lambda_1+p)/(p-1)} \tag{4.9}$$

by (4.7). This is true for all $x \in B(z, r_1)$. Next, we let $x \in B(z, r_2)$. Noting (4.6), we apply Lemma 3.2 on $B(x, 3r_2)$. As in (4.7), we have

$$\overline{u}(x) \le C\{1 + \mathbf{W}^{\mu}_{1,p}(x, 2r_2)\},$$

where a constant C may depend on r_2. Also, by (B2) and (4.9), we have

$$\mathcal{B}(y, \overline{u}(y), D\overline{u}(y)) \le Cd(y, E)^{\lambda_2} \quad \text{for all } y \in B(x, 2r_2).$$

Therefore Lemma 3.5 gives

$$\overline{u}(x) \le Cd(x, E)^{(\lambda_2+p)/(p-1)}.$$

This is true for all $x \in B(z, r_2)$. Repeating this process $N - 1$ times, we obtain

$$\overline{u}(x) \le Cd(x, E)^{(\lambda_{N-1}+p)/(p-1)} \quad \text{for all } x \in B(z, r_{N-1}),$$

which implies that if $x \in B(z, r_N)$, then

$$\mathcal{B}(y, \overline{u}(y), D\overline{u}(y)) \le Cd(y, E)^{\lambda_N} \quad \text{for all } y \in B(x, 2r_N).$$

Since $\lambda_N \ge -p$, it follows from Lemma 3.5 that for all $x \in B(z, r_N)$,

$$\overline{u}(x) \le \begin{cases} C\left(1 + \log^+ \dfrac{1}{d(x, E)}\right) & \text{if } \lambda_N = -p,\\ C & \text{if } \lambda_N > -p.\end{cases}$$

If $\lambda_N = -p$, then we apply Lemma 3.5 one more time to obtain the boundedness of $\overline{u}$ on $B(z, r_{N+1})$.

We have proved that if $z \in E$, then $\overline{u} \le C$ on $B(z, r_{N+1})$ for some constant C independent of z. Therefore we now get the boundedness of $\overline{u}$ on $E(r_{N+1})$ in the case $q > p - 1$.

Next, we consider the case $0 < q \le p - 1$. Take $Q \in \mathbb{R}$ with

$$p - 1 < Q < \frac{(n - m)(p - 1)}{n - m - p}.$$

With the same notation as above, we see that $d(x, E)^{-a} \le Cd(x, E)^{-b}$ for all $x \in B(z, 4r_1)$ if $0 \le a \le b$. Thus (4.8) holds for

$$\lambda_1 := (p - n + m)\frac{Q}{p - 1},$$

and the later estimates also hold for

$$\lambda_k := (p - n + m)\left(\frac{Q}{p - 1}\right)^k + p\left(\frac{Q}{p - 1}\right)^{k-1} + \cdots + p\left(\frac{Q}{p - 1}\right).$$

Therefore we can obtain the boundedness of $\overline{u}$ in this case as well. Thus the lemma is proved. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1 Let u be a nonnegative continuous solution of (2.1) in $\Omega\setminus E$ satisfying growth condition (2.6) at each $y \in E$. Let $\overline{u}$ be a nonnegative $\mathcal{A}$-superharmonic function on Ω obtained in Lemma 4.1. Then $\overline{u} \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ by Lemmas 3.1(iii) and 4.4, and we have $D\overline{u} = \nabla\overline{u}$. Thus $\overline{u}$ is a solution of (2.1) in Ω. As stated in Sect. 2, $\overline{u}$ has a continuous representative, so we may assume that $\overline{u}$ is continuous on Ω. Then $\overline{u} = u$ on $\Omega\setminus E$. This completes the proof. □

5 Proof of Theorem 2.2

This section gives the proof of Theorem 2.2. It should be noted that we cannot apply the known removability theorem for $\mathcal{A}$-super/$\mathcal{A}$-subharmonic functions (Lemma 3.9) to solutions of (2.2) in the first step of the proof, although the proof for case (2.1) was started from the use of that theorem (see the proof of Lemma 4.1). To overcome this, we need a removability theorem for $\mathcal{A}$-subharmonic functions fitting in our question.

Theorem 5.1 *Let $0 \le m < n-p$ and let E be a Lipschitz set in Ω of dimension m. If u is an $\mathcal{A}$-subharmonic function on $\Omega\setminus E$ satisfying at each $y \in E$,*

$$\limsup_{x\to y} d(x,E)^{(n-p-m)/(p-1)}u(x) \le 0, \tag{5.1}$$

then u can be extended to the whole of Ω as an $\mathcal{A}$-subharmonic function.

Proof Let ω be a bounded open set such that $E \subset \omega$ and $\overline{\omega} \subset \Omega$. For this ω, we take a positive $\mathcal{A}$-superharmonic function g on $\mathbb{R}^n$ with the properties in Lemma 3.7. Let $\varepsilon > 0$. By (5.1), (3.5) and the finite covering argument, we find $r_\varepsilon > 0$ such that $E(r_\varepsilon) \subset \omega$ and

$$u(x) \le \varepsilon g(x) \quad \text{for all } x \in \overline{E(r_\varepsilon)}\setminus E. \tag{5.2}$$

Taking Lemma 3.9 into account, it suffices to prove that u is bounded above on $\omega\setminus E$. To this end, noting that u is upper semicontinuous on $\partial\omega$, we let

$$\gamma := \max\{\max_{\partial\omega} u, 0\} \quad \text{and} \quad v := u - \gamma.$$

Then v is $\mathcal{A}$-subharmonic on $\Omega\setminus E$ satisfying $v \le u$ on $\Omega\setminus E$ and

$$v \le 0 \quad \text{on } \partial\omega. \tag{5.3}$$

By (5.2), we have $v \le \varepsilon g$ on $\overline{E(r_\varepsilon)}\setminus E$. Since $v \le 0 \le \varepsilon g$ on $\partial\omega$ by (5.3) and εg is $\mathcal{A}$-superharmonic on $\mathbb{R}^n$, it follows from the comparison principle that $v \le \varepsilon g$ on $\omega\setminus E(r_\varepsilon)$. Therefore $v \le \varepsilon g$ on $\omega\setminus E$. Since g is finite on $\omega\setminus E$ by (3.5), we obtain $v \le 0$ on $\omega\setminus E$ after $\varepsilon \to 0$. Hence $u \le \gamma$ on $\omega\setminus E$. Thus the theorem is proved. □

Corollary 5.2 *Let $0 \le m < n-p$ and let E be a Lipschitz set in Ω of dimension m. If u is an $\mathcal{A}$-harmonic function on $\Omega\setminus E$ satisfying growth condition (2.6) at each $y \in E$, then u can be extended to the whole of Ω as an $\mathcal{A}$-harmonic function.*

Proof of Theorem 2.2 Let u be a nonnegative continuous solution of (2.2) in $\Omega\setminus E$ satisfying growth condition (2.6) at each $y \in E$. Since u is a continuous subsolution of (3.1) in $\Omega\setminus E$ by (B1), we see from Lemma 3.1(i), (iv) that u is $\mathcal{A}$-subharmonic on $\Omega\setminus E$. Theorem 5.1 shows that u has a nonnegative $\mathcal{A}$-subharmonic extension $\overline{u}$ to Ω. Moreover, Lemma 3.1(iii) and the upper semicontinuity imply that $\overline{u} \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$, which is also continuous on Ω as stated in Sect. 2. By Lemma 4.1, we have $\mu(E) = 0$, where μ is the Riesz measure associated with $\overline{u}$. By the same reasoning as at the end of the proof of Lemma 4.1, we conclude that $\overline{u}$ is a solution of (2.2) in Ω. □

6 On the case $m = n - p$

This section gives comments on the case $m = n - p$. In this case, we can obtain

$$\frac{1}{C}\log\frac{\min\{r, r_0\}}{d(x, E)} \le \mathbf{W}^{\mathcal{H}^m|_E}_{1,p}(x, r) \le C\log\frac{\min\{r, r_0\}}{d(x, E)}$$

instead of (3.3). Then a function g in Lemma 3.7 can be estimated like

$$\frac{1}{C}\left(1 + \log^+\frac{1}{d(x, E)}\right) \le g(x) \le C\left(1 + \log^+\frac{1}{d(x, E)}\right)$$

for all $x \in \omega\setminus E$, where $\log^+ t = \max\{0, \log t\}$. Therefore we obtain the following theorem corresponding to Theorem 5.1 by the same argument.

Theorem 6.1 *Let E be a Lipschitz set in Ω of dimension $m = n - p$. If u is an $\mathcal{A}$-subharmonic function on $\Omega\setminus E$ satisfying at each $y \in E$,*

$$\limsup_{x\to y}\frac{u(x)}{-\log d(x, E)} \le 0,$$

then u can be extended to the whole of Ω as an $\mathcal{A}$-subharmonic function.

This yields the following theorem corresponding to Theorem 2.2.

Theorem 6.2 *Let E be a Lipschitz set in Ω of dimension $m = n - p$. Then E is removable for nonnegative continuous solutions of (2.2) in $\Omega\setminus E$ satisfying the growth condition: at each $y \in E$,*

$$u(x) = o(-\log d(x, E)) \quad \text{as } x \to y. \tag{6.1}$$

For equation $-\Delta_p u = e^u$, we could not obtain a result corresponding to Theorem 2.1 under condition (6.1) in the case $m = n - p$ because it is difficult to show that the associated Riesz measure does not charge on E without assuming $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$.

Theorem 6.3 *Let E be a uniform Minkowski set in Ω of dimension $m = n - p$. Then E is removable for nonnegative continuous solutions of $-\operatorname{div}\mathcal{A}(x, \nabla u) = e^u$ in $\Omega\setminus E$ satisfying $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and (6.1) at each $y \in E$.*

Proof By Lemmas 3.9 and 3.10, u has a nonnegative $\mathcal{A}$-superharmonic extension $\overline{u}$ to Ω and the associated Riesz measure μ satisfies $\mu(E) = 0$. Therefore $d\mu(x) = e^{\overline{u}(x)}\,dx$ on Borel subsets of Ω, which implies that $\overline{u}$ satisfies $-\operatorname{div}\mathcal{A}(x, \nabla\overline{u}) = e^{\overline{u}}$ in Ω in the weak sense. Then (6.1) and Lemma 3.5 with $\lambda = -1$ give $\mathbf{W}^{\mu}_{1,p}(x, r) \le C$. Thus we get $\overline{u} \in L^{\infty}_{\mathrm{loc}}(\Omega)$ from Lemma 3.2. This concludes that $\overline{u}$ is a nonnegative continuous solution of $-\operatorname{div}\mathcal{A}(x, \nabla\overline{u}) = e^{\overline{u}}$ in Ω such that $\overline{u} = u$ on $\Omega\setminus E$. □

7 Examples of Lipschitz sets

We say that E is a *Lipschitz manifold* in $\mathbb{R}^n$ of dimension $m \in \mathbb{N}$ if for each $x \in E$ there exist an open neighborhood U of x in $\mathbb{R}^n$, an open set V in $\mathbb{R}^n$ and a bi-Lipschitz mapping $\phi : U \to V$ such that $\phi(E \cap U) = \mathbb{R}_0^m \cap V$, where $\mathbb{R}_0^m := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_{m+1} = \cdots = x_n = 0\}$. It is easy to see that a compact Lipschitz manifold of dimension m is a Lipschitz set of dimension m.

There are nontrivial examples including fractal sets. Let us recall the definition of self-similar sets and some known facts. A mapping $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a *similarity with ratio* $c > 0$ if it satisfies for any $x, y \in \mathbb{R}^n$,

$$\|\psi(x) - \psi(y)\| = c\|x - y\|.$$

Let $N \in \mathbb{N}$. It is known that given a family of N-similarities $\Psi = \{\psi_1, \ldots, \psi_N\}$ with the same ratio $c < 1$, there exists a unique nonempty compact set E in $\mathbb{R}^n$ such that

$$E = \psi_1(E) \cup \cdots \cup \psi_N(E). \tag{7.1}$$

See Falconer [3] and Hutchinson [7] for details and generalizations. This set E is called *self-similar* with respect to Ψ. By the Moran-Hutchinson theorem, the Hausdorff dimension of the self-similar set E with respect to Ψ is given by

$$\dim_{\mathrm{H}} E = \frac{\log N}{-\log c},$$

if Ψ satisfies the following open set condition. Moreover, $0 < \mathcal{H}^m(E) < \infty$ with $m = \log N/(-\log c)$.

Definition 7.1 We say that $\Psi = \{\psi_1, \ldots, \psi_N\}$ satisfies the *open set condition* (with an open set V) if there exists a nonempty bounded open set V in $\mathbb{R}^n$ such that

(V1) $\psi_1(V) \cup \cdots \cup \psi_N(V) \subset V$,
(V2) $\psi_i(V) \cap \psi_j(V) = \emptyset$ whenever $i \neq j$.

As is well known, there is no inclusion between V and the self-similar set E, but we have $E \subset \overline{V}$.

Theorem 7.2 *Let $\Psi = \{\psi_1, \ldots, \psi_N\}$ be a family of N-similarities on $\mathbb{R}^n$ with the common ratio $0 < c < 1$ satisfying the open set condition. Then the self-similar set E with respect to Ψ is a Lipschitz set of dimension $m = \log N/(-\log c)$.*

From this theorem, we can see that the Cantor set, the Cantor dust, the Sierpinski triangle, the von Koch curve, etc., known as typical examples of self-similar sets are Lipschitz sets with appropriate dimensions.

To prove Theorem 7.2, we prepare several lemmas. In what follows, we suppose that $\Psi = \{\psi_1, \ldots, \psi_N\}$ is a family of N-similarities on $\mathbb{R}^n$ with the common ratio $0 < c < 1$. For $k \in \mathbb{N}$, let

$$\Lambda_k := \{(i_1, \ldots, i_k) : 1 \leq i_j \leq N \, (1 \leq j \leq k)\}.$$

Lemma 7.3 *Let E be the self-similar set with respect to Ψ and let K be a nonempty compact set in $\mathbb{R}^n$ satisfying $\psi_1(K) \cup \cdots \cup \psi_N(K) \subset K$. Then*

$$\bigcap_{j=1}^{\infty} \bigcup_{(i_1,\ldots,i_j)\in\Lambda_j} \psi_{i_1} \circ \cdots \circ \psi_{i_j}(K) = E.$$

Moreover, we have for all $j \in \mathbb{N}$,

$$\bigcup_{(i_1,\ldots,i_j)\in\Lambda_j} \psi_{i_1}\circ\cdots\circ\psi_{i_j}(E) = E.$$

Proof The first statement is found in Falconer's book [3, Theorem 9.1]. The second statement can be proved by (7.1) and induction. □

Lemma 7.4 *Suppose that* $\Psi = \{\psi_1, \ldots, \psi_N\}$ *satisfies the open set condition with an open set* V. *Let* $j \in \mathbb{N}$. *Then* $\{\psi_{i_1}\circ\cdots\circ\psi_{i_j}(V) : (i_1, \ldots, i_j) \in \Lambda_j\}$ *is mutually disjoint.*

Proof This follows from (V2). See Hutchinson [7, p. 736]. □

For a countable set A, we denote by $\sharp A$ the number of elements in A.

Lemma 7.5 *Suppose that* $\Psi = \{\psi_1, \ldots, \psi_N\}$ *satisfies the open set condition with an open set* V. *Let* $\ell \in \mathbb{N}$ *and let* B *be an open ball with* $c^\ell < \operatorname{diam} B \le c^{\ell-1}$. *Then there exists a positive constant* C *depending only on* c, V *and* n *such that for any* $k \ge \ell$ *and* $0 \le r \le c^{\ell-1}$,

$$\sharp\{(i_1, \ldots, i_k) \in \Lambda_k : [\psi_{i_1}\circ\cdots\circ\psi_{i_k}(\overline{V})](r) \cap B \ne \emptyset\} \le CN^{k-\ell}.$$

Proof Let $0 < r \le c^{\ell-1}$. For simplicity, we write

$$\Lambda_j^B := \{(i_1, \ldots, i_j) \in \Lambda_j : [\psi_{i_1}\circ\cdots\circ\psi_{i_j}(\overline{V})](r) \cap B \ne \emptyset\}.$$

Let $k \ge \ell$. Since $\psi_i(\overline{V}) \subset \overline{V}$ for any $i \in \{1, \ldots, N\}$ by (V1) and the continuity of ψ_i, it follows that for each $(i_1, \ldots, i_k) \in \Lambda_k$,

$$\psi_{i_1}\circ\cdots\circ\psi_{i_k}(\overline{V}) \subset \psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(\overline{V}).$$

Therefore

$$\sharp\Lambda_k^B \le \sharp\{(i_1, \ldots, i_k) \in \Lambda_k : (i_1, \ldots, i_\ell) \in \Lambda_\ell^B\} \le N^{k-\ell}\cdot\sharp\Lambda_\ell^B.$$

To obtain the required estimate, it suffices to show that $\sharp\Lambda_\ell^B \le C$. Take an open ball $B_1 \subset V$ and let ρ be its radius. Let $(i_1, \ldots, i_\ell) \in \Lambda_\ell^B$. Then $\psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(B_1)$ is the ball of radius $c^\ell\rho$ contained in $\psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(V)$. Also, $\psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(V)$ is contained in the ball B_2 with the same center as B of radius $3c^{\ell-1} + c^\ell \operatorname{diam} \overline{V}$, because $[\psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(\overline{V})](r) \cap B \ne \emptyset$ and

$$\begin{aligned}\operatorname{diam}[\psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(\overline{V})](r) &\le \operatorname{diam} \psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(\overline{V}) + 2r\\ &\le c^\ell \operatorname{diam} \overline{V} + 2c^{\ell-1}.\end{aligned}$$

Since $\{\psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(B_1) : (i_1, \ldots, i_\ell) \in \Lambda_\ell\}$ is mutually disjoint by Lemma 7.4, we have

$$\sum_{(i_1,\ldots,i_\ell)\in\Lambda_\ell^B} |\psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(B_1)| \le \left|\bigcup_{(i_1,\ldots,i_\ell)\in\Lambda_\ell^B} \psi_{i_1}\circ\cdots\circ\psi_{i_\ell}(B_1)\right| \le |B_2|,$$

and so

$$\sharp\Lambda_\ell^B \le \left(\frac{3c^{-1} + \operatorname{diam} \overline{V}}{\rho}\right)^n.$$

Thus the lemma is proved. □

Let us prove Theorem 7.2.

Proof of Theorem 7.2 First, we show (2.4). Let $0 < r, R < 1$ and $x \in E$. If $R \leq r$, then

$$|E(r) \cap B(x, R)| = |B(x, R)| = a_n R^n \leq a_n r^{n-m} R^m,$$

where $a_n := |B(0, 1)|$. Consider the case $r < R$. Take $k, \ell \in \mathbb{N}$ with $c^k < r \leq c^{k-1}$ and $c^\ell < R \leq c^{\ell-1}$. Then $k \geq \ell$. Let V be a nonempty bounded open set appearing in the open set condition. Then $\psi_1(\overline{V}) \cup \cdots \cup \psi_N(\overline{V}) \subset \overline{V}$ by (V1) and the continuity of ψ_i. Since

$$E(r) \subset \bigcup_{(i_1,\ldots,i_k)\in\Lambda_k} [\psi_{i_1} \circ \cdots \circ \psi_{i_k}(\overline{V})](r)$$

by Lemma 7.3 and

$$\begin{aligned}\left|[\psi_{i_1} \circ \cdots \circ \psi_{i_k}(\overline{V})](r)\right| &\leq a_n(\operatorname{diam} \psi_{i_1} \circ \cdots \circ \psi_{i_k}(\overline{V}) + r)^n \\ &= a_n(c^k \operatorname{diam} \overline{V} + r)^n \\ &\leq a_n(\operatorname{diam} \overline{V} + 1)^n r^n,\end{aligned}$$

it follows from Lemma 7.5 that

$$\begin{aligned}|E(r) \cap B(x, R)| &\leq \sum_{(i_1,\ldots,i_k)\in\Lambda_k} \left|[\psi_{i_1} \circ \cdots \circ \psi_{i_k}(\overline{V})](r) \cap B(x, R)\right| \\ &\leq C N^{k-\ell} r^n \leq C r^{n-m} R^m,\end{aligned}$$

where the last inequality is by $c^{-m} = N$.

Next, we show upper bound estimate in (2.3). Let $0 < r < 1/2$ and $x \in E$. Take $j \in \mathbb{N}$ with $c^j < 2r \leq c^{j-1}$. Then, by Lemma 7.3,

$$\mathcal{H}^m(E \cap B(x, r)) \leq \sum_{(i_1,\ldots,i_j)\in\Lambda_j} \mathcal{H}^m(\psi_{i_1} \circ \cdots \circ \psi_{i_j}(E) \cap B(x, r)).$$

Since $E \subset \overline{V}$, it follows from Lemma 7.5 that

$$\sharp\{(i_1, \ldots, i_j) \in \Lambda_j : \psi_{i_1} \circ \cdots \circ \psi_{i_j}(E) \cap B(x, r) \neq \emptyset\} \leq C,$$

and $\mathcal{H}^m(\psi_{i_1} \circ \cdots \circ \psi_{i_j}(E)) = c^{mj}\mathcal{H}^m(E)$ for $(i_1, \ldots, i_j) \in \Lambda_j$. Therefore

$$\mathcal{H}^m(E \cap B(x, r)) \leq C c^{mj} \mathcal{H}^m(E) \leq C r^m \mathcal{H}^m(E).$$

Finally, we show lower bound estimate in (2.3). Let $0 < r < \operatorname{diam} E$. Take $\ell \in \mathbb{N}$ with $c^\ell \operatorname{diam} E < r \leq c^{\ell-1} \operatorname{diam} E$. Let $x \in E$. By Lemma 7.3, there exists $(i_1, \ldots, i_\ell) \in \Lambda_\ell$ such that

$$x \in \psi_{i_1} \circ \cdots \circ \psi_{i_\ell}(E) \subset E$$

and

$$\operatorname{diam} \psi_{i_1} \circ \cdots \circ \psi_{i_\ell}(E) = c^\ell \operatorname{diam} E < r.$$

These imply $E \cap B(x, r) \supset \psi_{i_1} \circ \cdots \circ \psi_{i_\ell}(E)$. Therefore

$$\begin{aligned}\mathcal{H}^m(E \cap B(x, r)) &\geq \mathcal{H}^m(\psi_{i_1} \circ \cdots \circ \psi_{i_\ell}(E)) = c^{m\ell}\mathcal{H}^m(E) \\ &\geq \left(\frac{cr}{\operatorname{diam} E}\right)^m \mathcal{H}^m(E).\end{aligned}$$

Thus the lower bound estimate in (2.3) holds. The proof is complete. □

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