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A necessary condition for the existence of a doubly connected minimal surface

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Abstract Given two circles contained in parallel planes, it is expectable that there does not exist a doubly connected minimal surface bounded by both circles if these circles are either laterally or vertically far away. In this paper, we give a quantitative estimate of this separation. We also obtain bounds for the height of a Riemann minimal example in terms of a catenoid with the same boundary radii and waist.

Keywords Minimal surface · Riemann minimal example · Elliptic integral

Mathematics Subject Classification 53A10 · 53C42

1 Introduction and motivation of the results

If we dip a wire contour into a solution of a soapy water and then pull it out, the soap film formed by the frame is a surface with the least possible area among all surfaces having the same contour. The energy of the soap film is due to the forces of attraction between the molecules (surface tension) and where the gravity is neglected (small wire contours). Thus this energy is proportional to the area of the soap film. Mathematically, a surface that locally minimizes the area has the property that its mean curvature is zero everywhere and it is called a minimal surface.

In this paper, we are interested in the case that the wire contour is formed by two circles C_1 and C_2 and it is motivated by the experiments that one can easily do with the soap films framed by two circles. Firstly, we dip two coaxial circles $C_1 \cup C_2$ in parallel planes $\Pi_1 \cup \Pi_2, C_i \subset \Pi_i, i = 1, 2$. If C_1 is sufficiently close to C_2, the soap film obtained is

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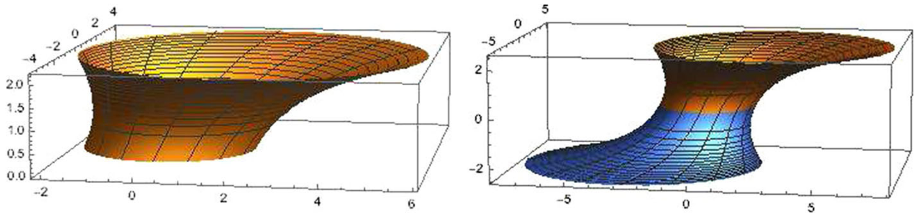

Fig. 1 *left*: A piece of a Riemann minimal example. *right*: the surface is obtained by the reflection across the plane of equation $z = 0$ of the piece on the left side. Here $a = 4$ and $b = 2$

a catenoid, which is, besides the plane, the only rotational minimal surface. From early works of Plateau, Goldschmidt and Lindelöf, it is known that after a critical value of the vertical distance between the circles, the catenoid breaks into two disks [2,4,14]. In fact, and depending on the distance h between the two circles, there are zero, one or two catenoids, one of which is unstable in the latter case. For example, if C_1 and C_2 have the same radius r, there exists $h_0 \simeq 1.325r$ such that if $h < h_0$, there are two catenoids spanning C_1 and C_2 and only one is physically realizable; if $h = h_0$ there is exactly one catenoid bounded by $C_1 \cup C_2$; and if $h > h_0$, there is not a catenoid joining C_1 with C_2.

Once that we have formed a catenoid we now displace slightly the circles C_1 and C_2 sideways in a direction parallel to Π_i without breaking the soap film. In such a case, we go obtaining a family of minimal surfaces with the topology of an annulus, also called in topology, a doubly connected surface. Shiffman proved in [17] that a minimal annulus spanning two circles in parallel planes is, indeed, foliated by circles in parallel planes, that is, the intersection of the surface with a parallel plane to Π_i is a circle. This surface belongs to a family of minimal surfaces discovered by Riemann in 1860s and called in the literature a *Riemann minimal example* [15]. A Riemann minimal example is a surface with zero mean curvature constructed by a uniparametric family of circles in parallel planes. See Fig. 1. Recall here that Enneper proved that if a minimal surface is foliated by a uniparametric family of circles, then the foliating planes must be parallel [1] and thus, the surface is rotational or it is one of the Riemann minimal examples. More properties of these surfaces will appear in Sect. 2.

After constructing a catenoid, and then a piece of a Riemann minimal example, we follow by displacing C_1 and C_2 sideways by keeping the vertical distance h. If C_1 and C_2 are sufficiently far, the surface breaks into two disks, namely the two disks bounded by each circle C_i in Π_i. If d denotes the lateral distance between the centers of C_1 and C_2, the above experiments can summarize as follows. If $d = 0$, the surface is the catenoid. For $d > 0$ close to 0, we obtain a Riemann minimal example and if d is sufficiently big, the surface leaves to be connected. It is a problem in classical theory of minimal surfaces to estimate the value $d_1 = d_1(r_1, r_2, h)$ such that no doubly connected minimal surface bounded by C_1 and C_2 exists for $d > d_1$. The control of the number $d_1(h)$ appears as 'Problem 33' in [8]. A similar question can be considered for two Jordan curves Γ_1 and Γ_2 contained in Π_1 and Π_2, respectively. We point out the main results related with this paper.

1. If there exists a connected minimal surface bounded by $C_1 \cup C_2$, then the orthogonal projection on a plane parallel to Π_i of C_1 and C_2 must overlap [10]. This result is more general, and it is known that if there exists a plane orthogonal to $\Pi_1 \cup \Pi_2$ separating C_1 and C_2, then $C_1 \cup C_2$ cannot bound a minimal surface of annulus type (see also [16] when C_1 and C_2 are convex curves by using an argument with the touching principle of minimal surfaces).

2. Denote by δ_i the diameter of Γ_i. If there exists a (connected) minimal surface, then $h \leq \max\{\delta_1, \delta_2\}$ ([16]). In particular, the surface is contained in a vertical slab of width $\max\{\delta_1, \delta_2\}$. This generalizes previous works of Nitsche for doubly connected surfaces [6].
3. For a doubly connected minimal surface we have:

$$h \leq \sqrt{\frac{(\delta_1 + \delta_2)^2}{4} - \frac{d^2}{2}}. \tag{1}$$

See [7]. Inequality (1) is generalized in [13] for surfaces in $\mathbb{R}^n$ without requiring that Γ_1 and Γ_2 lie in parallel hyperplanes.
4. There does not exist a doubly connected minimal surface if Γ_1 and Γ_2 lie in different components of the cone $x^2 + y^2 < z^2 \sinh^2 \tau$, where τ is the unique positive solution of $\cosh \tau - \tau \sinh \tau = 0$: see [13], extending previous results of Hildebrandt [3].

Motivated by these results, we pose the next two problems:

1. Problem 1. By moving C_1 and C_2 sideways along a fix direction, can we displace C_1 exactly until just before the orthogonal projection of C_1 onto Π_2 is tangent to C_2 obtaining during this displacement a Riemann minimal example?
2. Problem 2. Determine a function $M(r_1, r_2, d)$ such that the curves Γ_1 and Γ_2 cannot bound a doubly connected minimal surface whenever $h > M(r_1, r_2, d)$. This appears as 'Theorem' in [12].

In Sect. 2 we will give a universal bound for the overlapping distance of the circles C_1 and C_2 in such a way that if this distance is bigger than this bound, then there does not exist a doubly connected minimal surface spanning $C_1 \cup C_2$ (Theorem 1). This result generalizes for two Jordan curves in parallel planes. In Sect. 3 we compare the height of a piece of a Riemann minimal example bounded by two circles with the one of a catenoid with the same boundary radii and waist. Finally in Sect. 4, numerical results are presented that will indicate the approximation of the proposed estimate. Motivated by the experiments, the calculations fix the radii of the circles as well as the vertical distance.

2 The Riemann minimal examples

We review the description of the Riemann minimal examples. Here, we follow [11]. Let (x, y, z) be the standard coordinates in Euclidean space $\mathbb{R}^3$, where (x, y) indicate the horizontal coordinates and z is the vertical direction. Let Π be the plane of equation $z = 0$. A Riemann minimal example S foliated by circles in parallel horizontal planes is parametrized as $X(u, \theta) = (c(u), 0, u) + r(u)(\cos\theta, \sin\theta, 0)$, where $(c(u), 0, u))$, $u \in I \subset \mathbb{R}$, is the planar curve of centers of the circles. The minimality of S writes as

$$-rr'' + 1 + r'^2 + c'^2 + (2c'r' - rc'')\cos\theta = 0$$

for all $u \in I$ and $\theta \in \mathbb{R}$. Hence, we have two differential equations, namely

$$2c'r' - rc'' = 0, \quad -rr'' + 1 + r'^2 + c'^2 = 0.$$

We point out that c is not a constant function because on the contrary S would be a catenoid (a solution of $rr'' = 1 + r'^2$). Since $c' \neq 0$, the first equation gives $c' = \lambda r^2$ for some positive constant λ and it follows that the second equation is $-rr''+1+r'^2+\lambda^2r^4=0$. After some manipulations, there exists $\mu\in\mathbb{R}$ such that

$$r'=\pm\sqrt{\lambda^2r^4+2\mu r^2-1}. \tag{2}$$

By changing the variable u by the parameter radius r, the parametrization of S is

$$X(u,\theta)=\left(\int^u\frac{\lambda t^2\,dt}{\sqrt{\lambda^2t^4+2\mu t^2-1}},0,\int^u\frac{\mathrm{d}t}{\sqrt{\lambda^2t^4+2\mu t^2-1}}\right)+u\,(\cos\theta,\sin\theta,0)\,.$$

Let

$$a^2=\frac{\mu+\sqrt{\lambda^2+\mu^2}}{\lambda^2},\qquad b^2=\frac{-\mu+\sqrt{\lambda^2+\mu^2}}{\lambda^2}.$$

Up to reversing the values of a^2 and b^2 if necessary, each circle of the foliation of radius u ($u\geq b$) at height $z(u)$ is characterized in terms of elliptic integrals by

$$c(u)=\int_b^u\frac{t^2\,\mathrm{d}t}{\sqrt{\Delta}},\qquad z(u)=\int_b^u\frac{ab\,\mathrm{d}t}{\sqrt{\Delta}}, \tag{3}$$

where

$$\Delta=(t^2+a^2)(t^2-b^2),\qquad 0<b\leq a.$$

The parametrization of S is

$$X(u,\theta)=\left(\int_b^u\frac{t^2\,\mathrm{d}t}{\sqrt{\Delta}},0,\int_b^u\frac{ab\,\mathrm{d}t}{\sqrt{\Delta}}\right)+u(\cos\theta,\sin\theta,0), \tag{4}$$

where $u\geq b$ and $\theta\in\mathbb{R}$. The parametrization (4) is obtained when we choose the positive branch of the square root in (2) and corresponds with the part of the surface above the plane Π. In the half-space $z\leq 0$, the choice in (2) is the negative branch, and the surface parametrizes as

$$Y(u,\theta)=-\left(\int_b^u\frac{t^2\,\mathrm{d}t}{\sqrt{\Delta}},0,\int_b^u\frac{ab\,\mathrm{d}t}{\sqrt{\Delta}}\right)+u(\cos\theta,\sin\theta,0). \tag{5}$$

Denote by $\mathcal{R}=\mathcal{R}(a,b)$ the Riemann minimal example parametrized by (4) and (5). Some properties of $\mathcal{R}$ are:

1. The intersection of $\mathcal{R}(a,b)$ with the plane Π is the circle of the foliation of minimum radius $r=b$, called the *waist* of $\mathcal{R}(a,b)$. See Fig. 2.

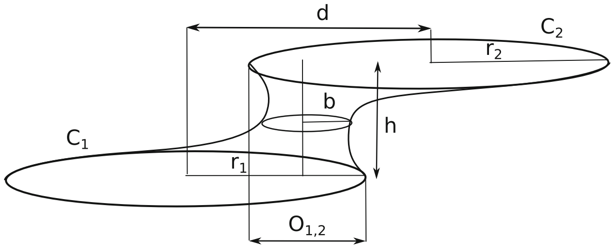

Fig. 2 A Riemann minimal surface bounded by the circles $C_1\cup C_2$. We indicate by h and d the vertical and the lateral distance between C_1 and C_2. The circle of the foliation of minimum radius is the waist

2. The surface is symmetric about the origin of coordinates and by a homothety, we have $\lambda\mathcal{R}(a,b) = \mathcal{R}_{\lambda a,\lambda b}$ for $\lambda > 0$.
3. The surface lies in a horizontal slab $-z_0 < z < z_0$ such that $\lim_{r\to\infty} z(r) = z_0$. Thus as r tends to ∞, the limit of circles converge to a straight line orthogonal to the xz-plane and by the Schwarz reflection principle, the surface can extend by rotating 180° about this line. Repeating this process and by successive reflections, we obtain a periodic embedded surface foliated by circles and, at a discrete set of heights, the intersection of the surface with horizontal planes is a straight line.

We will only work with compact pieces of a Riemann minimal example. We precise this definition.

Definition 1 A Riemann minimal surface is a compact sub-surface of a Riemann minimal example bounded by two circles.

In particular, we exclude that the surface can contain a straight line. We denote $\mathcal{R}_{1,2} = \mathcal{R}_{1,2}(a,b)$ to indicate that the boundary is formed by the circles C_1 and C_2 of radii r_1 and r_2, respectively, with $z(r_1) < z(r_2)$. In the symmetric case, that is, $r = r_1 = r_2$ (and consequently, $z(r_1) = -z(r_2) < 0$ and $c(r_1) = -c(r_2) < 0$, we stand for $\mathcal{R}_r$.

We know that the orthogonal projection onto Π of C_1 and C_2 must overlap. This property can express by saying that the lateral distance of C_1 and C_2 is less than the sum of their radii [11, pp. 88–89]. In this section, we want to estimate the amount of overlapping. We define the *overlapping distance* between C_1 and C_2 as

$$\mathcal{O}_{1,2} = \mathcal{O}_{1,2}(a,b) = c(r_1) - c(r_2) + r_1 + r_2.$$

See Fig. 2. In the symmetric case, $\mathcal{O}_{1,2}$ is simply $2(r - c(r))$. Denote by $h = z(r_2) - z(r_1)$ and by $d = c(r_2) - c(r_1)$ the *vertical distance* and the *lateral distance* between C_1 and C_2, respectively. Under this notation, the Problem 1 expresses by saying: if r_1, r_2 and h are fix, is it possible to have $\mathcal{O}_{1,2}(a,b) \to 0$ for all a and b?

Theorem 1 *There exists a universal constant $M = \sqrt{2} - 1$ such that it holds*

$$\mathcal{O}_{1,2}(a,b) > Mh, \tag{6}$$

for any Riemann minimal surface $\mathcal{R}_{1,2}(a,b)$.

Proof Consider the part of $\mathcal{R}(a,b)$ bounded by the waist and a circle C of radius $r > 0$ that lies above the plane Π. Then $c(r) > 0$ and we have

$$\begin{aligned} r - c(r) > \sqrt{r^2 - b^2} - c(r) &= \int_b^r \frac{t}{\sqrt{t^2 - b^2}}\,\mathrm{d}t - \int_b^r \frac{t^2}{\sqrt{\Delta}}\,\mathrm{d}t \\ &= a^2 \int_b^r \frac{t}{(t + \sqrt{t^2 + a^2})\sqrt{\Delta}}\,\mathrm{d}t. \end{aligned}$$

The function $t/(t + \sqrt{t^2 + a^2})$ is increasing on t and then its minimum value is attained at $t = b$. Thus (3) gives

$$\begin{aligned} r - c(r) &> \frac{a^2 b}{b + \sqrt{a^2 + b^2}} \int_b^{r_2} \frac{\mathrm{d}t}{\sqrt{\Delta}} = \frac{az(r)}{b + \sqrt{a^2 + b^2}} \\ &\geq \frac{az(r)}{a + \sqrt{a^2 + a^2}} = (\sqrt{2} - 1)z(r) = Mz(r). \end{aligned}$$

If $\mathcal{R}_{1,2}$ has points in both sides of Π, then we apply the above estimate for each part of $\mathcal{R}_{1,2}$, and finally we sum both estimates, obtaining (6).

Finally suppose now that $\mathcal{R}_{1,2}$ lies in one side of Π, for example, above Π. A similar argument yields

$$\begin{aligned} r_2 + r_1 + c(r_1) - c(r_2) &> \sqrt{r_2^2 - b^2} - \sqrt{r_1^2 - b^2} + c(r_1) - c(r_2) \\ &= \int_{r_1}^{r_2} \left(\frac{t}{\sqrt{t^2 - b^2}} - \frac{t^2}{\sqrt{\Delta}} \right) \mathrm{d}t \\ &= \frac{a^2 b}{b + \sqrt{a^2 + b^2}} \int_{r_1}^{r_2} \frac{dt}{\sqrt{\Delta}} = \frac{a(z(r_2) - z(r_1))}{a + \sqrt{a^2 + a^2}} \\ &= M(z(r_2) - z(r_1)) = Mh. \end{aligned}$$

□

The estimate (6) can view as a restriction on the vertical separation between two circles to bound a Riemann minimal surface. As we pointed out in the Introduction, it is known experimentally that if we move vertically C_1 and C_2 in opposite vertical directions, there exists a critical separation $h_0 = h_0(r_1, r_2, d)$ where the surface breaks into the two disks bounded by C_1 and C_2. By Theorem 1, we now can estimate the value h_0.

Corollary 1 *Given $r_1, r_2, d > 0$, then*

$$h_0 < (\sqrt{2} + 1)(r_1 + r_2 - d). \tag{7}$$

Proof If $\mathcal{R}_{1,2}(a, b)$ is Riemann minimal surface bounded by C_1 and C_2, then $h \leq (r_1 + r_2 - d)/M$. Since this holds for any $a \geq b > 0$, it follows the result. □

We can extend Theorem 1 for a doubly connected minimal surface spanning two Jordan curves. Let Γ_1 and Γ_2 be two Jordan curves and let C_i be the circumscribed circle of radius r_i for Γ_i, $i = 1, 2$. We define the lateral distance d and the overlapping distance $\mathcal{O}_{1,2}$ between Γ_1 and Γ_2 to be the ones between the circles C_1 and C_2. In this context, we formulate Theorem 1 in terms of a nonexistence result.

Corollary 2 *If*

$$h \geq (\sqrt{2} + 1)(r_1 + r_2 - d), \tag{8}$$

then the curves Γ_1 and Γ_2 cannot bound a doubly connected minimal surface.

Proof By contradiction, suppose that S is a doubly connected minimal surface bounded by $\Gamma_1 \cup \Gamma_2$. By the inclusion theorem that appears in [9], it is proved that the circumscribed circles C_1 and C_2 also bound a minimal surface $\tilde{S}$ of the type of the annulus. By the Shiffman's theorem, $\tilde{S}$ must be a Riemann minimal surface. However the same estimate (8) holds for $\tilde{S}$, which it is a contradiction with (6). □

This result gives some information on the function $M(r_1, r_2, d)$ proposed by Nitsche [8, Sec. IV.1] and by Nitsche and Leavitt in [12]. From (8), we have $M(r_1, r_2, d) \leq (\sqrt{2} + 1)(r_1 + r_2 - d)$.

We now compare the Nitsche's estimate (1) with the bound (6) in Theorem 1. The overlapping distance writes in terms of d and r_i as $\mathcal{O}_{1,2} = r_1 + r_2 - d$. Then (1) gives two types of inequalities for $\mathcal{O}_{1,2}$, namely,

$$\mathcal{O}_{1,2} \geq -d + \sqrt{h^2 + \frac{d^2}{2}}$$

and

$$\mathcal{O}_{1,2} \geq r_1 + r_2 - \sqrt{2(r_1 + r_2)^2 - h^2}.$$

In particular, both estimates depend on h and d or on h and $r_1 + r_2$. However, the inequality (6) gives a universal lower bound of $\mathcal{O}_{1,2}$ in terms of the vertical and lateral distances. Recall that the estimate (1) was proved with techniques of complex analysis and strongly using that the surface has the topology of an annulus.

3 Height estimates comparing with catenoids

Consider a catenoid whose rotational axis is the z-axis and suppose that the waist ω of the catenoid is a circle in the plane Π centered at the origin. Then the catenoid parametrizes as

$$Z(x, \theta) = \left(x \cos\theta, x \sin\theta, \omega \operatorname{arc cosh} \frac{x}{\omega}\right), \qquad x \geq \omega,\ \theta \in \mathbb{R}.$$

Denote by $\mathcal{C}_{r_1,r_2,\omega}$ the part of the catenoid bounded by two circles of radii r_1 and r_2. The height $h_{\text{cat}}(r_1, r_2, \omega)$ of $\mathcal{C}_{r_1,r_2,\omega}$ is the distance between the two boundaries circles and this value is

$$h_{\text{cat}}(r_1, r_2, \omega) = \begin{cases} \omega\left(\operatorname{arc cosh}\left(\frac{r_2}{\omega}\right) - \operatorname{arc cosh}\left(\frac{r_1}{\omega}\right)\right), & \text{if } 0 \leq z(r_1) < z(r_2) \\ \omega\left(\operatorname{arc cosh}\left(\frac{r_2}{\omega}\right) + \operatorname{arc cosh}\left(\frac{r_1}{\omega}\right)\right), & \text{if } z(r_1) < 0 < z(r_2). \end{cases} \tag{9}$$

If $\mathcal{C}_{r_1,r_2,\omega}$ is a symmetric catenoid, then $r_1 = r_2 = r$ and the height is $h_{\text{cat}}(r, \omega) = 2\omega \operatorname{arc cosh}(r/\omega)$. We compare the height $h_{\text{cat}}(r_1, r_2, \omega)$ of the catenoid $\mathcal{C}_{r_1,r_2,\omega}$ with the one of a Riemann minimal surface with the same radii and waist.

Theorem 2 *The height $h(r_1, r_2) = z(r_2) - z(r_1)$ of a Riemann minimal surface $\mathcal{R}_{1,2}(a, b)$ satisfies:*

1. *If $0 \leq z(r_1) < z(r_2)$, then*
$$\frac{a}{\sqrt{r_2^2 + a^2}} h_{\text{cat}}(r_1, r_2, b) \leq h(r_1, r_2) \leq \frac{a}{\sqrt{r_1^2 + a^2}} h_{\text{cat}}(r_1, r_2, b).$$
2. *If $z(r_1) < 0 < z(r_2)$, then for $i = 1, 2$ we have*
$$\frac{a}{2\sqrt{r_i^2 + a^2}} h_{\text{cat}}(r_i, b) \leq |z(r_i)| \leq \frac{a}{2\sqrt{b^2 + a^2}} h_{\text{cat}}(r_i, b).$$
3. *In the symmetric case, we have*
$$\frac{a}{\sqrt{r^2 + a^2}} h_{\text{cat}}(r, b) \leq h(r) \leq \frac{a}{\sqrt{b^2 + a^2}} h_{\text{cat}}(r, b).$$
In particular, $h < h_{\text{cat}}(r, b)$.

The inequality $h < h_{\text{cat}}(r, b)$ can formulate by saying that the height of a symmetric Riemann minimal surface is strictly less than the height of a catenoid of the same boundary radius and waist. Experimentally this inequality says that if we have formed a catenoid by a soap film framed by two given circles $C_1 \cup C_2$ of the same radius and we displace C_1 and C_2 sideways to produce Riemann minimal surfaces, if we want to keep the same waist of the catenoid, we have to reduce the vertical distance between C_1 and C_2 of the Riemann minimal surface.

Proof In a first case, suppose $0 \leq z(r_1) < z(r_2)$. For $b \leq r_1 \leq t \leq r_2$ we have $r_1^2 + a^2 \leq t^2 + a^2 \leq r_2^2 + a^2$ and this yields

$$\int_{r_1}^{r_2} \frac{\mathrm{d}t}{\sqrt{\Delta}} \leq \frac{1}{\sqrt{r_1^2 + a^2}} \int_{r_1}^{r_2} \frac{\mathrm{d}t}{\sqrt{t^2 - b^2}} = \frac{1}{\sqrt{r_1^2 + a^2}} \left(\operatorname{arc\,cosh} \frac{r_2}{b} - \operatorname{arc\,cosh} \frac{r_1}{b}\right),$$

and similarly,

$$\frac{1}{\sqrt{r_2^2 + a^2}} \left(\operatorname{arc\,cosh} \frac{r_2}{b} - \operatorname{arc\,cosh} \frac{r_1}{b}\right) \leq \int_{r_1}^{r_2} \frac{\mathrm{d}t}{\sqrt{\Delta}}.$$

The first item in Theorem 2 follows from the value $z(r)$ in (3), the height of $\mathcal{R}_{1,2}(a, b)$, namely, $z(r_2) - z(r_1)$ and the expression of $h_{\mathrm{cat}}(r_1, r_2, b)$ in (9).

Suppose now $z(r_1) < 0 < z(r_2)$. For the part of $\mathcal{R}_{1,2}(a, b)$ that lies above the plane Π, we have

$$\frac{1}{\sqrt{r_2^2 + a^2}} \operatorname{arc\,cosh} \frac{r_2}{b} \leq \int_{b}^{r_2} \frac{\mathrm{d}t}{\sqrt{\Delta}} \leq \frac{1}{\sqrt{b^2 + a^2}} \operatorname{arc\,cosh} \frac{r_2}{b}.$$

Then (3) gives the estimate for $z(r_2)$ in the second item. For $z(r_1) < 0$ the argument is similar.

The third item is proved by letting $r_1 = r_2 = r$ with $z(r_1) = -z(r_2)$, and using the previous item. □

With the same ideas than in Theorem 2, we can estimate the center of each circle of the foliation.

Corollary 3 *For a Riemann minimal surface $\mathcal{R}(a, b)$, consider the center $(c(r), 0, z(r))$ of the circle of $\mathcal{R}(a, b)$ at height $z = z(r)$. Then*

$$\frac{r\sqrt{r^2 - b^2} + b^2 \operatorname{arc\,cosh}(r/b)}{2\sqrt{r^2 + a^2}} \leq |c(r)| \leq \frac{r\sqrt{r^2 - b^2} + b^2 \operatorname{arc\,cosh}(r/b)}{2\sqrt{b^2 + a^2}},$$

$$|c(r)| \leq \frac{1}{2\sqrt{2}} \left(r\sqrt{\left(\frac{r}{b}\right)^2 - 1} + \frac{1}{2} h_{\mathrm{cat}}(r, b) \right).$$

Proof Without loss of generality, we suppose $z(r) \geq 0$, so $c(r) \geq 0$. For each $b \leq t \leq r$, we have $b^2 + a^2 \leq t^2 + a^2$. Then the expression of $c(r)$ in (3) gives

$$c(r) = \int_{b}^{r} \frac{t^2\,\mathrm{d}t}{\sqrt{\Delta}} \leq \frac{1}{\sqrt{b^2 + a^2}} \int_{b}^{r} \frac{t^2\,\mathrm{d}t}{\sqrt{t^2 - b^2}}$$
$$= \frac{1}{2\sqrt{b^2 + a^2}} \left(r\sqrt{r^2 - b^2} + b^2 \operatorname{arc\,cosh} \frac{r}{b} \right).$$

The other inequality for $c(r)$ uses $t^2 + a^2 \leq r^2 + a^2$. The estimate for $|c(r)|$ is a consequence that $a \geq b$. □

4 Numerical computations on the overlapping distance and the vertical distance

The estimate (6) is not sharp as we see when $d \to 0$, where the limit surface is a catenoid. In such a case, and when the two circles have the same radius r, we have $\mathcal{O}_{1,2} = 2r$, and thus $\mathcal{O}_{1,2} \geq 1.509h$, in contrast to the estimate (6) that gives $\mathcal{O}_{1,2} \geq 0.4142h$. However, the importance of our estimate is based that it holds for any radii r_1 and r_2 and any vertical distance h. The aim of this section is showing some computations of $\mathcal{O}_{1,2}$ in order to measure how amount of sharpness is the bound $M = \sqrt{2} - 1$. We also give some computations for the bound (7) of the vertical distance given in Corollary 1.

We begin with the estimate (6). Consider the symmetric case, that is, a Riemann minimal surface $\mathcal{R}_r$ bounded by two circles with the same radius $r > 0$. After an homothety of the space, we assume that the vertical distance is $h = 2$. Thus the value of $z(r)$ is $z(r) = 1$. For this value of h, the estimate (6) becomes $\mathcal{O}_{1,2} \geq 0.82843$. We explain the steps to follow in the next computations.

1. Fix the value of the radius r.
2. For each value of the parameter a, compute the value of $z(r)$ in (3) depending on the parameter b until that we get $z(r) = 1$.
3. The equality $z(r) = 1$ provides the value of the waist b.
4. Compute the x-coordinate $c(r)$ of the center of the circle C_2 by using (3).
5. Compute the overlapping distance $\mathcal{O}_{1,2} = 2(r - c(r))$.

Here we use the software `Mathematica` where the calculations have an approximation of 5 decimal digits.

In the above scheme, there appears the problem of whether any given circle C_2 and the corresponding symmetric circle C_1 at the height $-z(r)$, bound a symmetric Riemann minimal surface $\mathcal{R}_r$. In a first step, we begin for small values for r, namely, $r = 3$ and $r = 5$ (Table 1). In each step of computation, and once fixed the value of a, we do not know *a priori* if there exists a value of b so the height of $\mathcal{R}_r$ is 2.

Denote $z = z(r, a, b)$ the height of the circle C_2 indicating the dependence on the three parameters. Fixing r and b, we have

$$z(r, a, b) \geq \frac{ab}{\sqrt{a^2 + b^2}} \int_b^r \frac{\mathrm{d}t}{\sqrt{t^2 - b^2}} = \frac{ab}{\sqrt{a^2 + b^2}} \operatorname{arc\,cosh} \frac{r}{b}.$$

The function $b \mapsto b \operatorname{arc\,cosh}(r/b)$ defined in the interval $(0, r)$ has a unique maximum at a value b_0 which satisfies $\operatorname{arc\,cosh}(r/b_0) = r/\sqrt{r^2 - b_0^2}$. The value of $b_0 \operatorname{arc\,cosh}(r/b_0)$ is $rb_0/\sqrt{r^2 - b_0^2}$ which increases until ∞ as $r \to \infty$. Thus we can begin with a value a sufficiently big so $z(r, a, b)$ is bigger than $z = 1$, that is, the height where we want to place the circle C_2. Once obtained the value of a, we decrease the value of b until to get the height of the circle C_2 at $z = 1$.

By the values obtained in Table 1, we conclude the next facts:

1. Fixing r, the overlapping distance $\mathcal{O}_{1,2}$ decreases with the parameter a.
2. The value b that gives the desired height $z = 1$ is a decreasing function on the parameter a.
3. Fixing r, there exists a number α which is the infimum for the values a and the supremum for the values b, where the height $z = 1$ is attained. For this α, the corresponding overlapping distance $\mathcal{O}_{1,2}(r)$ is the infimum of all $\mathcal{O}_{1,2}(a, b)$ where r and h are keeping fix.
4. As r increases, the value $\mathcal{O}_{1,2}(r)$ decreases.

By Theorem 1 we know that if

$$\frac{\mathcal{O}_{1,2}(r)}{h} \leq M, \tag{10}$$

Table 1 Computation of $\mathcal{O}_{1,2}$ for for $r = 3, 5, 10, 20$

a	b	$c(r)$	$\mathcal{O}_{1,2}$
$r = 3$, $Mh = 0.82843$			
2	0.44804	1.69192	2.61617
1.5	0.52120	1.98096	2.03809
1.2	0.64592	2.20914	1.58172
1.1	0.75473	2.30265	1.39471
1.05	0.88490	2.35586	1.28828
$r = 5$, $Mh = 0.82843$			
2	0.37837	3.46157	3.07685
1	0.64224	4.31227	1.37547
0.9	0.81386	4.44295	1.11409
0.89	0.84655	4.45783	1.08435
0.885	0.86550	4.46545	1.06910
$r = 10$, $Mh = 0.82843$			
2	0.33947	8.26831	3.46339
1	0.54960	9.24573	1.50854
0.9	0.64195	9.37760	1.24481
0.82	0.79263	9.50102	0.99796
0.815	0.80755	9.50963	0.98074
$r = 20$, $Mh = 0.82843$			
2	0.32307	18.16736	3.66539
1	0.51733	19.21384	1.57244
0.8	0.75014	19.50452	0.99110
0.79	0.77603	19.52230	0.95547
0.787	0.78445	19.52770	0.94457

Table 2 Computation of $\mathcal{O}_{1,2}$ for $r = 50$

a	b	$c(r)$	$\mathcal{O}_{1,2}$
$r = 50$, $Mh = 0.82843$			
1	0.50057	49.19490	1.61019
0.8	0.70574	49.48690	1.02615
0.79	0.72687	49.55045	0.95547
0.772	0.77144	49.53730	0.92534
0.7719	0.77171	49.53750	0.92497

then there is not a (symmetric) Riemann minimal surface spanning $C_1 \cup C_2$. Our calculations cannot compute this quotient for large values of r because `Mathematica` uses the elliptic integrals of type E and F and the errors appear for large values of r. We have obtained the value of $\mathcal{O}_{1,2}$ at least for $r = 50$, where the constants a and b differ up to an accuracy of ten thousandths. See Table 2.

It is an open problem to estimate a sharper upper bound (if there is) for the quotient $\mathcal{O}_{1,2}(r)/h$ in (10).

Table 3 Comparison between the estimate (7) and the real value of the height h of a symmetric Riemann surface $\mathcal{R}_r(a, b)$

b	a	h	$\frac{2(r-c(r))}{M}$
$r = 2$, $d = 1$, $c(r) = 0.5$			
1.8	2.66677	1.37785	7.24264
1.5	4.0990	2.21032	7.24264
1.2	4.52332	2.50766	7.24264
0.9	4.51030	2.48868	7.24264
0.7	4.3663	2.31696	7.24264
$r = 2$, $d = 2$, $c(r) = 1$			
1.4	1.63133	1.79571	4.82843
1.2	1.8375	2.06828	4.82843
1	1.90354	2.16727	4.82843
0.8	1.88183	2.1264	4.82843
0.6	1.80280	1.95105	4.82843
$r = 2$, $d = 3$, $c(r) = 1.5$			
0.8	0.82120	1.47998	2.41421
0.75	0.824235	1.48700	2.41421
0.7	0.82302	1.48408	2.41421
0.65	0.81807	1.47157	2.41421
0.6	0.80990	1.44977	2.41421
$r = 2$, $d = 3.8$, $c(r) = 1.9$			
0.15	0.16141	0.39033	0.48284
0.16	0.16471	0.40208	0.48284
0.165	0.16630	0.40767	0.48284
0.166	0.16662	0.40878	0.48284
0.1666	0.16680	0.40943	0.48284

Here the radius is $r = 2$ and the lateral distance is $d = 2c(r) = 1, 2, 3$ and 3.8

In the last part of this section, we study the estimate for the vertical distance given in (7). It is not expectable a sharp bound for a general situation because inequality (7) holds for any Riemann minimal surface. Again we consider the symmetric case and after a homothety we suppose that the radii of the boundary circles are $r = 2$. Here we fix two circles C_1 and C_2 of radii $r = 2$ situated in each side of Π. Let d be the lateral distance which it agrees with the value $2c(r)$. Now the steps to follow in the calculations have been:

1. Fix the lateral distance d, that is, fix the value $c(r)$.
2. Take a value b, the waist, where we know that $b < r$.
3. For each value of b, calculate the parameter of a so we have $c(r) = d/2$.
4. Once we know a and b, compute the height of $\mathcal{R}_r(a, b)$ which agrees with $2z(r)$ and compare with the estimate (7).

See Table 3. We have observed that fixed the value of b, the value of $c(r)$ is decreasing on a. Thus, when we fix the value of d, for each value of b, we take $a = b$ such that $c(r) > d/2$. Then, we increase the parameter a so $c(r)$ decreases until we get the desired lateral distance $2c(r)$. Finally we compute the height h of the Riemann minimal surface and we compare the estimate (7) with $2(r - c(r))/M$.

Remark 1 By the computations obtained in Table 3 we point out that for two given circles $C_1 \cup C_2$ (in this case of radius $r = 2$) separated a fixed vertical distance h, there exist two Riemann minimal surfaces spanning $C_1 \cup C_2$. Indeed, in each one of the cases $d = 1$, $d = 2$ and $d = 3$, we observe that there exists a maximum of the vertical distance h_{max} (for $d = 1$, $d = 2$ and $d = 3$, the value h_{max} it is approximately 2.50766, 2.16727 and 1.48700, respectively). For values of h close to h_{max} with $h < h_{max}$ we have two pairs (a_1, b_1) and (a_2, b_2) such that $\mathcal{R}_r(a_1, b_1)$ and $\mathcal{R}_r(a_2, b_2)$ have the same boundary. This is expectable by the results of Meeks and White in [5] that assert under certain configurations that given two curves lying in parallel planes, these curves are the boundary of exactly two minimal annuli, one stable and one unstable. As in the case of the catenoid, it is expectable that the Riemann minimal surface with least waist is unstable while the Riemann minimal surface with biggest waist is stable.

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