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Partial regularity results for non-autonomous functionals with Φ-growth conditions

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Abstract We prove the partial Hölder continuity of the local minimizers of non-autonomous integral functionals of the type

$$\int_\Omega \Phi\left(\left(A_{ij}^{\alpha\beta}(x,u)D_i u^\alpha D_j u^\beta\right)^{1/2}\right) \mathrm{d}x,$$

where Φ is an Orlicz function satisfying both the Δ_2 and ∇_2 conditions and the function $A(x,s) = \left(A_{ij}^{\alpha\beta}(x,s)\right)$ is uniformly elliptic, bounded and continuous. Assuming in addition that the function $A(x,s) = \left(A_{ij}^{\alpha\beta}(x,s)\right)$ is Hölder continuous, we prove the partial Hölder continuity also of the gradient of the local minimizers.

Keywords Partial regularity · Nonstandard growth · Non-autonomous functionals

Mathematics Subject Classification 49N60 · 35J50

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1 Introduction

The aim of this paper is to establish the partial Hölder continuity of the local minimizers of a class of integral functionals satisfying the so-called Φ- growth conditions. More precisely, we consider functionals of the type

$$\mathcal{F}(u,\Omega) := \int_\Omega \Phi\left(\left(A_{ij}^{\alpha\beta}(x,u)D_i u^\alpha D_j u^\beta\right)^{1/2}\right)\mathrm{d}x, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u : \Omega \to \mathbb{R}^N$, $n, N \geq 2$ and $\Phi : [0,+\infty) \to [0,+\infty)$ is a strictly convex function of class C^2 with $\Phi(0) = 0$. In order to state and comment our results precisely, we now introduce our hypotheses.

We assume that the function $A(x,s) = \left(A_{ij}^{\alpha\beta}(x,s)\right) : \Omega \times \mathbb{R}^N \to \mathbb{R}^{nN}$ is uniformly continuous and satisfies, for some positive constants λ and Λ

$$\lambda|\xi|^2 \leq A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j, \quad |A_{ij}^{\alpha\beta}(x,s)| \leq \Lambda, \tag{1.2}$$

for every $(x,s,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$. Moreover, we shall assume that there exists a concave, continuous, non-decreasing modulus of continuity $\omega : [0,\infty) \to [0,M)$, $M > 0$, with $\omega(0) = 0$ such that

$$|A(x,s_1) - A(y,s_2)| \leq \omega(|x-y| + |s_1 - s_2|), \tag{1.3}$$

for every $(x,y,s_1,s_2) \in \Omega \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N$.

Concerning the function $\Phi : [0,+\infty) \to [0,+\infty)$, we assume that it satisfies the so-called Δ_2 condition and ∇_2 condition. Namely Φ and its Orlicz conjugate Φ^* satisfy for positive constant $C_{1,\Phi}$ and $C_{2,\Phi}$ that

$$\Phi(2t) \leq C_{1,\Phi}\Phi(t) \quad \text{and} \quad \Phi^*(2t) \leq C_{2,\Phi}\Phi^*(t) \tag{1.4}$$

for all $t > 0$. Moreover, we shall assume that

$$t\Phi'(t) \sim \Phi(t) \quad \text{and} \quad t\Phi''(t) \sim \Phi'(t). \tag{1.5}$$

Here, it is worth mentioning that either of the relations of (1.5) implies nonnegativity of Φ'.

For further needs, we observe that the conditions in (1.4) are equivalent to the existence of two positive exponents $1 < p < q < +\infty$ such that

$$\frac{\Phi(t)}{t^p} \text{ is increasing} \quad \text{and} \quad \frac{\Phi(t)}{t^q} \text{ is decreasing}, \tag{1.6}$$

and also that (1.4) implies the existence of positive constants m and C for which

$$\Phi(kt) \leq Ck^m\Phi(t) \tag{1.7}$$

holds for any $t > 0$ and $k \geq 1$.

We notice that from (1.4) and (1.2) it follows

$$\Phi(|\xi|) \sim \Phi(\|\xi\|_A), \tag{1.8}$$

where we used the notation

$$\|\xi\|_A = \left(\langle A(x,s)\xi,\xi\rangle\right)^{1/2} := \left(A_{ij}^{\alpha\beta}(x,s)\xi_\alpha^i\xi_\beta^j\right)^{1/2}.$$

The model case we have in mind is

$$\int_\Omega a(x,u)|Du|^p \log^\alpha(e+|Du|)\,\mathrm{d}x, \quad 1<p\le n,\ \alpha>0,$$

with $a(x,u)$ a bounded continuous and positive coefficient.

If $\Phi(t)=t^p$, our study trivially reduces to the classical setting of functionals satisfying the so-called standard growth conditions. In this case, the regularity of minimizers has been widely investigated over the last 50 years and a vast literature is available (for an exhaustive treatment we refer to the monographs [21,28]).

Our general growth assumptions become part of the setting of functionals

$$\int_\Omega f(x,u,Du)\,\mathrm{d}x$$

with nonstandard growth conditions, i.e., with integrands $f(x,s,z)$ such that

$$|z|^p \le f(x,s,z)\le C(1+|z|^q), \quad 1<p\le q$$

introduced by Marcellini in the pioneeristic papers [30–32]. It is well known that, in this case, the regularity of the minimizers depends on the distance between the growth and the ellipticity exponents and that the dependence of the integrand on (x,u) can give substantial difficulties since the Lavrentiev phenomenon may appear. Moreover, in the general vectorial setting, only few contributions are available (see for example [1,9,10,16,17,35]), unless some additional structure assumptions are imposed on the integrand f. We refer to [4] for an overview and detailed references on this subject.

An intermediate case between the standard and the nonstandard growth conditions is the case of the so-called $p(x)$-growth conditions, i.e.,

$$|z|^{p(x)} \le f(x,s,z)\le C(1+|z|^{p(x)}),$$

where the function $p(x)>1$ is continuous with a modulus of continuity that verifies suitable assumptions. The study of the regularity of the local minimizers of such functionals started with the paper by Zhykov ([38]), and then it was widely investigated (see for example [36,37] and [2,3,6,14,15,18–20] for the case of integrands $f=f(x,\xi)$). More recently, in [19] the Hölder continuity of functionals with integrand of the form

$$f(x,s,z)=|z|^{p(x)}\varphi(|z|),$$

where

$$\varphi(|z|)\sim \log(e+|z|),$$

has been established. For $p(x)$ constant, such functionals are a particular case of those satisfying the so-called Φ-growth conditions, i.e.,

$$f(x,s,\xi)\sim \Phi(|\xi|)$$

that has been proposed by Marcellini in [32] and for which many contributions are available ([5,7,8,11,27,29,34]). Recently, Marcellini and Papi ([33]) proved the Lipschitz continuity of the minimizers of autonomous functionals, i.e., depending only on the gradient variable, with growth conditions general enough to cover the cases of almost linear and exponential growth. Once the Lipschitz continuity is established, the Hölder continuity of the gradient of the minimizers follows by classical arguments by the C^1 regularity of the integrand.

Another approach to the $C^{1,\alpha}$ regularity of the minimizers of functionals with Φ-growth has been used in [13], where a suitable decay estimate for the excess function of the gradient is the key tool in the proof.
However, as far as we know, the Hölder continuity of the minimizers and of their gradients, in the case of functionals with Φ-growth, has been proven only in the case of autonomous integrands. The aim of this paper is to fill this gap, establishing the Hölder continuity of the local minimizers of $\mathcal{F}(u, \Omega)$, also in view of further applications. Actually, this is the first step in the investigation of more general functionals, whose prototype is

$$\int_\Omega a(x,u)|Du|^{p(x)}\varphi(|Du|),$$

for some bounded positive coefficient $a(x,u)$. Here, we are going to prove the following

Theorem 1.1 *Let $\Phi : [0,+\infty) \to [0,+\infty)$ be a strictly convex function of class C^2 with $\Phi(0) = 0$ satisfying* (1.4) (*equivalently* (1.6)) *and* (1.5)*, and $p > 1$ a constant for which* (1.6) *holds. Assume that $A(x,s) = \big(A_{ij}^{\alpha\beta}(x,s)\big) : \Omega \times \mathbb{R}^N \to \mathbb{R}^{nN}$ satisfies* (1.2) *and* (1.3) *and let $u \in W^{1,\Phi}(\Omega)$ be a local minimizer of $\mathcal{F}$. Then there exists an open subset $\Omega_0 \subset \Omega$ such that $u \in C^{0,\alpha}(\Omega_0)$ for any $\alpha \in (0,1)$. Moreover,*

$$\Omega\backslash\Omega_0 \subset \{x \in \Omega \;;\quad \liminf_{r\to 0} r^{p-n}\int_{B_r(x)} \Phi(|Du|)\,dy > 0\},$$

and $\dim_{\mathcal{H}}(\Omega\backslash\Omega_0) \le n - p$.

Note that the interesting case is $p \le n$ since, for $p > n$ we get the Hölder continuity by the first assumption in (1.6) and by the Sobolev embedding theorem.
It is well known that the regularity of the integrand with respect to the (x,u)-variable reflects on the regularity of the minimizers. Also in our setting, when ω is Hölder continuous, we are able to establish the following partial Hölder continuity result for the gradient of the local minimizers.

Theorem 1.2 *Let $u \in W^{1,\Phi}(\Omega)$ be a local minimizer of $\mathcal{F}$ under all the assumptions in Theorem 1.1 Assume moreover that* (1.3) *holds true for a Hölder continuous function ω. Then $u \in C^{1,\zeta}(\Omega_0)$ for some $\zeta \in (0,1)$ and with Ω_0 given in Theorem 1.1*

Our proof relies on a comparison argument, introduced in [22,24]. Actually, we compare in small balls the minimizer u of our functional with the minimizer v of a suitable “frozen” one for which good decay estimates are available. The core of the proof consists in showing that u and v are close enough, in an integral sense, to have that u shares with v the same decay estimates. Here, with respect to the classical setting, new difficulties arise because of the Φ-growth of the functional. We have to combine classical tools in the theory of the regularity with new results for local minimizers of autonomous functionals with Φ-growth and use the assumptions on the function Φ to obtain the decay estimates in the setting of Lebesgue spaces. Then the results follow by the characterization of the Hölder continuous function due to Campanato.

We conclude noting that the dependence on u of our energy densities prevents us to obtain everywhere regularity, as it is shown already for the case $\Phi(t) = t^p$ (see [28]).

2 Preliminary results

In this section, we recall some standard definitions and collect several Lemmas that we shall need to establish our main result.
We shall follow the usual convention and denote by c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. All the norms we use on $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{nN}$ will be the standard euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices $\xi, \eta \in \mathbb{R}^{nN}$ we use the notation $\langle \xi, \eta\rangle := \mathrm{trace}(\xi^T \eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi\rangle^{\frac{1}{2}}$ for the corresponding euclidean norm.

In what follows, $B(x, r) = B_r(x) = \{y \in \mathbb{R}^n : \ |y-x| < r\}$ will denote the ball centered at x of radius r. The integral mean of a function u over the ball $B_r(x)$ will be denoted by

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,\mathrm{d}y = \fint_{B_r(x)} u(y)\,\mathrm{d}y = (u)_{x,r}.$$

We shall omit the dependence on the center when no confusion arises.

We recall that, if $\Psi : [0, +\infty) \to [0, +\infty)$ is a strictly convex function with $\Psi(0) = 0$, the Orlicz space $L^{\Psi}(\Omega; \mathbb{R}^N)$ consists of the measurable functions $u : \Omega \to \mathbb{R}^N$ such that

$$\int_{\Omega} \Psi(|u|)\,\mathrm{d}x < +\infty$$

and, equipped with the Luxenburg norm,

$$||u||_{L^{\Psi}(\Omega;\mathbb{R}^N)} = \inf\left\{\lambda > 0 : \ \int_{\Omega} \Psi\left(\frac{|u|}{\lambda}\right)\,\mathrm{d}x \le 1\right\}$$

it becomes a Banach space. In addition, the Orlicz–Sobolev space $W^{1,\Psi}(\Omega; \mathbb{R}^N)$ consists of the functions $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ such that $u,\ Du \in L^{\Psi}$ and is equipped with the norm

$$||u||_{W^{1,\Psi}(\Omega;\mathbb{R}^N)} = ||u||_{L^{\Psi}(\Omega;\mathbb{R}^N)} + ||Du||_{L^{\Psi}(\Omega;\mathbb{R}^{nN})}.$$

The following Lemma will be useful for technical reasons and extends to general Orlicz functions Lemma 2.1 in [25]. See also [28, Lemma 8.3].

Lemma 2.1 *Let $\Phi \in C^2([0, +\infty)$ be a nonnegative function satisfying assumptions* (1.4) *and* (1.5)*. There exists a constant C_1 such that*

$$\int_0^1 (1-t)\Phi''(|t\xi + (1-t)\eta|)\mathrm{d}t \ge C_1\Phi''(|\xi| + |\eta|). \tag{2.1}$$

Proof For $t \in [0, 1]$, let us write $\xi_t = t\xi + (1-t)\eta$. We treat the two cases $|\xi| \ge |\eta|$ and $|\xi| \le |\eta|$ separately.

Case (i) $|\xi| \ge |\eta|$.
For $t \in (3/4, 1)$, we see that

$$|\xi_t| \ge t|\xi| - (1-t)|\eta| \ge \frac{3}{4}|\xi| - \frac{1}{4}|\eta| \ge \frac{1}{2}|\xi| \ge \frac{1}{4}(|\xi| + |\eta|).$$

On the other hand

$$|\xi_t| \le t|\xi| + (1-t)|\eta| \le |\xi| + |\eta|.$$

Therefore, by using assumption (1.5), the Δ_2-condition and the monotonicity of Φ, we have

$$\Phi''(|\xi_t|) \sim \frac{1}{|\xi_t|^2}\Phi(|\xi_t|) \geq \frac{1}{(|\xi|+|\eta|)^2}\Phi\Big(\frac{1}{4}(|\xi|+|\eta|)\Big)$$
$$\geq c\frac{1}{(|\xi|+|\eta|)^2}\Phi(|\xi|+|\eta|) \sim \Phi''(|\xi|+|\eta|),$$

and so

$$\int_0^1 (1-t)\Phi''(|\xi_t|)\mathrm{d}t \geq c\int_{3/4}^1 (1-t)\Phi''(|\xi|+|\eta|)\mathrm{d}t = C_2\Phi''(|\xi|+|\eta|),$$

for some C_2.

Case (ii) $|\xi| \leq |\eta|$.
For $t \in (0, 1/4)$, we observe that

$$|\xi_t| \geq (1-t)|\eta| - t|\xi| \geq \frac{3}{4}|\eta| - \frac{1}{4}|\xi| \geq \frac{1}{2}|\eta| \geq \frac{1}{4}(|\xi|+|\eta|)$$

holds. Hence, arguing as we did above, we see that

$$\int_0^1 (1-t)\Phi''(|\xi_t|)\mathrm{d}t \geq \int_0^{1/4} c(1-t)\Phi''(|\xi|+|\eta|)\mathrm{d}t \geq C_3\Phi''(|\xi|+|\eta|),$$

for some C_3. Now, taking $C_1 = \min\{C_2, C_3\}$ and recalling that $t \in (0,1)$, we get the assertion. □

Let us introduce the following auxiliary function

$$V_\Phi(\xi) := \Big(\frac{\Phi'(|\xi|)}{|\xi|}\Big)^{1/2}\xi, \tag{2.2}$$

and recall the following Lemma, whose proof is a direct consequence of Lemma 3 and Lemma 21 in [11].

Lemma 2.2 *For every $\xi, \eta \in \mathbb{R}^{nN}$, we have that*

$$|V_\Phi(\xi)|^2 \sim \Phi(|\xi|)$$

and

$$|V_\Phi(\xi) - V_\Phi(\eta)|^2 \sim |\xi-\eta|^2\Phi''(|\xi|+|\eta|).$$

The following lemma finds an important application in the so-called hole-filling method. Its proof can be found for example in [28, Lemma 6.1] .

Lemma 2.3 *Let $h : [r, R_0] \to \mathbb{R}$ be a nonnegative bounded function and $0 < \vartheta < 1$, $A, B \geq 0$ and $m > 0$. Assume that*

$$h(s) \leq \vartheta h(t) + \frac{A}{(t-s)^m} + B,$$

for all $r \leq s < t \leq R_0$. Then

$$h(r) \leq \frac{cA}{(R_0-r)^m} + cB,$$

where $c = c(\vartheta, m) > 0$.

Now, let us recall the Sobolev–Poincaré inequality for Orlicz–Sobolev functions (see for example [11, Theorem 7]).

Theorem 2.4 *Let Φ satisfy (1.4). Further, let $Q \in \mathbb{R}^n$ be some cube with side length R and let $\omega \in L^\infty(Q)$ with $\omega \geq 0$ and $\int_Q \omega(x)\mathrm{d}x = 1$. Then there exists $0 < \theta < 1$, which only depends on the constants in (1.4) and $R^n\|\omega\|_\infty$, such that for all $v \in W^{1,\Phi}(Q)$ it holds*

$$\fint_Q \Phi\left(\frac{|v-\langle v\rangle_\omega|}{R}\right)\mathrm{d}x \leq K\left(\fint_Q \left(\Phi(|Dv|)\right)^\theta \mathrm{d}x\right)^{1/\theta},$$

where $\langle v\rangle_\omega := \int_Q v(x)\omega(x)\mathrm{d}x$.

From the above theorem (with B_R instead of Q), we get the following corollaries.

Corollary 2.5 *Le Φ be as above, and let D be a subset of B_R of positive measure. Then there exist a constant $K_1 = K_1(\Phi, D) > 0$ and $\theta \in (0,1)$ such that the following inequality holds for every $v \in W^{1,\Phi}(B_R)$ with $v \equiv 0$ on D*

$$\fint_{B_R} \Phi\left(\frac{|v|}{R}\right)\mathrm{d}x \leq K_1\left(\fint_{B_R} \left(\Phi(|Dv|)\right)^\theta \mathrm{d}x\right)^{1/\theta}. \tag{2.3}$$

Proof Choosing ω so that $\omega = 0$ on $B_R\backslash D$ and applying Theorem 2.4, we get the assertion. □

Corollary 2.6 *Le Φ be as above. Then there exists $0 < \theta < 1$, which only depends on the Δ_2 constants of Φ and Φ^*, such that for all $v \in W_0^{1,\Phi}(B_R)$ the following inequality holds*

$$\fint_{B_R} \Phi\left(\frac{|v|}{R}\right)\mathrm{d}x \leq K_2\left(\fint_{B_R} \left(\Phi(|Dv|)\right)^\theta \mathrm{d}x\right)^{1/\theta}. \tag{2.4}$$

Proof Extending v, Dv as 0 outside B_R and using Theorem 2.4 on B_{2R}, we see that

$$\fint_{B_{2R}} \Phi\left(\frac{|v-\langle v\rangle_\omega|}{R}\right)\mathrm{d}x \leq K\left(\fint_{B_{2R}} \left(\Phi(|Dv|)\right)^\theta \mathrm{d}x\right)^{1/\theta},$$

where

$$\langle v\rangle_\omega := \int_{B_{2R}} v(x)\omega(x)\mathrm{d}x.$$

Let us choose ω so that $\omega = 0$ on B_R, then $\langle v\rangle_\omega = 0$. So, we have

$$\fint_{B_{2R}} \Phi\left(\frac{|v|}{R}\right)\mathrm{d}x \leq K\left(\fint_{B_{2R}} \left(\Phi(|Dv|)\right)^\theta \mathrm{d}x\right)^{1/\theta}.$$

Now, remembering that v, $Dv = 0$ outside B_R and that $\Phi(0) = 0$, we get the assertion with suitably changed constant K. □

The higher integrability of the minimizers of the functional $\mathcal{F}(u,\Omega)$ has been widely investigated. We recall the following result due to Diening and Ettwein [11, Theorem 9].

Theorem 2.7 *Let $u \in W^{1,\Phi}(\Omega)$ be a local minimizer of $\mathcal{F}$. Then there exists $\delta_0 > 0$ such that for all $\delta \in [0,\delta_0)$ we have $\left(\Phi(|Du|)\right)^{1+\delta} \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, for some positive constant C for all $B_r(y)$ with $B_{2r}(y) \subset \Omega$ and all $\delta \in [0,\delta_0)$, it holds that*

$$\fint_{B_r(y)} \Phi(|Du|)^{1+\delta}\mathrm{d}x \leq C\left(\fint_{B_{2r}(y)} \Phi(|Du|)\mathrm{d}x\right)^{1+\delta}. \tag{2.5}$$

In order to employ the comparison argument needed to establish our main results, we consider the so-called frozen functional $\mathcal{F}_0$ defined for $x_0 \in \Omega$ and $0 < R < \operatorname{dist}(x_0, \partial\Omega)/2$, as follows

$$\mathcal{F}_0(w) := \int_{B_R(x_0)} \Phi(\|Dw(x)\|_{A_0})\mathrm{d}x, \tag{2.6}$$

where

$$\|\xi\|_{A_0} = \left(A_{ij}^{\alpha\beta}(x_0, (u)_R)\xi_\alpha^i \xi_\beta^j\right)^{1/2}.$$

Let v be a minimizer of $\mathcal{F}_0$ in the class

$$u + W_0^{1,\Phi}(B_R) := \left\{w \in W_0^{1,\Phi}(B_R) : w - u \in W_0^{1,\Phi}(B_R)\right\}.$$

We shall need also the following higher integrability result up to boundary, whose proof is analogous to that of [19, Theorem 3.4]. We give it for the reader's convenience.

Theorem 2.8 *Let $u \in W^{1,\Phi}(\Omega)$ be a minimizer of the functional $\mathcal{F}(u, \Omega)$ and let v be a local minimizer of the functional $\mathcal{F}_0(w)$ in the class $u + W_0^{1,\Phi}(B_R)$. Then there exists a positive constant C such that*

$$\fint_{B_R} \Phi(|Dv|)^{1+\delta}\,\mathrm{d}x \le C\left(\fint_{B_R} \Phi(|Dv|)\,\mathrm{d}x\right)^{1+\delta} + C\fint_{B_R} \Phi(|Du|)^{1+\delta}\,\mathrm{d}x,$$

for every $0 < \delta < \delta_0$, where δ_0 is given by Theorem 2.7

Proof Consider the function

$$w(x) = \begin{cases} v(x) & \text{if } x \in B_R \\ u(x) & \text{if } x \in B_{2R}\setminus B_R \end{cases} \tag{2.7}$$

Let $x_1 \in B_R$ and let $B_{2\rho}(x_1) \subset B_R$. By Theorem 3.1 in [13], we have that there exists a constant $C > 0$ such that

$$\fint_{B_\rho(x_1)} \Phi(|Dv|)\,\mathrm{d}x \le C\fint_{B_{2\rho}(x_1)} \Phi\left(\left|\frac{v - (v)_{x_1,2\rho}}{\rho}\right|\right)\mathrm{d}x$$

Hence, by the Poincaré inequality of Theorem 2.4, also

$$\fint_{B_\rho(x_1)} \Phi(|Dv|)\,\mathrm{d}x \le C\left(\fint_{B_{2\rho}(x_1)} \Phi(|Dv|)^{\vartheta}\,\mathrm{d}x\right)^{\vartheta}$$

for some $\vartheta \in (0, 1)$. The higher integrability in this case immediately follows by the so-called *reverse Hölder inequality with increasing supports* due to Giaquinta–Modica [24] (see also [28, p.203, Theorem 6.6] or [26, p.299, Theorem 3]).

Note that by the minimality of v, assumption (1.2) and the equivalence in (1.8), we get

$$\begin{aligned}\lambda \int_{B_R} \Phi(|Dv|)\,\mathrm{d}x &\le C\int_{B_R} \Phi(\|Dv\|_{A_0})\,\mathrm{d}x \le C\int_{B_R} \Phi(\|Du\|_{A_0})\,\mathrm{d}x \\ &\le C(\Lambda)\int_{B_R} \Phi(|Du|)\,\mathrm{d}x\end{aligned}$$

and so

$$\int_{B_R} \Phi(|Dv|)\,\mathrm{d}x \le C(\lambda, \Lambda)\int_{B_R} \Phi(|Du|)\,\mathrm{d}x, \tag{2.8}$$

i.e., v is a quasi-minimizer of the functional $\int_{B_R} \Phi(|Dw|)\,\mathrm{d}x$.
Suppose now that $B_{2\rho}(x_1) \subset B_{2R}$ with $x_1 \in \partial B_R$ and fix $\rho \le t < s \le 2\rho$.
Consider a cut-off function $\eta \in C_0^\infty(B_s(x_1))$, such that $\eta \equiv 1$ on $B_t(x_1)$ and $|D\eta| \le \frac{2}{s-t}$.
By the minimality of v, using $g = v - \eta(v-u)$ as a test function, we obtain

$$\int_{B_t\cap B_R} \Phi(|Dv|)\,\mathrm{d}x \le \int_{B_s\cap B_R} \Phi(|Dv|)\,\mathrm{d}x \le c\int_{B_s\cap B_R} \Phi(|Dg|)\,\mathrm{d}x$$
$$= c\int_{B_s\cap B_R} \Phi(|(1-\eta)Dv + \eta Du + D\eta(u-v)|)\,\mathrm{d}x$$

The properties of η, the Δ_2 condition of Φ and (1.7) yield for some positive constant m

$$\begin{aligned}\int_{B_t(x_1)\cap B_R} \Phi(|Dv|)\,\mathrm{d}x &\le c\int_{(B_s(x_1)\setminus B_t(x_1))\cap B_R} \Phi(|Dv|)\mathrm{d}x \\ &\quad + c\int_{(B_s(x_1)\setminus B_t(x_1))\cap B_R} \Phi\left(\left|\frac{u-v}{s-t}\right|\right)\mathrm{d}x \\ &\quad + c\int_{B_s(x_1)\cap B_R} \Phi(|Du|)\mathrm{d}x \\ &\le c\int_{(B_s(x_1)\setminus B_t(x_1))\cap B_R} \Phi(|Dv|)\mathrm{d}x \\ &\quad + \frac{c\rho^m}{(s-t)^m}\int_{(B_s(x_1)\setminus B_t(x_1))\cap B_R} \Phi\left(\left|\frac{u-v}{\rho}\right|\right)\mathrm{d}x \\ &\quad + c\int_{B_s(x_1)\cap B_R} \Phi(|Du|)\,\mathrm{d}x. \end{aligned} \tag{2.9}$$

Now filling the hole, i.e., adding the quantity

$$c\int_{B_t(x_1)\cap B_R} \Phi\;(|Dv|)\;\mathrm{d}x$$

to both sides of (2.9) and dividing the obtained inequality by $c+1$, we can apply Lemma 2.3, thus getting

$$\begin{aligned}\int_{B_\rho(x_1)\cap B_R} \Phi(|Dv|)\,\mathrm{d}x &\le c\int_{B_{2\rho}(x_1)\cap B_R} \Phi\left(\left|\frac{u-v}{\rho}\right|\right)\mathrm{d}x + c\int_{B_{2\rho}(x_1)\cap B_R} \Phi(|Du|)\,\mathrm{d}x \\ &= c\int_{B_{2\rho}(x_1)\cap B_R} \Phi\left(\left|\frac{u-w}{\rho}\right|\right)\mathrm{d}x + c\int_{B_{2\rho}(x_1)\cap B_R} \Phi(|Du|)\,\mathrm{d}x,\end{aligned}$$

where in the last equality we used that $w = v$ on B_R. It follows that

$$\begin{aligned}\int_{B_\rho(x_1)} \Phi(|Dw|)\,\mathrm{d}x &= \int_{B_\rho(x_1)\cap B_R} \Phi(|Dv|)\,\mathrm{d}x + \int_{B_\rho(x_1)\setminus B_R} \Phi(|Du|)\,\mathrm{d}x \\ &\le c\int_{B_{2\rho}(x_1)} \Phi\left(\left|\frac{u-w}{\rho}\right|\right)\mathrm{d}x + c\int_{B_{2\rho}(x_1)} \Phi(|Du|)\,\mathrm{d}x.\end{aligned}$$

By the definition of w, we have that $u - w = 0$ on $B_{2R}\setminus B_R$ and therefore on $B_{2\rho}(x_1)\setminus B_R$.
Hence, we can use the Sobolev imbedding inequality at (2.3), thus getting

$$\fint_{B_\rho(x_1)} \Phi(|Dw|)\,\mathrm{d}x \le c\left(\fint_{B_{2\rho}(x_1)} \Phi(|Du-Dw|)^\vartheta\,\mathrm{d}x\right)^{\frac{1}{\vartheta}} + c\fint_{B_{2\rho}(x_1)} \Phi(|Du|)\,\mathrm{d}x,$$

for some $0 < \vartheta < 1$. Hence,

$$\fint_{B_\rho(x_1)} \Phi(|Dw|)\,dx \le c\left(\fint_{B_{2\rho}(x_1)} \Phi(|Dw|)^\vartheta\,dx\right)^{\frac{1}{\vartheta}} + c\fint_{B_{2\rho}(x_1)} \Phi(|Du|)\,dx.$$

Since, thanks to Theorem 2.7, we have that there exists δ_0 such that $\Phi(|Du|)^{1+\delta} \in L^1_{\mathrm{loc}}(\Omega)$ for every $\delta < \delta_0$, by virtue of reverse Hölder inequality with increasing supports due to Giaquinta–Modica [24] , we have that $\Phi(|Dw|)^{1+\gamma} \in L^1(B_\rho(x_1))$ for every $\gamma < \delta < \delta_0$ and then $\Phi(|Dv|)^{1+\gamma} \in L^1(B_\rho(x_1) \cap B_R)$, for every $\gamma < \delta_0$. The conclusion follows by a simple covering argument. □

Next theorem has been proven in [13] for a local minimizer of the functional $\int \Phi(|Du|)dx$. Since $\|\cdot\|_{A_0}$ gives a norm which is equivalent to the standard norm on $\mathbb{R}^{nN}$, by suitable modifications of constants depending on the largest and the smallest eigenvalues of the matrix $A_0 = (A^{\alpha\beta}_{ij}(x_0, (u)_R))$, we can see that it holds for $\mathcal{F}_0(u) = \int \Phi(\|Du\|_{A_0})dx$.

Theorem 2.9 *Let v be a local minimizer of the functional $\mathcal{F}_0(w) = \int \Phi(\|Dw\|_{A_0})dx$. Then there exist an exponent $\sigma \in (0, 1)$ and a positive constant C such that*

$$\fint_{B_\rho} \left|V_\Phi(Dv) - \left(V_\Phi(Dv)\right)_\rho\right|^2 dx \le C\left(\frac{\rho}{r}\right)^\sigma \fint_{B_r} \left|V_\Phi(Dv) - \left(V_\Phi(Dv)\right)_r\right|^2 dx$$

and

$$\sup_{B_\rho} \Phi(|Dv|) \le C\fint_{B_r} \Phi(|Dv|)\,dx,$$

for every balls $B_\rho \subset B_r \subseteq B_R$.

3 The proof of Theorem 1.1

This section is devoted to the proof of the partial Hölder continuity of the local minimizers of the functional $\mathcal{F}(u, \Omega)$ stated in Theorem 1.1

Proof of Theorem 1.1 Let v be a minimizer of $\mathcal{F}_0$ in the class $u + W^{1,\Phi}_0(B_R)$. By using the minimality of u, since v is an admissible test function, we estimate $\mathcal{F}_0(u) - \mathcal{F}_0(v)$ as follows.

$$\begin{aligned}\mathcal{F}_0(u) - \mathcal{F}_0(v) =&\mathcal{F}_0(u) - \mathcal{F}(u) + \mathcal{F}(u) - \mathcal{F}(v) + \mathcal{F}(v) - \mathcal{F}_0(v)\\ \le&\mathcal{F}_0(u) - \mathcal{F}(u) + \mathcal{F}(v) - \mathcal{F}_0(v),\end{aligned} \tag{3.1}$$

By the definition of $\mathcal{F}_0$ and by virtue of assumption (1.3), we get

$$\begin{aligned}\mathcal{F}(v) - \mathcal{F}_0(v) &= \int_{B_R} \left(\Phi(\|Dv\|_A) - \Phi(\|Dv\|_{A_0})\right)dx\\ &= \int_{B_R} \Big(\int_0^1 \frac{d}{dt}\Phi(t\|Dv\|_A - (1-t)\|Dv\|_{A_0})dt\Big)dx\\ &= \int_{B_R} \Big(\int_0^1 \Phi'(t\|Dv\|_A - (1-t)\|Dv\|_{A_0})\left(\|Dv\|_A - \|Dv\|_{A_0}\right)dt\Big)dx\end{aligned}$$

$$\leq C\int_{B_R}\Phi'(|Dv|)\omega^{1/2}(|x-x_0|+|v-(u)_R|)|Dv|\mathrm{d}x$$
$$\leq C\int_{B_R}\Phi'(|Dv|)\omega^{1/2}(|x-x_0|+|v-u|+|u-(u)_R|)\mathrm{d}x \tag{3.2}$$

where we used the monotonicity of Φ', the second assumption in (1.2), the Δ_2-condition and the first equivalence in (1.5). By virtue of Theorem 2.8, we have that there exists $\delta_0 > 0$ such that $\Phi(|Dv|)^{1+\delta} \in L^1(B_R)$ for every $\delta < \delta_0$ and so, the Hölder's inequality with exponents $1+\delta$ and $\frac{1+\delta}{\delta}$ implies

$$\int_{B_R}\Phi(|Dv|)\omega^{1/2}(|x-x_0|+|v-u|+|u-(u)_R|)\mathrm{d}x$$
$$\leq R^n\left(\fint_{B_R}\omega^{(1+\delta)/2\delta}\mathrm{d}x\right)^{\delta/(1+\delta)}\left(\fint_{B_R}\left(\Phi(|Dv|)^{1+\delta}\mathrm{d}x\right)^{1/1+\delta}\right. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\mathcal{F}(v)-\mathcal{F}_0(v)\leq R^n\left(\fint_{B_R}\omega^{(1+\delta)/2\delta}\mathrm{d}x\right)^{\delta/(1+\delta)}\left(\fint_{B_R}\left(\Phi(|Dv|)^{1+\delta}\mathrm{d}x\right)^{1/1+\delta}\right.. \tag{3.4}$$

Using the estimate of Theorem 2.8 in the right-hand side of previous inequality, then the estimate of Theorem 2.7 to bound the term involving the gradient of u and the minimality of v, we obtain

$$\begin{aligned}
&\mathcal{F}(v)-\mathcal{F}_0(v)\\
&\leq CR^n\left(\fint_{B_R}\omega^{(1+\delta)/2\delta}\mathrm{d}x\right)^{\delta/(1+\delta)}\left[\left(\fint_{B_R}(\Phi(|Du|)^{(1+\delta)}\mathrm{d}x\right)^{1/(1+\delta)}+\fint_{B_R}\Phi(|Dv|)\,\mathrm{d}x\right]\\
&\leq CR^n\left(\fint_{B_R}\omega^{(1+\delta)/2\delta}\mathrm{d}x\right)^{\delta/(1+\delta)}\fint_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x\\
&\leq C\left(\fint_{B_R}\omega^{(1+\delta)/2\delta}(|x-x_0|+|u-v|+|u-(u)_R|)\mathrm{d}x\right)^{\delta/(1+\delta)}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x.
\end{aligned} \tag{3.5}$$

Similarly, we have

$$\mathcal{F}_0(u)-\mathcal{F}(u)\leq C\left(\fint_{B_R}\omega^{(1+\delta)/2\delta}(|x-x_0|+|u-(u)_R|)\mathrm{d}x\right)^{\delta/(1+\delta)}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x. \tag{3.6}$$

Inserting (3.5) and (3.6) in (3.1), we get

$$\begin{aligned}
\mathcal{F}_0(u)-\mathcal{F}_0(v)\leq C\Bigg(\fint_{B_R}&\omega^{(1+\delta)/2\delta}(|x-x_0|+|v-u|\\
&+|u-(u)_R|)\mathrm{d}x\Bigg)^{\delta/(1+\delta)}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x.
\end{aligned} \tag{3.7}$$

Without loss of generality, we can assume that $\delta \leq 1$ so that $(1+\delta)/2\delta \geq 1$. Now, we observe that

$$
\begin{aligned}
&\fint_{B_R} \omega^{(1+\delta)/2\delta}(|x-x_0|+|(u)_R-u|+|u-v|)\mathrm{d}x \\
&\leq C\fint_{B_R} \omega(R+|(u)_R-u|+|u-v|)\mathrm{d}x \\
&\leq C\omega\left(R+CR\left(\fint_{B_R}|Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}}+CR\left(\fint_{B_R}|Dv|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\right) \\
&\leq C\omega\left(R+CR\left(\fint_{B_R}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}+CR\left(\fint_{B_R}\Phi(|Dv|)\,\mathrm{d}x\right)^{\frac{1}{p}}\right) \\
&\leq C\omega\left(R+C\left(R^{p-n}\int_{B_R}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}\right), \qquad (3.8)
\end{aligned}
$$

where we used that ω is a bounded concave function, Jensen's inequality, the Sobolev–Poincaré inequality, the assumption (1.6) and the minimality of v. Inserting estimate (3.8) in (3.7), we get

$$
\mathcal{F}_0(u)-\mathcal{F}_0(v) \leq C\omega^{\frac{\delta}{1+\delta}}\left(R+C\left(R^{p-n}\int_{B_R}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}\right)\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x. \quad (3.9)
$$

Now, we claim that

$$
\mathcal{F}_0(u)-\mathcal{F}_0(v) \geq \int_{B_R}|Du-Dv|^2\Phi''(|Du|+|Dv|)\mathrm{d}x. \quad (3.10)
$$

Put

$$
g(\xi) := \Phi(\|\xi\|_{A_0}) \quad \text{so that} \quad \mathcal{F}_0(w) := \int_{B_R} g(Dw)\mathrm{d}x.
$$

Since v is a minimizer of $\mathcal{F}_0$ with $v|_{\partial B_R} = u|_{\partial B_R}$, then v satisfies the Euler–Lagrange equation of $\mathcal{F}_0$, namely

$$
\int_{B_R} D_{\xi^i_\alpha} g(Dv) D_\alpha \varphi^i\,\mathrm{d}x = 0, \quad \forall \varphi \in W_0^{1,\Phi}(B_R).
$$

So we have

$$
\int_{B_R} D_{\xi^i_\alpha} g(Dv)\big(D_\alpha u^i - D_\alpha v^i\big)\mathrm{d}x = 0. \quad (3.11)
$$

On the other hand, Taylor's theorem yields

$$
\begin{aligned}
\mathcal{F}_0(u)-\mathcal{F}_0(v) = &\int_{B_R} D_{\xi^i_\alpha} g(Dv)\big(D_\alpha u^i - D_\alpha v^i\big)\mathrm{d}x \\
&+ \int_{B_R}\mathrm{d}x\int_0^1 (1-t) D_{\xi^j_\beta} D_{\xi^i_\alpha} g(tDu \\
&+ (1-t)Dv)\big(D_\alpha u^i - D_\alpha v^i\big)\big(D_\beta u^j - D_\beta v^j\big)\mathrm{d}t. \qquad (3.12)
\end{aligned}
$$

Using (3.11) in (3.12), we obtain

$$\mathcal{F}_0(u) - \mathcal{F}_0(v) = \int_{B_R} \mathrm{d}x \int_0^1 (1-t) D_{\xi^j_\beta} D_{\xi^i_\alpha} g(tDu + (1-t)Dv)\big(D_\alpha u^i - D_\alpha v^i\big)\big(D_\beta u^j - D_\beta v^j\big)\mathrm{d}t. \tag{3.13}$$

Now, let us calculate DDg.

$$\begin{aligned} D_{\xi^j_\beta} D_{\xi^i_\alpha} g(\xi) =& D_{\xi^j_\beta} D_{\xi^i_\alpha} \Phi(\|\xi\|_{A_0}) = D_{\xi^j_\beta}\Big(\Phi'(\|\xi\|_{A_0}) D_{\xi^i_\alpha}(\|\xi\|_{A_0})\Big) \\ =& D_{\xi^j_\beta}\left(\frac{(A_0)^{\alpha\gamma}_{ik}\xi^k_\gamma}{\|\xi\|_{A_0}}\Phi'(\|\xi\|_{A_0})\right) \\ =& \frac{\Phi'(\|\xi\|_{A_0})}{\|\xi\|_{A_0}}\left((A_0)^{\alpha\beta}_{ij} - \frac{(A_0)^{\alpha\gamma}_{ik}\xi^k_\gamma}{\|\xi\|_{A_0}}\frac{(A_0)^{\beta\delta}_{jl}\xi^l_\delta}{\|\xi\|_{A_0}}\right) \\ &+ \Phi''(\|\xi\|_{A_0})\frac{(A_0)^{\alpha\gamma}_{ik}\xi^k_\gamma}{\|\xi\|_{A_0}}\frac{(A_0)^{\beta\delta}_{jl}\xi^l_\delta}{\|\xi\|_{A_0}}, \end{aligned}$$

where we used the notation $(A_0)^{\alpha\gamma}_{ik} = A^{\alpha\gamma}_{ik}(x_0, (u)_R)$. So, we have

$$\begin{aligned} D_{\xi^j_\beta} D_{\xi^i_\alpha} g(\xi)\eta^i_\alpha\eta^j_\beta =& \frac{\Phi'(\|\xi\|_{A_0})}{\|\xi\|_{A_0}}\left(\|\eta\|^2_{A_0} - \left\langle\frac{\xi}{\|\xi\|_{A_0}}, \eta\right\rangle^2_{A_0}\right) \\ &+ \Phi''(\|\xi\|_{A_0})\left\langle\frac{\xi}{\|\xi\|_{A_0}}, \eta\right\rangle^2_{A_0} =: (*), \end{aligned} \tag{3.14}$$

where we used the notation $\langle\xi, \eta\rangle_{A_0} = (A_0)^{\alpha\beta}_{ij}\xi^i_\alpha\eta^j_\beta$.

We can estimate the above quantity as follows:

Case (i) $\left\langle\frac{\xi}{\|\xi\|_{A_0}}, \eta\right\rangle^2_{A_0} \le \frac{1}{2}\|\eta\|^2_{A_0}$.

For this case, by the assumptions on Φ, we have

$$(*) \ge \frac{\Phi'(\|\xi\|_{A_0})}{\|\xi\|_{A_0}}\frac{1}{2}\|\eta\|^2_{A_0} \sim \Phi''(\|\xi\|_{A_0})\|\eta\|^2_{A_0}.$$

Case (ii) $\left\langle\frac{\xi}{\|\xi\|_{A_0}}, \eta\right\rangle^2_{A_0} \ge \frac{1}{2}\|\eta\|^2_{A_0}$.

Since the first term of $(*)$ is always nonnegative, it is nothing to see

$$(*) \ge \frac{1}{2}\Phi''(\|\xi\|_{A_0})\|\eta\|^2_{A_0}.$$

So, we always have

$$D_{\xi^j_\beta} D_{\xi^i_\alpha} g(\xi)\eta^i_\alpha\eta^j_\beta \ge c\Phi''(\|\xi\|_{A_0})\|\eta\|^2_{A_0}, \tag{3.15}$$

for some positive constant c.

From (3.13) and (3.15), we deduce that

$$\mathcal{F}_0(u) - \mathcal{F}_0(v) \ge \int_{B_R} \mathrm{d}x \int_0^1 (1-t)\Phi''(\|tDu + (1-t)Dv)\|_{A_0})\mathrm{d}t\|Du - Dv\|^2_{A_0}, \tag{3.16}$$

and so, using Lemma 2.1 in the right-hand side of the above estimate, we get (3.10).

Combining (3.9) and (3.10), we have

$$
\begin{aligned}
\int_{B_R} |Du - Dv|^2 \Phi''(|Du| + |Dv|) \mathrm{d}x &\le \mathcal{F}_0(u) - \mathcal{F}_0(v) \\
&\le C\tilde{\omega}\Big(R + C\left(R^{p-n}\int_{B_R}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}\Big)\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x
\end{aligned}
\tag{3.17}
$$

where, in the last line, we put $\tilde{\omega} = \omega^{\frac{\delta}{1+\delta}}$. On the other hand, recalling the properties of the function V_Φ of Lemma 2.2, for every $\rho \le R$, we have

$$
\begin{aligned}
\int_{B_\rho} \Phi(|Du|)\,\mathrm{d}x &\le C\int_{B_\rho}|V_\Phi(|Du|)|^2\,\mathrm{d}x \\
&\le C\int_{B_\rho}|V_\Phi(|Du|) - V_\Phi(|Dv|)|^2\,\mathrm{d}x + C\int_{B_\rho}|V_\Phi(|Dv|)|^2\,\mathrm{d}x \\
&\le C\int_{B_\rho}|Du - Dv|^2\Phi''(|Du| + |Dv|)\,\mathrm{d}x + C\int_{B_\rho}\Phi(|Dv|)\,\mathrm{d}x \\
&\le C\tilde{\omega}\left(R + C\left(R^{p-n}\int_{B_R}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}\right)\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x \\
&\quad + C\int_{B_\rho}\Phi(|Dv|)\,\mathrm{d}x,
\end{aligned}
$$

where we used estimate (3.17). Using the second inequality in Theorem 2.9 to estimate the last term in the previous inequality, we get

$$
\begin{aligned}
\int_{B_\rho}\Phi(|Du|)\,\mathrm{d}x &\le C\tilde{\omega}\left(2R + C\left((2R)^{p-n}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}\right)\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x \\
&\quad + C\rho^n\sup_{B_\rho}\Phi(|Dv|) \\
&\le C\tilde{\omega}\left(2R + C\left((2R)^{p-n}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}\right)\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x \\
&\quad + C\Big(\frac{\rho}{R}\Big)^n\int_{B_R}\Phi(|Dv|)\,\mathrm{d}x \\
&\le C\Bigg[\Big(\frac{\rho}{2R}\Big)^n + \tilde{\omega}\,(2R \\
&\quad + C\left((2R)^{p-n}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}\Bigg)\Bigg]\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x,
\end{aligned}
$$

where, in the last line, we used again the minimality of v. Putting $r = 2R$, we have for any $0 < \rho < r < \operatorname{dist}(x_0, \partial\Omega)$

$$
\int_{B_\rho}\Phi(|Du|)\,\mathrm{d}x \le C_0\left[\Big(\frac{\rho}{r}\Big)^n + \tilde{\omega}\left(r + C_1\left(r^{p-n}\int_{B_r}\Phi(|Du|)\,\mathrm{d}x\right)^{\frac{1}{p}}\right)\right]\int_{B_r}\Phi(|Du|)\,\mathrm{d}x,
\tag{3.18}
$$

for some positive constants C_0 and C_1.

Now, by a standard iteration argument we get the partial regularity of u as follows.

In (3.18), let $\rho = \tau r$ for some $\tau \in (0, 1)$ which will be determined later. Then we have

$$\int_{B_{\tau r}} \Phi(|Du|)\,\mathrm{d}x \le C_0 \left[\tau^n + \tilde{\omega}\left(r + C_1 \left(r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x \right)^{\frac{1}{p}} \right) \right] \int_{B_r} \Phi(|Du|)\,\mathrm{d}x$$

Dividing both sides of the above estimate by $(\tau r)^{n-p}$, we get

$$\begin{aligned}
&(\tau r)^{p-n} \int_{B_{\tau r}} \Phi(|Du|)\,\mathrm{d}x \\
&\quad \le C_0 \left[\tau^p + \tau^{p-n} \tilde{\omega}\left(r + C_1 \left(r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x \right)^{\frac{1}{p}} \right) \right] r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x \\
&\quad \le C_0 \tau^p \left[1 + \tau^{-n} \tilde{\omega}\left(r + C_1 \left(r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x \right)^{\frac{1}{p}} \right) \right] r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x
\end{aligned}$$

For any $\alpha \in (0, 1)$, choosing $\tau \in (0, 1)$ so that $C_0 \tau^{p-p\alpha} \le 1/2$, we obtain

$$\begin{aligned}
&(\tau r)^{p-n} \int_{B_{\tau r}} \Phi(|Du|)\,\mathrm{d}x \\
&\quad \le \frac{1}{2} \tau^{p\alpha} \left[1 + \tau^{-n} \tilde{\omega}\left(r + C_1 \left(r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x \right)^{\frac{1}{p}} \right) \right] r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x.
\end{aligned} \tag{3.19}$$

For such τ, there exist positive constants ε_0 and r_0 such that

$$\tau^{-n} \tilde{\omega}(r_0 + C_1 \varepsilon_0^{1/p}) \le 1,$$

and assume that for some $r \in (0, r_0)$

$$r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x \le \varepsilon_0 \tag{3.20}$$

holds. Then, for such r we have

$$(\tau r)^{p-n} \int_{B_{\tau r}} \Phi(|Du|)\,\mathrm{d}x \le \frac{1}{2} \tau^{p\alpha} [1 + 1] r^{p-n} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x \le \varepsilon_0. \tag{3.21}$$

From (3.21) we have

$$\int_{B_{\tau r}} \Phi(|Du|)\,\mathrm{d}x \le \tau^{n-p+p\alpha} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x.$$

On the other hand, (3.21) allows us to repeat the above procedure for $\tau r, \tau^2 r, \tau^3 r, \ldots$ and so we obtain, for any $k \in \mathbb{N}$,

$$\int_{B_{\tau^k r}} \Phi(|Du|)\,\mathrm{d}x \le (\tau^k)^{n-p+p\alpha} \int_{B_r} \Phi(|Du|)\,\mathrm{d}x.$$

Thus, we have

$$\rho^{-n+p-p\alpha} \int_{B_\rho} \Phi(|Du|)\,\mathrm{d}x \le C \tag{3.22}$$

and so, for every $\rho < r$,

$$\rho^{-n+p-p\alpha} \int_{B_\rho} |Du|^p \, dx \le C \tag{3.23}$$

under assumption (3.20).

Now, if we set

$$\Omega_0 := \{x \in \Omega; \quad \liminf_{r \to 0} r^{p-n} \int_{B_r} \Phi(|Du|) \, dx = 0\},$$

we deduce that $u \in C^{0,\alpha}(\Omega_0)$ by virtue of Morrey's theorem (see for example [23, Theorem 5.7]). By the continuity of the integral, we have that Ω_0 is an open set and, by well-known property of Hausdorff measure that $\dim_{\mathcal{H}}(\Omega \backslash \Omega_0) < n - p$. □

4 Proof of Theorem 1.2

In this section, we shall prove that $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega_0)$ under an Hölder continuity assumption on ω. Indeed we are ready to give the

Proof of Theorem 1.2 In the following, we assume Hölder continuity of ω, i.e., ω satisfies

$$\omega(t) \le C t^\theta \tag{4.1}$$

for some $\theta \in (0, 1)$. As before, we shall denote by u a local minimizer of $\mathcal{F}(u, \Omega)$ and by v the unique minimizer of $\mathcal{F}_0(v, B_R)$ such that $v = u$ on ∂B_R.
Let $B_{2R} \subset \Omega_0$ and $0 < \rho < \frac{R}{2}$, and observe that

$$\begin{aligned}
\int_{B_\rho} \left|V_\Phi(Du) - \left(V_\Phi(Du)\right)_\rho\right|^2 dx &\le C \int_{B_\rho} \left|V_\Phi(Du) - V_\Phi(Dv)\right|^2 dx \\
&\quad + C \int_{B_\rho} \left|V_\Phi(Dv) - \left(V(Dv)\right)_\rho\right|^2 dx \\
&\le C \int_{B_\rho} |V_\Phi(Du) - V_\Phi(Dv)|^2 dx \\
&\quad + C\rho^n \left(\frac{\rho}{R}\right)^\sigma \fint_{B_R} \left|V_\Phi(Dv) - \left(V_\Phi(Dv)\right)_R\right|^2 dx \\
&\le C \int_{B_\rho} |V_\Phi(Du) - V_\Phi(Dv)|^2 dx + \\
&\quad C\rho^n \left(\frac{\rho}{R}\right)^\sigma \fint_{B_R} \Phi(|Dv|) \, dx \\
&\le C \int_{B_\rho} |V_\Phi(Du) - V_\Phi(Dv)|^2 dx \\
&\quad + C\rho^n \left(\frac{\rho}{R}\right)^\sigma \fint_{B_R} \Phi(|Du|) \, dx,
\end{aligned} \tag{4.2}$$

where we used the decay estimate for $V_\Phi(Dv)$ given by the first estimate of Theorem 2.9, assumption (1.6) and the minimality of v.

From (3.17), (3.22) and assumption (4.1) we see also that for any $\alpha \in (0,1)$

$$\int_{B_R} |Du-Dv|^2\Phi''(|Du|+|Dv|)\mathrm{d}x \le C(R+R^\alpha)^{\theta\delta/(\delta+1)}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x \\ \le CR^{\alpha\theta\delta/(\delta+1)}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x, \tag{4.3}$$

for every $0<\delta<\delta_0$, where δ_0 is defined in Theorem 2.7
By virtue of Lemma 2.2 and the estimate in (4.3),
we have that

$$\int_{B_\rho} |V_\Phi(Du)-V_\Phi(Dv)|^2\,\mathrm{d}x \le c\int_{B_\rho}|Du-Dv|^2\Phi''(|Du|+|Dv|)\,\mathrm{d}x \\ \le CR^{\alpha\theta\delta/(\delta+1)}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x.$$

Therefore, inserting previous estimate in (4.2), we obtain

$$\int_{B_\rho}\left|V_\Phi(Du)-\left(V_\Phi(Du)\right)_\rho\right|^2\mathrm{d}x \le CR^{\alpha\theta\delta/(\delta+1)}\int_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x \\ +\rho^n\left(\frac{\rho}{R}\right)^\sigma ⨍_{B_R}\Phi(|Du|)\,\mathrm{d}x \\ \le C\left(R^{\alpha\theta\delta/(\delta+1)+n}+\rho^n\left(\frac{\rho}{R}\right)^\sigma\right)⨍_{B_{2R}}\Phi(|Du|)\,\mathrm{d}x \\ \le C\left(R^{\alpha\theta\delta/(\delta+1)+n}+\rho^n\left(\frac{\rho}{R}\right)^\sigma\right)R^{p(\alpha-1)}, \tag{4.4}$$

where we used again the decay estimate at (3.22). Since previous estimate holds true for every $\rho<\frac{R}{2}$, we may choose $\rho=\frac{R^{\gamma+1}}{2}$ with $\gamma=\frac{\alpha\theta\delta}{(\delta+1)(n+\sigma)}$ to obtain

$$\int_{B_\rho}\left|V_\Phi(Du)-\left(V_\Phi(Du)\right)_\rho\right|^2\mathrm{d}x \le C\left(R^{\alpha\theta\delta/(\delta+1)+n}\right)R^{p(\alpha-1)} = CR^{\alpha\theta\delta/(\delta+1)+n+p(\alpha-1)} \\ \le C\rho^{\frac{1}{\gamma+1}(\alpha\theta\delta/(\delta+1)+n+p(\alpha-1))} = C\rho^{n+\frac{1}{\gamma+1}(\delta\alpha\theta/(\delta+1)-\gamma n+p(\alpha-1))}.$$

Here, by the choice of γ, we have

$$\frac{\delta}{\delta+1}\alpha\theta-\gamma n+p(\alpha-1) = \frac{\delta}{\delta+1}\alpha\theta-\frac{\alpha\theta\delta}{(\delta+1)(n+\sigma)}n+p(\alpha-1) \\ =\frac{\delta}{\delta+1}\alpha\theta\left(1-\frac{n}{n+\sigma}\right)+p(\alpha-1),$$

that is positive for $\alpha\in(0,1)$ sufficiently close to 1. Therefore, we can conclude that

$$\int_{B_\rho}\left|V_\Phi(Du)-\left(V(Du)\right)_\rho\right|^2\mathrm{d}x \le C\rho^{n+\nu}$$

for some $\nu>0$. So, the Hölder continuity of $V_\Phi(Du)$ follows by the Campanato's theorem (see [28, Theorem 2.9]). On the other hand, $V_\Phi:\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ is invertible and, as shown in [12, Lemma 2.10], V_Φ^{-1} is of class $C^{0,\gamma}$ for some $\gamma\in(0,1)$ depending only on the properties of Φ. Thus, we see that $u\in C^{1,\zeta}(\Omega_0)$, where $\zeta=\gamma\nu\in(0,1)$. □

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