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Blow-up analysis for a class of higher-order viscoelastic inverse problem with positive initial energy and boundary feedback

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Abstract In this paper we consider a nonlinear higher-order viscoelastic inverse problem with memory in the boundary. Under some suitable conditions on the coefficients, relaxation function and initial data, we proved a blow-up result for the solution with positive initial energy.

Keywords Inverse problem · Higher-order · Blow-up · Viscoelastic · Boundary feedback

Mathematics Subject Classification 35B44 · 35G31 · 65N21 · 74D10

1 Introduction

In this paper, we are concerned with the following higher-order viscoelastic inverse problem of determining a pair of functions $\{u(x,t), f(t)\}$

$$u_{tt} + (-\Delta)^m u - \int_0^t g(t-\tau)(-\Delta)^m u(\tau)\mathrm{d}\tau - |u|^{p-2}u = f(t)\omega(x), \qquad x \in \Omega, t > 0 \tag{1.1}$$

$$\begin{cases} \frac{\partial^i u}{\partial \nu^i}(x,t) = 0, & i = 0, 1, \ldots, m-2, \; x \in \Gamma_0, t > 0 \\ \frac{\partial^m u}{\partial \nu^m} = \int_0^t g(t-\tau)\frac{\partial^m u}{\partial \nu^m}(\tau)\mathrm{d}\tau - a\frac{\partial^{m-1} u}{\partial \nu^{m-1}}, & x \in \Gamma_1, t > 0 \end{cases} \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \tag{1.3}$$

$$\int_\Omega u(x,t)\omega(x)\mathrm{d}x = 1, \qquad t > 0 \tag{1.4}$$

where Ω is a bounded domain of $R^n (n \geq 1)$ with smooth boundary $\Gamma_0 \cup \Gamma_1 = \partial\Omega$ so that the divergence theorem can be applied, ν is unit outward normal vector on $\partial\Omega$, and $\frac{\partial^i u}{\partial \nu^i}$ denotes

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the i-order normal derivation of u. Here $m \geq 1$ is a natural number, and $p > 2$, a are real positive numbers. Moreover, $g(t)$ and $\omega(x)$ are functions satisfying specific conditions to be enunciated later.

It is familiar that viscoelastic materials indicate natural damping, which is due to the special property of these substances to keep memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators.

In several mathematical models we face higher-order partial differential equations. For example it can be found in Fluid Dynamics, Mechanics, Biology, Electromagnetism, image processing, where three-dimensional problems are represented on surfaces, for instance in the case of thin geometries, modeled as membranes, plates or shells, depending on the structure of the original domain. This leads to defining surface partial differential equations which often involve high-order differential operators [3,15].

The problem of proving asymptotic stability and blow-up of solutions for the equations with boundary conditions has recently attracted a lot of attention, and various results are available (see [1,2,5,6,8,14,16,18,22] and references therein).

When $m = 2$, $g(.) \neq 0$ and $\omega(x) = 0$, the equation in (1.1) becomes the following Petrovsky equation with memory term and nonlinear source term:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\mathrm{d}s = |u|^{p-2}u, \quad (x,t) \in \Omega \times R^+. \tag{1.5}$$

Tahamtani and Shahrouzi [22] prove the existence of weak solutions of equation (1.5) with initial-boundary value conditions. Meanwhile, they show that there are solutions under some conditions on initial data which blow up in finite time with nonpositive initial energy as well as positive initial energy and give the life span estimates of solutions.

In [18], Shahrouzi studied the following fourth-order nonlinear wave equation with dissipative boundary condition

$$u_{tt} + \Delta[(a_0 + a|\Delta u|^{m-2})\Delta u] - b\Delta u_t = g(x,t,u,\Delta u) + |u|^{p-2}u, \quad x \in \Omega, \ t > 0$$

the author showed that there are solutions under some conditions on initial data which blow up in finite time with positive initial energy.

On the other hand we less know about the global behavior of solutions for inverse problems; the readers are referred to [4,7,9–13,17,19–21].

In elastography, the displacement field in the interior of tissue in response to an excitation is measured using either ultrasound or magnetic resonance imaging (MRI). A viscoelastic inverse problem is focused on solving the subsequent inverse problem of determining the spatial distribution of the viscoelastic parameters of the tissue given the knowledge of the displacement fields in its interior. Such problem is motivated by applications in biomechanical imaging, where the material modulus distributions are used to detect and/or diagnose cancerous tumors [23,26].

In the absence of relaxation function, Tahamtani and Shahrouzi [20,21] investigated global behavior of solutions to some class of inverse source problems. In [20], they studied the global in time behavior of solutions for an inverse problem of determining a pair of functions $\{u, f\}$ satisfying the equation

$$u_{tt} + \Delta^2 u - |u|^p u + a(x,t,u,\nabla u,\Delta u) = f(t)\omega(x), \quad x \in \Omega, \quad t > 0,$$

the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,$$

the boundary conditions

$$u(x,t)=\partial_\nu u(x,t)=0, \quad x\in\partial\Omega, \quad t>0,$$

and the overdetermination condition

$$\int_\Omega u(x,t)\omega(x)\mathrm{d}x=1, \quad t>0,$$

also, the asymptotic stability result has been established with the opposite sign of power-type nonlinearities.

Later, in [21], Tahamtani and Shahrouzi considered

$$\begin{aligned}
&u_{tt}+\Delta^2 u-\alpha_1\Delta u+\alpha_2 u_t+\alpha_3|u|^p u+b(x,t,u,\nabla u,\Delta u)=f(t)\omega(x),\ x\in\Omega, t>0,\\
&u(x,t)=0, \qquad \Delta u=-c_0\partial_\nu u(x,t), \qquad x\in\Gamma, t>0,\\
&u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x), \qquad x\in\Omega,\\
&\int_\Omega u(x,t)\omega(x)\mathrm{d}x=\phi(t), \qquad t>0.
\end{aligned}$$

They showed that the solutions of this problem under some suitable conditions are stable if α_1,α_2 are large enough, $\alpha_3\geq 0$ and $\phi(t)$ tends to zero as time goes to infinity and also established a blow-up result, if $\alpha_3<0$ and $\phi(t)=k$ be a constant. Their approaches are based on the Lyapunov function and perturbed energy method for stability result and concavity argument for blow-up result.

Shahrouzi [19] investigated the asymptotic behavior of solutions for the following viscoelastic inverse problem

$$\begin{aligned}
&u_{tt}+\Delta^2 u-\int_0^t g(t-\tau)\Delta^2 u(\tau)\mathrm{d}\tau-a_1\Delta u+a_2 u_t=f(t)\omega(x), \qquad x\in\Omega, t>0,\\
&\begin{cases} u(x,t)=0, & x\in\Gamma_0, t>0\\ \Delta u(x,t)=\int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau-a_3|\nabla u|^p\nabla u, & x\in\Gamma_1, t>0\end{cases}\\
&u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega,\\
&\int_\Omega u(x,t)\omega(x)\mathrm{d}x=\phi(t), \qquad t>0,
\end{aligned}$$

He obtained sufficient conditions on relaxation function and initial data for which the solutions of problem are asymptotically stable when the integral overdetermination tends to zero as time goes to infinity.

Recently, when $\omega(x)=0$ and with homogeneous Dirichlet boundary conditions, the following initial-boundary value problem was investigated in [24]:

$$\begin{aligned}
&u_{tt}+(-\Delta)^m u-\int_0^t g(t-\tau)(-\Delta)^m u(\tau)\mathrm{d}\tau=|u|^{p-2}u, \qquad x\in\Omega, t>0\\
&u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega\\
&\frac{\partial^i u}{\partial\nu^i}(x,t)=0, \qquad i=0,1,\dots,m-1,\ x\in\partial\Omega, t>0.
\end{aligned}$$

Ye proved the existence of global weak solutions by using the Galerkin method. Moreover, he showed that under some suitable conditions on relaxation function and the positive initial energy as well as nonpositive initial energy, the solution blows up in a finite time and the life span estimates of solutions are also given (see also [25]).

Motivated by the aforementioned works, we consider in this paper the blow-up of solutions for problem (1.1)–(1.4). We show that if we take initial data in the appropriate domain, then solutions of (1.1)–(1.4) blow up in a finite time. Our approaches are based on the modified concavity argument method.

2 Preliminaries and main results

In this section, we present some material needed in the proof of our main results. Throughout this paper all the functions considered are real-valued. We adopt the usual notations and convention. Let $H^m(\Omega)$ denote the Sobolev space with the usual scalar products and norm. $H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notations, hereafter we denote by $\|.\|_q$ the L^q-norm over Ω . In particular, the L^2-norm is denoted $\|.\|$ in Ω and $\|.\|_{\Gamma_i}$ in Γ_i; we write equivalent norm $\|D^m.\|$ instead of $H^m(\Omega)$ norm $\|.\|_{H^m(\Omega)}$ and $\|D^m.\|_{\Gamma_i}$ in Γ_i, where D denotes the gradient operator, that is $D. = \nabla. = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$. Moreover, $D^m. = \Delta^j.$ if $m = 2j$ and $D^m. = D\Delta^j.$ if $m = 2j + 1$.

We sometimes use the Young's inequality

$$ab \le \beta a^q + C(\beta, q) b^{q'}, \quad a, b \ge 0, \quad \beta > 0, \quad \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.1}$$

where $\theta = \theta(\Omega, n)$ and $C(\beta, q) = \frac{1}{q'}(\beta q)^{-\frac{q'}{q}}$ are constants. We recall the trace Sobolev embedding

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1) \qquad \text{for} \qquad 2 \le q < \frac{2(n-1)}{n-2}$$

where

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$$

and the embedding inequality

$$\|u\|_{q,\Gamma_1} \le B_q \|\nabla u\|_2, \tag{2.2}$$

where B_q is the optimal constant.

The following lemma was introduced in [9]; it will be used in Sect. 3 in order to prove the blow-up result.

Lemma 1 *Let $\mu > 0, c_1 > 0$. Assume that $\psi(t)$ is a twice differentiable positive function such that*

$$\psi''\psi - (1+\mu)[\psi']^2 \ge -2c_1\psi\psi', \tag{2.3}$$

for all $t \ge 0$. If

$$\psi(0) > 0 \qquad \textit{and} \qquad \psi'(0) - 2c_1\mu^{-1}\psi(0) > 0, \tag{2.4}$$

then

$$\psi(t) \to +\infty \qquad \textit{as} \qquad t \to t_1 \le t_2 = \frac{1}{2c_1} \log \frac{\mu\psi'(0)}{-2c_1\psi(0) + \mu\psi'(0)}. \tag{2.5}$$

To prove our main result, we make the following assumptions

(A1) $$\omega \in H^{2m}(\Omega) \cap H_0^{2m-1}(\Omega) \cap L^p(\Omega), \qquad \int_\Omega \omega^2(x)\mathrm{d}x = 1$$

(A2)
$$g(t) \geq 0, \quad g'(t) \leq \lambda g(t)$$

(A3)
$$1 - \int_0^\infty e^{-\lambda t} g(t)\mathrm{d}t = l > 0$$

We consider the following problem by substituting $u(x, t) = e^{\lambda t} v(x, t)$ in (1.1)–(1.4):

$$v_{tt} + (-\Delta)^m v + \lambda^2 v + 2\lambda v_t - \int_0^t e^{-\lambda(t-\tau)} g(t-\tau)\mathrm{d}\tau = e^{\lambda(p-2)t}|v|^{p-2}v$$
$$+ e^{-\lambda t} f(t)\omega(x), \qquad (x, t) \in \Omega \times R^+ \tag{2.6}$$
$$\begin{cases} \frac{\partial^i v}{\partial \nu^i}(x, t) = 0, & i = 0, 1, \ldots, m-2, \ x \in \Gamma_0, t > 0 \\ \frac{\partial^m v}{\partial \nu^m} = \int_0^t e^{-\lambda(t-\tau)} g(t-\tau) \frac{\partial^m v}{\partial \nu^m}(\tau)\mathrm{d}\tau - a\frac{\partial^{m-1} v}{\partial \nu^{m-1}}, & x \in \Gamma_1, t > 0 \end{cases} \tag{2.7}$$
$$v(x, 0) = u_0(x), \qquad v_t(x, 0) = u_1(x) + \lambda u_0(x), \qquad x \in \Omega \tag{2.8}$$
$$\int_\Omega v(x, t)\omega(x)\mathrm{d}x = e^{-\lambda t}. \qquad t \in R^+ \tag{2.9}$$

Multiplying equation (2.6) by $\omega(x)$ and using (A1) we obtain

$$f(t) = e^{\lambda t}((-\Delta)^m v, \omega) - e^{\lambda t}\int_0^t e^{-\lambda(t-\tau)} g(t-\tau)((-\Delta)^m v(\tau), \omega)\mathrm{d}\tau$$
$$- e^{\lambda(p-1)t}(|v|^{p-2}v, \omega). \tag{2.10}$$

Adapting the idea of Prilepko et.al [17] the key observation is that problem (2.6)–(2.9) is equivalent to problem (2.6)–(2.8) in which the unknown function $f(t)$ in (2.6) is replaced by (2.10) (the value of the parameter λ will be prescribed later).

Once the unknown function $f(t)$ is eliminated, the standard theory of nonlinear hyperbolic equations also becomes applicable to deduce the local existence of solutions.

The energy associated with problem (2.6)–(2.8) is given by

$$E_\lambda(t) = \frac{e^{\lambda(p-2)t}}{p}\|v\|_p^p - \frac{1}{2}\Big(\|v_t\|^2 + \lambda^2\|v\|^2 + \Big(1 - \int_0^t g_1(s)\mathrm{d}s\Big)\|D^m v\|^2$$
$$+ (g_1 * D^m v)(t) + a\|D^{m-1}v\|_{\Gamma_1}^2\Big), \tag{2.11}$$

where

$$g_1(s) = e^{-\lambda s} g(s), \qquad (g_1 * v)(t) = \int_0^t g_1(t-s)\|v(t) - v(s)\|^2\mathrm{d}s.$$

Now we are in a position to state a local existence of solutions for (2.6)–(2.8):

Theorem 1 (Local existence) *Assume that (A1)–(A3) hold. If*

$$u_0 \in H_0^{m-1}(\Omega) \cap H^{2m}(\Omega) \cap L^p(\Omega), \ u_1 \in L^2(\Omega), \ \int_\Omega u_0(x, t)\omega(x)dx = 1$$

then there exists $T > 0$ such that problem (2.6)–(2.8) has a unique local solution $u(t)$ which satisfies

$$u \in C([0, T); H_0^{m-1}(\Omega)), \qquad u_t \in C([0, T); L^2(\Omega)).$$

We conclude this section by stating the blow-up result as follows.

Theorem 2 *Under the conditions of Theorem 1, we assume*

$$\|u_0\| > 0, \qquad E_\lambda(0) \geq \frac{D_1}{4\lambda} + \frac{2D_2}{p+4}, \tag{2.12}$$

where

$$D_1 = \big(\frac{5\lambda^2}{4} + \frac{\lambda^2}{l}\big)\|D^m\omega\|^2 + \frac{\lambda}{p\left(\frac{p-4}{p-1}\right)^{p-1}}\|\omega\|_p^p, \tag{2.13}$$

$$D_2 = \big(1 + \frac{12}{lp}\big)\|D^m\omega\|^2 + \frac{1}{p\left(\frac{p-4}{2p-2}\right)^{p-1}}\|\omega\|_p^p. \tag{2.14}$$

If $\lambda \geq \max\{c, \frac{l+4(1-l)^2}{4l}\}$ *for a positive constant c, and*

$$a \leq \frac{lp}{2B_2^2(p+8)}, \quad \int_0^{+\infty} g_1(s)ds \leq \frac{p^2+4p-8c}{p(p+4)}, \quad p \geq \max\left\{4, \ \frac{\sqrt{16+32c}-4}{2}\right\},$$

then there exists a finite time $t_1 \in [0, T)$ *such that the solution of problem* (1.1)–(1.4) *blows up in a finite time, that is*

$$\|u(t)\| \to +\infty \ \ as \ \ t \to t_1. \tag{2.15}$$

3 Blow-up

In this section we are going to prove that for appropriate initial data some of the solutions blow up in a finite time. To prove the blow-up result (Theorem 2) for certain solutions with positive initial energy, we need the following lemma for problem (2.6)–(2.9).

Lemma 2 *Let the conditions of Theorem 1 are satisfied and* $\lambda \geq \max\{c, \frac{l+4(1-l)^2}{4l}\}$ *for a positive constant c. Then*

$$E_\lambda(t) \geq E_\lambda(0) - \frac{1}{4\lambda}D_1,$$

Proof A multiplication of equation (2.6) by v_t and integrating over Ω gives

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(t) \geq 2\lambda\|v_t\|^2 + \frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\|v\|_p^p + \lambda e^{-2\lambda t}f(t), \tag{3.1}$$

where condition (A2) has been used.

Plugging definition of $f(t)$, (2.10), into (3.1) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(t) &\geq 2\lambda\|v_t\|^2 + \frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\|v\|_p^p + \lambda e^{-\lambda t}(D^m v, D^m\omega) \\ &\quad - \lambda e^{-\lambda t}\int_0^t g_1(t-\tau)(D^m v(\tau), D^m\omega)\mathrm{d}\tau - \lambda e^{\lambda(p-3)t}(|v|^{p-2}v, \omega). \end{aligned}\tag{3.2}$$

And so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(t) - 2\lambda E_\lambda(t) &\geq 3\lambda\|v_t\|^2 + \frac{\lambda(p-4)}{p}e^{\lambda(p-2)t}\|v\|_p^p + \lambda^3\|v\|^2 + a\lambda\|D^{m-1}v\|_{\Gamma_1}^2 \\ &\quad + \lambda\big(1 - \int_0^t g_1(s)\mathrm{d}s\big)\|D^m v\|^2 + \lambda(g_1 * D^m v)(t) + \lambda e^{-\lambda t}(D^m v, D^m\omega)\end{aligned}$$

$$- \lambda e^{-\lambda t} \int_0^t g_1(t-\tau)(D^m v(\tau), D^m \omega) \mathrm{d}\tau - \lambda e^{\lambda(p-3)t}(|v|^{p-2}v, \omega). \tag{3.3}$$

Now, by using the Young's inequality, the terms on the right-hand side of (3.3) can be estimated as follows

$$\lambda e^{-\lambda t}|(D^m v, D^m \omega)| \leq \frac{l}{4}\|D^m v\|^2 + \frac{\lambda^2}{l} e^{-\lambda t}\|D^m \omega\|^2, \tag{3.4}$$

$$\lambda e^{\lambda(p-3)t}|(|v|^{p-2}v, \omega)| \leq \frac{\lambda(p-4)}{p} e^{\lambda(p-2)t}\|v\|_p^p + \frac{\lambda e^{-2\lambda t}}{p[\frac{p-4}{p-1}]^{p-1}}\|\omega\|_p^p, \tag{3.5}$$

where we take

$$a = e^{\frac{(p-1)(p-2)}{p}} t\|v\|_p^{p-1}, \quad b = e^{\frac{-2\lambda t}{p}}\|\omega\|_p, \quad q = \frac{p}{p-1}, \quad q' = p,$$

with $\beta = \frac{p-4}{p}$ in the Young's inequality (2.1).

Also there exists a positive constant c such that

$$\lambda e^{-\lambda t} \int_0^t g_1(t-\tau)(D^m v(\tau), D^m \omega) \mathrm{d}\tau \leq (1-l)^2\|D^m v\|^2 + c(g_1 * D^m v)(t)$$
$$+ \frac{5\lambda^2}{4} e^{-2\lambda t}\|D^m \omega\|^2, \tag{3.6}$$

Taking into account estimates (3.4)–(3.6) in relation with (3.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\lambda(t) - 2\lambda E_\lambda(t) \geq 3\lambda\|v_t\|^2 + a\lambda\|D^{m-1}v\|_{\Gamma_1}^2 + (\lambda - c)(g_1 * D^m v)(t)$$
$$+ \left(\lambda l - \frac{l}{4} - (1-l)^2\right)\|D^m v\|^2 - e^{-2\lambda t} D_1, \tag{3.7}$$

where D_1 satisfies (2.13).

At this point if we choose $\lambda \geq \max\{c, \frac{l+4(1-l)^2}{4l}\}$, then we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\lambda(t) \geq 2\lambda E_\lambda(t) - e^{-2\lambda t} D_1, \tag{3.8}$$

by integrating (3.8) between 0 and t, we observe that

$$E_\lambda(t) \geq E_\lambda(0) - \frac{1}{4\lambda} D_1, \quad \forall t \geq 0, \tag{3.9}$$

and proof of Lemma 2 is complete.

Proof of Theorem 2 To obtain the blow-up result, the choice of the following functional is standard (see [9])

$$\psi(t) = \|v(t)\|^2, \tag{3.10}$$

then

$$\psi'(t) = 2(v, v_t), \tag{3.11}$$
$$\psi''(t) = 2(v, v_{tt}) + 2\|v_t\|^2. \tag{3.12}$$

A multiplication of equation (2.6) by v and integrating over Ω gives

$$(v_{tt}, v) = -2\lambda(v_t, v) - \lambda^2\|v\|^2 - \big(1 - \int_0^t g_1(s)\mathrm{d}s\big)\|D^m v\|^2 - a\|D^{m-1}v\|_{\Gamma_1}^2 + \int_\Omega D^m v \int_0^t g_1(t-\tau)(D^m v(\tau) - D^m v)\mathrm{d}\tau\mathrm{d}x + e^{\lambda(p-2)t}\|v\|_p^p + e^{-2\lambda t} f(t). \tag{3.13}$$

By virtue of trace embedding inequality (2.2), it is easy to see that

$$(v_{tt}, v) \geq -2\lambda(v_t, v) - \lambda^2\|v\|^2 - \big(1 - \int_0^t g_1(s)\mathrm{d}s + aB_2^2\big)\|D^m v\|^2 + e^{\lambda(p-2)t}\|v\|_p^p + \int_\Omega D^m v \int_0^t g_1(t-\tau)(D^m v(\tau) - D^m v)\mathrm{d}\tau\mathrm{d}x + e^{-2\lambda t} f(t). \tag{3.14}$$

Consequently, from definition of $E_\lambda(t)$ we get

$$\begin{aligned}(v_{tt}, v) \geq{} & \left(2 + \frac{p}{2}\right) E_\lambda(t) - 2\lambda(v_t, v) + \frac{p-4}{2p} e^{\lambda(p-2)t}\|v\|_p^p - a\left(1 + \frac{p}{4}\right)\|D^{m-1}v\|_{\Gamma_1}^2 \\ & + \frac{\lambda^2 p}{4}\|v\|^2 + \left(1 + \frac{p}{4}\right)(g_1 * D^m v)(t) - \left(1 - \int_0^t g_1(s)\mathrm{d}s + aB_2^2\right)\|D^m v\|^2 \\ & + \left(1 + \frac{p}{4}\right)\big(1 - \int_0^t g_1(s)\mathrm{d}s\big)\|D^m v\|^2 + \left(1 + \frac{p}{4}\right)\|v_t\|^2 \\ & + \int_\Omega D^m v \int_0^t g_1(t-\tau)(D^m v(\tau) - D^m v)\mathrm{d}\tau\mathrm{d}x + e^{-2\lambda t} f(t).\end{aligned} \tag{3.15}$$

Taking into account definition of unknown function (2.10) and trace embedding inequality (2.3), we obtain

$$\begin{aligned}(v_{tt}, v) \geq{} & \left(2 + \frac{p}{2}\right) E_\lambda(t) - 2\lambda(v_t, v) + \frac{p-4}{2p} e^{\lambda(p-2)t}\|v\|_p^p + \left(1 + \frac{p}{4}\right)\|v_t\|^2 \\ & + \frac{\lambda^2 p}{4}\|v\|^2 + \left(1 + \frac{p}{4}\right)(g_1 * D^m v)(t) + \big(\frac{lp}{4} - aB_2^2\big(2 + \frac{p}{4}\big)\big)\|D^m v\|^2 \\ & + \int_\Omega D^m v \int_0^t g_1(t-\tau)(D^m v(\tau) - D^m v)\mathrm{d}\tau\mathrm{d}x \\ & + e^{-\lambda t}((-\Delta)^m v, \omega) - e^{-\lambda t}\int_0^t g_1(t-\tau)((-\Delta)^m v(\tau), \omega)\mathrm{d}\tau \\ & - e^{\lambda(p-3)t}(|v|^{p-2}v, \omega).\end{aligned} \tag{3.16}$$

To estimate the terms on the right-hand side of (3.16), we start with the memory term. Using Cauchy–Schwartz inequality and the Young's inequality, we get

$$\left|\int_\Omega D^m v \int_0^t g_1(t-\tau)(D^m v(\tau) - D^m v)\mathrm{d}\tau\mathrm{d}x\right| \leq \frac{lp}{24}\|D^m v\|^2 + \frac{c}{lp}(g_1 * D^m v)(t), \tag{3.17}$$

$$e^{-\lambda t}|((-\Delta)^m v, \omega)| \leq \frac{lp}{24}\|D^m v\|^2 + \frac{6e^{-2\lambda t}}{lp}\|D^m \omega\|^2, \tag{3.18}$$

$$e^{-\lambda t}|\int_0^t g_1(t-\tau)((-\Delta)^m v(\tau), \omega)\mathrm{d}\tau| \leq \frac{lp}{24}\|D^m v\|^2 + \frac{c}{lp}(g_1 * D^m v)(t)$$

$$+\left(1+\frac{6}{lp}\right)e^{-2\lambda t}\|D^m\omega\|^2, \tag{3.19}$$

$$e^{\lambda(p-3)t}|(|v|^{p-2}v,\omega)| \le \frac{p-4}{2p}e^{\lambda(p-2)t}\|v\|_p^p + \frac{e^{-2\lambda t}}{p(\frac{p-4}{2p-2})^{p-1}}\|\omega\|_p^p. \tag{3.20}$$

Substituting (3.17)–(3.20) into (3.16) gives

$$(v_{tt},v) \ge \left(2+\frac{p}{2}\right)E_\lambda(t) + \left(1+\frac{p}{4}\right)\|v_t\|^2 + \left(\frac{lp}{8} - aB_2^2\left(2+\frac{p}{4}\right)\right)\|D^m v\|^2 - 2\lambda(v_t,v)$$
$$+\left(1+\frac{p}{4}-\frac{2c}{lp}\right)(g_1 * D^m v)(t) - e^{-2\lambda t}D_2, \tag{3.21}$$

where D_2 satisfies (2.14).

At this point we choose

$$a \le \frac{lp}{2B_2^2(p+8)}, \quad \int_0^{+\infty} g_1(s)\mathrm{d}s \le \frac{p^2+4p-8c}{p(p+4)}, \quad p \ge \max\left\{4, \ \frac{\sqrt{16+32c}-4}{2}\right\}.$$

This implies that

$$(v_{tt},v) \ge \left(2+\frac{p}{2}\right)E_\lambda(t) - 2\lambda(v_t,v) + \left(1+\frac{p}{4}\right)\|v_t\|^2 - e^{-2\lambda t}D_2. \tag{3.22}$$

Hence from (3.12) we have

$$\psi''(t) \ge (p+4)E_\lambda(t) - 4\lambda(v_t,v) + 4\left(1+\frac{p}{8}\right)\|v_t\|^2 - 2D_2. \tag{3.23}$$

To this end, by substituting (3.10),(3.11) in (3.23), using Lemma 2 and (2.13) we arrive at

$$\psi''(t) \ge -2\lambda\psi'(t) + 4\left(1+\frac{p}{8}\right)\|v_t\|^2,$$

finally we get

$$\psi(t)\psi''(t) \ge \left(1+\frac{p}{8}\right)[\psi'(t)]^2 - 2\lambda\psi(t)\psi'(t).$$

Hence we see that the hypotheses of Lemma 1 are fulfilled with $\mu = \frac{p}{8}$, $c_1 = \lambda$, and the conclusion of Lemma 1 gives us that some solutions of problem (2.6)–(2.9) blow up in a finite time and since this system is equivalent to (1.1)–(1.4), the proof is complete.

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