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Intrinsic geometry and analysis of Finsler structures

Chang-Yu Guo[1,2]

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Abstract In this short note, we prove that if F is a weak upper semicontinuous admissible Finsler structure on a domain in $\mathbb{R}^n, n \geq 2$, then the intrinsic distance and differential structures coincide.

Keywords Finsler structure · Dual Finsler structure · Intrinsic distance · Lipschitz constant

Mathematics Subject Classification 58J60 · 46E99

1 Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a domain and F an admissible Finsler structure on Ω (the precise definition is given in Sect. 2 below). Associated with F, we have the following intrinsic distance defined by

$$\delta_F(x, y) = \sup_u \{u(x) - u(y) : u \text{ is Lipschitz and } \|F(x, \mathrm{d}u(x))\|_\infty \leq 1\}. \tag{1.1}$$

Above, $\mathrm{d}u(x)$ denotes the differential of the Lipschitz function u at a point x. Recall that the well-known Rademacher's theorem implies that $\mathrm{d}u(x)$ exists at almost every $x \in \Omega$, and thus the above definition makes sense. The ellipticity condition on F implies that δ_F is locally comparable to the standard Euclidean distance. We define the pointwise Lipschitz constant of a Lipschitz function $u : \Omega \to \mathbb{R}$ by setting

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✉ Chang-Yu Guo
changyu.guo@unifr.ch

1 Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, 40014 Jyväskylä, Finland

2 Department of Mathematics, University of Fribourg, 1700 Fribourg, Switzerland

$$\mathrm{Lip}_{\delta_F} u(x) = \limsup_{y\to x} \frac{|u(y)-u(x)|}{\delta_F(x,y)}.$$

Given a subset K of $\mathbb{R}^n$, we set

$$\mathrm{Lip}_{\delta_F}(u,K) = \sup_{x,y\in K, x\neq y} \frac{|u(x)-u(y)|}{\delta_F(x,y)}$$

and denote by $\mathrm{Lip}_{\delta_F}(K)$ the collection of all functions $u : K \to \mathbb{R}$ with $\mathrm{Lip}_{\delta_F}(u,K) < \infty$.

Sturm asked the following interesting question in [12]: Is a diffusion process determined by the intrinsic distance? Mathematically, Sturm's question can be formulated as follows: Is it true that for all $u \in \mathrm{Lip}_{\delta_F}(\Omega)$,

$$F(x, \mathrm{d}u(x)) = \mathrm{Lip}_{\delta_F} u(x)$$

almost everywhere with $F(x,v) = \sqrt{\langle A(x)v, v\rangle}$?

The answer to the question is yes when A is supposed to be continuous, as shown by Sturm [12, Proposition 4]. He also pointed out that the answer to this question is not always positive [12, Theorem 2]: For $F(x,v) = \sqrt{\langle A(x)v, v\rangle}$, where A is a diffusion matrix, there exists $\tilde{F}(x,v) = \sqrt{\langle \tilde{A}(x)v, v\rangle}$ such that $\delta_F = \delta_{\tilde{F}}$ but

$$F(x,v) < \tilde{F}(x,v)$$

for all $v \in \mathbb{R}^n \backslash \{0\}$; see also [11] for a different example.

The case $F(x,v) = \sqrt{\langle A(x)v, v\rangle}$ gained deeper understanding in a recent paper [10], where the authors enhanced Sturm's result by showing that if the diffusion matrix A is weak upper semicontinuous, then the differential and distance structures coincide. They also constructed an example, which shows that if A fails to be upper semicontinuous on a set of positive measure, then the differential and distance structure may fail to coincide.

The main purpose of this paper is to generalize the above result of [10] to more general Finsler structures. More precisely, we are going to prove the following result.

Theorem 1.1 *Let $n \geq 2$ and F be an admissible Finsler structure on a domain $\Omega \subset \mathbb{R}^n$. If F is weak upper semicontinuous on Ω, then the intrinsic distance and differential structure coincide. That is given a Lipschitz function u on Ω (with respect to the Euclidean distance), for almost every $x \in \Omega$, we have*

$$\mathrm{Lip}_{\delta_F} u(x) = F(x, \mathrm{d}u(x)).$$

The proof of [10, Theorem 2] relies heavily on the structure of $F(x,v) = \sqrt{\langle A(x)v, v\rangle}$. It seems that there is little hope to adapt their proofs in the greater generality of this paper.

To see an example where Theorem 1.1 applies more generally than [10, Theorem 2], we may choose suitable weighted L^p-norm with $1 \leq p < \infty$. For instance, consider $F(x,v) = (\sum_{i=1}^n w(x)|v_i|^p)^{1/p}$, where the weight function w is upper semicontinuous and satisfies the ellipticity condition $0 < c \leq w(x) \leq C < \infty$ for all $x \in \mathbb{R}^n$.

Theorem 1.1 can be regarded as an improved version of [8, Proposition 2.4] from L^∞-norm to pointwise equality.

Our proof of Theorem 1.1 completely differs from that used in [10] and it is simpler than [10], even in their setting. The crucial observation is Proposition 3.1 below, a special case of a result due to De Cecco and Palmieri [6], which states that the intrinsic distance δ_F (infinitesimally) coincides with d_c^*, where d_c^* is the cc-distance induced by the Finsler structure F. The weak upper semicontinuity is crucial for our proof, since it implies that the "metric density" of a curve with respect to the metric length coincides with its "differential density"; see Sect. 4 below for the precise meaning. Our approach is more geometric and was influenced a lot by the recent studies in Finsler geometry [2,4,6,7]. Some of the ideas from this paper were successfully used in our companion paper [9] on certain L^∞-variational problems associated with measurable Finsler structures. It is known (e.g., [1,11]) that the intrinsic distance and differential structures coincide even for abstract Dirichlet forms on metric measure spaces. It would be interesting to know that whether a version of Theorem 1.1 holds in the abstract setting as there.

This paper is organized as follows. Section 2 contains all the preliminaries related to Finsler structures. Sections 3 and 4 contain an overview of the necessary background that are needed for our proof of Theorem 1.1. In Sect. 5, we prove Theorem 1.1. "Appendix" contains a separate proof of Proposition 3.1 under the weak upper semicontinuity assumption.

2 Preliminaries on Finsler structures

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain, i.e., an open connected set.

Definition 2.1 (*Finsler structures*) We say that a function $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ is a Finsler structure on Ω if

- $F(\cdot, v)$ is Borel measurable for all $v \in \mathbb{R}^n$, $F(x, \cdot)$ is continuous for a.e. $x \in \Omega$;
- $F(x, v) > 0$ for a.e. x if $v \neq 0$;
- $F(x, \lambda v) = |\lambda| F(x, v)$ for a.e. $x \in \Omega$ and for all $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$.

Definition 2.2 (*Admissible Finsler structures*) A Finsler structure F is said to be admissible if

- $F(x, \cdot)$ is convex for a.e. $x \in \Omega$;
- F is locally equivalent to the Euclidean norm or elliptic, i.e., there exists a continuous function $\lambda : \Omega \to [1, \infty)$ such that
$$\frac{1}{\lambda(x)}|v| \leq F(x, v) \leq \lambda(x)|v|$$
for a.e. $x \in \Omega$ and for all $v \in \mathbb{R}^n$.

It is straightforward to verify that the standard L^p-norm ($1 \leq p < \infty$), i.e., $F(x, v) = (\sum_{i=1}^n v_i^p)^{1/p}$, is an admissible Finsler structure on $\mathbb{R}^n$. From the geometric point of view, there are many other interesting examples and we refer the interested readers to [2] for the details.

Recall that a function $u : \Omega \to \mathbb{R}$ is said to be upper semicontinuous at $x \in \Omega$ if
$$u(x) \geq \limsup_{y \to x} u(y).$$
Following [10], we say that u is weak upper semicontinuous in Ω if u is upper semicontinuous at almost every $x \in \Omega$. Let F be an admissible Finsler structure on Ω. We say that F is weak upper semicontinuous on Ω if for each $v \in \mathbb{R}^n$, the function $F(\cdot, v)$ is weak upper semicontinuous on Ω.

Similarly a function $u : \Omega \to \mathbb{R}$ is said to be lower semicontinuous at $x \in \Omega$ if
$$u(x) \leq \liminf_{y \to x} u(y),$$

and u is weak lower semicontinuous in Ω if u is lower semicontinuous at almost every $x \in \Omega$. Let F be an admissible Finsler structure on Ω. We say that F is weak lower semicontinuous on Ω if for each $v \in \mathbb{R}^n$, the function $F(\cdot, v)$ is weak lower semicontinuous on Ω.

Let F be an admissible Finsler structure for Ω. We introduce the dual of $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ in the standard way.

Definition 2.3 (*Dual Finsler structures*) The dual F^* of an admissible Finsler structure $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ is defined as

$$F^*(x, w) = \sup_{v \in \mathbb{R}^n} \{\langle v, w\rangle : F(x, v) \le 1\}$$
$$= \max_{v \neq 0} \left\{ \left\langle w, \frac{v}{F(x, v)} \right\rangle \right\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.

The following proposition follows immediately from Definition 2.3; see for instance [8, Section 1.2] or [3, Section 2] for more information.

Proposition 2.4 (Basic properties of a dual Finsler structure) *Let F be an admissible Finsler structure on Ω. Then the dual function F^* satisfies the following properties*

- *$F^*(\cdot, v)$ is Borel measurable and $F^*(x, \cdot)$ is Lipschitz;*
- *$F^*(x, \cdot)$ is a norm;*
- *$F^*(x, \cdot)$ is locally equivalent to the Euclidean norm, i.e.*

$$\frac{1}{\lambda(x)}|v| \le F^*(x, v) \le \lambda(x)|v|.$$

- *$(F^*)^*(x, v) = F(x, v)$;*
- *F is weak upper (lower) semicontinuous if and only if F^* is weak lower (upper) semicontinuous.*

3 Comparison of intrinsic distances

Let $(\Omega, F(x, \cdot), d_c^F, \delta_F)$ be a Finsler manifold with an admissible Finsler structure F. For an admissible Finsler structure F on Ω, we may associate a cc-distance in the standard way by setting

$$d_c^*(x, y) := \sup_N \inf_{\gamma \in \Gamma_N^{x,y}} \left\{ \int_0^1 F^*\left(\gamma(t), \gamma'(t)\right) \mathrm{d}t \right\},$$

where the supremum is taken over all subsets N of Ω such that $|N| = 0$ and $\Gamma_N^{x,y}(\Omega)$ denotes the set of all Lipschitz curves in Ω with end points x and y transversal to N, i.e., $\mathscr{H}^1(N \cap \gamma) = 0$. For an admissible Finsler metric F, d_c^* is indeed an intrinsic distance; for the definition of an intrinsic distance and this fact, see [6,7]. Above, we use $|E|$ to denote the n-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$ and $\mathcal{H}^1$ the one-dimensional Hausdorff measure.

The following fundamental result, which relates δ_F and d_c^*, was a special case of [6, Theorem 3.7].

Proposition 3.1 *Let F be an admissible Finsler structure on Ω. Then for almost every $x \in \Omega$, it holds*

$$\lim_{y \to x} \frac{\delta_F(x, y)}{d_c^*(x, y)} = 1. \tag{3.1}$$

Since we have assumed the weak upper semicontinuity on our admissible Finsler structure in our main result Theorem 1.1, we give a separate proof of Proposition 3.1 under this extra assumption in "Appendix."

4 Comparison of metric derivatives

For any distance d on Ω and each Lipschitz (with respect to d) curve $\gamma : [a, b] \to \Omega$, the length of γ with respect to d is denoted by $\mathcal{L}_d(\gamma)$, i.e.,

$$\mathcal{L}_d(\gamma) := \sup \left\{ \sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i+1})) \right\},$$

where the supremum is taken over all partitions $\{[t_i, t_{i+1}]\}$ of $[a, b]$.

Given a curve γ, the metric derivative of γ at t is defined to be

$$\left|\gamma'(t)\right|_d := \limsup_{s \to 0} \frac{d(\gamma(t+s), \gamma(t))}{s}.$$

If $\gamma : [a, b] \to \Omega$ is Lipschitz with respect to d, then its length can be computed by integrating the metric derivative, i.e.

$$\mathcal{L}_d(\gamma) = \int_a^b |\gamma'(t)|_d \mathrm{d}t.$$

In other words, for a Lipschitz curve, the metric derivative is the metric density of its length.

For any intrinsic distance d, which is locally bi-Lipschitz equivalent to the Euclidean distance, we may associate a Finsler structure Δ_d in the following manner. For each $x \in \Omega$ and for every direction v, we define

$$\Delta_d(x, v) := \limsup_{t \to 0^+} \frac{d(x, x + tv)}{t}. \tag{4.1}$$

It can be proved that for every Lipschitz curve $\gamma : [a, b] \to \Omega$, we have

$$\mathcal{L}_d(\gamma) = \int_a^b \Delta_d\left(\gamma(t), \gamma'(t)\right) \mathrm{d}t.$$

In particular, $\Delta_d(\gamma(t), \gamma'(t)) = |\gamma'(t)|_d$ for a.e. $t \in [a, b]$.

Remark 4.1 For any admissible Finsler structure F, one always has

$$\Delta_{d_c^*}(x, v) \leq F^*(x, v) \quad \text{for a.e. } x \in \Omega \quad \text{and all } v \in \mathbb{R}^n; \tag{4.2}$$

see [8, Proposition 1.6]. However, the equality does not necessary hold; see [7, Example 5.1] for a counterexample.

In addition, for an admissible Finsler structure F, the dual Finsler structure F^* always induces a lower semicontinuous length structure; see [4, Section 2.4.2]. Moreover, if the Finsler metric F is weak upper semicontinuous on Ω, then the following stronger result holds.

Proposition 4.2 ([3, Proposition 2.9]) *If the Finsler structure F is weak upper semicontinuous on Ω, then for a.e. $x \in \Omega$ and all $v \in \mathbb{R}^n$, it holds*

$$\Delta_{d_c^*}(x, v) = F^*(x, v).$$

5 Coincidence of distance structure and differential structure

In this section, we are ready to prove our main result Theorem 1.1.

Proposition 5.1 *For each $u \in \mathrm{Lip}_{\delta_F}(\Omega)$, $F(x, \mathrm{d}u(x)) \le \mathrm{Lip}_{\delta_F} u(x)$ for a.e. $x \in \Omega$.*

Proof Since both sides are positively 1-homogeneous with respect to u, we only need to show that for a.e. $x \in \Omega$, if $\mathrm{Lip}_{\delta_F} u(x) = 1$, then $F(x, \mathrm{d}u(x)) \le 1$.

Note that by Proposition 3.1, for a.e. $x \in \Omega$, $\mathrm{Lip}_{\delta_F} u(x) = \mathrm{Lip}_{d_c^*} u(x)$. Fix such an x. For each $v \in \mathbb{R}^n$, we have

$$\begin{aligned}\mathrm{d}u(x)v &= \lim_{t\to 0} \frac{u(x+tv)-u(x)}{t}\\ &\le \limsup_{t\to 0} \frac{d_c^*(x, x+tv)}{t} \cdot \limsup_{t\to 0} \frac{u(x+tv)-u(x)}{d_c^*(x, x+tv)}\\ &\le \Delta_{d_c^*}(x, v)\,\mathrm{Lip}_{d_c^*} u(x) \le F^*(x, v),\end{aligned}$$

where in the last inequality, we have used the inequality (4.2).

Therefore,

$$\begin{aligned}F(x, \mathrm{d}u(x)) &= F^{**}(x, \mathrm{d}u(x))\\ &= \max_{v\neq 0} \left\{ \mathrm{d}u(x) \left(\frac{v}{F^*(x, v)} \right) \right\} \le 1\end{aligned}$$

as desired. This completes our proof.

Theorem 5.2 *Let F be an admissible Finsler structure on Ω. If F is weak upper semicontinuous on Ω, then for any Lipschitz function u in (Ω, δ_F),*

$$\mathrm{Lip}_{\delta_F} u(x) \le F(x, \mathrm{d}u(x))$$

for a.e. $x \in \Omega$.

Proof First, note that our assumption on F implies that F satisfies the following uniform upper semicontinuity property, for a.e. $x \in \Omega$,

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : F(y, v) \le (1+\varepsilon)F(x, v) \quad \text{for all } y \in B(x, \delta), \quad v \in \mathbb{R}^n. \tag{5.1}$$

By homogeneity of F (with respect to v), it suffices to prove (5.1) for all $v \in \mathbb{S}$ (the unit sphere). Suppose by contradiction, that (5.1) fails. Then there exist some $x \in \Omega$ and some $\varepsilon_0 > 0$ such that for each $k \in \mathbb{N}$, there exist some $y_k \in B\left(x, \frac{1}{k}\right)$ and $v_k \in \mathbb{S}$ so that

$$F(y_k, v_k) > (1+\varepsilon_0)F(x, v_k). \tag{5.2}$$

By compactness of $\mathbb{S}$, we may assume (up to another subsequence if necessary) $v_k \to v \in \mathbb{S}$ as $k \to \infty$. Then

$$\begin{aligned} F(x, v) &= \limsup_{k\to\infty} F(x, v_k) \ge \limsup_{k\to\infty} \limsup_{y\to x} F(y, v_k) \\ &\ge \limsup_{k\to\infty} F(y_k, v_k) \ge \limsup_{k\to\infty} (1+\varepsilon_0) F(x, v_k) \\ &= (1+\varepsilon_0) F(x, v), \end{aligned}$$

which is a contradiction.

Secondly, by Rademacher's theorem, it suffices to prove Theorem 5.2 when $u(x) = \langle v, x\rangle$ is linear. We may additionally assume that $v \neq 0$. By the fundamental theorem of calculus and the definition of F^*, we have

$$\begin{aligned} |u(x) - u(y)| = |\langle v, y - x\rangle| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) \mathrm{d}t \right| \\ &= \left| \int_0^1 \langle v, \gamma'(t)\rangle \mathrm{d}t \right| \le (1+\varepsilon) F(x, v) \int_0^1 F^* \left(\gamma(t), \gamma'(t)\right) \mathrm{d}t \end{aligned}$$

whenever x, y and $\gamma(t)$ belongs to the "δ-neighborhood of x where (5.1) holds; it follows that

$$\frac{|\langle v, y - x\rangle|}{d_{\mathrm{c}}^*(x, y)} \le (1+\varepsilon) F(x, v),$$

whenever $|x - y| < \delta$. Letting $y \to x$ and $\varepsilon \to 0$ concludes our proof. □

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Appendix: Proof of Proposition 3.1 when *F* is weak upper semicontinuous

Proof The inequality $\delta_F(x, y) \le d_{\mathrm{c}}^*(x, y)$ follows directly from definitions. Indeed, for each Lipschitz function u with $\|F(\cdot, \mathrm{d}u(\cdot))\|_{L^\infty(\Omega)} \le 1$, each $x, y \in \Omega$, for each Lipschitz curve γ joining x and y that is transversal to the zero measure set $N = \{x \in \Omega : F(x, \mathrm{d}u(x)) > 1\}$,

$$\begin{aligned} u(x) - u(y) &= \int_0^1 \mathrm{d}u(\gamma(t)) \left(\gamma'(t)\right) \mathrm{d}t \\ &\le \int_0^1 F^* \left(\gamma(t), \gamma'(t)\right) \mathrm{d}t = \mathcal{L}_{d_{\mathrm{c}}^*}(\gamma), \end{aligned}$$

where $\mathcal{L}_{d_{\mathrm{c}}^*}$ denotes the length of the curve γ with respect to the metric d_{c}^*. Taking infimum over all admissible curves on the right-hand side and then supermum over all admissible functions over the left-hand side, we obtain via Proposition 4.2 that

$$\delta_F(x, y) \le d_{\mathrm{c}}^*(x, y).$$

In particular,

$$\limsup_{y\to x} \frac{\delta_F(x,y)}{d_c^*(x,y)} \le 1.$$

We are left to prove that

$$\liminf_{y\to x} \frac{\delta_F(x,y)}{d_c^*(x,y)} \ge 1. \tag{5.3}$$

We divide the proof of this equation into two steps.

Step 1 Assume that $F(\cdot, v)$ is continuous.

Fix $x \in \Omega$ and $\varepsilon > 0$. Since $F(\cdot, v)$ and $F^*(\cdot, v)$ are continuous in $B(x,\delta)$, we may assume that for all $z \in B(x,\delta)$,

$$(1-\varepsilon)F(z,v) \le F(x,v) \le (1+\varepsilon)F(z,v)$$

and

$$(1-\varepsilon)F^*(z,v) \le F^*(x,v) \le (1+\varepsilon)F^*(z,v).$$

Note that the issue is local, we are now restricting ourselves to the ball $B(x,\delta)$.

Consider the curve $\gamma(t) = x + t(y-x)$, we have

$$d_c^*(x,y) \le \mathcal{L}_{d_c^*}(\gamma) = \int_0^1 F^*\left(\gamma(t), \gamma'(t)\right) \mathrm{d}t \le (1+\varepsilon)F^*(x, y-x).$$

By the definition of a dual Finsler structure, we know that there exists some $\tilde{v} \ne 0$ such that $F^*(x, y-x) = \langle y-x, \frac{\tilde{v}}{F(x,\tilde{v})}\rangle$. Set

$$v := \frac{\tilde{v}}{(1+\varepsilon)F(x,\tilde{v})}.$$

Then $F(x,v) = \frac{1}{1+\varepsilon}$ and $\langle v, y-x\rangle = \frac{1}{1+\varepsilon}F^*(x,y-x)$. Note that for all $z \in B(x,\delta)$, $F(z,v) \le (1+\varepsilon)F(x,v) \le 1$ and so the function $u(z) := \langle v, z\rangle$ is an admissible function for $\delta_F(x,y)$. This means that

$$\delta_F(x,y) \ge u(y) - u(x) = 1/(1+\varepsilon)F^*(x,y-x) \ge \frac{1}{(1+\varepsilon)^2} d_c^*(x,y).$$

It is clear that (5.3) follows from the above inequality by letting $\varepsilon \to 0$.

Step 2 Assume that $F(\cdot, v)$ is weak upper semicontinuous.

In this case, F^* is weak lower semicontinuous, it is a well-known fact that there exists a sequence of admissible Finsler norms $F_n^*(\cdot, v)$, which is continuous in the first variable, such that

$$F_n(x,v)^* \le F_{n+1}^*(x,v) \le \cdots \to F^*(x,v);$$

and $d_c^{*n} \to d_c^*$ as $n \to \infty$, where d_c^{*n} is the cc-distance induced by the Finsler structure F_n; see for instance [5, Section 4]. Let $F_n = F_n^{**}$ denote the dual of F_n^*, then it is easy to check from our definition that

$$F_n(x,v) \ge F_{n+1}(x,v) \ge \cdots \to F(x,v).$$

It follows that

$$\frac{\delta_F(x,y)}{d_c^*(x,y)} = \lim_{n\to\infty} \frac{\delta_{F_n}(x,y)}{d_c^{*n}(x,y)},$$

where δ_{F_n} is the intrinsic distance induced by F_n similar as δ_F. Given $\varepsilon > 0$, there exists N_0 such that for all $n \geq N_0$,

$$\frac{\delta_F(x,y)}{d_c^*(x,y)} \geq (1-\varepsilon)\frac{\delta_{F_n}(x,y)}{d_c^{*n}(x,y)}.$$

On the other hand, by step 1,

$$\liminf_{y\to x} \frac{\delta_{F_n}(x,y)}{d_c^{*n}(x,y)} \geq 1.$$

We thus obtain

$$\liminf_{y\to x} \frac{\delta_{F_n}(x,y)}{d_c^{*n}(x,y)} \geq \liminf_{y\to x}(1-\varepsilon)\frac{\delta_{F_n}(x,y)}{d_c^{*n}(x,y)} \geq 1-\varepsilon.$$

The claim follows by letting $\varepsilon \to 0$. □

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