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Multiple standing waves for the nonlinear Helmholtz equation concentrating in the high frequency limit

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Abstract This paper studies for large frequency number $k > 0$ the existence and multiplicity of solutions of the semilinear problem

$$-\Delta u - k^2 u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad N \geq 2.$$

The exponent p is subcritical, and the coefficient Q is continuous, nonnegative and satisfies the condition

$$\limsup_{|x|\to\infty} Q(x) < \sup_{x\in\mathbb{R}^N} Q(x).$$

In the limit $k \to \infty$, sequences of solutions associated with ground states of a dual equation are shown to concentrate, after rescaling, at global maximum points of the function Q.

Keywords Nonlinear Helmholtz equation · Concentration of solutions · Dual variational method · Lusternik–Schnirelmann category

Mathematics Subject Classification Primary 35J20; Secondary 35J05

1 Introduction and main results

The existence of solutions of semilinear elliptic PDEs on $\mathbb{R}^N$, concentrating at single points or on higher-dimensional sets, has a long history. In their pioneering papers, Floer and Weinstein [18] and Rabinowitz [20] studied this question for positive solutions of the nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1}$$

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in the case where $Q \equiv 1$ and assuming $\inf V > 0$. Under the global condition

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x), \tag{2}$$

it was proved in [20] that a ground state (i.e., positive least-energy solution) of (1) exists for small $\varepsilon > 0$. In the limit $\varepsilon \to 0$, Wang [24] showed that sequences of ground states concentrate at a global minimum point x_0 of V and converge, after rescaling, toward the ground state of the limit problem

$$-\Delta u + V(x_0)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{3}$$

Extensions of these results were obtained by many authors, and the interested reader may consult the monograph by Ambrosetti and Malchiodi [2] for a precise list of references. Among the recent papers on this topic, let us point out the work of Byeon, Jeanjean and Tanaka [8,9] where the right-hand side is replaced by a very broad class of autonomous nonlinearities, and the paper by Bonheure and Van Schaftingen [6] in which V is allowed to vanish at infinity and Q may have singularities.

In the present paper, we focus on the nonlinear Helmholtz equation

$$-\Delta u - k^2 u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{4}$$

where $Q \geq 0$ is a bounded function. Our aim is to investigate the existence of real-valued solutions for $k > 0$ large, as well as their behavior as $k \to \infty$. Setting $\varepsilon = k^{-1}$ and $w = \varepsilon^{\frac{2}{p-2}}u$, we find that w solves the problem (1) with $V \equiv -1$, and it is therefore natural to ask, whether the concentration results mentioned above can also be obtained for this equation. But when trying to adapt the previous methods to the present case, several obstacles appear. First, the structure of the limit problem

$$-\Delta u - u = Q(x_0)|u|^{p-2}u \quad \text{in } \mathbb{R}^N \tag{5}$$

is more complex than (3). In particular, all solutions of (5) change sign infinitely many times, and no uniqueness result is known. Second, there is no direct variational formulation available for the problems (4)–(5) and therefore no natural concept of ground state associated with them. Nevertheless, we will show that variational arguments in the spirit of [20,24] can be used to obtain existence and concentration results for solutions of the nonlinear Helmholtz equation (4).

Our method relies on the dual variational framework established in the recent paper [17] which consists in inverting the linear part and the nonlinearity. More precisely, setting $\varepsilon = k^{-1}$ and $Q_\varepsilon(x) = Q(\varepsilon x)$, we look at the integral equation

$$|v|^{p'-2}v = Q_\varepsilon^{\frac{1}{p}}\mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}}v\right), \tag{6}$$

where $p' = \frac{p}{p-1}$ and where $\mathbf{R}$ denotes the real part of the Helmholtz resolvent operator. The solutions of this equation are critical points of the so-called dual energy functional $J_\varepsilon : L^{p'}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\varepsilon(v) = \frac{1}{p'}\int_{\mathbb{R}^N} |v|^{p'}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}}v\mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}}v\right)\mathrm{d}x.$$

Furthermore, every critical point v of J_ε gives rise to a strong solution u of (4) with $k = \frac{1}{\varepsilon}$, by setting

$$u(x) = k^{\frac{2}{p-2}}\mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}}v\right)(kx), \quad x \in \mathbb{R}^N. \tag{7}$$

This correspondence allows us to define a notion of ground state for (4) as follows. If $\varepsilon = \frac{1}{k}$ and v is a nontrivial critical point for J_ε at the mountain pass level, the function u given by (7) will be called a *dual ground state* of (4).

A motivation behind this definition is given by considering (4) on a bounded domain with Dirichlet boundary condition. For this problem, Szulkin and Weth [23, Sect. 3] proved that the ground state level for the direct functional is attained by a nontrivial critical point. In the case where the linear operator $-\Delta - k^2$ is invertible, one can show that it is also a critical point of the dual energy functional at the mountain pass level.

The first main result of this paper concerns the existence and concentration, up to rescaling, of sequences of dual ground states.

Theorem 1.1 *Let $N \geq 2$, $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2}$ (resp. $6 < p < \infty$ if $N = 2$) and consider a bounded continuous function $Q \geq 0$ such that*

$$Q_\infty := \limsup_{|x|\to\infty} Q(x) < Q_0 := \sup_{x\in\mathbb{R}^N} Q(x). \tag{8}$$

(i) There is $k_0 > 0$ such that for all $k > k_0$ the problem (4) admits a dual ground state.

(ii) Let $(k_n)_n \subset (k_0, \infty)$ satisfy $\lim_{n\to\infty} k_n = \infty$ and consider for each n, a dual ground state u_n of

$$-\Delta u - k_n^2 u = Q(x)|u|^{p-2}u \quad in \ \mathbb{R}^N.$$

Then there is a maximum point x_0 of Q, a dual ground state u_0 of

$$-\Delta u - u = Q_0|u|^{p-2}u \quad in \ \mathbb{R}^N \tag{9}$$

and a sequence $(x_n)_n \subset \mathbb{R}^N$ such that (up to a subsequence) $\lim_{n\to\infty} x_n = x_0$ and

$$k_n^{-\frac{2}{p-2}} u_n\left(\frac{\cdot}{k_n} + x_n\right) \to u_0 \quad in \ L^p(\mathbb{R}^N), \ \ as\ n \to \infty.$$

For the Schrödinger equation (1) with $V \equiv 1$, Wang and Zeng [25] noticed that (8) plays the same role as the Rabinowitz condition (2). As a consequence of Theorem 1.1, we see that this condition also ensures the concentration, in the L^p-sense, for (1) with $V \equiv -1$. To the best of our knowledge, this is the first concentration result for semilinear problems where 0 lies in the interior of the essential spectrum of the linearization.

The proof of the above theorem is given in Sect. 3. It relies on the fact that, due to (8), the dual energy functional satisfies the Palais–Smale condition at all levels strictly below the least among all possible energy levels for the problem at infinity. In contrast to similar problems where the dual method is used (see, e.g., [1]), we have no sign information about the nonlocal term appearing in the dual energy functional, since the resolvent Helmholtz operator is not positive definite. In order to handle this term, we derive a new energy estimate (Lemma 2.4) for the nonlocal interaction between functions with disjoint support, which we believe to be of independent interest. The proof of the L^p-concentration in Part (ii) of the above theorem is given in Theorem 3.5. The main ingredients are an energy comparison with the limit problem (9) and a representation lemma for Palais–Smale sequences (Lemma 2.3) in the spirit of and Benci and Cerami [3].

The second main result in this paper is the following multiplicity result for (4) with $k > 0$ large. Here, $M = \{x \in \mathbb{R}^N : Q(x) = Q_0\}$ denotes the set of maximum points of Q, and for $\delta > 0$ we let $M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\}$. Also, for a closed subset Y of a metric space X we denote by $\text{cat}_X(Y)$ the Lusternik–Schnirelmann category of Y with respect to X, i.e., the least number of closed contractible sets in X which cover Y.

Theorem 1.2 *Let $N \geq 2$, $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2}$ (resp. $6 < p < \infty$ if $N = 2$) and consider a bounded and continuous function $Q \geq 0$ satisfying (8). For every $\delta > 0$, there exists $k(\delta) > 0$ such that (4) has at least $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions for all $k > k(\delta)$.*

In the case where $Q_\infty = 0$, Palais–Smale sequences for the dual functional are relatively compact and a mountain pass argument was used in [17] to obtain the existence of infinitely many solutions. When $Q_\infty > 0$, only Palais–Smale sequences below the least-energy level at infinity are relatively compact. This loss of compactness has to be handled in order to prove the existence of multiple solutions. Our proof uses topological arguments close to the ones developed by Cingolani and Lazzo [11] for (1) (see also [12]) and based on ideas of Benci, Cerami and Passaseo [4,5] for problems on bounded domains. The main point lies in the construction of two maps whose composition is homotopic to the inclusion $M \hookrightarrow M_\delta$. For more results concerning the multiplicity of solutions for small $\varepsilon > 0$ of the Schrödinger equation (1) with $\inf V > 0$, the interested reader may consult the recent paper by Cingolani et al. [10] and the references therein.

The paper is organized as follows. In Sect. 2, we describe the dual variational framework set up in [17] for the study of the problem (4) with fixed k and discuss the basic properties of the associated Nehari manifold. Next, we establish a representation lemma for Palais–Smale sequences of the dual energy functional in the case of constant Q. The section concludes with the proof of the Palais–Smale condition for the dual energy functional on the Nehari manifold below some limit energy level. A crucial element in the proof of this result is the decay estimate given in Lemma 2.4, for the nonlocal interaction induced by the Helmholtz resolvent operator. In Sect. 3, we start by proving that for small $\varepsilon = k^{-1} > 0$ the least-energy level for critical points of the dual energy functional is attained (Proposition 3.3). As a consequence of this, we obtain Part (i) in Theorem 1.1. In a second part, the concentration in the limit $\varepsilon = k^{-1} \to 0$ is established for sequences of ground states in the dual formulation (Proposition 3.4), and this allows us to prove Part (ii) in Theorem 1.1. The last section, Sect. 4, is devoted to the proof of Theorem 1.2.

2 The variational framework

2.1 Notation and preliminaries

Throughout the paper, we let $N \geq 2$ and consider a nonnegative function $Q \in L^\infty(\mathbb{R}^N)$, $Q \not\equiv 0$. Setting $2_* := \frac{2(N+1)}{N-1}$ and $2^* := \frac{2N}{N-2}$ if $N \geq 3$, resp. $2^* := \infty$ if $N = 2$, we fix an exponent $p \in (2_*, 2^*)$ and we let $p' = \frac{p}{p-1}$ denote its conjugate exponent. For $1 \leq q \leq \infty$, we write $\|\cdot\|_q$ instead of $\|\cdot\|_{L^q(\mathbb{R}^N)}$ for the standard norm of the Lebesgue space $L^q(\mathbb{R}^N)$. In addition, for $r > 0$ and $x \in \mathbb{R}^N$, we denote by $B_r(x)$ the open ball in $\mathbb{R}^N$ of radius r centered at x, and let $B_r = B_r(0)$.

With this notation, we consider for $k > 0$ the equation

$$-\Delta u - k^2 u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{10}$$

Setting $\varepsilon = k^{-1}$, $u_\varepsilon(x) = \varepsilon^{\frac{2}{p-2}} u(\varepsilon x)$ and $Q_\varepsilon(x) = Q(\varepsilon x)$, $x \in \mathbb{R}^N$, (10) can be rewritten as

$$-\Delta u_\varepsilon - u_\varepsilon = Q_\varepsilon(x)|u_\varepsilon|^{p-2}u_\varepsilon \quad \text{in } \mathbb{R}^N. \tag{11}$$

Consider the fundamental solution of the Helmholtz equation $-\Delta u - u = \delta_0$,

$$\Phi(x) = \frac{i}{4}(2\pi|x|)^{\frac{2-N}{2}} H^{(1)}_{\frac{N-2}{2}}(|x|), \quad x \in \mathbb{R}^N \backslash \{0\}, \tag{12}$$

where $H^{(1)}_{\nu}$ denotes the Hankel function of the first kind of order ν. As a consequence of estimates by Kenig, Ruiz and Sogge [19], the operator $\mathbf{R}$, defined on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing functions by the convolution

$$\mathbf{R}f = \mathrm{Re}(\Phi) * f, \quad f \in \mathcal{S}(\mathbb{R}^N),$$

has a continuous extension $\mathbf{R} : L^{p'}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$. Using this operator, we define the C^1-functional

$$J_\varepsilon : L^{p'}(\mathbb{R}^N) \to \mathbb{R}, \quad J_\varepsilon(v) := \frac{1}{p'}\int_{\mathbb{R}^N} |v|^{p'}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} v\right) \mathrm{d}x$$

(for more details on the construction of $\mathbf{R}$ and J_ε, see [17]). Every critical point of J_ε corresponds to a solution of (11) in the following way. A function $v \in L^{p'}(\mathbb{R}^N)$ satisfies $J'_\varepsilon(v) = 0$ if and only if it solves the integral equation

$$|v|^{p'-2}v = Q_\varepsilon^{\frac{1}{p}} \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} v\right).$$

Setting $u = \mathbf{R}(Q_\varepsilon^{\frac{1}{p}} v)$, it is equivalent to

$$u = \mathbf{R}(Q_\varepsilon |u|^{p-2}u) \tag{13}$$

and since $\mathbf{R}$ is a right inverse for the Helmholtz operator $-\Delta - 1$, it follows that u is a strong solution of (11) (see [17, Lemma 4.3 and Theorem 4.4] concerning the regularity and asymptotic behavior of u). Conversely, if u solves (13), then $v = Q_\varepsilon^{\frac{1}{p'}} |u|^{p-2}u$ is a critical point of J_ε. Notice that distinct critical points correspond to distinct solutions of (13) and therefore of (11).

Let us recall some properties of the dual functional, obtained in [15–17]. Since $p' < 2$ and since the kernel of the operator $\mathbf{R}$ is positive close to the origin, the geometry of the functional J_ε is of mountain pass type:

$$\exists\, \alpha > 0 \text{ and } \rho > 0 \text{ such that } J_\varepsilon(v) \geq \alpha > 0, \quad \forall v \in L^{p'}(\mathbb{R}^N) \text{ with } \|v\|_{p'} = \rho. \tag{14}$$

$$\exists\, v_0 \in L^{p'}(\mathbb{R}^N) \text{ such that } \|v_0\|_{p'} > \rho \text{ and } J_\varepsilon(v_0) < 0. \tag{15}$$

As a consequence, the Nehari set associated to J_ε:

$$\mathcal{N}_\varepsilon := \{v \in L^{p'}\left(\mathbb{R}^N\right) \backslash \{0\} \ : \ J'_\varepsilon(v)v = 0\},$$

is not empty. More precisely, by (15), the set

$$U_\varepsilon^+ := \left\{ v \in L^{p'}\left(\mathbb{R}^N\right) \ : \ \int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} v\right) \mathrm{d}x > 0 \right\}$$

is not empty and for each $v \in U_\varepsilon^+$ there is a unique $t_v > 0$ such that $t_v v \in \mathcal{N}_\varepsilon$ holds. It is given by

$$t_v^{2-p'} = \frac{\int_{\mathbb{R}^N} |v|^{p'}\,\mathrm{d}x}{\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} v\right) \mathrm{d}x}. \tag{16}$$

In addition, t_v is the unique maximum point of $t \mapsto J_\varepsilon(tv)$, $t \geq 0$. Using (14), we obtain in particular

$$c_\varepsilon := \inf_{\mathcal{N}_\varepsilon} J_\varepsilon = \inf_{v \in U_\varepsilon^+} J_\varepsilon(t_v v) > 0.$$

Moreover, for every $v \in \mathcal{N}_\varepsilon$ we have $c_\varepsilon \leq J_\varepsilon(v) = \left(\frac{1}{p'} - \frac{1}{2}\right) \|v\|_{p'}^{p'}$. Hence, 0 is isolated in the set $\{v \in L^{p'}(\mathbb{R}^N) \,:\, J_\varepsilon'(v)v = 0\}$ and, as a consequence, the C^1-submanifold $\mathcal{N}_\varepsilon$ of $L^{p'}(\mathbb{R}^N)$ is complete.

We recall that $(v_n)_n \subset L^{p'}(\mathbb{R}^N)$ is termed a Palais–Smale sequence, or a (PS)-sequence, for J_ε if $(J_\varepsilon(v_n))_n$ is bounded and $J_\varepsilon'(v_n) \to 0$ as $n \to \infty$. Also, for $d > 0$, we say that $(v_n)_n$ is a $(\mathrm{PS})_d$-sequence for J_ε if it is a (PS)-sequence and if $J_\varepsilon(v_n) \to d$ as $n \to \infty$. The following properties hold (see [16, Sect. 2]).

Lemma 2.1 *Let $(v_n)_n \subset L^{p'}(\mathbb{R}^N)$ be a Palais–Smale sequence for J_ε. Then $(v_n)_n$ is bounded and there exists $v \in L^{p'}(\mathbb{R}^N)$ such that $J_\varepsilon'(v) = 0$ and, up to a subsequence, $v_n \rightharpoonup v$ weakly in $L^{p'}(\mathbb{R}^N)$ and $J_\varepsilon(v) \leq \liminf_{n\to\infty} J_\varepsilon(v_n)$.*

Moreover, for every bounded and measurable set $B \subset \mathbb{R}^N$, $1_B v_n \to 1_B v$ strongly in $L^{p'}(\mathbb{R}^N)$.

As a consequence, we obtain the following characterization of the infimum c_ε of J_ε over the Nehari manifold $\mathcal{N}_\varepsilon$ (see [16, Sect. 4]).

Lemma 2.2 *(i) c_ε coincides with the mountain pass level, i.e.,*

$$c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\varepsilon(\gamma(t)), \quad \textit{where}$$
$$\Gamma = \left\{\gamma \in C([0,1], L^{p'}(\mathbb{R}^N)) \,:\, \gamma(0) = 0 \textit{ and } J_\varepsilon(\gamma(1)) < 0\right\}.$$

(ii) If c_ε is attained, then $c_\varepsilon = \min\{J_\varepsilon(v) \,:\, v \in L^{p'}(\mathbb{R}^N)\backslash\{0\},\ J_\varepsilon'(v) = 0\}$.
(iii) If Q_ε is constant or $\mathbb{Z}^N$-periodic, then c_ε is attained.

In view of the preceding results, we introduce the following terminology.

If $v \in L^{p'}(\mathbb{R}^N)\backslash\{0\}$ is a critical point for J_ε at the mountain pass level, i.e., $J_\varepsilon'(v) = 0$ and $J_\varepsilon(v) = c_\varepsilon$, we call the function u given by

$$u(x) = k^{\frac{2}{p-2}} \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} v\right)(kx), \quad x \in \mathbb{R}^N, \tag{17}$$

where $k = \varepsilon^{-1}$, a *dual ground state* of (10). More generally, if v is a nontrivial critical point of J_ε, the function u obtained from v by (17) will be called a *dual bound state* of (10).

2.2 Representation lemma and Palais–Smale condition

We now take a closer look at the Palais–Smale sequences of the functional J_ε and first prove a representation lemma in the case where the coefficient Q is a positive constant. A crucial ingredient related to the nonlocal quadratic part of the energy functional is the nonvanishing theorem proved in [17, Sect. 3].

For simplicity, and since the next result is independent of ε, we drop the subscript ε.

Lemma 2.3 *Suppose $Q \equiv Q(0) > 0$ on $\mathbb{R}^N$. Consider for some $d > 0$ a (PS)$_d$-sequence $(v_n)_n \subset L^{p'}(\mathbb{R}^N)$ for J. Then there is an integer $m \geq 1$, critical points $w^{(1)}, \ldots, w^{(m)}$ of J and sequences $(x_n^{(1)})_n, \ldots, (x_n^{(m)})_n \subset \mathbb{R}^N$ such that (up to a subsequence)*

$$\begin{cases} \left\| v_n - \sum_{j=1}^{m} w^{(j)}(\cdot - x_n^{(j)}) \right\|_{p'} \to 0 \quad \text{as } n \to \infty, \\ |x_n^{(i)} - x_n^{(j)}| \to \infty \quad \text{as } n \to \infty, \text{ if } i \neq j, \\ \sum_{j=1}^{m} J(w^{(j)}) = d. \end{cases} \tag{18}$$

Proof Since $(v_n)_n$ is a (PS)$_d$-sequence for J, it is bounded and there holds

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} Q^{\frac{1}{p}} v_n \mathbf{R}\left(Q^{\frac{1}{p}} v_n\right) \mathrm{d}x = \frac{2p'}{2-p'} \lim_{n\to\infty} \left[J(v_n) - \frac{1}{p'} J'(v_n)v_n\right] = \frac{2p'd}{2-p'} > 0.$$

By the nonvanishing theorem [17, Theorem 3.1], there are $R, \zeta > 0$ and a sequence $(x_n^{(1)})_n$ such that, up to a subsequence,

$$\int_{B_R(x_n^{(1)})} |v_n|^{p'} \mathrm{d}x \geq \zeta > 0 \quad \text{for all } n.$$

Replacing $(v_n)_n$ by the corresponding subsequence and setting $v_n^{(1)} = v_n(\cdot + x_n^{(1)})$, we find that $(v_n^{(1)})_n$ is also a (PS)$_d$-sequence for J, since this functional is invariant under translations. By Lemma 2.1, going to a further subsequence, we may assume $v_n^{(1)} \rightharpoonup w^{(1)}$ weakly, $1_{B_R} v_n^{(1)} \to 1_{B_R} w^{(1)}$ strongly in $L^{p'}(\mathbb{R}^N)$, and $J(w^{(1)}) \leq \lim_{n\to\infty} J(v_n^{(1)}) = d$. These last properties and the definition of $v_n^{(1)}$ imply that $w^{(1)}$ is a nontrivial critical point of J.

If $J(w^{(1)}) = d$, we obtain

$$\begin{aligned} \left(\frac{1}{p'} - \frac{1}{2}\right) \|w^{(1)}\|_{p'}^{p'} &= J(w^{(1)}) - \frac{1}{2} J'(w^{(1)})w^{(1)} \\ &= d = \lim_{n\to\infty} \left[J(v_n) - \frac{1}{2} J'(v_n)v_n\right] = \left(\frac{1}{p'} - \frac{1}{2}\right) \lim_{n\to\infty} \|v_n\|_{p'}^{p'}, \end{aligned}$$

i.e., $v_n^{(1)} \to w^{(1)}$ strongly in $L^{p'}(\mathbb{R}^N)$, and the lemma is proved.

Otherwise, $J(w^{(1)}) < d$ and we set $v_n^{(2)} = v_n^{(1)} - w^{(1)}$. The weak convergence $v_n^{(1)} \rightharpoonup w^{(1)}$ then implies

$$\begin{aligned} \int_{\mathbb{R}^N} Q^{\frac{1}{p}} v_n^{(2)} \mathbf{R}\left(Q^{\frac{1}{p}} v_n^{(2)}\right) \mathrm{d}x &= \int_{\mathbb{R}^N} Q^{\frac{1}{p}} v_n^{(1)} \mathbf{R}\left(Q^{\frac{1}{p}} v_n^{(1)}\right) \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} Q^{\frac{1}{p}} w^{(1)} \mathbf{R}\left(Q^{\frac{1}{p}} w^{(1)}\right) \mathrm{d}x + o(1), \end{aligned}$$

as $n \to \infty$. Moreover, by the Brézis-Lieb Lemma [7],

$$\int_{\mathbb{R}^N} |v_n^{(2)}|^{p'} \mathrm{d}x = \int_{\mathbb{R}^N} |v_n^{(1)}|^{p'} \mathrm{d}x - \int_{\mathbb{R}^N} |w^{(1)}|^{p'} \mathrm{d}x + o(1), \quad \text{as } n \to \infty.$$

These properties and the translation invariance of J together give

$$J(v_n^{(2)}) = J(v_n^{(1)}) - J(w^{(1)}) + o(1) = d - J(w^{(1)}) + o(1), \quad \text{as } n \to \infty.$$

Since by Lemma 2.1, $1_{B_r} v_n^{(1)} \to 1_{B_r} w^{(1)}$ strongly in $L^{p'}(\mathbb{R}^N)$ for all $r > 0$, we find

$$1_{B_r}|v_n^{(2)}|^{p'-2}v_n^{(2)} - 1_{B_r}|v_n^{(1)}|^{p'-2}v_n^{(1)} + 1_{B_r}|w^{(1)}|^{p'-2}w^{(1)} \to 0 \ \text{ in } L^p(\mathbb{R}^N), \ \text{as } n \to \infty.$$

Furthermore, since $\big||a|^{q-1}a - |b|^{q-1}b\big| \le 2^{1-q}|a-b|^q$ for all $a, b \in \mathbb{R}$ and $0 < q < 1$, it follows that

$$\int_{\mathbb{R}^N \setminus B_r} \Big||v_n^{(2)}|^{p'-2}v_n^{(2)} - |v_n^{(1)}|^{p'-2}v_n^{(1)}\Big|^p \,\mathrm{d}x \le 2^{(2-p')p} \int_{\mathbb{R}^N \setminus B_r} |w^{(1)}|^{p'} \,\mathrm{d}x \to 0,$$

as $r \to \infty$, uniformly in n. Combining these two facts, we arrive at the strong convergence

$$|v_n^{(2)}|^{p'-2}v_n^{(2)} - |v_n^{(1)}|^{p'-2}v_n^{(1)} + |w^{(1)}|^{p'-2}w^{(1)} \to 0 \quad \text{in } L^p(\mathbb{R}^N), \quad \text{as } n \to \infty,$$

and therefore,

$$J'(v_n^{(2)}) = J'(v_n^{(1)}) - J'(w^{(1)}) + o(1) = o(1), \quad \text{as } n \to \infty.$$

We conclude that $(v_n^{(2)})_n$ is a (PS)-sequence for J at level $d - J(w^{(1)}) > 0$. Thus, the nonvanishing theorem gives the existence of $R_1, \zeta_1 > 0$ and of a sequence $(y_n)_n \subset \mathbb{R}^N$ such that, going to a subsequence,

$$\int_{B_{R_1}(y_n)} |v_n^{(2)}|^{p'} \,\mathrm{d}x \ge \zeta_1 > 0 \quad \text{for all } n.$$

By Lemma 2.1, there is a critical point $w^{(2)}$ of J such that (taking a further subsequence) $v_n^{(2)}(\cdot + y_n) \rightharpoonup w^{(2)}$ weakly and $1_B v_n^{(2)}(\cdot + y_n) \to 1_B w^{(2)}$ strongly in $L^{p'}(\mathbb{R}^N)$, for all bounded and measurable sets $B \subset \mathbb{R}^N$. In particular, $w^{(2)} \neq 0$ and since $v_n^{(2)} \rightharpoonup 0$, we see that $|y_n| \to \infty$ as $n \to \infty$.

Setting $x_n^{(2)} = x_n^{(1)} + y_n$, we obtain $|x_n^{(2)} - x_n^{(1)}| \to \infty$ as $n \to \infty$, and

$$v_n - \Big(w^{(1)}(\cdot - x_n^{(1)}) + w^{(2)}(\cdot - x_n^{(2)})\Big) = v_n^{(2)}(\cdot + y_n - x_n^{(2)}) - w^{(2)}(\cdot - x_n^{(2)}) \rightharpoonup 0,$$

weakly in $L^{p'}(\mathbb{R}^N)$. In addition, the same arguments as before show that

$$J(w^{(2)}) \le \liminf_{n\to\infty} J(v_n^{(2)}) = d - J(w^{(1)})$$

with equality if and only if $v_n^{(2)}(\cdot + y_n) \to w^{(2)}$ strongly in $L^{p'}(\mathbb{R}^N)$. If the inequality is strict, we can iterate the procedure. Since for every nontrivial critical point w of J we have $J(w) \ge c = \inf_{\mathcal{N}} J > 0$, the iteration has to stop after finitely many steps, and we obtain the desired result. □

We now turn to investigate the Palais–Smale condition for J_ε and first note that if $Q(x) \to 0$ as $|x| \to \infty$, it holds at every level, i.e., every Palais–Smale sequence has a convergent subsequence (see [17, Sect. 5]). To treat the case where

$$Q_\infty := \limsup_{|x|\to\infty} Q(x) > 0, \tag{19}$$

we consider the energy functional $J_\infty : L^{p'}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\infty(v) = \frac{1}{p'}\int_{\mathbb{R}^N} |v|^{p'} \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} Q_\infty^{\frac{1}{p}} v \mathbf{R}\left(Q_\infty^{\frac{1}{p}} v\right) \mathrm{d}x, \quad v \in L^{p'}(\mathbb{R}^N).$$

The corresponding Nehari manifold

$$\mathcal{N}_\infty := \{v \in L^{p'}(\mathbb{R}^N)\backslash\{0\} \ : \ J'_\infty(v)v = 0\},$$

has the same structure as $\mathcal{N}_\varepsilon$ and, since Q_∞ is constant, Lemma 2.2 implies that $c_\infty := \inf_{\mathcal{N}_\infty} J_\infty$ is attained and coincides with the least-energy level for nontrivial critical points of J_∞. As the last result in this section shows, the Palais–Smale condition holds for J_ε on the Nehari manifold $\mathcal{N}_\varepsilon$ at every energy level strictly below c_∞. The proof is inspired by the papers of Cingolani and Lazzo [11,12]. A new feature here is the fact that the quadratic part of the functional is nonlocal, and this induces a nonzero interaction between functions with disjoint supports. In order to handle this, we first prove an estimate on this nonlocal interaction in terms of the distance between the supports of the two functions. It is based on a decomposition of the fundamental solution already introduced in [17, Sect. 3]. Having obtained the estimate, we establish the Palais–Smale condition for J_ε on $\mathcal{N}_\varepsilon$ below the level c_∞.

Lemma 2.4 *There exists a constant $C = C(N, p) > 0$ such that for any $R > 0, r \geq 1$ and $u, v \in L^{p'}(\mathbb{R}^N)$ with $\mathrm{supp}(u) \subset B_R$ and $\mathrm{supp}(v) \subset \mathbb{R}^N\backslash B_{R+r}$,*

$$\left|\int_{\mathbb{R}^N} u\mathbf{R}v \,\mathrm{d}x\right| \leq Cr^{-\lambda_p}\|u\|_{p'}\|v\|_{p'}, \quad \textit{where} \ \ \lambda_p = \frac{N-1}{2} - \frac{N+1}{p}.$$

Proof We prove the lemma for the nonlocal term $\int_{\mathbb{R}^N} v\mathcal{R}u \,\mathrm{d}x$, where $\mathcal{R}$ denotes the resolvent operator given (for Schwartz functions) by the convolution with the kernel Φ in (12) (see [17, Sect. 2] for more details). Since $\mathbf{R}$ is the real part of $\mathcal{R}$ and since u, v are real-valued, this will imply the desired result. By density, it suffices to prove the estimate for Schwartz functions. Let $M_{R+r} := \mathbb{R}^N\backslash B_{R+r}$ and let $u, v \in \mathcal{S}(\mathbb{R}^N)$ be such that $\mathrm{supp}(u) \subset B_R$ and $\mathrm{supp}(v) \subset M_{R+r}$. The symmetry of the operator $\mathcal{R}$ and Hölder's inequality gives

$$\left|\int_{\mathbb{R}^N} u\mathcal{R}v \,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^N} v\mathcal{R}u \,\mathrm{d}x\right| \leq \|v\|_{p'}\|\Phi * u\|_{L^p(M_{R+r})}, \tag{20}$$

and it remains to estimate the second factor on the right-hand side. For this, we decompose Φ as follows. Fix $\psi \in \mathcal{S}(\mathbb{R}^N)$ such that $\widehat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ is radial, $0 \leq \widehat{\psi} \leq 1$, $\widehat{\psi}(\xi) = 1$ for $||\xi| - 1| \leq \frac{1}{6}$ and $\widehat{\psi}(\xi) = 0$ for $||\xi| - 1| \geq \frac{1}{4}$. Writing $\Phi = \Phi_1 + \Phi_2$ with

$$\Phi_1 := (2\pi)^{-\frac{N}{2}}(\psi * \Phi), \qquad \Phi_2 := \Phi - \Phi_1,$$

we recall the following estimates obtained in [15,17]:

$$|\Phi_1(x)| \leq C_0(1 + |x|)^{\frac{1-N}{2}} \qquad \text{for } x \in \mathbb{R}^N \tag{21}$$

$$\text{and} \quad |\Phi_2(x)| \leq C_0|x|^{-N} \qquad \text{for } x \neq 0. \tag{22}$$

Since the support of u is contained in B_R, we find

$$\begin{aligned}\|\Phi_2 * u\|_{L^p(M_{R+r})} &\leq \left[\int_{|x|\geq R+r}\left(\int_{|y|\leq R} |\Phi_2(x-y)||u(y)|\,\mathrm{d}y\right)^p \mathrm{d}x\right]^{\frac{1}{p}} \\ &\leq \left[\int_{\mathbb{R}^N}\left(\int_{|x-y|\geq r} |\Phi_2(x-y)||u(y)|\,\mathrm{d}y\right)^p \mathrm{d}x\right]^{\frac{1}{p}} \\ &= \|(1_{M_r}|\Phi_2|) * |u|\,\|_p \leq \|1_{M_r}\Phi_2\|_{\frac{p}{2}}\|u\|_{p'}.\end{aligned}$$

Moreover, (22) gives

$$\|1_{M_r}\Phi_2\|_{\frac{p}{2}} \le C_0\left(\omega_N\int_r^\infty s^{N-1-\frac{Np}{2}}\,\mathrm{d}s\right)^{\frac{2}{p}} \le Cr^{-\frac{N(p-2)}{p}} \le Cr^{-\lambda_p},$$

since $r \ge 1$, and therefore

$$\|\Phi_2 * u\|_{L^p(M_{R+r})} \le Cr^{-\lambda_p}\|u\|_{p'}. \tag{23}$$

To prove the estimate for Φ_1, let us fix a radial function $\phi \in \mathcal{S}(\mathbb{R}^N)$ such that $\widehat{\phi} \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ is radial, $0 \le \widehat{\phi} \le 1$, $\widehat{\phi}(\xi) = 1$ for $||\xi|-1| \le \frac{1}{4}$ and $\widehat{\phi}(\xi) = 0$ for $||\xi|-1| \ge \frac{1}{2}$. Moreover, let $\tilde{u} := \phi * u \in \mathcal{S}(\mathbb{R}^N)$. We then have $\Phi_1 * u = (2\pi)^{-\frac{N}{2}}(\Phi_1 * \tilde{u})$, since $\widehat{\Phi_1}\widehat{\phi} = \widehat{\Phi_1}$ by construction. We now write

$$\Phi_1 * \tilde{u} = \left[1_{B_{\frac{r}{2}}}\Phi_1\right] * \tilde{u} + \left[1_{M_{\frac{r}{2}}}\Phi_1\right] * \tilde{u}$$

and let $g_r := [1_{B_{\frac{r}{2}}}\Phi_1] * \phi$. Since $\mathrm{supp}(u) \subset B_R$, we find as above

$$\|\left[1_{B_{\frac{r}{2}}}\Phi_1\right] * \tilde{u}\|_{L^p(M_{R+r})} = \|g_r * u\|_{L^p(M_{R+r})} \le \|(1_{M_r}|g_r|) * |u|\,\|_p \le \|1_{M_r}g_r\|_{\frac{p}{2}}\|u\|_{p'}.$$

Using (21) and the fact that $\phi \in \mathcal{S}(\mathbb{R}^N)$, we may estimate

$$\begin{aligned}
\|1_{M_r}g_r\|_{\frac{p}{2}}^{\frac{p}{2}} &\le C_0^{\frac{p}{2}}\int_{|x|\ge r}\left(\int_{|y|\le\frac{r}{2}}|\phi(x-y)|\,\mathrm{d}y\right)^{\frac{p}{2}}\mathrm{d}x\\
&\le C\int_{|x|\ge r}\left(\int_{|y|\le\frac{r}{2}}|x-y|^{-m}\,\mathrm{d}y\right)^{\frac{p}{2}}\mathrm{d}x \le C|B_{\frac{r}{2}}|^{\frac{p}{2}}\int_{|x|\ge r}\left(|x|-\frac{r}{2}\right)^{-\frac{mp}{2}}\mathrm{d}x\\
&= Cr^{\frac{(N-m)p}{2}+N}\int_{|z|\ge 1}\left(|z|-\frac{1}{2}\right)^{-\frac{mp}{2}}\mathrm{d}z = Cr^{\frac{(N-m)p}{2}+N},
\end{aligned}$$

where C is independent of r and where m may be fixed so large that $\frac{(m-N)p}{2} - N \ge \lambda_p$. As a consequence of [17, Proposition 3.3], we have moreover

$$\|\left[1_{M_{\frac{r}{2}}}\Phi_1\right] * \tilde{u}\|_{L^p(M_{R+r})} \le \|\left[1_{M_{\frac{r}{2}}}\Phi_1\right] * \tilde{u}\|_p \le Cr^{-\lambda_p}\|\tilde{u}\|_{p'} \le Cr^{-\lambda_p}\|u\|_{p'}$$

and we conclude that

$$\|\Phi_1 * u\|_{L^p(M_{R+r})} \le Cr^{-\lambda_p}\|u\|_{p'}. \tag{24}$$

Combining (20), (23) and (24) yields the claim. □

Lemma 2.5 *Let $\varepsilon > 0$ and assume $Q_\infty > 0$ and $c_\varepsilon < c_\infty$. Then J_ε satisfies the Palais–Smale condition on $\mathcal{N}_\varepsilon$ at every level below c_∞, i.e., every sequence $(v_n)_n \subset \mathcal{N}_\varepsilon$ such that $J_\varepsilon(v_n) \to d < c_\infty$ and $(J_\varepsilon|_{\mathcal{N}_\varepsilon})'(v_n) \to 0$ as $n \to \infty$ has a convergent subsequence.*

Proof First note that by assumption, $\{v \in \mathcal{N}_\varepsilon : J_\varepsilon(v) < c_\infty\}$ is not empty. If $d < c_\varepsilon$, there is nothing to prove. Let therefore $c_\varepsilon \le d < c_\infty$ and consider a $(\mathrm{PS})_d$-sequence $(v_n)_n$ for $J_\varepsilon|_{\mathcal{N}_\varepsilon}$. Since $\mathcal{N}_\varepsilon$ is a natural constraint and a C^1-manifold, we find that $(v_n)_n$ is a $(\mathrm{PS})_d$-sequence for the unconstrained functional J_ε. Using Lemma 2.1, we obtain that (up to a subsequence) $v_n \rightharpoonup v$ and $1_{B_R}v_n \to 1_{B_R}v$ in $L^{p'}(\mathbb{R}^N)$ for all $R > 0$, where $v \in L^{p'}(\mathbb{R}^N)$ is a critical point of J_ε with $J_\varepsilon(v) \le d$. In order to conclude that $v_n \to v$ strongly in $L^{p'}(\mathbb{R}^N)$, it suffices to show that

$$\forall\,\zeta > 0, \quad \exists\, R > 0 \quad \text{such that} \quad \int_{|x|>R}|v_n|^{p'}\,\mathrm{d}x < \zeta, \quad \forall\, n. \tag{25}$$

As a first step, we claim that this holds true in annular regions, in the following sense:

$$\forall\, \eta > 0 \text{ and } \forall\, R > 0, \quad \exists\, r > R \quad \text{such that} \quad \liminf_{n\to\infty} \int_{r<|x|<2r} |v_n|^{p'}\,\mathrm{d}x < \eta. \tag{26}$$

Suppose not, then we find $\eta_0, R_0 > 0$ with the property that for every $m > R_0$ there is $n_0 = n_0(m)$ such that $\int_{m<|x|<2m} |v_n|^{p'}\,\mathrm{d}x \ge \eta_0$ for all $n \ge n_0$. Without loss of generality, we assume that $n_0(m+1) \ge n_0(m)$ for all m. Hence, for every $\ell \in \mathbb{N}$ there is $N_0 = N_0(\ell)$ such that

$$\int_{\mathbb{R}^N} |v_n|^{p'}\,\mathrm{d}x \ge \sum_{k=0}^{\ell-1} \int_{2^k([R_0]+1)<|x|<2^{k+1}([R_0]+1)} |v_n|^{p'}\,\mathrm{d}x \ge \ell\eta_0, \quad \forall\, n \ge N_0.$$

Letting $\ell \to \infty$, we obtain a contradiction to the fact that $(v_n)_n$ is bounded and this gives (26).

We now prove (25) by contradiction. Assuming that it does not hold, we find $\zeta_0 > 0$ and a subsequence $(v_{n_k})_k$ such that

$$\int_{|x|>k} |v_{n_k}|^{p'}\,\mathrm{d}x \ge \zeta_0, \quad \forall\, k. \tag{27}$$

Fix $0 < \eta < \min\{1, (\frac{\zeta_0}{3C_1})^{p'}\}$, where $C_1 = 2C(N,p)\|Q\|_\infty^{\frac{2}{p}} \max\{1, \sup_{k\in\mathbb{N}} \|v_{n_k}\|_{p'}^2\}$, the constant $C(N,p)$ being chosen such that Lemma 2.4 holds and $\|\mathbf{R}v\|_p \le C(N,p)\|v\|_{p'}$ for all $u \in L^{p'}(\mathbb{R}^N)$. By definition of Q_∞ and since $\varepsilon > 0$ is fixed, there exists $R(\eta) > 0$ such that

$$Q_\varepsilon(x) \le Q_\infty + \eta \quad \text{for all } |x| \ge R(\eta).$$

Also, from (26), we can find $r > \max\{R(\eta), \eta^{-\frac{1}{\lambda_p}}\}$ and a subsequence, still denoted by $(v_{n_k})_k$, such that

$$\int_{r<|x|<2r} |v_{n_k}|^{p'}\,\mathrm{d}x < \eta \quad \text{for all } k.$$

Setting $w_{n_k} := 1_{\{|x|\ge 2r\}} v_{n_k}$ we can write for all k,

$$\begin{aligned}
\left| J_\varepsilon'(v_{n_k})w_{n_k} - J_\varepsilon'(w_{n_k})w_{n_k} \right| &= \left| \int_{|x|<r} Q_\varepsilon^{\frac{1}{p}} v_{n_k} \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} w_{n_k}\right)\mathrm{d}x \right.\\
&\quad \left. + \int_{r<|x|<2r} Q_\varepsilon^{\frac{1}{p}} v_{n_k} \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} w_{n_k}\right)\mathrm{d}x \right|\\
&\le C(N,p) r^{-\lambda_p} \|Q\|_\infty^{\frac{2}{p}} \|v_{n_k}\|_{p'}^2\\
&\quad + C(N,p)\|Q\|_\infty^{\frac{2}{p}} \|v_{n_k}\|_{p'} \left(\int_{r<|x|<2r} |v_{n_k}|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}\\
&\le C_1 \eta^{\frac{1}{p'}},
\end{aligned}$$

using Lemma 2.4. In addition, by (27) and the definition of w_{n_k}, there holds

$$\int_{\mathbb{R}^N} |w_{n_k}|^{p'}\,\mathrm{d}x \ge \zeta_0 \quad \text{for all } k \ge 2r.$$

Recalling our choice of η, we know that $C_1\eta^{\frac{1}{p'}} < \frac{\zeta_0}{3}$ and we find some $k_0 = k_0(r,\eta,\zeta_0) \geq 2r$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} w_{n_k} \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} w_{n_k}\right) \mathrm{d}x &= \int_{\mathbb{R}^N} |w_{n_k}|^{p'} \mathrm{d}x - J_\varepsilon'(v_{n_k})w_{n_k} \\ &\quad + [J_\varepsilon'(v_{n_k})w_{n_k} - J_\varepsilon'(w_{n_k})w_{n_k}] \\ &\geq \int_{\mathbb{R}^N} |w_{n_k}|^{p'} \mathrm{d}x - |J_\varepsilon'(v_{n_k})w_{n_k}| - C_1\eta^{\frac{1}{p'}} \\ &\geq \frac{\zeta_0}{2}, \quad \text{for all } k \geq k_0,\end{aligned} \tag{28}$$

since $J_\varepsilon'(v_{n_k})w_{n_k} \to 0$ as $k \to \infty$. We note also that, since $v_{n_k} \in \mathcal{N}_\varepsilon$, there holds

$$\int_{\mathbb{R}^N} |w_{n_k}|^{p'} \mathrm{d}x \leq \int_{\mathbb{R}^N} |v_{n_k}|^{p'} \mathrm{d}x = \left(\frac{1}{p'} - \frac{1}{2}\right)^{-1} J_\varepsilon(v_{n_k}). \tag{29}$$

For $k \geq k_0$, let now $\tilde{w}_k := \left(\frac{Q_\varepsilon}{Q_\infty}\right)^{\frac{1}{p}} w_{n_k}$ and notice that $|\tilde{w}_k| \leq \left(1 + \frac{\eta}{Q_\infty}\right)^{\frac{1}{p}} |w_{n_k}|$. In view of (28), there is $t_k^\infty > 0$ for which $t_k^\infty \tilde{w}_k \in \mathcal{N}_\infty$ and there holds

$$\begin{aligned}(t_k^\infty)^{2-p'} &\leq \frac{\left(1+\frac{\eta}{Q_\infty}\right)^{p'-1} \int_{\mathbb{R}^N} |w_{n_k}|^{p'} \mathrm{d}x}{\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} w_{n_k} \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} w_{n_k}\right) \mathrm{d}x} \\ &\leq \left(1+\frac{\eta}{Q_\infty}\right)^{p'-1} \left(1 + \frac{|J_\varepsilon'(v_{n_k})w_{n_k}| + C_1\eta^{\frac{1}{p'}}}{\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} w_{n_k} \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} w_{n_k}\right) \mathrm{d}x}\right) \\ &\leq \left(1+\frac{\eta}{Q_\infty}\right)^{p'-1} \left(1 + \frac{2|J_\varepsilon'(v_{n_k})w_{n_k}| + 2C_1\eta^{\frac{1}{p'}}}{\zeta_0}\right).\end{aligned}$$

Consequently, the above estimate and (29) together give for all $k \geq k_0$,

$$\begin{aligned}c_\infty &\leq J_\infty(t_k^\infty \tilde{w}_k) \\ &\leq \left(\frac{1}{p'} - \frac{1}{2}\right)(t_k^\infty)^{p'} \left(1+\frac{\eta}{Q_\infty}\right)^{p'-1} \int_{\mathbb{R}^N} |w_{n_k}|^{p'} \mathrm{d}x \\ &\leq \left(1+\frac{\eta}{Q_\infty}\right)^{\frac{2(p'-1)}{2-p'}} \left(1 + \frac{2|J_\varepsilon'(v_{n_k})w_{n_k}| + 2C_1\eta^{\frac{1}{p'}}}{\zeta_0}\right)^{\frac{p'}{2-p'}} J_\varepsilon(v_{n_k}).\end{aligned}$$

Letting $k \to \infty$, we find

$$c_\infty \leq \left(1+\frac{\eta}{Q_\infty}\right)^{\frac{2(p'-1)}{2-p'}} \left(1 + \frac{2C_1\eta^{\frac{1}{p'}}}{\zeta_0}\right)^{\frac{p'}{2-p'}} d,$$

and letting $\eta \to 0$ we obtain

$$c_\infty \leq d,$$

which contradicts the assumption $d < c_\infty$ and proves (25). From this, we deduce the strong convergence $v_n \to v$ in $L^{p'}(\mathbb{R}^N)$ and the assertion follows. □

Remark 2.6 Under the stronger assumption $Q_\infty = \lim_{|x|\to\infty} Q(x)$, the proof of the preceding result simplifies. Indeed, having extracted a weakly converging subsequence and a critical point v of J_ε, the sequence $w_n = v_n - v$ can be shown to be a Palais–Smale sequence for J_∞ at a level lying strictly below c_∞. The representation lemma (Lemma 2.3) can then be used to conclude that $w_n \to 0$ strongly in $L^{p'}(\mathbb{R}^N)$.

3 Existence and concentration of dual ground states

In this and the next section, we work under the following assumptions on Q.

(Q0) Q is continuous, bounded and $Q \geq 0$ on $\mathbb{R}^N$;
(Q1) $Q_\infty := \limsup_{|x|\to\infty} Q(x) < Q_0 := \sup_{x\in\mathbb{R}^N} Q(x)$.

Consider the functional

$$J_0(v) := \frac{1}{p'} \int_{\mathbb{R}^N} |v|^{p'} \,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v \mathbf{R}\left(Q_0^{\frac{1}{p}} v\right) \mathrm{d}x, \quad v \in L^{p'}(\mathbb{R}^N)$$

and the corresponding Nehari manifold

$$\mathcal{N}_0 := \{v \in L^{p'}(\mathbb{R}^N)\backslash\{0\} \;:\; J_0'(v)v = 0\},$$

associated to the limit problem

$$-\Delta u - u = Q_0|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{30}$$

Lemma 2.2 implies that the level $c_0 := \inf_{\mathcal{N}_0} J_0$ is attained and coincides with the least-energy level, i.e.,

$$c_0 = \inf\{J_0(v) \;:\; v \in L^{p'}(\mathbb{R}^N),\ v \neq 0 \text{ and } J_0'(v) = 0\}.$$

Our first goal will be to show, comparing the energy level c_ε with c_0, that for small $\varepsilon > 0$, c_ε is attained. For this, let us denote the set of maximum points of Q by

$$M := \{x \in \mathbb{R}^N \;:\; Q(x) = Q_0\}.$$

Notice that $M \neq \varnothing$, since (Q0) and (Q1) are assumed. We start by studying the projection on the Nehari manifold of truncations of translated and rescaled ground states of J_0. Take a cutoff function $\eta \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$, such that $\eta \equiv 1$ in $B_1(0)$ and $\eta \equiv 0$ in $\mathbb{R}^N\backslash B_2(0)$. For $y \in M$, $\varepsilon > 0$ we let

$$\varphi_{\varepsilon,y}(x) := \eta(\varepsilon x - y)\, w(x - \varepsilon^{-1}y), \tag{31}$$

where $w \in L^{p'}(\mathbb{R}^N)$ is some fixed least-energy critical point of J_0.

Lemma 3.1 *There is* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon^*$, $y \in M$, *a unique* $t_{\varepsilon,y} > 0$ *satisfying* $t_{\varepsilon,y}\varphi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$ *exists. Moreover,*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(t_{\varepsilon,y}\varphi_{\varepsilon,y}) = c_0, \text{ uniformly for } y \in M.$$

Proof We start by remarking that $Q(y+\varepsilon\cdot)\eta(\varepsilon\cdot)w \to Q_0 w$ in $L^{p'}(\mathbb{R}^N)$ as $\varepsilon \to 0^+$, uniformly with respect to $y \in M$, since M is compact and Q is continuous by assumption. Consequently, as $\varepsilon \to 0^+$,

$$\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} \varphi_{\varepsilon,y} \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} \varphi_{\varepsilon,y}\right) \mathrm{d}x = \int_{\mathbb{R}^N} Q^{\frac{1}{p}}(y+\varepsilon z)\eta(\varepsilon z)w(z)\mathbf{R}\left(Q^{\frac{1}{p}}(y+\varepsilon\cdot)\eta(\varepsilon\cdot)w\right)(z)\,\mathrm{d}z$$
$$\longrightarrow \int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} w \mathbf{R}\left(Q_0^{\frac{1}{p}} w\right) \mathrm{d}z = \left(\frac{1}{p'}-\frac{1}{2}\right)^{-1} c_0 > 0,$$

uniformly for $y \in M$. Therefore, $\varphi_{\varepsilon,y} \in U_\varepsilon^+$ for all $y \in M$ and $\varepsilon > 0$ small enough, which shows the first assertion with $t_{\varepsilon,y}$ given by (16). In addition, for all $y \in M$,

$$\int_{\mathbb{R}^N} |\varphi_{\varepsilon,y}|^{p'} \mathrm{d}x = \int_{\mathbb{R}^N} |\eta(\varepsilon z)w(z)|^{p'} \mathrm{d}z \to \int_{\mathbb{R}^N} |w|^{p'} \mathrm{d}z = \left(\frac{1}{p'}-\frac{1}{2}\right)^{-1} c_0, \text{ as } \varepsilon \to 0^+.$$

As a consequence, $t_{\varepsilon,y} \to 1$ as $\varepsilon \to 0^+$, uniformly for $y \in M$, and we obtain $J_\varepsilon(t_{\varepsilon,y}\varphi_{\varepsilon,y}) \to c_0$ as $\varepsilon \to 0^+$, uniformly for $y \in M$. The second assertion follows. □

Lemma 3.2 *For all $\varepsilon > 0$ there holds $c_\varepsilon \geq c_0$. Moreover, $\lim_{\varepsilon\to 0^+} c_\varepsilon = c_0$.*

Proof Consider $v_\varepsilon \in \mathcal{N}_\varepsilon$ and set $v_0 := \left(\frac{Q_\varepsilon}{Q_0}\right)^{\frac{1}{p}} v_\varepsilon$. Notice that $|v_0| \leq |v_\varepsilon|$ a.e. on $\mathbb{R}^N$. Since $v_\varepsilon \in U_\varepsilon^+$, we find

$$\int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v_0 \mathbf{R}\left(Q_0^{\frac{1}{p}} v_0\right) \mathrm{d}x = \int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v_\varepsilon \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} v_\varepsilon\right) \mathrm{d}x > 0,$$

i.e., $v_0 \in U_0^+$. Hence, with

$$t_\varepsilon^{2-p'} = \frac{\int_{\mathbb{R}^N} |v_0|^{p'} \mathrm{d}x}{\int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v_0 \mathbf{R}\left(Q_0^{\frac{1}{p}} v_0\right) \mathrm{d}x} \leq \frac{\int_{\mathbb{R}^N} |v_\varepsilon|^{p'} \mathrm{d}x}{\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v_\varepsilon \mathbf{R}\left(Q_\varepsilon^{\frac{1}{p}} v_\varepsilon\right) \mathrm{d}x} = 1,$$

it follows that $t_\varepsilon v_0 \in \mathcal{N}_0$, and we obtain

$$c_0 \leq J_0(t_\varepsilon v_0) = \left(\frac{1}{p'}-\frac{1}{2}\right) t_\varepsilon^{p'} \int_{\mathbb{R}^N} |v_0|^{p'} \mathrm{d}x \leq \left(\frac{1}{p'}-\frac{1}{2}\right) \int_{\mathbb{R}^N} |v_\varepsilon|^{p'} \mathrm{d}x = J_\varepsilon(v_\varepsilon).$$

Since $v_\varepsilon \in \mathcal{N}_\varepsilon$ was arbitrarily chosen, we conclude that $c_\varepsilon = \inf_{\mathcal{N}_\varepsilon} J_\varepsilon \geq c_0$. On the other hand, Lemma 3.1 gives for $y \in M$, $c_\varepsilon \leq J_\varepsilon(t_{\varepsilon,y}\varphi_{\varepsilon,y}) \to c_0$ as $\varepsilon \to 0^+$. Hence, $\lim_{\varepsilon\to 0^+} c_\varepsilon = c_0$ and the lemma is proven. □

Proposition 3.3 *There is $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ the least-energy level c_ε is attained.*

Proof By Lemma 3.2 and Condition (Q1), there is $\varepsilon_0 > 0$ such that $c_\varepsilon < c_\infty$ for all $0 < \varepsilon < \varepsilon_0$. For such ε, using the fact that $\mathcal{N}_\varepsilon$ is a C^1-submanifold of $L^{p'}(\mathbb{R}^N)$, we obtain from Ekeland's variational principle [14, Theorem 3.1] the existence of a Palais–Smale sequence for J_ε on $\mathcal{N}_\varepsilon$, at level c_ε, and Lemma 2.5 concludes the proof. □

Setting $k_0 = \varepsilon_0^{-1}$, the assertion (i) in Theorem 1.1 from the Introduction is a direct consequence of the above result. Our next goal is to examine the behavior of critical points of J_ε in the limit $\varepsilon \to 0^+$.

Proposition 3.4 *Let $(\varepsilon_n)_n \subset (0,\infty)$ satisfy $\varepsilon_n \to 0$ as $n \to \infty$. Consider for each n some $v_n \in \mathcal{N}_{\varepsilon_n}$ and assume that $J_{\varepsilon_n}(v_n) \to c_0$ as $n \to \infty$. Then, there is $x_0 \in M$, a critical point w_0 of J_0 at level c_0 and a sequence $(y_n)_n \subset \mathbb{R}^N$ such that (up to a subsequence)*

$$\varepsilon_n y_n \to x_0 \quad \textit{and} \quad \|v_n(\cdot + y_n) - w_0\|_{p'} \to 0 \;\; \textit{as} \;\; n \to \infty.$$

Proof For each $n \in \mathbb{N}$, set $v_{0,n} := \left(\frac{Q_{\varepsilon_n}}{Q_0}\right)^{\frac{1}{p}} v_n$. It follows that $|v_{0,n}| \leq |v_n|$ a.e. on $\mathbb{R}^N$ and that

$$\int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v_{0,n} \mathbf{R}\left(Q_0^{\frac{1}{p}} v_{0,n}\right) \mathrm{d}x = \int_{\mathbb{R}^N} Q_{\varepsilon_n}^{\frac{1}{p}} v_n \mathbf{R}\left(Q_{\varepsilon_n}^{\frac{1}{p}} v_n\right) \mathrm{d}x > 0.$$

Therefore, setting

$$t_{0,n}^{2-p'} = \frac{\int_{\mathbb{R}^N} |v_{0,n}|^{p'} \mathrm{d}x}{\int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v_{0,n} \mathbf{R}\left(Q_0^{\frac{1}{p}} v_{0,n}\right) \mathrm{d}x}$$

we find that $t_{n,0} v_{0,n} \in \mathcal{N}_0$ and $0 < t_{0,n} \leq 1$. As a consequence, we can write

$$\begin{aligned} c_0 \leq J_0(t_{0,n} v_{0,n}) &= \left(\frac{1}{p'} - \frac{1}{2}\right) t_{0,n}^2 \int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v_{0,n} \mathbf{R}\left(Q_0^{\frac{1}{p}} v_{0,n}\right) \mathrm{d}x \\ &= \left(\frac{1}{p'} - \frac{1}{2}\right) t_{0,n}^2 \int_{\mathbb{R}^N} Q_{\varepsilon_n}^{\frac{1}{p}} v_n \mathbf{R}\left(Q_{\varepsilon_n}^{\frac{1}{p}} v_n\right) \mathrm{d}x \\ &= t_{0,n}^2 J_{\varepsilon_n}(v_n) \leq J_{\varepsilon_n}(v_n) \to c_0, \quad \text{as } n \to \infty. \end{aligned}$$

In particular, we find

$$\lim_{n\to\infty} t_{0,n} = 1,$$

and $(t_{0,n} v_{0,n})_n \subset \mathcal{N}_0$ is thus a minimizing sequence for J_0 on $\mathcal{N}_0$. Using Ekeland's variational principle [14] and the fact that $\mathcal{N}_0$ is a natural constraint, we obtain the existence of a $(\mathrm{PS})_{c_0}$-sequence $(w_n)_n \subset L^{p'}(\mathbb{R}^N)$ for J_0 with the property that $\|v_{0,n} - w_n\|_{p'} \to 0$, as $n \to \infty$.

By Lemma 2.3, there exists a critical point w_0 for J_0 at level c_0 and a sequence $(y_n)_n \subset \mathbb{R}^N$ such that (up to a subsequence) $\|w_n(\cdot + y_n) - w_0\|_{p'} \to 0$, as $n \to \infty$. Therefore,

$$v_{0,n}(\cdot + y_n) \to w_0 \quad \text{strongly in } L^{p'}(\mathbb{R}^N), \;\text{as } n \to \infty.$$

We now claim that $(\varepsilon_n y_n)_n$ is bounded. Suppose by contradiction that some subsequence (which we still call $(\varepsilon_n y_n)_n$) has the property $\lim\limits_{n\to\infty} |\varepsilon_n y_n| = \infty$. We distinguish two cases.

(1) If $Q_\infty = 0$, then $Q(\varepsilon_n \cdot + \varepsilon_n y_n) \to 0$, as $n \to \infty$, holds uniformly on bounded sets of $\mathbb{R}^N$. From the definition of $v_{0,n}$, we infer that $v_{0,n}(\cdot + y_n) \rightharpoonup 0$ and therefore $w_0 = 0$, in contradiction to $J_0(w_0) = c_0 > 0$. Hence, $(\varepsilon_n y_n)_n$ is bounded in this case.

(2) If $Q_\infty > 0$ instead, Fatou's lemma and the strong convergence $v_{0,n}(\cdot + y_n) \to w_0$ together imply

$$\begin{aligned}
c_0 &= \lim_{n\to\infty} J_{\varepsilon_n}(v_n) = \lim_{n\to\infty} \left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} |v_n|^{p'}\,\mathrm{d}x \\
&= \lim_{n\to\infty} \left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} |v_n(x+y_n)|^{p'}\,\mathrm{d}x \\
&= \liminf_{n\to\infty} \left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \left(\frac{Q_0}{Q(\varepsilon_n x + \varepsilon_n y_n)}\right)^{p'-1} |v_{0,n}(x+y_n)|^{p'}\,\mathrm{d}x \\
&\geq \left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \left(\frac{Q_0}{Q_\infty}\right)^{p'-1} |w_0|^{p'}\,\mathrm{d}x \\
&= \left(\frac{Q_0}{Q_\infty}\right)^{p'-1} c_0,
\end{aligned}$$

and this contradicts (Q1). Therefore, $(\varepsilon_n y_n)_n$ is a bounded sequence, and we may assume (going to a subsequence) that $\varepsilon_n y_n \to x_0 \in \mathbb{R}^N$. Since $Q(\varepsilon_n x + \varepsilon_n y_n) \to Q(x_0)$, as $n \to \infty$, uniformly on bounded sets, the argument of Case (1) above gives $Q(x_0) > 0$ and, using the Dominated Convergence Theorem, we see that $Q(x_0) = Q_0$, since the following holds.

$$\begin{aligned}
c_0 &= \lim_{n\to\infty} J_{\varepsilon_n}(v_n) = \lim_{n\to\infty} \left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} |v_n|^{p'}\,\mathrm{d}x \\
&= \lim_{n\to\infty} \left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \left(\frac{Q_0}{Q(\varepsilon_n x + \varepsilon_n y_n)}\right)^{p'-1} |v_{0,n}(x+y_n)|^{p'}\,\mathrm{d}x \\
&= \left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \left(\frac{Q_0}{Q(x_0)}\right)^{p'-1} |w_0|^{p'}\,\mathrm{d}x \\
&= \left(\frac{Q_0}{Q(x_0)}\right)^{p'-1} c_0.
\end{aligned}$$

Going back to the original sequence we obtain

$$v_n(\cdot + y_n) = \left(\frac{Q_0}{Q(\varepsilon_n \cdot + \varepsilon_n y_n)}\right)^{\frac{1}{p}} v_{0,n}(\cdot + y_n) \to \left(\frac{Q_0}{Q(x_0)}\right)^{\frac{1}{p}} w_0 = w_0, \quad \text{as } n \to \infty,$$

strongly in $L^{p'}(\mathbb{R}^N)$, using again the Dominated Convergence Theorem. The proof is complete. □

In the next result, we prove the assertion (ii) in Theorem 1.1 from the Introduction. For the reader's convenience, let us recall its formulation.

Theorem 3.5 *Let $k_0 := \varepsilon_0^{-1} > 0$, where $\varepsilon_0 > 0$ is given by Proposition 3.3. For every sequence $(k_n)_n \subset (k_0, \infty)$ satisfying $k_n \to \infty$ as $n \to \infty$, and every sequence $(u_n)_n$ such that u_n is a dual ground state of*

$$-\Delta u - k_n u = Q(x)|u|^{p-2}u \quad \textit{in } \mathbb{R}^N,$$

there is $x_0 \in M$, *a dual ground state* u_0 *of* (30) *and a sequence* $(x_n)_n \subset \mathbb{R}^N$ *such that (up to a subsequence)* $\lim_{n\to\infty} x_n = x_0$ *and*

$$k_n^{-\frac{2}{p-2}} u_n\left(\frac{\cdot}{k_n} + x_n\right) \to u_0 \quad \text{in } L^p(\mathbb{R}^N), \quad \text{as } n \to \infty.$$

Proof For each n, the dual ground state u_n can be represented as

$$u_n(x) = k_n^{\frac{2}{p-2}} \mathbf{R}\left(Q_{\varepsilon_n}^{\frac{1}{p}} v_n\right)(k_n x), \quad x \in \mathbb{R}^N,$$

where $\varepsilon_n = k_n^{-1}$ and $v_n \in L^{p'}(\mathbb{R}^N)$ is a least-energy critical point of J_{ε_n}, i.e., $J'_{\varepsilon_n}(v_n) = 0$ and $J_{\varepsilon_n}(v_n) = c_{\varepsilon_n}$. By Lemma 3.2 and Proposition 3.4, there is $x_0 \in M$ and a sequence $(y_n)_n \subset \mathbb{R}^N$ such that, as $n \to \infty$, $x_n := \varepsilon_n y_n \to x_0$ and, going to a subsequence, $v_n(\cdot + y_n) \to w_0$ in $L^{p'}(\mathbb{R}^N)$ for some least-energy critical point w_0 of J_0. Since for $x \in \mathbb{R}^N$,

$$k_n^{-\frac{2}{p-2}} u_n\left(\frac{x}{k_n} + x_n\right) = \mathbf{R}\left(Q_{\varepsilon_n}^{\frac{1}{p}} v_n\right)(x + y_n) = \mathbf{R}\Big(Q_{\varepsilon_n}^{\frac{1}{p}}(\cdot + y_n) v_n(\cdot + y_n)\Big)(x),$$

we obtain, using the continuity of $\mathbf{R}$ and the pointwise convergence $Q_{\varepsilon_n}(x+y_n) \to Q(x_0) = Q_0$ as $n \to \infty$ for all $x \in \mathbb{R}^N$, the strong convergence

$$k_n^{-\frac{2}{p-2}} u_n\left(\frac{x}{k_n} + x_n\right) \to \mathbf{R}\left(Q_0^{\frac{1}{p}} w_0\right) \quad \text{in } L^p(\mathbb{R}^N).$$

Setting $u_0 = \mathbf{R}(Q_0^{\frac{1}{p}} w_0)$, the properties $J_0(w_0) = c_0$ and $J_0'(w_0) = 0$ imply that u_0 is a dual ground state solution of (30) and this concludes the proof. ◻

Remark 3.6 (i) The conclusion of the preceding theorem holds more generally for every sequence of dual bound states. Indeed, in view of Proposition 3.4 it is enough to have $u_n(x) = k_n^{\frac{2}{p-2}} \mathbf{R}(Q_{\varepsilon_n}^{\frac{1}{p}} v_n)(k_n x)$, where v_n is a critical point of J_{ε_n}, and to require $J_{\varepsilon_n}(v_n) \to c_0$ as $n \to \infty$.

(ii) Elliptic estimates imply that the convergence toward u_0 holds in $W^{2,q}(\mathbb{R}^N)$ for all $\frac{2N}{N-1} < q < \infty$. In particular, the convergence holds in $L^\infty(\mathbb{R}^N)$ and since $u_0 \in W^{2,p}(\mathbb{R}^N)$ we find that for every $\delta > 0$ there is $R_\delta > 0$ such that for large n,

$$k_n^{-\frac{2}{p-2}} |u_n(x)| < \delta \quad \text{for all } |x - x_n| \geq \frac{R_\delta}{k_n},$$

whereas $k_n^{-\frac{2}{p-2}} \|u_n\|_\infty \to \|u_0\|_\infty > 0$ as $n \to \infty$. In addition, if $\tilde{x}_n$ denotes any global maximum point of $|u_n|$, then $\tilde{x}_n \to x_0$ as $n \to \infty$.

4 Multiplicity of dual bound states

As before, we work under the assumptions (Q0) and (Q1) and let M denote the set of maximum points of Q. In addition, for $\delta > 0$ we consider the closed neighborhood $M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\}$ of M.

The purpose of this section is to prove the multiplicity result stated in the Introduction, relating the number of solutions of (10) and the topology of M. We recall it for the reader's convenience.

Theorem 4.1 *Suppose (Q0) and (Q1) holds. For every $\delta > 0$, there exists $k(\delta) > 0$ such that the problem* (10) *has at least* $\mathrm{cat}_{M_\delta}(M)$ *distinct dual bound states for all $k > k(\delta)$.*

To prove this result, we shall construct two maps whose composition is homotopic to the inclusion $M \hookrightarrow M_\delta$. We start by introducing some notation.

For fixed $\delta > 0$, we consider the family of rescaled barycenter type maps

$$\beta_\varepsilon \;:\; L^{p'}(\mathbb{R}^N)\backslash\{0\} \;\to\; \mathbb{R}^N, \quad \varepsilon > 0,$$

given as follows. Let $\rho > 0$ be such that $M_\delta \subset B_\rho(0)$ and define $\Xi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\Xi(x) = \begin{cases} x & \text{if } |x| < \rho \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

For $v \in L^{p'}(\mathbb{R}^N)\backslash\{0\}$, we set

$$\beta_\varepsilon(v) := \frac{1}{\|v\|_{p'}^{p'}} \int_{\mathbb{R}^N} \Xi(\varepsilon x)|v(x)|^{p'}\,\mathrm{d}x.$$

Moreover, as in the previous section, we consider for $\varepsilon > 0$ and $y \in M_\delta$ the function $\varphi_{\varepsilon,y} \in L^{p'}(\mathbb{R}^N)$ defined by (31), where $\eta \in C_c^\infty(\mathbb{R}^N)$ is a cutoff function satisfying $0 \leq \eta \leq 1$ in $\mathbb{R}^N$, $\eta \equiv 1$ in $B_1(0)$ and $\eta \equiv 0$ in $\mathbb{R}^N\backslash B_2(0)$, and where $w \in L^{p'}(\mathbb{R}^N)$ is any fixed least-energy critical point of J_0.

We note that, due to the compactness of M_δ, the following holds uniformly in $y \in M_\delta$.

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\varphi_{\varepsilon,y}) = \lim_{\varepsilon\to 0^+} \frac{\int_{\mathbb{R}^N} \Xi(y+\varepsilon z)\eta(\varepsilon z)|w(z)|^{p'}\,\mathrm{d}z}{\int_{\mathbb{R}^N} \eta(\varepsilon z)|w(z)|^{p'}\,\mathrm{d}z} = \Xi(y) = y. \tag{32}$$

Before proving the main result in this section, we need the following preparatory lemma.

Lemma 4.2 *Let $\delta > 0$ and let $\nu : (0,\infty) \to (0,\infty)$ satisfy $\lim\limits_{\varepsilon\to 0^+} \nu(\varepsilon) = 0$ and $\nu(\varepsilon) > c_\varepsilon - c_0$ for all $\varepsilon > 0$. Considering the sublevel set*

$$\Sigma_\varepsilon := \{v \in \mathcal{N}_\varepsilon \;:\; J_\varepsilon(v) \leq c_0 + \nu(\varepsilon)\},$$

we have

$$\lim_{\varepsilon\to 0^+} \sup_{v\in\Sigma_\varepsilon} \inf_{y\in M_{\frac{\delta}{2}}} |\beta_\varepsilon(v) - y| = 0.$$

Proof Notice that $\Sigma_\varepsilon \neq \varnothing$, since $c_\varepsilon < c_0 + \nu(\varepsilon)$ by assumption. Let $(\varepsilon_n)_n \subset (0,\infty)$ be any sequence such that $\varepsilon_n \to 0$ as $n \to \infty$, and choose for each n some $v_n \in \Sigma_{\varepsilon_n}$ such that

$$\inf_{y\in M_{\frac{\delta}{2}}} |\beta_{\varepsilon_n}(v_n) - y| \geq \sup_{v\in\Sigma_{\varepsilon_n}} \inf_{y\in M_{\frac{\delta}{2}}} |\beta_{\varepsilon_n}(v) - y| - \frac{1}{n}. \tag{33}$$

By Proposition 3.4, there is $x_0 \in M$, a least-energy critical point w_0 of J_0 and a sequence $(y_n)_n \subset \mathbb{R}^N$ such that, up to a subsequence, $\varepsilon_n y_n \to x_0$ and $v_n(\cdot + y_n) \to w_0$ in $L^{p'}(\mathbb{R}^N)$, as $n \to \infty$. Therefore, similar to (32) we obtain

$$\beta_{\varepsilon_n}(v_n) = \frac{\int_{\mathbb{R}^N} \Xi(\varepsilon_n x + \varepsilon_n y_n)|v_n(x+y_n)|^{p'}\,\mathrm{d}x}{\int_{\mathbb{R}^N} |v_n(x+y_n)|^{p'}\,\mathrm{d}x} \to \Xi(x_0) = x_0, \quad \text{as } n \to \infty.$$

From (33), we deduce that (up to a subsequence) $\sup_{v\in\Sigma_{\varepsilon_n}} \inf_{y\in M_{\frac{\delta}{2}}} |\beta_{\varepsilon_n}(v) - y| \to 0$ as $n \to \infty$. Since the sequence $(\varepsilon_n)_n$ was arbitrarily chosen, the conclusion follows by a contradiction argument. □

Proof of Theorem 4.1 Let $\delta > 0$. According to Lemma 3.1, Lemma 3.2 and the assumption (Q1), we can find $\bar{\varepsilon} > 0$ and a function $\nu : (0,\infty) \to (0,\infty)$ such that $\nu(\varepsilon) > c_\varepsilon - c_0$ for all $\varepsilon > 0$, $\nu(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and $J_\varepsilon(t_{\varepsilon,y}\varphi_{\varepsilon,y}) < c_0 + \nu(\varepsilon) < c_\infty$, for all $y \in M$ and all $0 < \varepsilon < \bar{\varepsilon}$. Moreover, let us assume without loss of generality that, for every $0 < \varepsilon < \bar{\varepsilon}$, the level $c_0 + \nu(\varepsilon)$ is not critical for J_ε.

Consider for $0 < \varepsilon < \bar{\varepsilon}$ the set Σ_ε given in Lemma 4.2. Then $t_{\varepsilon,y}\varphi_{\varepsilon,y} \in \Sigma_\varepsilon$ and there exists $\varepsilon_1 \le \bar{\varepsilon}$ such that for all $0 < \varepsilon < \varepsilon_1$,

$$\sup_{v\in\Sigma_\varepsilon} \inf_{y\in M_{\frac{\delta}{2}}} |\beta_\varepsilon(v) - y| < \frac{\delta}{2}. \tag{34}$$

In particular, $\beta_\varepsilon(\Sigma_\varepsilon) \subset M_\delta$ and by (32) the map $y \mapsto \beta_\varepsilon(\varphi_{\varepsilon,y}) = \beta_\varepsilon(t_{\varepsilon,y}\varphi_{\varepsilon,y})$ is homotopic to the inclusion $M \hookrightarrow M_\delta$ in M_δ. Therefore, [12, Lemma 2.2] gives $\mathrm{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \ge \mathrm{cat}_{M_\delta}(M)$ for all $0 < \varepsilon < \varepsilon_1$.

Since $\mathcal{N}_\varepsilon$ is a complete C^1-manifold and since by Lemma 2.5, J_ε satisfies the Palais–Smale condition on Σ_ε, the Lusternik–Schnirelmann theory for C^1-manifolds from [21] (see also [13,22]) ensures the existence of at least $\mathrm{cat}_{M_\delta}(M)$ distinct critical points of J_ε for all $0 < \varepsilon < \varepsilon_1$.

The transformation (17) gives for each critical point of J_ε a dual bound state of (10) with $k = \varepsilon^{-1}$ and, since distinct critical points correspond to distinct bound states, the theorem follows by setting $k(\delta) = \varepsilon_1^{-1}$. □

Remark 4.3 According to Remark 3.6(i), the solutions given by Theorem 4.1 concentrate as $k \to \infty$ in the sense of Theorem 3.5.

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