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Towards the classification of odd-dimensional homogeneous reversible Finsler spaces with positive flag curvature

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Abstract In this paper, we use the flag curvature formula for homogeneous Finsler spaces in our previous work to classify odd-dimensional smooth coset spaces admitting positively curved reversible homogeneous Finsler metrics. We will show that most important features of L. Bérard-Bergery's classification results for odd-dimensional positively curved Riemannian homogeneous spaces can be generalized to reversible Finsler spaces.

Keywords Finsler metric · Flag curvature · Homogeneous Finsler space · Compact Lie group · Compact Lie algebra

Mathematics Subject Classification 22E46 · 53C30

1 Introduction

Finding new examples of compact manifolds admitting Riemannian metrics of positive sectional curvature is one of the central problems in Riemannian geometry. In the homogeneous setting, the problem is to classify positively curved Riemannian homogeneous spaces, and this has been achieved in several classical works in this field; see [1–3,16]. Notice that in [2], Berger missed one in his classification of positively curved normal homogeneous spaces,

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as pointed out by Wilking [17]. In the classification of odd-dimensional positively curved Riemannian homogeneous spaces by Bérard-Bergery [3], a gap was recently found by Xu and Wolf, and it has been corrected by Wilking; see [25]. Based on some more advanced methods developed in [18], Wilking and Ziller provided an alternative and modern proof of the classification in [3] in their recent preprint [19].

In homogeneous Finsler geometry, the following problem is of great significance:

Problem 1.1 Classify the smooth coset spaces G/H admitting a G-invariant Finsler metric with positive flag curvature.

For simplicity, we will call a homogeneous space *positively curved* when it admits an invariant Finsler metric with positive flag curvature, or if it has been endowed with such a metric. By the Bonnet–Myers Theorem for Finsler spaces, a positively curved homogeneous space must be compact.

Problem 1.1 was first studied by Deng and Hu [13], where they classified homogeneous Randers metrics with positive flag curvature and vanishing S-curvature. Note that their classification is also valid for homogeneous (α, β)-spaces with positive flag curvature and vanishing S-curvature [22].

Recently, big progress has been made on the classification with more generality. In [20], we classified positively curved normal homogeneous Finsler spaces, generalizing the classical results of [2]. In the joint work of the authors with Huang and Hu [23], we classified even-dimensional positively curved homogeneous Finsler spaces, generalizing the results of [16].

It should be noted that a very useful homogeneous flag curvature formula has been established in [23] (see Theorem 3.5 below). In this paper, we will apply this formula to the classification of odd-dimensional positively curved homogeneous Finsler spaces.

The general theme for the classification has been set up in [20]. Recall that for a positively curved homogeneous Finsler space $(G/H, F)$ with a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie group $\mathfrak{g}$, and a fundamental Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ (i.e. $\mathfrak{t} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$), we call H a *regular subgroup* of G if each root plane of $\mathfrak{h} = \mathrm{Lie}(H)$ with respect to $\mathfrak{t} \cap \mathfrak{h}$ is also a root plane of $\mathfrak{g}$ with respect to $\mathfrak{g}$. Otherwise, we call H *not regular* or *irregular*. We will divide our discussion into three cases (see Sect. 3.3), where in Case I H is regular in G and in Case II and III H is not.

The classification is only up to local isometry. So we introduce an equivalence relation (see Sect. 2.5) for homogeneous Finsler spaces to specify some typical procedures which results local isometries, such as changing G and H to their covering groups, cancelling common product factors from G and H, changing the pair G and H by an isomorphism of G, and so on. This technical terminology greatly reduces the complexity of the statement and the proofs of the classification.

In this paper, we shall consider the classification of odd-dimensional reversible homogeneous Finsler spaces with positive flag curvature. Our motivation to consider reversible metrics is twofold. On one hand, our application of the homogeneous flag curvature formula can only be carried out with the assumption that the metric is reversible. On the other hand, restricting our discussion to reversible Finsler metrics will not lose much generality. It includes the Riemannian case and many other important types of Finsler metrics.

For the case when H is irregular in G, we get a complete classification, which can be summarized as the following main theorem.

Theorem 1 *Let $(G/H, F)$ be an odd-dimensional positively curved reversible homogeneous Finsler space. If H is not regular in G, then G/H admits a G-invariant Riemannian metric with positive curvature.*

Theorem 1 follows immediately from Theorems 3 and 6.2, which deal with Case III and II, respectively. To be precise, the classification list is the following. When G/H belongs to Case II, it is equivalent to the homogeneous spheres $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$, or $S^{4n-1} = \mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n-1)\mathrm{Sp}(1), n > 1$, or Wilking's space $\mathrm{SU}(3) \times \mathrm{SO}(3)/\mathrm{U}(2)$. When G/H belongs to Case III, it is equivalent to the homogeneous spheres $S^{2n-1} = \mathrm{SO}(2n)/\mathrm{SO}(2n-1), n > 2$, $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$, $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$, or one of the Berger's spaces $\mathrm{SU}(5)/\mathrm{Sp}(2)\mathrm{U}(1)$ and $\mathrm{Sp}(2)/\mathrm{SU}(2)$. It is obvious that all these coset spaces admit positively curved Riemannian homogeneous metrics.

Notice that any invariant Finsler metric on the coset space $S^{2n-1} = \mathrm{SO}(2n)/\mathrm{SO}(2n-1)$ or $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$ must be the standard Riemannian metric of positive constant curvature. On the other hand, as pointed out in [13,22], the Aloff–Wallach's spaces admit non-Riemannian homogeneous Randers metrics or (α, β)-metrics with positive flag curvature and vanishing S-curvature. Moreover, any of the other coset spaces listed in Theorem 1 admits a non-Riemannian positively curved normal homogeneous Finsler metric; see [20].

In the case that H is regular in G, we prove the following theorem:

Theorem 2 *Let $(G/H, F)$ be an odd-dimensional positively curved reversible homogeneous Finsler space. If H is a regular subgroup of G, then there are only the following two cases:*

(1) *G/H is equivalent to the homogeneous spheres $S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1)$, $S^{4n-1} = \mathrm{Sp}(n)\mathrm{U}(1)/\mathrm{Sp}(n-1)\mathrm{U}(1), n > 1$, or the $\mathrm{U}(3)$-homogeneous Aloff–Wallach's spaces.*
(2) *G/H is equivalent to an odd-dimensional reversible positively curved homogeneous Finsler space G'/H' such that G' is compact simple and H' is a regular subgroup in G'.*

To finish this classification, we need to discuss the case (2) in Theorem 2. This will be further studied in [26]. Besides the homogeneous spheres $S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1)$, $S^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$, and SU(3)-homogeneous Aloff–Wallach's spaces, which are known to admit positively curved homogeneous Riemannian metrics (as well as non-Riemannian positively curved homogeneous Randers metrics), there are several undetermined potential candidates.

This work is organized as following. In Sect. 2, we give a brief summary of basic notions in Finsler geometry and homogeneous Finsler geometry and define the notion of equivalence which will be used throughout this paper. In Sect. 3, we present the general theme for the classification of odd-dimensional positively curved homogeneous Finsler spaces, including the homogeneous flag curvature formula, the rank equality, and some useful lemmas. In Sects. 4 and 5, we discuss the classification of odd-dimensional positively curved reversible homogeneous Finsler spaces in Case III. In Sect. 6, we discuss the classification of odd-dimensional positively curved reversible homogeneous Finsler spaces in Case II and I.

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2 Preliminaries

In this section, we summarize some definitions and fundamental results in Finsler geometry; see [6,7] for more details. In this paper, we will only consider connected smooth manifolds and connected Lie groups.

2.1 Minkowski norm and Finsler metric

A *Minkowski norm* on a real vector space $\mathbf{V}$, $\dim \mathbf{V} = n$, is a continuous real-valued function $F : \mathbf{V} \to [0, +\infty)$ satisfying the following conditions:

(1) F is positive and smooth on $\mathbf{V}\backslash\{0\}$;
(2) $F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$;
(3) With respect to any linear coordinates $y = y^i e_i$, the Hessian matrix

$$(g_{ij}(y)) = \left(\frac{1}{2}[F^2]_{y^i y^j}\right) \tag{2.1}$$

is positive definite at any nonzero y.

The Hessian matrix $(g_{ij}(y))$ and its inverse $(g^{ij}(y))$ can be used to move up and down indices of relevant tensors in Finsler geometry.

Given a nonzero vector y, the Hessian matrix $(g_{ij}(y))$ defines an inner product $\langle\cdot,\cdot\rangle_y$ on $\mathbf{V}$ by

$$\langle u, v\rangle_y = g_{ij}(y)u^i v^j,$$

where $u = u^i e_i$ and $v = v^i e_i$. In the literature, the above inner product is also denoted as $\langle\cdot,\cdot\rangle_y^F$ to specify the norm. Sometimes it is shortened as g_y or g_y^F. This inner product can also be expressed as

$$\langle u, v\rangle_y = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s=t=0}. \tag{2.2}$$

It is easy to check that the above definition is independent of the choice of linear coordinates.

Let M be a smooth manifold of dimension n. A *Finsler metric* F on M is a continuous function $F : TM \to [0, +\infty)$ such that it is positive and smooth on the slit tangent bundle $TM\backslash 0$, and its restriction to each tangent space is a Minkowski norm. Generally, (M, F) is called a *Finsler manifold* or a *Finsler space*.

Here are some important examples.

Riemannian metrics are a special class of Finsler metrics such that the Hessian matrix only depends on $x \in M$. For a Riemannian manifold, the metric is often referred to as the global smooth section $g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ of $\mathrm{Sym}^2(T^*M)$. Unless otherwise stated, we mainly deal with non-Riemannian metrics in this paper.

Randers metrics are the simplest and the most important class of non-Riemannian metrics in Finsler geometry. A Randers metric can be written as $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form. The notion of Randers metrics can be naturally generalized to (α, β)-metrics. An (α, β)-metric is a Finsler metric of the form $F = \alpha\phi(\beta/\alpha)$, where ϕ is a positive smooth real function, α is a Riemannian metric and β is a 1-form. In recent years, there have been a lot of research works concerning (α, β)-metrics as well as Randers metrics.

Recently, we have defined and studied (α_1, α_2)-metrics and introduced the more generalized class of $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-metrics; see [10,23]. Such metrics naturally appear in the study of homogeneous Finsler geometry.

A Minkowski norm or a Finsler metric is called *reversible* if $F(y) = F(-y)$ for any $y \in \mathbf{V}$ or $F(x, y) = F(x, -y)$ for any $x \in M$ and $y \in T_xM$. Obviously, a Riemannian metric is reversible, and a non-Riemannian Randers metric must be non-reversible. Note that a non-Riemannian (α, β)-metric is reversible if the function ϕ is an even function, and there exist many non-reversible (α, β)-metrics.

2.2 Geodesic spray and geodesics

Let (M, F) be a Finsler space. A local coordinate system $\{x = (x^i) \in M; y = y^j \partial_{x^j} \in T_x M\}$ on TM is called a *standard local coordinates system*. The geodesic spray is a vector field $\mathbf{G}$ globally defined on $TM\backslash 0$. On a standard local coordinate system, it can be expressed as

$$\mathbf{G} = y^i \partial_{x^i} - 2\mathbf{G}^i \partial_{y^i}, \tag{2.3}$$

in which

$$\mathbf{G}^i = \frac{1}{4} g^{il} \left([F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right). \tag{2.4}$$

A non-constant curve $c(t)$ on M is called a geodesic if $(c(t), \dot{c}(t))$ is an integral curve of G, in which the tangent field $\dot{c}(t) = \frac{\mathrm{d}}{\mathrm{d}t} c(t)$ along the curve gives the speed. On a standard local coordinate, a geodesic $c(t) = (c^i(t))$ can be characterized by the equations

$$\ddot{c}^i(t) + 2\mathbf{G}^i(c(t), \dot{c}(t)) = 0. \tag{2.5}$$

It is well known that $F(c(t), \dot{c(t)})$ is a constant function, or in other words, a geodesic defined by the above equations must be of nonzero constant speed.

2.3 Riemann curvature and flag curvature

In Finsler geometry, there is a similar notion of curvature as in the Riemannian case, which is called the Riemann curvature. It can be defined either by the Jacobi field or the structure equation for the curvature of the Chern connection.

On a standard local coordinate system, the Riemann curvature is a linear map $R_y = R^i_k(y)\partial_{x^i} \otimes \mathrm{d}x^k : T_x M \to T_x M$, defined by

$$R^i_k(y) = 2\partial_{x^k}\mathbf{G}^i - y^j \partial^2_{x^j y^k}\mathbf{G}^i + 2\mathbf{G}^j \partial^2_{y^j y^k}\mathbf{G}^i - \partial_{y^j}\mathbf{G}^i \partial_{y^k}\mathbf{G}^j. \tag{2.6}$$

When the metric needs to be specified, the Riemann curvature is denoted as ${R^F}_y = (R^F)^i_k(y)\partial_{x^i} \otimes \mathrm{d}x^k$. From Proposition 6.2.2 of [14], it is easily seen that the Riemann curvature R_y is self-adjoint with respect to the inner product $\langle\cdot, \cdot\rangle_y$.

Using the Riemann curvature, we can generalize the notion of sectional curvature to Finsler geometry, called the flag curvature. Let $y \in T_x M$ be a nonzero tangent vector and $\mathbf{P}$ a tangent plane in $T_x M$ containing y, and suppose it is linearly spanned by y and v. Then the flag curvature of the pair $(y, \mathbf{P})$ is defined by

$$K(x, y, y \wedge v) = K(x, y, \mathbf{P}) = \frac{\langle R_y v, v\rangle_y}{\langle y, y\rangle_y \langle v, v\rangle_y - \langle y, v\rangle_y^2}. \tag{2.7}$$

Obviously, the flag curvature in (2.7) does not depend on the choice of v but only on y and $\mathbf{P}$. Sometimes we also write the flag curvature of a Finsler metric F as $K^F(x, y, y \wedge v)$ or $K^F(x, y, \mathbf{P})$ to indicate the metric explicitly.

2.4 Totally geodesic submanifolds

A submanifold N of a Finsler space (M, F) can be naturally endowed with a submanifold Finsler metric, denoted as $F|_N$. At each point $x \in N$, the Minkowski norm $F|_N(x, \cdot)$ is just the restriction of the Minkowski norm $F(x, \cdot)$ to $T_x N$. We say that $(N, F|_N)$ is a *Finsler submanifold* or a *Finsler subspace*.

A Finsler subspace $(N, F|_N)$ of (M, F) is called *totally geodesic* if any geodesic of $(N, F|_N)$ is also a geodesic of (M, F). On a standard local coordinate system (x^i, y^j) such that N is locally defined by $x^{k+1} = \cdots = x^n = 0$, and the totally geodesic condition can be expressed as

$$\mathbf{G}^i(x, y) = 0, \quad k < i \leq n, x \in N, y \in T_x N.$$

A direct calculation shows that in this case the Riemann curvature $R_y^{F|_N} : T_x N \to T_x N$ of $(N, F|_N)$ is just the restriction of the Riemann curvature R_y^F of (M, F), where y is a nonzero tangent vector of N at $x \in N$. Therefore, we have .„

Proposition 2.1 *Let $(N, F|_N)$ be a totally geodesic submanifold of (M, F). Then for any $x \in N$, $y \in T_x N \backslash 0$, and a tangent plane $\mathbf{P} \subset T_x N$ containing y, we have*

$$K^{F|_N}(x, y, \mathbf{P}) = K^F(x, y, \mathbf{P}). \tag{2.8}$$

As in Riemannian geometry, the local properties of exponential maps implies any connected component N of the common fixed points for a set of isometries $\{\rho_a, a \in \mathcal{A}\}$ of (M, F) is a totally geodesic submanifolds of (M, F). To be more precise, for each point $x \in N$,

$$T_x N = \{y \in T_x M | \rho_{a*} y = y, \forall a \in \mathcal{A}\}$$

and N contains a small neighbourhood of x in $\exp_x T_x N$.

2.5 Homogeneous Finsler geometry

Let (M, F) be a connected Finsler manifold. If the full group $I(M, F)$ of isometries of (M, F) (or equivalently, the identity component $I_0(M, F)$ of $I(M, F)$) acts transitively on M, then we say that (M, F) is a *homogeneous Finsler space*, or F is a *homogeneous Finsler metric*. By [8,15], $I(M, F)$ (hence $G = I_0(M, F)$) is a Lie transformation group on M, which can be identified as the isometry group for the (possibly irreversible) distance function $d_F(\cdot, \cdot)$ that F defines on M. Let H be the compact isotropic subgroup of G at a point $o \in M$. Then M is diffeomorphic to the smooth coset space G/H, associated with a canonical smooth projection map $\pi : G \to M = G/H$ such that $\pi(e) = o$. The tangent space $T_o M$ can be naturally identified with $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$, in which $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively. The isotropy action of H on $T_o M$ coincides with the induced $\mathrm{Ad}(H)$-action on $\mathfrak{m}$. In the cases we will consider in this paper, $\mathfrak{m}$ can be realized as a complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$ which is preserved by $\mathrm{Ad}(H)$-actions. Then we have an $\mathrm{Ad}(H)$-*invariant decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ satisfying the reductive condition $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

If (M, F) is positively curved, then by the Bonnet–Myers Theorem, M must be compact, and hence $G = I_0(M, F)$ is also compact. Fix a bi-invariant inner product on $\mathfrak{g}$. Then we can realize $\mathfrak{m}$ as the bi-invariant orthogonal complement of $\mathfrak{h}$. In this case, the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is called a *bi-invariant orthogonal decomposition* for the homogeneous space G/H.

Notice that for any closed connected subgroup G of $I_0(M, F)$ which acts transitively on M, we have a corresponding representation $M = G/H$. The most typical example is the nine classes of homogeneous spheres; see [4]. For convenience, we will consider a slightly more general situation, namely for a positively curved homogeneous Finsler space $M = G/H$, we only require that the Lie algebra $\mathfrak{g}$ of G is compact (i.e. G is quasi-compact). The notion of bi-invariant orthogonal decomposition is still valid in this case.

To simplify the discussion and avoid unnecessary iterance in the classification, we will not distinguish homogeneous Finsler spaces which are locally isometric to each other. In particular, we will call $(G_1/H_1, F_1)$ and $(G_2/H_2, F_2)$ (with corresponding bi-invariant orthogonal decompositions for the compact Lie groups $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively) *equivalent* if one of the following conditions is satisfied

(1) G_1 is a covering group of G_2, with the connected components of the isotropy subgroups $(H_1)_0$ covering $(H_2)_0$, and F_1 is naturally induced from F_2, up to a positive scalar;
(2) $G_1 = G_2 \times G'$, $H_1 = H_2 \times G'$, and F_1 and F_2 are induced from the same Minkowski norm, when $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are naturally identified as the same vector space;
(3) There exists a group isomorphism from G_1 to G_2, which maps H_1 onto H_2 and induces an isometry from F_1 to F_2.

The above notion actually defines an *equivalent relation* on the set of compact homogeneous Finsler spaces G/H with $\mathfrak{g} = \mathrm{Lie}(G)$ compact. In the following, compact homogeneous Finsler spaces in the same equivalent class will not be distinguished. Thus our classification will be local, or in other words, on the Lie algebra level.

3 The general theme for the classification

In this section, we establish the theme for our classification.

3.1 The totally geodesic technique and the rank equality

Assume that $(G/H, F)$ is a positively curved homogeneous Finsler space, with a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie group $\mathfrak{g}$.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t}\cap\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$. For simplicity, we just call $\mathfrak{t}$ a *fundamental Cartan subalgebra*. Denote the tori T, T_H and T' such that $\mathrm{Lie}(T) = \mathfrak{t}$, $\mathrm{Lie}(T_H) = \mathfrak{t} \cap \mathfrak{h}$, and $\mathrm{Lie}(T') = \mathfrak{t}'$ is a subalgebra of $\mathfrak{t} \cap \mathfrak{h}$. Let G' be the connected group $(C_G(T'))_0 = (C_G(\mathfrak{t}'))_0$. Its Lie algebra $\mathfrak{g}' = \mathrm{Lie}(G')$ can be identified as $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{t}')$. Let $H' = G' \cap H$. Then $(G'/H', F|_{G'/H'})$ is a homogeneous submanifold of $(G/H, F)$.

We first prove the following useful lemma.

Lemma 3.1 *Keep all the above notation. Then $(G'/H', F|_{G'/H'})$ is totally geodesic in $(G/H, F)$. In particular, if G/H admits positively curved homogeneous Finsler metrics and* $\dim G'/H' > 1$, *then G'/H' also admits positively curved homogeneous Finsler metrics.*

Proof By Corollary II.5.7 of [5], the set of common fixed points of T' is a disconnected union of finite orbits of $N_G(T') = \{g \in G | g^{-1}T'g = T'\}$. Thus the connected component of $N_G(T')\cdot o$ containing $o = eH$, which coincides with G'/H', is a totally geodesic submanifold of $(G/H, F)$. Therefore, if $(G/H, F)$ is positively curved and $\dim G'/H' > 1$, then the homogeneous Finsler space $(G'/H', F|_{G'/H'})$ has positive flag curvature. □

Lemma 3.1 is valid when T' is changed to any closed subgroup of H. Using it, we can shorten some later argument. For simplicity, we call it the totally geodesic technique. Notice up to equivalence, G' and H' contains a common product factor T' which can be cancelled.

An immediate application of Lemma 3.1 is the rank inequality (see Theorem 5.2 of [23]) $\mathrm{rk}\mathfrak{g} \leq \mathrm{rk}\mathfrak{h} + 1$ for any positively curved homogeneous Finsler space G/H. Take $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{h}$, then $F' = F|_{G'/H'}$ induces a left invariant Finsler metric F'' on the compact Lie group G''

with $\mathrm{Lie}(G'') = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t} \cap \mathfrak{h}) \cap \mathfrak{m}$. Then the above lemma implies that if $\dim G'' > 1$, then F'' is positively curved. Thus by Theorem 5.1 of [9], we have $G'' = \mathrm{U}(1)$, $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. The rank inequality follows immediately. In the case that $\dim G/H$ is odd, we get the following rank equality

Corollary 3.2 *Let* $(G/H, F)$ *be an odd-dimensional positively curved homogeneous Finsler space with compact* $\mathfrak{g} = \mathrm{Lie}(G)$. *Then* $\mathrm{rk}\mathfrak{g} = \mathrm{rk}\mathfrak{h} + 1$.

3.2 Some notations for Lie algebras and root systems

We now introduce some notations for the relevant Lie algebras and root systems used in [24]. Let $(G/H, F)$ be an odd-dimensional positively curved homogeneous Finsler space with a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The orthogonal projections to the $\mathfrak{h}$-factor and $\mathfrak{m}$-factor are denoted as $\mathrm{pr}_{\mathfrak{h}}$ and $\mathrm{pr}_{\mathfrak{m}}$, respectively. Our conventions are as following. We will use a suitably chosen bi-invariant inner product of $\mathfrak{g}$ to identify the root system of $\mathfrak{g}$ as the subset of $\mathfrak{g}$ with the standard presentation for each of its simple factor. By the same bi-invariant inner product on $\mathfrak{g}$ (i.e. its restriction on $\mathfrak{h}$), the root system of $\mathfrak{h}$ is regarded as a subset of $\mathfrak{t} \cap \mathfrak{h}$. We will use α, β, γ, etc, to denote vectors in $\mathfrak{t}$, and particularly α', β', γ', etc, to denote vectors in $\mathfrak{t} \cap \mathfrak{h}$. No matter if the vector in $\mathfrak{t}$ (or $\mathfrak{t} \cap \mathfrak{h}$) is or is not a root of $\mathfrak{g}$ (or $\mathfrak{h}$), the root plane can be formally defined, and it is 0 when the vector is not a root.

Fix a fundamental Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ (i.e. $\mathfrak{t} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$). From now on, root systems, root planes, etc, for $\mathfrak{g}$ will be taken with respect to $\mathfrak{t}$, and those for $\mathfrak{h}$ will be taken with respect to $\mathfrak{t} \cap \mathfrak{h}$. It is easy to see that $\mathfrak{t}$ is a splitting Cartan subalgebra, i.e., $\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{h}) + (\mathfrak{t} \cap \mathfrak{m})$. By Corollary 3.2, we have $\dim(\mathfrak{t} \cap \mathfrak{m}) = 1$.

We have the maximal torus T (resp. T_H) of G (resp. H) corresponding to $\mathfrak{t}$ (resp. $\mathfrak{t} \cap \mathfrak{h}$). We now have the following decomposition of $\mathfrak{g}$ with respect to $\mathrm{Ad}(T)$-actions:

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta_{\mathfrak{g}}} \mathfrak{g}_{\pm\alpha},$$

where $\Delta_{\mathfrak{g}} \subset \mathfrak{t}$ is the root system of $\mathfrak{g}$, and for each $\alpha \in \Delta_{\mathfrak{g}}$, $\mathfrak{g}_{\pm\alpha}$ is a two-dimensional irreducible representation of $\mathrm{Ad}(T)$-actions, called a *root plane* (notice $\mathfrak{g}_{\pm\alpha} = \mathfrak{g}_{\pm\beta}$ when $\alpha = -\beta$).

For the compact Lie subalgebra $\mathfrak{h} = \mathrm{Lie}(H)$, we have a similar decomposition with respect to $\mathrm{Ad}(T_H)$-actions, i.e.,

$$\mathfrak{h} = \mathfrak{t} \cap \mathfrak{h} + \sum_{\alpha' \in \Delta_{\mathfrak{h}}} \mathfrak{h}_{\pm\alpha'},$$

where $\Delta_{\mathfrak{h}} \subset \mathfrak{t} \cap \mathfrak{h}$ is the root system of $\mathfrak{h}$, and for each root $\alpha' \in \Delta_{\mathfrak{h}}$, $\mathfrak{h}_{\pm\alpha'}$ is the two-dimensional root plane.

There is another decomposition of $\mathfrak{g}$ with respect to the $\mathrm{Ad}(T_H)$-action, namely

$$\mathfrak{g} = \sum_{\alpha' \in \mathfrak{t} \cap \mathfrak{h}} \hat{\mathfrak{g}}_{\pm\alpha'}, \tag{3.9}$$

where

$$\hat{\mathfrak{g}}_{\pm\alpha'} = \sum_{\mathrm{pr}_{\mathfrak{h}}(\alpha) = \alpha'} \mathfrak{g}_{\pm\alpha}, \quad \text{if } \alpha' \neq 0,$$

$\hat{\mathfrak{g}}_0 = \mathfrak{t} \cap \mathfrak{m} + \mathfrak{g}_{\pm\alpha}$, if there is a root α of $\mathfrak{g}$ contained in $\mathfrak{t} \cap \mathfrak{m}$, and $\hat{\mathfrak{g}}_0 = \mathfrak{t} \cap \mathfrak{m}$ otherwise. This $\mathrm{Ad}(T_H)$-*invariant decomposition* is compatible with the bi-invariant orthogonal decomposition in the sense that

$$\hat{\mathfrak{g}}_{\pm\alpha'} = (\hat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{h}) + (\hat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{m}).$$

To be more precise, we have the following easy lemma, which will be repeatedly used in the sequel.

Lemma 3.3 *Let α' be a vector of $\mathfrak{t} \cap \mathfrak{h}$. Then we have the following:*

(1) *if $\alpha' \in \Delta_{\mathfrak{h}}$, then we have $\hat{\mathfrak{g}}_{\pm\alpha'} = (\hat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{h}) + (\hat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{m})$, where $\hat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{h} = \mathfrak{h}_{\pm\alpha'}$;*
(2) *if $\alpha' \notin \Delta_{\mathfrak{h}}$, then we have $\hat{\mathfrak{g}}_{\pm\alpha'} \subset \mathfrak{m}$. In particular, $\hat{\mathfrak{g}}_0 \subset \mathfrak{m}$, and $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{m}$ if $\mathrm{pr}_{\mathfrak{h}}\alpha \notin \Delta_{\mathfrak{h}}$.*

For the bracket between root planes, we have the following well-known relation,

$$[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}] \subseteq \mathfrak{g}_{\pm(\alpha+\beta)} + \mathfrak{g}_{\pm(\alpha-\beta)}, \tag{3.10}$$

where $\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_{\pm\beta}$ are different root planes, i.e., $\alpha \neq \pm\beta$, and each term of the right side can be 0 when the corresponding vector is not a root of $\mathfrak{g}$. In fact, this is just a special case of the following general fact; see for example [11].

Lemma 3.4 *Keep all the above notation. We have*

(1) *For any root α of $\mathfrak{g}$, $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\alpha}] = \mathbb{R}\alpha$.*
(2) *Let α and β be two linearly independent roots of $\mathfrak{g}$. If none of the roots $\alpha \pm \beta$ is a root of $\mathfrak{g}$, then $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}] = 0$; if one of $\alpha \pm \beta$, say γ, is a root, and the other is not, then $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}] = \mathfrak{g}_{\pm\gamma}$; If both $\alpha \pm \beta$ are roots of $\mathfrak{g}$, then $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}]$ is a cone in $\mathfrak{g}_{\pm(\alpha+\beta)} + \mathfrak{g}_{\pm(\alpha-\beta)}$.*
(3) *In the second case of (2), for any nonzero vector $v \in \mathfrak{g}_{\pm\alpha}$, the linear map $\mathrm{ad}(v)$ is an isomorphism from $\mathfrak{g}_{\pm\beta}$ onto $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}] = \mathfrak{g}_{\pm\gamma}$.*

3.3 The three cases and the reversibility assumption

Keep all the above assumptions and notation. In [21], we established the general theme for our classification of positively curved normal homogeneous Finsler spaces. The main idea can be applied to this paper. In particular, we only need to consider the following three cases for the classification of odd-dimensional positively curved homogeneous Finsler spaces:

Case I. Each root plane of $\mathfrak{h}$ is a root plane of $\mathfrak{g}$.
Case II. There exists a root plane of $\mathfrak{h}$ which is not that of $\mathfrak{g}$. For the corresponding root α' of $\mathfrak{h}$, there are two roots α and β of $\mathfrak{g}$ from different simple factors such that $\mathrm{pr}_{\mathfrak{h}}(\alpha) = \mathrm{pr}_{\mathfrak{h}}(\beta) = \alpha'$ is a root of $\mathfrak{h}$.
Case III. The same as Case II except that the roots α and β are from the same simple factor of $\mathfrak{g}$.

Here we keep all notation of the previous subsection with respect to the chosen bi-invariant inner product (which determines the bi-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ as well) and a fixed fundamental Cartan subalgebra $\mathfrak{t}$. It is easy to see that H is regular in G in Case I, and H is not regular in Case II and III.

In the following sections, we will restrict our discussion to reversible Finsler metrics (i.e. $F(x, y) = F(x, -y)$ for any $y \in T_x(G/H)$). The reason for adding this condition for F will be explained in the next subsection.

It turns out that with the reversibility assumption for F, Case II is the easiest. Case III contains a lot of case-by-case discussions. But in this case we can use the root α' of $\mathfrak{h}$ to get the complete classification. Case I turns out to be very difficult, and we can only get some partial classification result for this case.

Adding the reversibility assumption will provide an alternative certification that the classification result in [3] is correct.

3.4 The homogeneous flag curvature formula and the key lemmas

First we quote the following theorem which gives a very useful homogeneous flag curvature formula.

Theorem 3.5 *Let* $(G/H, F)$ *be a connected homogeneous Finsler space, and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ *be an* $\mathrm{Ad}(H)$*-invariant decomposition for* G/H*. Then for any linearly independent commutative pair* u *and* v *in* $\mathfrak{m}$ *satisfying* $\langle [u, \mathfrak{m}], u\rangle_u^F = 0$*, we have*

$$K^F(o, u, u \wedge v) = \frac{\langle U(u, v), U(u, v)\rangle_u^F}{\langle u, u\rangle_u^F \langle v, v\rangle_u^F - \langle u, v\rangle_u^F \langle u, v\rangle_u^F},$$

where U *is the bilinear map from* $\mathfrak{m} \times \mathfrak{m}$ *to* $\mathfrak{m}$ *defined by*

$$\langle U(u, v), w\rangle_u^F = \frac{1}{2}\left(\langle [w, u]_\mathfrak{m}, v\rangle_u^F + \langle [w, v]_\mathfrak{m}, u\rangle_u^F\right), \text{ for any } w \in \mathfrak{m},$$

here $[\cdot,\cdot]_\mathfrak{m} = \mathrm{pr}_\mathfrak{m} \circ [\cdot,\cdot]$ *and* $\mathrm{pr}_\mathfrak{m}$ *is the projection with respect to the given* $\mathrm{Ad}(H)$*-invariant decomposition.*

Theorem 3.5 is a corollary of the more general homogeneous flag curvature formula of Huang in [12]. It can also be proven directly by the Finslerian submersion technique. All the details of its proof can be found in [23].

We will use this important homogeneous flag curvature formula to prove two key lemmas for odd-dimensional positively curved reversible homogeneous Finsler spaces. As the preparation for proving the key lemmas, we will present some results on the g_u^F-orthogonal (i.e. with respect to the inner product $\langle\cdot,\cdot\rangle_u^F$) decomposition of $\mathfrak{m}$. These lemmas will also be crucial for our later discussions.

Lemma 3.6 *Keep the above assumptions and notations.*

(1) *Let* u *be a nonzero vector in* $\hat{\mathfrak{g}}_0 \subset \mathfrak{m}$*. Then* $\mathfrak{m}$ *has a* g_u^F*-orthogonal decomposition as the sum of all* $\hat{\mathfrak{m}}_{\pm\alpha'} = \hat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{m}$, $\alpha' \in \mathfrak{t} \cap \mathfrak{h}$*. In particular,* $\hat{\mathfrak{m}}_0 = \hat{\mathfrak{g}}_0$*.*

(2) *If* $\dim \hat{\mathfrak{g}}_0 = 3$*, then there is a fundamental Cartan subalgebra* $\mathfrak{t}$*, such that for any nonzero vector* $u \in \mathfrak{t} \cap \mathfrak{m}$*, we have* $\langle \mathfrak{t} \cap \mathfrak{m}, \mathfrak{g}_{\pm\alpha}\rangle_u^F = 0$*, where* α *is the root in* $\mathfrak{t} \cap \mathfrak{m}$*.*

Proof (1) Let T_H be the torus in H with $\mathrm{Lie}(T_H) = \mathfrak{t} \cap \mathfrak{h}$. Since both F and $u \in \hat{\mathfrak{g}}_0$ are $\mathrm{Ad}(T_H)$-invariant, the inner product $\langle\cdot,\cdot\rangle_u^F$ is also $\mathrm{Ad}(T_H)$-invariant. The summands given in the decomposition correspond to different irreducible representations of T_H; thus, it is a g_u^F-orthogonal decomposition.

(2) Choose the F-unit vector $u \in \hat{\mathfrak{g}}_0$ such that $||u||_{\mathrm{bi}}$ reaches the maximum among all F-unit vectors in $\hat{\mathfrak{g}}_0$. Then $\mathfrak{t}_0 = \mathfrak{t} \cap \mathfrak{h} + \mathbb{R}u$ is also a fundamental Cartan subalgebra of $\mathfrak{g}$. Notice that for $\alpha' \in \mathfrak{t} \cap \mathfrak{h}$, the subspace $\hat{\mathfrak{g}}_{\pm\alpha'}$ does not change when $\mathfrak{t}$ is replaced with $\mathfrak{t}_0$. The bi-invariant orthogonal complement $u^\perp \cap \hat{\mathfrak{g}}_0$ of u in $\hat{\mathfrak{g}}_0$ is a root plane $\mathfrak{g}_{\pm\alpha}$ for $\mathfrak{t}_0$. Then our assumption on u implies that

$$\langle \mathfrak{t}_0 \cap \mathfrak{m}, \mathfrak{g}_{\pm\alpha}\rangle_u^F = \langle \mathbb{R}u, u^\perp \cap \hat{\mathfrak{g}}_0\rangle_u^F = 0.$$

This completes the proof of the lemma. □

Lemma 3.7 *Keep the above assumptions and notations. Let $u \in \mathfrak{m}$ be a nonzero vector in a root plane $\hat{\mathfrak{m}}_{\pm\alpha'}$ with $\alpha' \neq 0$. Denote the bi-invariant orthogonal complement of α' in $\mathfrak{t} \cap \mathfrak{h}$ as $\mathfrak{t}'$, and the bi-invariant orthogonal projection to $\mathfrak{t}'$ as $\mathrm{pr}_{\mathfrak{t}'}$. Then $\mathfrak{m}$ can be g_u^F-orthogonally decomposed as the sum of*

$$\begin{aligned}\hat{\hat{\mathfrak{m}}}_{\pm\gamma''} &= \left(\sum_{\mathrm{pr}_{\mathfrak{t}'}(\gamma)=\gamma''} \mathfrak{g}_{\pm\gamma}\right) \cap \mathfrak{m} = \sum_{\mathrm{pr}_{\mathfrak{t}'}(\gamma')=\gamma''} (\hat{\mathfrak{g}}_{\pm\gamma'} \cap \mathfrak{m}) \\ &= \left(\sum_{\gamma\in\tau+\mathbb{R}\alpha+\mathfrak{t}\cap\mathfrak{m}} \mathfrak{g}_{\pm\gamma}\right) \cap \mathfrak{m},\end{aligned}$$

where τ is a root of $\mathfrak{g}$ with $\mathrm{pr}_{\mathfrak{t}'}(\tau) = \gamma'' \neq 0$, and $\hat{\mathfrak{m}}_0 = \left(\sum_{\gamma\in\mathbb{R}\alpha+\mathfrak{t}\cap\mathfrak{m}} \mathfrak{g}_{\pm\gamma}\right) \cap \mathfrak{m} + \mathfrak{t} \cap \mathfrak{m}$.

Proof The existence of nonzero vector u in $\hat{\mathfrak{m}}_{\pm\alpha'}$ implies the existence of a root α of $\mathfrak{g}$ such that $\mathrm{pr}_{\mathfrak{h}}(\alpha) = \alpha'$. The subalgebra $\mathfrak{t}' = \alpha'^{\perp} \cap (\mathfrak{t} \cap \mathfrak{h})$ is the intersection of the Weyl wall bi-invariant orthogonal to α and the Cartan subalgebra $\mathfrak{t} \cap \mathfrak{h}$ of $\mathfrak{h}$. So there is a subtorus T' in T_H with $\mathrm{Lie}(T') = \mathfrak{t}'$. Since both F and u are $\mathrm{Ad}(T')$-invariant, the inner product $\langle\cdot,\cdot\rangle_u^F$ on $\mathfrak{m}$ is also $\mathrm{Ad}(T')$-invariant. The summands given in the decomposition correspond to different irreducible representations of T'; thus, it is an orthogonal decomposition with respect to $\langle\cdot,\cdot\rangle_u^F$. □

The following lemma does not hold in general without the reversibility assumption.

Lemma 3.8 *Keep the above assumptions and notations. Then for any nonzero vector $u \in \hat{\mathfrak{m}}_{\pm\alpha'} = \hat{\mathfrak{g}}_{\pm\alpha'} \cap \mathfrak{m}$ with $\alpha' \neq 0$, and any $\beta' \in \mathfrak{t} \cap \mathfrak{h}$ which is not an even multiple of α', we have*

$$\langle \hat{\mathfrak{m}}_{\pm\beta'}, \hat{\mathfrak{g}}_0 \rangle_u^F = 0.$$

In particular, we have

$$\langle \hat{\mathfrak{m}}_{\pm\alpha'}, \hat{\mathfrak{g}}_0 \rangle_u^F = 0.$$

Proof Without losing generality, we can assume that $\hat{\mathfrak{m}}_{\pm\beta'} \neq 0$. Then $\dim \hat{\mathfrak{m}}_{\pm\beta'} = 2k > 0$ is even. Hence there exists an element g in the maximal torus T_H of H, and a bi-invariant orthonormal basis $\{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ of $\hat{\mathfrak{g}}_{\pm\beta'} \cap \mathfrak{m}$ such that $\mathrm{Ad}(g)|_{\hat{\mathfrak{m}}_{\pm\alpha'}} = -\mathrm{Id}$, $\mathrm{Ad}(g)|_{\hat{\mathfrak{g}}_0} = \mathrm{Id}$, and for each i, $\mathrm{Ad}(g)|_{\mathbb{R}u_i+\mathbb{R}v_i}$ is the anticlockwise rotation $R(\theta)$ with angle $\theta \in (0, 2\pi)$.

Since F is $\mathrm{Ad}(g)$-invariant, for any $w_1 \in \mathbb{R}u_i + \mathbb{R}v_i$ and $w_2 \in\in \hat{\mathfrak{g}}_0 \cap \mathfrak{m}$, we have

$$\langle w_1, w_2\rangle_u^F = \langle \mathrm{Ad}(g)w_1, \mathrm{Ad}(g)w_2\rangle_{\mathrm{Ad}(g)u}^F = \langle R(\theta)w_1, w_2\rangle_{-u}^F = \langle R(\theta)w_1, w_2\rangle_u^F.$$

Repeating this procedure, we get $\langle w_1, w_2\rangle_u^F = \langle R(n\theta)w_1, w_2\rangle_u^F$ for each $n \in \mathbb{N}$. So

$$\langle w_1, w_2\rangle_u^F = \lim_{n\to\infty} \langle \frac{1}{n}(R(\theta)w_1 + \cdot + R(n\theta)w_1), w_2\rangle_u^F = 0.$$

Now the above argument holds for any i between 1 to k. This proves the lemma. □

Now we are ready to use the homogeneous flag curvature formula prove two key lemmas.

Lemma 3.9 *Let F be a positively curved homogeneous Finsler metric on the odd-dimensional coset space G/H. Keep all the relevant notation as before. If α is a root of $\mathfrak{g}$ contained in $\mathfrak{t}\cap\mathfrak{h}$, and it is the only root of $\mathfrak{g}$ contained in $\alpha+(\mathfrak{t}\cap\mathfrak{m})$, then it is a root of $\mathfrak{h}$ and we have $\mathfrak{h}_{\pm\alpha}=\hat{\mathfrak{g}}_{\pm\alpha}=\mathfrak{g}_{\pm\alpha}$.*

Proof We only need to prove that α is a root of $\mathfrak{h}$. The other statement follows easily.

Assume conversely that α is not a root of $\mathfrak{h}$. Then $\mathfrak{g}_{\pm\alpha}=\hat{\mathfrak{g}}_{\pm\alpha}$ is contained in $\mathfrak{m}$. By (2) of Lemma 3.6, if $\dim\hat{\mathfrak{g}}_0=3$, then there exists a fundamental Cartan subalgebra $\mathfrak{t}$ and a nonzero u in $\mathfrak{t}\cap\mathfrak{m}$, such that

$$\langle u^{\perp}\cap\hat{\mathfrak{g}}_0,u\rangle_u^F=0,\tag{3.11}$$

where $u^{\perp}\cap\hat{\mathfrak{g}}_0$ is the bi-invariant orthogonal complement of u in $\hat{\mathfrak{g}}_0$. Let v be a nonzero vector in $\mathfrak{g}_{\pm\alpha}$. Since $\alpha\in\mathfrak{t}\cap\mathfrak{h}$, it is easy to see that u and v are linearly independent and commutative.

Let $\alpha'=\mathrm{pr}_{\mathfrak{h}}(\alpha)$. Then a direct calculation shows that

$$[u,\mathfrak{m}]_{\mathfrak{m}}\subset u^{\perp}\cap\hat{\mathfrak{g}}_0+\sum_{\gamma'\neq\alpha',\gamma'\neq0}\hat{\mathfrak{g}}_{\pm\gamma'}.$$

Thus by (3.11) and (1) of Lemma 3.6, we have

$$\langle[u,\mathfrak{m}]_{\mathfrak{m}},u\rangle_u^F=\langle[u,\mathfrak{m}]_{\mathfrak{m}},v\rangle_u^F=0.\tag{3.12}$$

On the other hand, a direct calculation also shows that

$$[v,\mathfrak{m}]_{\mathfrak{m}}\subset\sum_{\gamma'\neq0}\hat{\mathfrak{g}}_{\pm\gamma'}.$$

Hence by (1) of Lemma 3.6, we have

$$\langle[v,\mathfrak{m}]_{\mathfrak{m}},u\rangle_u^F=0.\tag{3.13}$$

Taking the summation of (3.12) and (3.13), we get $U(u,v)=0$. Hence by Theorem 3.5, we have $K^F(o,u,u\wedge v)=0$. This is a contradiction. □

Lemma 3.10 *Let F be a reversible positively curved homogeneous Finsler metric on an odd-dimensional coset space G/H. Keep all the relevant notations as before. Then there does not exist a pair of linearly independent roots α and β of $\mathfrak{g}$ such that the following (1)–(4) hold simultaneously:*

(1) *Neither α nor β is a root of $\mathfrak{h}$;*
(2) *None of $\alpha\pm\beta$ is a root of $\mathfrak{g}$;*
(3) *$\pm\alpha$ are the only roots of $\mathfrak{g}$ in $\mathbb{R}\alpha+\mathfrak{t}\cap\mathfrak{m}$;*
(4) *$\pm\beta$ are the only roots of $\mathfrak{g}$ in $\mathbb{R}\alpha\pm\beta+\mathfrak{t}\cap\mathfrak{m}$.*

Though (2) is implied by (4), we prefer to list it separately because in some cases of our later discussion, (4) is not satisfied but (1)–(3) are.

Proof Assume conversely that there are roots α and β of $\mathfrak{g}$ satisfying (1)–(4) of the lemma. Denote $\alpha'=\mathrm{pr}_{\mathfrak{h}}(\alpha)$ and $\beta'=\mathrm{pr}_{\mathfrak{h}}(\beta)$. Then $\mathfrak{g}_{\pm\alpha}=\hat{\mathfrak{g}}_{\pm\alpha'}$ must be contained in $\mathfrak{m}$, otherwise by (3) of the lemma, $\mathfrak{g}_{\pm\alpha}$ is a root plane in $\mathfrak{h}$, and hence $\alpha\subset[\mathfrak{g}_{\pm\alpha},\mathfrak{g}_{\pm\alpha}]\subset\mathfrak{h}$ is a root of $\mathfrak{h}$, which is a contradiction to (1). Similarly, by (1) and (4) of the lemma, $\mathfrak{g}_{\pm\beta}=\hat{\mathfrak{g}}_{\pm\beta'}$ is also contained in $\mathfrak{m}$.

First we consider the case that $\alpha' \neq 0$, i.e., α is not contained by $\mathfrak{t}\cap\mathfrak{m}$. Let u and v be any nonzero vectors in $\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_{\pm\beta}$, respectively. By (1) of the lemma and the above argument, they must be linearly independent and commutative.

Let u' be another nonzero vector in $\mathfrak{g}_{\pm\alpha}$ such that $\langle u, u'\rangle_{\mathrm{bi}}$=0. Because of the $\mathrm{Ad}(T_H)$-invariance of F, $F|_{\mathfrak{g}_{\pm\alpha}}$ coincides with the restriction of the bi-invariant inner product up to a scalar. So we have

$$\langle u^\perp \cap \mathfrak{g}_{\pm\alpha}, u\rangle_u^F = \langle \mathbb{R}u', u\rangle_u^F = 0, \tag{3.14}$$

where $u^\perp \cap \mathfrak{g}_{\pm\alpha} = \mathbb{R}u'$ is the bi-invariant orthogonal complement of u in $\mathfrak{g}_{\pm\alpha}$.

Let $\mathfrak{t}'$ be the bi-invariant orthogonal complement of α in $\mathfrak{h}$, and $\mathrm{pr}_{\mathfrak{t}'}$ be the orthogonal projection to $\mathfrak{t}'$ with respect to the bi-invariant inner product. By Lemma 3.7, $\mathfrak{m}$ can be g_u^F-orthogonally decomposed as the sum of

$$\hat{\hat{\mathfrak{m}}}_{\pm\gamma''} = \left(\sum_{\mathrm{pr}_{\mathfrak{t}'}(\gamma)=\gamma''} \mathfrak{g}_\gamma\right) \cap \mathfrak{m}$$

for all different $\{\pm\gamma''\} \subset \mathfrak{t}'$. In particular, (3) and (4) of the lemma indicates that

$$\hat{\mathfrak{g}}_0 = \mathfrak{t}\cap\mathfrak{m},\ \ \hat{\hat{\mathfrak{m}}}_0 = \mathfrak{t}\cap\mathfrak{m} + \mathfrak{g}_{\pm\alpha},\ \text{ and } \hat{\hat{\mathfrak{m}}}_{\pm\beta''} = \mathfrak{g}_{\pm\beta}, \tag{3.15}$$

where $\beta'' = \mathrm{pr}_{\mathfrak{t}'}(\beta)$.

Now (1), (2) of the lemma and a direct calculation implies that

$$[u, \mathfrak{m}] \subset \mathfrak{t}\cap\mathfrak{m} + u^\perp \cap \mathfrak{g}_{\pm\alpha} + \sum_{\gamma''\neq 0, \gamma''\neq\pm\beta'',} \hat{\hat{\mathfrak{m}}}_{\pm\gamma''}.$$

So by Lemmas 3.8, 3.7 and (3.14), we have

$$\langle [u, \mathfrak{m}]_{\mathfrak{m}}, u\rangle_u^F = \langle [u, \mathfrak{m}]_{\mathfrak{m}}, v\rangle_u^F = 0. \tag{3.16}$$

On the other hand, a direct calculation also shows that

$$[v, \mathfrak{m}]_{\mathfrak{m}} \subset \hat{\mathfrak{g}}_0 + \sum_{\gamma''\neq 0} \hat{\hat{\mathfrak{m}}}_{\pm\gamma''}.$$

Thus by Lemmas 3.8 and 3.7, we have

$$\langle [v, \mathfrak{m}]_{\mathfrak{m}}, u\rangle_u^F = 0. \tag{3.17}$$

Taking the summation of (3.16) and (3.17), we get $U(u, v) = 0$. Hence by Theorem 3.5, we have $K^F(o, u, u\wedge v) = 0$. This is a contradiction. □

Notice that Lemmas 3.6, 3.7 and 3.9 does not require F to be reversible. For most cases in later discussions, the key lemmas will be enough to deduce our classification. But in some cases (Sect. 5.5 for example), we need to use Theorem 3.5 more carefully to complete the proofs.

4 Case III: The general reduction and the classical groups

In this section, we consider the Case III for classical groups.

4.1 The general reduction

Assume that $(G/H, F)$ is an odd-dimensional positively curved reversible homogeneous Finsler space in Case III. We have chosen the bi-invariant inner product for the compact Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ such that the root system of $\mathfrak{g}$ is presented as the subset of $\mathfrak{t}$. Then for the bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and a fundamental Cartan subalgebra $\mathfrak{t}$, there exists a pair of roots α and β of $\mathfrak{g}$ from the same simple factor, with $\alpha \neq \pm\beta$, such that $\mathrm{pr}_{\mathfrak{h}}(\alpha) = \mathrm{pr}_{\mathfrak{h}}(\beta) = \alpha'$ is a root of $\mathfrak{h}$. Obviously, in this case $\mathfrak{t} \cap \mathfrak{m}$ is spanned by $\alpha - \beta$.

We first prove the following lemma.

Lemma 4.1 *Let $(G/H, F)$ be an odd-dimensional positively curved reversible homogeneous Finsler space in Case III. Keep all the relevant notations. Then $(G/H, F)$ is equivalent to a positively curved reversible homogeneous Finsler space $(G'/H', F')$ in Case III with a compact simple Lie group G'.*

Proof Suppose $\mathfrak{g}$ has a direct sum decomposition as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \cdots \oplus \mathfrak{g}_n,$$

where $\mathfrak{g}_0$ is an abelian subalgebra, and for $i > 0$, $\mathfrak{g}_i$ is a simple ideal of $\mathfrak{g}$. Let α and β be two roots of $\mathfrak{g}_1$. So $\mathfrak{g}_0$ and $\mathfrak{t} \cap \mathfrak{g}_i$ for $i > 1$ are contained in $\mathfrak{h}$.

Let γ be any root of $\mathfrak{g}_i$ with $i > 1$. Then γ is the only root contained in $\gamma + \mathfrak{t} \cap \mathfrak{m}$. Thus by Lemma 3.9, γ is a root of $\mathfrak{h}$ and $\mathfrak{g}_{\pm\gamma} = \mathfrak{h}_{\pm\gamma}$ is contained in $\mathfrak{h}$. So we have $\mathfrak{g}_i \subset \mathfrak{h}$ for each $i > 1$. Let G'/H' be the homogeneous space corresponding to the pair $\mathfrak{g}_1$ and $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1$. Then G'/H' admits a homogeneous Finsler metric F' naturally induced by F, such that $(G/H, F)$ is equivalent to $(G'/H', F')$. This completes the proof of the lemma. □

Since Lemma 3.9 holds without the reversible assumption, Lemma 4.1 is also valid for non-reversible metrics.

In the following, we will start a case-by-case consideration of the compact simple Lie algebras. However, there are some common subcases which can be uniformly dealt with. We summarize them as the following lemma.

Lemma 4.2 *Let $(G/H, F)$ be an odd-dimensional positively curved reversible homogeneous Finsler space in Case III, with compact simple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Then for any two different roots α and β such that $\mathrm{pr}_{\mathfrak{h}}(\alpha) = \mathrm{pr}_{\mathfrak{h}}(\beta) = \alpha'$ is a root of $\mathfrak{h}$, the angle between α and β can not be $\frac{\pi}{3}$ or $\frac{2\pi}{3}$.*

Proof First we assume that $\mathfrak{g} \neq G_2$ and prove that the angle between α and β can not be $\frac{\pi}{3}$. Assume conversely that the angle between α and β is $\frac{\pi}{3}$. Let $\mathfrak{t}' = \alpha'^{\perp} \cap \mathfrak{t} \cap \mathfrak{h} = (\mathbb{R}\alpha + \mathbb{R}\beta)^{\perp} \cap \mathfrak{t}$ be the bi-invariant orthogonal complement of α' in $\mathfrak{t} \cap \mathfrak{h}$, and T' be the corresponding torus in H. Notice that there is a decomposition $\mathrm{Lie}(C_G(T')) = \mathfrak{t}' \oplus A_2$ such that α and β are roots of the A_2-factor. By Lemma 3.1, there is a positively curved homogeneous Finsler space $(G''/H'', F'')$, where $\mathfrak{g}'' = \mathrm{Lie}(G'') = \mathfrak{su}(3)$, and $\mathfrak{h}'' = \mathrm{Lie}(H'') = A_1$ is linearly spanned by

$$w_1 = \sqrt{-1}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \sqrt{-1}\begin{pmatrix} 0 & \bar{a} & \bar{b} \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \text{ and}$$

$$w_3 = \frac{1}{3}[w_1, w_2] = \begin{pmatrix} 0 & \bar{a} & \bar{b} \\ -a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix},$$

where $a, b \in \mathbb{C}$ and $(a, b) \neq (0, 0)$. But then $[w_3, w_1]$ is not contained in $\mathfrak{h}''$. This is a contradiction.

Now we prove that the angle between α and β can not be $\frac{2\pi}{3}$. Assume conversely that it is $\frac{2\pi}{3}$. Then $\alpha' = \frac{1}{2}(\alpha + \beta)$ is a root of $\mathfrak{h}$. But then $\gamma = 2\alpha' = \alpha + \beta$ is a root of $\mathfrak{g}$ contained in $\mathfrak{t} \cap \mathfrak{h}$, and it is the only root contained in $\gamma + (\mathfrak{t} \cap \mathfrak{m})$. So by Lemma 3.9, $\gamma = 2\alpha'$ is also a root of $\mathfrak{h}$. This is a contradiction.

Finally, we assume that $\mathfrak{g} = G_2$ and prove that the angle between α and β can not be $\frac{\pi}{3}$. If α and β are short roots, then they can be replaced with two long roots with angle $\frac{2\pi}{3}$, which has already been proven to be impossible. If α and β are long roots, then $\alpha' = \frac{1}{2}(\alpha + \beta)$ is a root of $\mathfrak{h}$. By Lemma 3.9 and a similar argument as above, the short root $\gamma = \frac{1}{3}(\alpha + \beta) = \frac{2}{3}\alpha'$ is also a root of $\mathfrak{h}$. This is a contradiction. □

Now we start the case-by-case discussion. Notice that in the following, we always assume that the relevant coset space has been endowed with an invariant reversible Finsler metric with positive flag curvature. If a contradiction arises, then we can conclude that the coset space cannot be positively curved in the reversible homogeneous sense. In each case, we use the standard presentation of the root systems and divide the discussion into subcases with respect to the rank of G, the long/short roots choices of α and β and the angle between α and β. Using the Weyl group actions and more outer automorphisms for D_n and E_6, the subcases can be reduced to the following.

4.2 The case $\mathfrak{g} = A_n$

We only need to consider the following subcases.

Subcase 1 $n = 3$, and $\alpha = e_1 - e_4$, $\beta = e_3 - e_2$.

In this case, we have $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 - e_3 - e_4)$ and $\alpha' = \frac{1}{2}(e_1 - e_2 + e_3 - e_4)$ is a root of $\mathfrak{h}$. By Lemma 3.9, $e_1 - e_2$ and $e_3 - e_4$ are roots of $\mathfrak{h}$. Notice that $\hat{\mathfrak{g}}_{\pm(e_1-e_2)} = \mathfrak{g}_{\pm(e_1-e_2)}$ is a root plane of $\mathfrak{h}$. Let $\beta' = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4) \in \mathfrak{t} \cap \mathfrak{h}$. Then any non zero $u \in \mathfrak{g}_{\pm(e_1-e_2)} \subset \mathfrak{h}$ defines a linear isomorphism

$$\mathrm{ad}(u) : \hat{\mathfrak{g}}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1-e_4)} + \mathfrak{g}_{\pm(e_2-e_3)} \to \hat{\mathfrak{g}}_{\pm\beta'} = \mathfrak{g}_{\pm(e_2-e_4)} + \mathfrak{g}_{\pm(e_1-e_3)}. \tag{4.18}$$

Since $u \in \mathfrak{h}$, $\mathrm{ad}(u)$ preserves the bi-invariant orthogonal decomposition. So $\beta' = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4)$ is also a root of $\mathfrak{h}$. It follows that $\mathfrak{h} = B_2$ and its root system is

$$\{\pm(e_1 - e_2), \pm(e_3 - e_4), \pm\alpha', \pm\beta'\}.$$

Now we prove that up to conjugation, $\mathfrak{h}$ is uniquely determined. By (4.18), it is easy to see that $\mathfrak{h}$ is uniquely determined by $\mathfrak{h}_{\pm\alpha'}$. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ isomorphic to $A_1 \oplus A_1$, defined by

$$\mathfrak{g}' = \mathbb{R}\alpha + \mathbb{R}\beta + \mathfrak{g}_{\pm(e_1-e_4)} + \mathfrak{g}_{\pm(e_2-e_3)},$$

and let $\mathfrak{h}'$ be the subalgebra of $\mathfrak{g}'$ defined by $\mathfrak{h}' = \mathbb{R}\alpha' + \mathfrak{h}_{\pm\alpha'}$. Suppose $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{g}' = \mathbb{R}\alpha + \mathbb{R}\beta$ is a fundamental Cartan subalgebra of $\mathfrak{g}'$. Then we also have the induced bi-invariant orthogonal decomposition $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$ such that $\mathfrak{m}' = \mathfrak{m} \cap \mathfrak{g}'$ and $\mathfrak{t}' \cap \mathfrak{m}' = \mathfrak{t} \cap \mathfrak{m}$. Notice also that $\mathfrak{h}'$ can not have nonzero intersection with either of the two simple factors of $\mathfrak{g}'$, otherwise, by $\mathrm{Ad}(\exp \mathfrak{h}')$-actions, the whole subalgebra $\mathfrak{h}'$ coincides with that factor, which is a contradiction with the fact that $\alpha, \beta \notin \mathfrak{h}'$.

The following lemma will be useful.

Lemma 4.3 *Let* $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2 = A_1 \oplus A_1$ *be endowed with a bi-invariant inner product. Assume that* $\mathfrak{t}'$ *is a Cartan subalgebra, and* $\mathfrak{h}'$ *and* $\mathfrak{h}''$ *are subalgebras of* $\mathfrak{g}'$ *isomorphic to* A_1 *satisfying the following conditions:*

(1) $\mathfrak{h}' \cap \mathfrak{t}' = \mathfrak{h}'' \cap \mathfrak{t}'$ *is one dimensional.*
(2) $\mathfrak{h}' \cap \mathfrak{g}_i = \mathfrak{h}'' \cap \mathfrak{g}_i = 0, i = 1, 2.$
(3) $\mathfrak{h}' \cap (\mathfrak{h}' \cap \mathfrak{t}')^\perp \subset \mathfrak{t}'^\perp$, *and* $\mathfrak{h}'' \cap (\mathfrak{h}'' \cap \mathfrak{t}')^\perp \subset \mathfrak{t}'^\perp$, *where the orthogonal complements are taken with respect to the chosen bi-invariant inner product on* $\mathfrak{g}$.

Then there is an $\mathrm{Ad}(\exp \mathfrak{t}')$*-action which maps* $\mathfrak{h}'$ *onto* $\mathfrak{h}''$.

Proof We first give a definition. For a compact Lie algebra of type A_1 endowed with a bi-invariant inner product, we call the bi-invariant orthogonal basis $\{u_1, u_2, u_3\}$ standard, if $[u_i, u_j] = u_k$ for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ or $(3, 1, 2)$. Then all u_i's have the same length c which only depends on the scale of the bi-invariant inner product. The bracket $u'_3 = [u'_1, u'_2]$ of any two orthogonal vectors with length c is also a vector with length c, and $\{u'_1, u'_2, u'_3\}$ is a standard basis as well.

Now we go back to the proof. Let c_1 and c_2 be the length of the standard basis vectors for $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Then we can choose standard bases $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ for $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, as follows. First, we choose vectors u_1 and v_1 from $\mathfrak{t}' \cap \mathfrak{g}_1$ and $\mathfrak{t}' \cap \mathfrak{g}_2$ with length c_1 and c_2, respectively. Then we freely choose any vectors u_2 of length c_1 from $\mathfrak{t}'^\perp \cap \mathfrak{g}_1$ and set $u_3 = [u_1, u_2]$. By (2) and (3) in the lemma, we can find a vector of $\mathfrak{h}'$ from $u_2 + \mathfrak{g}_2 \cap \mathfrak{t}'^\perp$. Then its $\mathfrak{g}_2$-factor is not 0, which can be positively scaled to a vector v_2 with the length c_2. Then $v_1, v_2, v_3 = [v_1, v_2]$ form a standard basis for $\mathfrak{g}_2$.

Now suppose $\mathfrak{h}'$ is linearly spanned by $u_1 + av_1, u_2 + bv_2$, and their bracket can be expressed as

$$[u_1 + av_1, u_2 + bv_2] = u_3 + abv_3,$$

where a is a fixed nonzero constant and $b > 0$. As a Lie algebra, $\mathfrak{h}'$ contains $[u_2 + bv_2, u_3 + abv_3] = u_1 + ab^2 v_1$, hence $b = 1$.

With $\mathfrak{h}'$ changed to $\mathfrak{h}''$, the same argument above can also give standard bases $\{u'_1, u'_2, u'_3\}$ and $\{v'_1, v'_2, v'_3\}$ for $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, such that $u'_i = u_i$ for each i, and $v'_1 = v_1$. Then it is easy to see that there exists a real number t such that $\mathrm{Ad}(\exp(tv_1))$ maps v_2 to v'_2, v_3 to v'_3, and keep v_1 and all the vectors u_i unchanged. So it maps $\mathfrak{h}'$ isomorphically to $\mathfrak{h}''$. □

By Lemma 4.3, it is easy to see that, up to the $\mathrm{Ad}(\exp \mathfrak{t})$-actions which preserve all the roots and root planes of $\mathfrak{g}$, $\mathfrak{h}_{\pm\alpha'}$ is uniquely determined. So $\mathfrak{h}$ is conjugate to the standard subalgebra $\mathfrak{sp}(2)$ in $\mathfrak{su}(4)$ which makes G/H a symmetric space. Since $A_3 = D_3$, G/H is equivalent to the standard Riemannian sphere $S^5 = \mathrm{SO}(6)/\mathrm{SO}(5)$ with constant positive curvature.

In this subcase, we can also directly prove that G/H is a symmetric homogeneous space, that is, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, and then apply the classification of symmetric homogeneous spaces to get the classification. However, this argument is not valid for some other subcases below.

Subcase 2 $n = 4$, and $\alpha = e_1 - e_4$, $\beta = e_3 - e_2$.

In this case, we have $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 - e_3 - e_4)$, and $\alpha' = \frac{1}{2}(e_1 - e_2 + e_3 - e_4)$ is a unit root of $\mathfrak{h}$. Notice that $\mathrm{pr}_\mathfrak{h}(e_i - e_5)$, $1 \leq i \leq 4$, can not be a root of $\mathfrak{h}$ since it is not orthogonal to α' and its length is $\frac{\sqrt{7}}{2}$. Thus any root of $\mathfrak{h}$ must be of the form $\mathrm{pr}_\mathfrak{h}(e_i - e_j)$ with $1 \leq i < j \leq 4$. A similar argument as in Subcase 1 then shows that the root system of $\mathfrak{h}$ is of type $B_2 = C_2$, i.e., up to the $\mathrm{Ad}(\mathrm{SU}(4))$-actions, $\mathfrak{h} = \mathbb{R}(e_1 + e_2 + e_3 + e_4 - 4e_5) \oplus \mathfrak{h}'$, where $\mathfrak{h}'$ is the standard subalgebra $\mathfrak{sp}(2)$ in $\mathfrak{su}(4)$ corresponding to e_i with $1 \leq i \leq 4$. So G/H is equivalent to the Berger's space SU(5)/Sp(2)U(1), which admits positive curved normal homogeneous (Riemannian) metrics.

Subcase 3 $n > 4$, and $\alpha = e_1 - e_4$, $\beta = e_3 - e_2$.

We have $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 - e_3 - e_4)$ and $\alpha' = \frac{1}{2}(e_1 - e_2 + e_3 - e_4)$ is a unit root of $\mathfrak{h}$. Then it is easy to check that the roots $\gamma_1 = e_1 - e_5$ and $\gamma_2 = e_2 - e_6$ satisfy the conditions (1)–(4) of Lemma 3.10; hence, the corresponding coset space does not admit any invariant reversible Finsler metric with positive flag curvature.

4.3 The case $\mathfrak{g} = B_n$ with $n > 1$

We only need to consider the following subcases.

Subcase 1 $\alpha = e_1 + e_2$, $\beta = e_2$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_1$ and $\alpha' = e_2$ is a root of $\mathfrak{h}$, with

$$\mathfrak{h}_{\pm e_2} \subset \hat{\mathfrak{g}}_{\pm e_2} = \mathfrak{g}_{\pm(e_2 - e_1)} + \mathfrak{g}_{\pm e_2} + \mathfrak{g}_{\pm(e_2 + e_1)}.$$

Denote $\mathfrak{g}' = \mathbb{R}e_1 + \mathbb{R}e_2 + \sum_{a,b} \mathfrak{g}_{\pm(ae_1 + be_2)}$ and $\mathfrak{g}'' = \mathbb{R}e_1 + \mathfrak{g}_{\pm e_1}$. Then $\mathfrak{g}'$, $\mathfrak{g}''$ are Lie algebras of types $B_2 = \mathfrak{so}(5)$ and A_1, respectively. The subalgebra $\mathfrak{h} \cap \mathfrak{g}'$ of type A_1 is linearly spanned by

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}e_2, \quad v = \begin{pmatrix} 0 & 0 & 0 & -a & -a' \\ 0 & 0 & 0 & -b & -b' \\ 0 & 0 & 0 & -c & -c' \\ a & b & c & 0 & 0 \\ a' & b' & c' & 0 & 0 \end{pmatrix} \in \mathfrak{h}_{\pm e_2},$$

and

$$w = [u, v] = \begin{pmatrix} 0 & 0 & 0 & a' & -a \\ 0 & 0 & 0 & b' & -b \\ 0 & 0 & 0 & c' & -c \\ -a' & -b' & -c' & 0 & 0 \\ a & b & c & 0 & 0 \end{pmatrix},$$

in which (a, b, c, a', b', c') is a nonzero vector in $\mathbb{R}^6$. Since $[v, w] \in \mathfrak{h} \cap \mathfrak{g}'$, (a, b, c) and (a', b', c') are linearly dependent vectors. Using a suitable isomorphism $l \in \mathrm{Ad}(\exp \mathfrak{g}'')$ of $\mathfrak{g}$, we can make $b = c = b' = c' = 0$, i.e., up to equivalence, we can assume that $\mathfrak{h}_{\pm e_2} = \mathfrak{g}_{\pm e_2}$. Thus $\mathfrak{g}_{\pm(e_2 \pm e_1)} \in \mathfrak{m}$.

By Lemma 3.9, any root $\pm e_i \pm e_j$ of $\mathfrak{g}$ with $1 < i < j$ must be a root of $\mathfrak{h}$ and we have $\mathfrak{h}_{\pm(e_i \pm e_j)} = \mathfrak{g}_{\pm(e_i \pm e_j)} = \hat{\mathfrak{g}}_{\pm(e_i \pm e_j)}$. By the linear isomorphism $\mathrm{ad}(w)$ between $\hat{\mathfrak{g}}_{\pm e_2}$ and $\hat{\mathfrak{g}}_{\pm e_i}$, for any nonzero vector $w \in \mathfrak{g}_{\pm(e_2 - e_i)}$ with $i > 2$, we have $\mathfrak{g}_{\pm e_i} \subset \mathfrak{h}$. Moreover, for any $i \geq 2$, we have $\mathfrak{g}_{\pm(e_i \pm e_1)} \subset \mathfrak{m}$. To summarize, we have

$$\mathfrak{m} = \mathbb{R}e_1 + \mathfrak{g}_{\pm e_1} + \sum_{i=2}^{n} (\mathfrak{g}_{\pm(e_i + e_1)} + \mathfrak{g}_{\pm(e_i - e_1)}). \tag{4.19}$$

Let $\{u, u'\}$ be a bi-invariant orthonormal basis of $\mathfrak{g}_{\pm(e_1 + e_2)}$ and choose a nonzero vector v from $\mathfrak{g}_{\pm(e_1 - e_2)}$ such that $\langle u', v \rangle_u^F = 0$. Since the Minkowski norm $F|_{\mathfrak{g}_{\pm(e_1 + e_2)}}$ is

$\mathrm{Ad}(\exp(\mathbb{R}e_2))$-invariant, it coincides with the restriction of the bi-invariant inner product up to scalar changes. So we have

$$\langle u', u\rangle_u^F = \langle [u, e_1], u\rangle_u^F = \langle [u, e_2], u\rangle_u^F = 0. \tag{4.20}$$

Now a direct calculation shows that

$$[u, \mathfrak{m}]_\mathfrak{m} \subset \mathbb{R}[e_1, u] + \mathbb{R}e_1 \subset \mathbb{R}u' + \hat{\mathfrak{g}}_0.$$

By Lemma 3.8,

$$\langle v, \hat{\mathfrak{g}}_0\rangle_u^F = \langle u, \hat{\mathfrak{g}}_0\rangle_u^F = 0,$$

so by our assumptions on u and v, we have

$$\langle [u, \mathfrak{m}]_\mathfrak{m}, u\rangle_u^F = \langle \mathbb{R}u', u\rangle_u^F = 0, \tag{4.21}$$

and

$$\langle [u, \mathfrak{m}]_\mathfrak{m}, v\rangle_u^F = \langle \mathbb{R}u', u\rangle_u^F + \langle \hat{\mathfrak{g}}_0, v\rangle_u^F = 0. \tag{4.22}$$

Since $e_2 \in \mathfrak{h}$, by Theorem 1.3 of [8], we have

$$\langle [e_2, v], u\rangle_u^F = -\langle [e_2, u], v\rangle_u^F - 2C_u^F(u, v, [e_2, u]). \tag{4.23}$$

By (4.20), the first term of the right side of above equation vanishes. By the property of Cartan tensor, $C_u^F(u, \cdot, \cdot) \equiv 0$, so the second term also vanishes. Thus we have

$$\langle [e_1, v], u\rangle_u^F = \langle [e_2, v], u\rangle_u^F = 0. \tag{4.24}$$

A direct calculation then shows that

$$[v, \mathfrak{m}]_\mathfrak{m} \subset \mathbb{R}[e_1, v] + \hat{\mathfrak{g}}_0.$$

By Lemma 3.8 and (4.24), we have

$$\langle [v, \mathfrak{m}]_\mathfrak{m}, u\rangle_u^F = \langle \mathbb{R}[e_1, v], u\rangle_u^F + \langle \hat{\mathfrak{g}}_0, u\rangle_u^F = 0. \tag{4.25}$$

Taking the summation of (4.21), (4.22) and (4.25), we get $U(u, v) = 0$. Hence by Theorem 3.5, we have $K^F(o, u, u \wedge v) = 0$. Therefore, the corresponding coset space does not admit any invariant reversible Finsler metric with positive flag curvature.

Subcase 2 $\alpha = e_1 + e_2, \beta = e_2 - e_1$.

This subcase has been covered by Subcase 1.

Subcase 3 $n = 4$, and $\alpha = e_1 + e_2, \beta = -e_3 - e_4$.

In this case, we have $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 + e_3 + e_4)$, and $\alpha' = \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$ is a root of $\mathfrak{h}$ with $\mathfrak{h}_{\pm\alpha'} \subset \hat{\mathfrak{g}}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} + \mathfrak{g}_{\pm(e_3+e_4)}$. The argument here is very similar to Subcase 1 for A_n. Obviously $\mathfrak{h}_{\pm\alpha'}$ is not a root plane of $\mathfrak{g}$. By Lemma 3.9, if $1 \le i < j \le 4$, then the root $e_i - e_j$ of $\mathfrak{g}$ is also a root of $\mathfrak{h}$ with $\mathfrak{h}_{\pm(e_i-e_j)} = \mathfrak{g}_{\pm(e_i-e_j)} = \hat{\mathfrak{g}}_{\pm(e_i-e_j)}$. Using the action $\mathrm{ad}(u)$, one easily shows that for any non zero vector $u \in \mathfrak{g}_{\pm(e_i-e_j)} \subset \mathfrak{h}$, with $(i, j) = (2, 3)$ or $(2, 4)$, both $\beta' = \frac{1}{2}(e_1 + e_3 - e_2 - e_4)$ and $\gamma' = \frac{1}{2}(e_1 + e_4 - e_2 - e_3)$ are also roots of $\mathfrak{h}$. Hence $\mathfrak{h}$ is of type B_3, and it is uniquely determined by the choice of $\mathfrak{h}_{\pm\alpha'}$. By Lemma 4.3, up to the $\mathrm{Ad}(G)$-action, the subalgebra $\mathfrak{h}$ is unique. So we can assume $\mathfrak{h}$ to be the standard subalgebra such that the pair $(\mathfrak{g}, \mathfrak{h})$ defines the homogeneous Finsler sphere $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$, i.e., in this subcase $(G/H, F)$ must be equivalent to the homogeneous sphere $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$ on which there exist positively curved homogeneous Riemannian metrics.

Subcase 4 $n > 4$, and $\alpha = e_1 + e_2$, $\beta = -e_3 - e_4$.

We have $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}(e_1+e_2+e_3+e_4)$, and it is easily seen that $\alpha' = \frac{1}{2}(e_1+e_2-e_3-e_4)$ is a unit root of $\mathfrak{h}$. Then the roots $\gamma_1 = e_1 + e_5$ and $\gamma_2 = e_1 - e_5$ satisfy (1)–(4) of Lemma 3.10. Hence the corresponding coset space does not admit any invariant reversible Finsler metric with positive flag curvature.

Subcase 5 $n = 3$, and $\alpha = e_1 + e_2$, $\beta = -e_3$.

Then we have $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}(e_1+e_2+e_3)$ and $\alpha' = \frac{1}{3}(e_1+e_2-2e_3)$ is a root of $\mathfrak{h}$. The argument here is similar to that of Subcase 3. By Lemmas 3.9 and 3.4, the root system of $\mathfrak{h}$ contains the roots

$$\pm(e_i - e_j),\ \ 1 \le i < j \le 3,$$

and

$$\frac{1}{3}(e_1 + e_2 + e_3) - e_i,\ \ 1 \le i \le 3.$$

The subalgebra $\mathfrak{h}$ is of type G_2 and is uniquely determined by the choice of

$$\mathfrak{h}_{\pm\alpha'} \subset \hat{\mathfrak{g}}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} + \mathfrak{g}_{\pm e_3}.$$

By Lemma 4.3, up to the $\mathrm{Ad}(G)$-action, there exists a unique $\mathfrak{h}$, and the corresponding coset space is the homogeneous sphere $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$. Notice that in this case the isotropy action is transitive, so any homogeneous Finsler metric on it must be Riemannian with positive constant curvature. Consequently in this subcase $(G/H, F)$ is equivalent to the Riemannian homogeneous sphere $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$ of positive constant curvature.

Subcase 6 $n > 3$, and $\alpha = e_1 + e_2$, $\beta = -e_3$.

In this case, $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}(e_1 + e_2 + e_3)$ and $\alpha' = \frac{1}{3}(e_1 + e_2 - 2e_3)$ is a root of $\mathfrak{h}$. The roots $\gamma_1 = e_1 + e_4$ and $\gamma_2 = e_1 - e_4$ satisfy the conditions (1)-(4) of Lemma 3.10. Hence the corresponding coset space does not admit any invariant reversible Finsler metric with positive flag curvature.

Subcase 7 $\alpha = e_1$, $\beta = e_2$.

In this case, $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}(e_1 - e_2)$ and $\alpha' = \frac{1}{2}(e_1 + e_2)$ is a root of $\mathfrak{h}$. By Lemma 3.9, $2\alpha' = e_1 + e_2$ is also a root of $\mathfrak{h}$. Hence the corresponding coset space does not admit any invariant reversible Finsler metric with positive flag curvature.

Subcase 8 $n = 2$, and $\alpha = e_1 + e_2$, $\beta = -e_1$.

In this case, $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}(2e_1 + e_2)$ and $\alpha' = -\frac{1}{5}e_1 + \frac{2}{5}e_2$ is a root of $\mathfrak{h}$. The subalgebra $\mathfrak{h}$ is of type A_1, and is uniquely determined by the choice of $\mathfrak{h}_{\pm\alpha'}$ in $\hat{\mathfrak{g}}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} + \mathfrak{g}_{\pm e_1}$. By Lemma 4.3, $\mathfrak{h}$ is uniquely determined up to $\mathrm{Ad}(G)$-actions. So G/H must be equivalent to unique known example in this subcase, i.e., the Berger's space $\mathrm{Sp}(2)/\mathrm{SU}(2)$. There exists positively curved normal homogeneous Riemannian metrics on it.

Subcase 9 $n > 2$, and $\alpha = e_1 + e_2$, $\beta = -e_1$.

In this case, $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}(2e_1 + e_2)$ and $\alpha' = -\frac{1}{5}e_1 + \frac{2}{5}e_2$ is a root of $\mathfrak{h}$. The roots $\gamma_1 = e_1 + e_3$ and $\gamma_2 = e_1 - e_3$ satisfy (1)-(3) but does not satisfy (4) of Lemma 3.10, i.e., $\pm\gamma_1$ are the only roots of $\mathfrak{g}$ in $\mathbb{R}\gamma_1 + \mathfrak{t}\cap\mathfrak{m}$, and all the roots of $\mathfrak{g}$ in $\pm\gamma_2 + \mathbb{R}\gamma_1 + \mathfrak{t}\cap\mathfrak{m}$ are $\pm\gamma_2 = \pm(e_1 - e_3)$

and $\pm\gamma_3 = \pm e_2$. Choosing u and v from $\mathfrak{g}_{\pm\gamma_1}$ and $\mathfrak{g}_{\pm\gamma_2}$ as in the proof for Lemma 3.10, we can similarly get

$$\langle [u, \mathfrak{m}]_{\mathfrak{m}}, u\rangle_u^F = \langle [v, \mathfrak{m}]_{\mathfrak{m}}, u\rangle_u^F = 0. \tag{4.26}$$

Notice that $\gamma_1 = e_1 + e_3$ and $\gamma_3 = e_2$ also satisfy (2) of Lemma 3.10, i.e., $\gamma_1 \pm \gamma_3$ are not roots of $\mathfrak{g}$, and Lemma 3.8 implies $\langle \hat{\mathfrak{g}}_0, v\rangle_u^F = 0$, so we can still get

$$\langle [u, \mathfrak{m}]_{\mathfrak{m}}, v\rangle_u^F = 0. \tag{4.27}$$

Taking the summation of (4.26) and (4.27), we get $U(u, v) = 0$. Thus by Theorem 3.5, we have $K^F(o, u, u \wedge v) = 0$. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

4.4 The case $\mathfrak{g} = C_n$ with $n > 2$

We only need to consider the following subcases.

Subcase 1 $\alpha = 2e_1, \beta = e_1 + e_2$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 - e_2)$ and $\alpha' = \beta = e_1 + e_2$ is a root of $\mathfrak{h}$. Let $\mathfrak{t}'$ be the subalgebra of $\mathfrak{t} \cap \mathfrak{h}$ spanned by $\{e_3, \ldots, e_n\}$, and T' the corresponding subtorus in $T \cap H$. Then the Lie algebra of $C_G(T')$ is $\mathfrak{t}' \oplus \mathfrak{g}''$, in which $\mathfrak{g}''$ is of type B_2. If the corresponding coset space can be positively curved, then Lemma 3.1 implies that the positively curved reversible homogeneous Finsler space SO(5)/SO(3) should appear in Subcase 1 for B_n, which is a contradiction. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

Subcase 2 $\alpha = 2e_1, \beta = 2e_2$.

This subcase has been covered by the previous one.

Subcase 3 $\alpha = 2e_1, \beta = -e_2 - e_3$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(2e_1 + e_2 + e_3)$ and $\alpha' = \frac{2}{3}e_1 - \frac{2}{3}e_2 - \frac{2}{3}e_3$ is a root of $\mathfrak{h}$. Then the roots $\gamma_1 = 2e_2$ and $\gamma_2 = 2e_3$ satisfy the conditions (1)–(4) of Lemma 3.10. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

Subcase 4 $\alpha = e_1 + e_2, \beta = e_1 - e_2$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_2$ and $\alpha' = e_1$ is a root of $\mathfrak{h}$. By Lemma 3.9, $2\alpha' = 2e_1$ is also a root of $\mathfrak{h}$. This is a contradiction. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

Subcase 5 $\alpha = e_1 + e_2, \beta = -e_3 - e_4$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 + e_3 + e_4)$ and $\alpha' = \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$ is a root of $\mathfrak{h}$. Then the roots $\gamma_1 = 2e_1$ and $\gamma_2 = 2e_2$ satisfy (1)–(4) of Lemma 3.10. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

Subcase 6 $\alpha = 2e_1, \beta = -e_1 - e_2$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(3e_1 + e_2)$ and $\alpha' = \frac{1}{5}e_1 - \frac{3}{5}e_2$ is a root of $\mathfrak{h}$. Then the roots $\gamma_1 = e_1 + e_3$ and $\gamma_2 = 2e_2$ satisfy the conditions (1)–(4) of Lemma 3.10. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

4.5 The case $\mathfrak{g} = D_n$ with $n > 3$

We only need to consider the following subcases.

Subcase 1 $\alpha = e_1 + e_2$, $\beta = e_2 - e_1$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_1$ and $\alpha' = e_2$ is a root of $\mathfrak{h}$. Then we can apply Lemmas 3.9, 3.4 and a similar argument as in Subcase 1 for A_n (which in fact is a special situation of this subcase), to show that $\mathfrak{h}$ is of type B_{n-1} with all the roots given by

$$\pm e_i \pm e_j \text{ for } 1 < i < j \leq n \text{ and } \pm e_i \text{ for } 1 < i \leq n.$$

We can also use Lemma 4.3 to show that, up to $\mathrm{Ad}(G)$-actions, $\mathfrak{h}$ is unique, i.e., $(G/H, F)$ must be equivalent to the Riemannian symmetric sphere $\mathrm{SO}(2n)/\mathrm{SO}(2n-1)$ of positive constant curvature.

Subcase 2 $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$.

First notice that D_4 has outer automorphisms. So the argument in the above subcase can be applied to this subcase for $n = 4$, i.e., we get the Riemannian symmetric sphere $S^7 = \mathrm{SO}(8)/\mathrm{SO}(7)$. If $n > 4$, then $\mathfrak{t}\cap\mathfrak{m} = \mathbb{R}(e_1+e_2+e_3+e_4)$ and $\alpha' = \frac{1}{2}(e_1+e_2-e_3-e_4)$ is a root of $\mathfrak{h}$. Then the roots $\gamma_1 = e_1 + e_5$ and $\gamma_2 = e_1 - e_5$ satisfy the conditions (1)–(4) of Lemma 3.10. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

5 Case III: The exceptional groups and summary

We continue the case-by-case discussion of the last section and summarize all the results of these two sections as a theorem at the end, which proves half of the first main theorem in Sect. 1. We still choose the suitable bi-invariant inner product on $\mathfrak{g}$ such that the root system $\Delta_{\mathfrak{g}}$ is viewed as a subset of $\mathfrak{g}$ with its standard presentation.

5.1 The case $\mathfrak{g} = E_6$

Without losing generality, we can assume that the orthogonal pair of the roots α and β are of the form $\pm e_i \pm e_j$. Up to the Weyl group action induced by D_5, there are two subcases: (1) $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$; (2) $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$. Using the outer automorphisms of E_6 as well as the Weyl group action, the second subcase can be reduced to the first one. So we can assume that $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$. Then $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_1$, and $\alpha' = e_2$ is a root of $\mathfrak{h}$. Then the roots

$$\gamma_1 = -\frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4 + \frac{1}{2}e_5 + \frac{\sqrt{3}}{2}e_6,$$

and

$$\gamma_2 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4 - \frac{1}{2}e_5 + \frac{\sqrt{3}}{2}e_6$$

satisfy the conditions (1)–(4) of Lemma 3.10. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

5.2 The case $\mathfrak{g} = E_7$

Given an orthogonal pair of roots α and β of $\mathfrak{g}$, we can use certain Weyl group action to change β to $\sqrt{2}e_7$. Since β is orthogonal to α, α must be then of the form $\pm e_i \pm e_j$ with $1 \leq i < j \leq 6$. Using Weyl group actions induced by D_6, we can change α to $e_1 + e_2$ while keeping $\beta = \sqrt{2}e_7$ fixed. So essentially there is only one subcase for the orthogonal pair of roots. The most convenient way is to choose $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$. Then $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_1$ and $\alpha' = e_2$ is a root of $\mathfrak{h}$. Now the pair of roots

$$\gamma_1 = -\frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4 + \frac{1}{2}e_5 + \frac{1}{2}e_6 + \frac{\sqrt{2}}{2}e_7,$$

and

$$\gamma_2 = \frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4 + \frac{1}{2}e_5 + \frac{1}{2}e_6 + \frac{\sqrt{2}}{2}e_7$$

satisfy the conditions (1)–(4) of Lemma 3.10. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

5.3 The case $\mathfrak{g} = E_8$

Up to the Weyl group action, we can assume that α and β are of the form $\pm e_i \pm e_j$. We only need to consider the following two subcases.

Subcase 1 $\alpha = e_1 + e_2, \beta = e_2 - e_1$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_1$ and $\alpha' = e_2$ is a root of $\mathfrak{h}$. The pair of roots

$$\gamma_1 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4 + \frac{1}{2}e_5 + \frac{1}{2}e_6 + \frac{1}{2}e_7 + \frac{1}{2}e_8,$$

and

$$\gamma_2 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4 + \frac{1}{2}e_5 + \frac{1}{2}e_6 + \frac{1}{2}e_7 + \frac{1}{2}e_8$$

satisfy the conditions (1)–(4) of Lemma 3.10. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

Subcase 2 $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 + e_3 + e_4)$ and $\alpha' = \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$ is a root of $\mathfrak{h}$. Then the pair of roots $\gamma_1 = e_1 + e_5$ and $\gamma_2 = e_2 + e_6$ satisfy the conditions (1)–(4) of Lemma 3.10. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

5.4 The case $\mathfrak{g} = F_4$

Notice that up to the Weyl group action, any short root of F_4 can be changed to e_1. This implies that any orthogonal pair of short roots of F_4 can be changed to the pairs e_1 and $-e_2$. On the other hand, using the reflections induced by the roots $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$, any orthogonal pair of long roots can be changed to the pair $e_1 \pm e_2$. Hence we only need to consider the following subcases.

Subcase 1 $\alpha = e_1 + e_2, \beta = e_2$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_1$ and $\alpha' = e_2$ is a root of $\mathfrak{h}$. Let $\mathfrak{t}'$ be the subalgebra of $\mathfrak{t} \cap \mathfrak{h}$ spanned by e_3 and e_4, and T' be the closed subtorus in $T \cap H$ with $\mathrm{Lie}(T') = \mathfrak{t}'$. Then applying Lemma 3.1 to T', we conclude that there should be a positively curved reversible homogeneous Finsler space $\mathrm{SO}(5)/\mathrm{SO}(3)$ in Subcase 1 for B_n, which is a contradiction. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

Subcase 2 $\alpha = e_1 + e_2, \beta = e_2 - e_1$.

This subcase has been covered by the previous one.

Subcase 3 $\alpha = e_1 + e_2, \beta = -e_3$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 + e_3)$ and $\alpha' = \frac{1}{3}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3$ is a root of $\mathfrak{h}$ with length $\sqrt{\frac{2}{3}}$, with $\mathfrak{h}_{\pm\alpha'} \subset \hat{\mathfrak{g}}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} + \mathfrak{g}_{\pm e_3}$. By Lemma 3.9, $\pm e_4$ are roots of $\mathfrak{h}$, and $\mathfrak{h}_{\pm e_4} = \mathfrak{g}_{\pm e_4} = \hat{\mathfrak{g}}_{\pm e_4}$. Notice that $\mathrm{pr}_{\mathfrak{h}}(e_4 - e_3)$ is not orthogonal to α' and has length $\sqrt{\frac{5}{3}}$. So $\mathrm{pr}_{\mathfrak{h}}(e_4 - e_3)$ is not a root of $\mathfrak{h}$. Thus $\mathfrak{g}_{\pm(e_4-e_3)} \subset \mathfrak{m}$. Therefore, we have

$$\mathfrak{g}_{\pm e_3} = [\mathfrak{g}_{\pm e_4}, \mathfrak{g}_{\pm(e_4-e_3)}] \subset \mathfrak{m}.$$

Hence $\mathfrak{h}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1+e_2)}$. Then we have

$$\alpha' = \frac{1}{3}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3 \subset [\mathfrak{h}_{\pm\alpha'}, \mathfrak{h}_{\pm\alpha'}] = [\mathfrak{g}_{\pm(e_1+e_2)}, \mathfrak{g}_{\pm(e_1+e_2)}] = \mathbb{R}(e_1 + e_2),$$

which is a contradiction. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

Subcase 4 $\alpha = e_1, \beta = -e_2$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2)$ and $\alpha' = \frac{1}{2}(e_1 - e_2)$ is a root of $\mathfrak{h}$. By Lemma 3.9, $e_1 - e_2 = 2\alpha'$ is also a root of $\mathfrak{h}$, which is a contradiction. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

Subcase 5 $\alpha = e_1 + e_2, \beta = -e_2$.

In this case, $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + 2e_2)$, and $\alpha' = \frac{2}{5}e_1 - \frac{1}{5}e_2$ is a root of $\mathfrak{h}$ of length $\sqrt{\frac{1}{5}}$, with $\mathfrak{h}_{\pm\alpha'} \subset \hat{\mathfrak{g}}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} + \mathfrak{g}_{\pm e_2}$. By Lemma 3.9, e_3 is a root of $\mathfrak{h}$ and $\mathfrak{h}_{\pm e_3} = \mathfrak{g}_{\pm e_3} = \hat{\mathfrak{g}}_{\pm e_3}$. The vector $\mathrm{pr}_{\mathfrak{h}}(e_2 + e_3)$ is not a root of $\mathfrak{h}$, since it is not orthogonal to α' and its length is $\sqrt{\frac{6}{5}}$. So $\mathfrak{g}_{\pm(e_2+e_3)} \subset \mathfrak{m}$. Then we have

$$\mathfrak{g}_{\pm e_2} = [\mathfrak{g}_{\pm(e_2+e_3)}, \mathfrak{g}_{\pm e_3}] \subset \mathfrak{m}.$$

This implies that $\mathfrak{h}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1+e_2)}$. Then we can deduce a contradiction by a similar argument as in Subcase 3 of this section.

There is another way to deduce the contradiction. Let $\mathfrak{t}' = \mathbb{R}e_4$ and T' be the corresponding closed one-parameter subgroup in H. Using Lemma 3.1, we get a positively curved reversible homogeneous Finsler space in Subcase 9 for B_n, which is impossible.

5.5 The case $\mathfrak{g} = G_2$

If the angle between α and β is $\frac{\pi}{6}$ or $\frac{\pi}{2}$, we can find a pair of short roots α_1 and β_1 of $\mathfrak{g}$ such that the angle between α_1 and β_1 is $\frac{\pi}{3}$, and $\alpha' = \mathrm{pr}_{\mathfrak{h}}(\alpha_1) = \mathrm{pr}_{\mathfrak{h}}(\beta_1)$ is a root of $\mathfrak{h}$. This is a contradiction to Lemma 4.2.

Therefore, we only need to consider the case that α is a long root, β is a short root, and the angle between them is $\frac{5\pi}{6}$, $\alpha' = \mathrm{pr}_{\mathfrak{h}}(\alpha) = \mathrm{pr}_{\mathfrak{h}}(\beta)$ is a root of $\mathfrak{h}$, and $\mathfrak{h}$ is of type A_1. Let $\gamma_1 = \alpha + 3\beta$ and $\gamma_2 = \alpha + \beta$. Select any two nonzero vectors $u \in \mathfrak{g}_{\pm\gamma_1}$ and $v \in \mathfrak{g}_{\pm\gamma_2}$. Then it is not hard to see that the long root γ_1 and the short root γ_2 are orthogonal to each other, and none of $\gamma_1 \pm \gamma_2$ is a root of $\mathfrak{g}$. So u and v are linearly independent and commutative. Denote the anticlockwise rotation with angle θ as $R(\theta)$. Then there exists $g \in T_H$, and suitable orthonormal bases for each of the subspaces of $\mathfrak{m}$ below, such that

$$\begin{aligned}
\mathrm{Ad}(g)|_{\mathfrak{t}\cap\mathfrak{m}=\hat{\mathfrak{g}}_0} &= \mathrm{Id},\\
\mathrm{Ad}(g)|_{\hat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{m}} &= R(\pi/4),\\
\mathrm{Ad}(g)|_{\mathfrak{g}_{\pm(\alpha+\beta)}=\hat{\mathfrak{g}}_{\pm2\alpha'}} &= R(\pi/2),\\
\mathrm{Ad}(g)|_{\mathfrak{g}_{\pm(\alpha+2\beta)}=\hat{\mathfrak{g}}_{\pm3\alpha'}} &= R(3\pi/4),\\
\mathrm{Ad}(g)|_{\mathfrak{g}_{\pm(\alpha+3\beta)}=\hat{\mathfrak{g}}_{\pm4\alpha'}} &= R(\pi) = -\mathrm{Id},\\
\mathrm{Ad}(g)|_{\mathfrak{g}_{\pm(2\alpha+3\beta)}=\hat{\mathfrak{g}}_{\pm5\alpha'}} &= R(5\pi/4).
\end{aligned}$$

Denote the above subspaces as $\mathfrak{m}_k$, $k = 0, 1, \ldots, 5$, i.e., the action of $\mathrm{Ad}(g)$ on $\mathfrak{m}_k$ is equal to $R(k\pi/4)$. In particular, $\mathfrak{m}_0 = \mathfrak{t} \cap \mathfrak{m}$, $\mathfrak{m}_2 = \mathfrak{g}_{\pm\gamma_2}$ and $\mathfrak{m}_4 = \mathfrak{g}_{\pm\gamma_1}$. By Lemma 3.8, we have

$$\langle \mathfrak{m}_4, \mathfrak{m}_i \rangle_u^F = 0, \quad \forall i \neq 4. \tag{5.28}$$

For any $v' \in \mathfrak{m}_2$ and $w' \in \mathfrak{m}_i$ with $i \neq 2$, we have

$$\begin{aligned}
\langle v', w' \rangle_u^F &= \langle \mathrm{Ad}(g)v', \mathrm{Ad}(g)w' \rangle_{\mathrm{Ad}(g)u}^F = \langle R(\pi/2)v', R(i\pi/4)w' \rangle_{-u}^F\\
&= \langle R(\pi/2)v', R(i\pi/4)w' \rangle_u^F = \langle R(\pi/2)^2 v', R(i\pi/4)^2 w' \rangle_u^F\\
&= \langle -v', R(i\pi/2)w' \rangle_u^F = \langle v', R((i-2)\pi/2)w' \rangle_u^F.
\end{aligned}$$

Using a similar argument as in the proof of Lemma 3.8, we get

$$\langle \mathfrak{m}_2, \mathfrak{m}_i \rangle_u^F = 0, \quad \forall i \neq 2. \tag{5.29}$$

By the $\mathrm{Ad}(T_H)$-invariance, the Minkowski norm $F|_{\mathfrak{m}_4}$ coincides with the restriction of the bi-invariant inner product up to a scalar change. Thus

$$\langle [u, \mathfrak{t}], u \rangle_u^F = 0. \tag{5.30}$$

Now a direct calculation shows that

$$[u, \mathfrak{m}]_{\mathfrak{m}} \subset \mathfrak{m}_0 + \mathfrak{m}_1 + \mathfrak{m}_3 + [u, \mathfrak{t}] + \mathfrak{m}_5,$$

and

$$[v, \mathfrak{m}]_{\mathfrak{m}} \subset \mathfrak{m}_0 + \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \mathfrak{m}_5.$$

So by (5.28), (5.29) and (5.30), we have

$$\langle [u, \mathfrak{m}]_{\mathfrak{m}}, u \rangle_u^F = \langle [u, \mathfrak{m}]_{\mathfrak{m}}, v \rangle_u^F = \langle [v, \mathfrak{m}]_{\mathfrak{m}}, u \rangle_u^F = 0. \tag{5.31}$$

Hence $U(u, v) = 0$. Then by Theorem 3.5, we get $K^F(o, u, u \wedge v) = 0$, which is a contradiction. Hence there does not exist any invariant reversible Finsler metric on the corresponding coset space with positive flag curvature.

5.6 Summary

We now summarize all the results in Sects. 4 and 5 as the following theorem, which gives a complete classification of odd-dimensional positively curved reversible homogeneous Finsler spaces in Case III.

Theorem 3 *Let $(G/H, F)$ be an odd-dimensional positively curved reversibly homogeneous Finsler space of Case III, i.e., with respect to a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie algebra $\mathfrak{g}$, and a fundamental Cartan subalgebra $\mathfrak{t}$, there are roots α and β of $\mathfrak{g}$ from the same simple factor, such that $\alpha \neq \pm\beta$ and $\mathrm{pr}_{\mathfrak{h}}(\alpha) = \mathrm{pr}_{\mathfrak{h}}(\beta) = \alpha'$ is a root of $\mathfrak{h}$. Then $(G/H, F)$ is equivalent to one of the following homogeneous Finsler spaces:*

(1) *The odd-dimensional Riemannian symmetric spheres $S^{2n-1} = \mathrm{SO}(2n)/\mathrm{SO}(2n-1)$ with $n > 2$;*
(2) *The homogeneous spheres $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$ and $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$;*
(3) *Berger's spaces $\mathrm{SU}(5)/\mathrm{Sp}(2)\mathrm{U}(1)$ and $\mathrm{Sp}(2)/\mathrm{SU}(2)$.*

6 The cases II and I

In this section, we will consider odd-dimensional positively curved reversible homogeneous Finsler spaces in Cases II and I.

6.1 The case II

Let $(G/H, F)$ be an odd-dimensional positively curved reversible homogeneous Finsler space in Case II, i.e., with respect to a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and a fundamental Cartan subalgebra $\mathfrak{t}$, there exists two roots α and β of $\mathfrak{g}$ from different simple factors such that $\mathrm{pr}_{\mathfrak{h}}(\alpha) = \mathrm{pr}_{\mathfrak{h}}(\beta) = \alpha'$ is a root of $\mathfrak{h}$. It is implied by this assumption that H is not a regular subgroup of G, i.e., $\mathfrak{h}_{\pm\alpha'} \subset \hat{\mathfrak{g}}_{\pm\alpha'} = \mathfrak{g}_{\pm\alpha} + \mathfrak{g}_{\pm\beta}$ is not a root plane of $\mathfrak{g}$, or equivalently, $\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_{\pm\beta}$ are not contained in $\mathfrak{h}$ or $\mathfrak{m}$.

First of all, we can find a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n \oplus \mathbb{R}^m,$$

such that each $\mathfrak{g}_i$ is a simple ideal of $\mathfrak{g}$, and α and β are roots of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Since $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(\alpha - \beta) \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the abelian factor of $\mathfrak{g}$ and $\mathfrak{t} \cap \mathfrak{g}_i$ for each $i > 2$ is contained in $\mathfrak{t} \cap \mathfrak{h}$. It is also obvious that for each root γ of $\mathfrak{g}$ with $\gamma \neq \pm\alpha$, $\gamma \neq \pm\beta$ and $\mathrm{pr}_{\mathfrak{h}}(\gamma) = \gamma'$, $\mathfrak{g}_{\pm\gamma} = \hat{\mathfrak{g}}_{\pm\gamma'}$ is contained either in $\mathfrak{h}$ or in $\mathfrak{m}$.

Now we prove that for any $i > 2$, $\mathfrak{g}_i$ is contained in $\mathfrak{t} \cap \mathfrak{h}$. We only need to prove that each root plane of $\mathfrak{g}_i$ is contained in $\mathfrak{h}$. Let γ be a root of $\mathfrak{g}_i$. Since $i > 2$, γ is contained in $\mathfrak{t} \cap \mathfrak{h}$ and it is the only root of $\mathfrak{g}$ in $\gamma + (\mathfrak{t} \cap \mathfrak{m})$. By Lemma 3.9, γ is a root of $\mathfrak{h}$ and $\mathfrak{g}_{\pm\gamma} = \hat{\mathfrak{g}}_{\pm\gamma} = \mathfrak{h}_{\pm\gamma} \subset \mathfrak{h}$.

Consider the roots of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Up to equivalence, we can assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let γ be a root of $\mathfrak{g}_1$ such that $\gamma \neq \pm\alpha$. Since it is the only root of $\mathfrak{g}_1$ contained in $\gamma + (\mathfrak{t} \cap \mathfrak{m})$, by Lemma 3.9, if $\gamma \in \mathfrak{t} \cap \mathfrak{h}$, then $\mathfrak{g}_{\pm\gamma} \subset \mathfrak{h}$. On the other hand, if γ is not bi-invariant orthogonal to α, then $\mathfrak{g}_{\pm\gamma} \subset \mathfrak{m}$, because otherwise $\mathfrak{g}_{\pm\gamma} \subset \mathfrak{h}$ and by Lemma 3.4, $\gamma \in [\mathfrak{g}_{\pm\gamma}, \mathfrak{g}_{\pm\gamma}] \subset \mathfrak{t} \cap \mathfrak{h}$, but γ and $\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(\alpha - \beta)$ are not bi-invariant orthogonal. The similar assertion is valid for any root of $\mathfrak{g}_2$.

Now we claim that there does not exist two roots γ_1 and γ_2 of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, such that $\gamma_1 \neq \pm\alpha$, $\gamma_2 \neq \pm\beta$, and their root planes are contained in $\mathfrak{m}$. If we have such a pair γ_1 and γ_2, then they satisfy the conditions (1)–(4) of Lemma 3.10, which is impossible.

Without loss of generality, we can assume that all roots of $\mathfrak{g}_1$ other than $\pm\alpha$ are roots of $\mathfrak{h}$. Thus they are bi-invariant orthogonal to $\pm\alpha$. Since $\mathfrak{g}_1$ is simple, $\mathfrak{g}_1$ is of type A_1 with the only roots $\pm\alpha$. Now we consider $\mathfrak{g}_2$. We first prove the following lemma

Lemma 6.1 *Keep the above assumptions and notations. Then there does not exist a pair of roots γ_1 and γ_2 of $\mathfrak{g}_2$ satisfying the following conditions:*

(1) $\gamma_1 \neq \pm\gamma_2$, $\gamma_1 \neq \pm\beta$ *and* $\gamma_2 \neq \pm\beta$*;*
(2) *Neither γ_1 nor γ_2 is a root of $\mathfrak{h}$;*
(3) *None of $\gamma_1 \pm \gamma_2$ is a root of $\mathfrak{g}$.*

Proof Assume conversely that there are two roots γ_1 and γ_2 of $\mathfrak{g}_2$ satisfying (1)–(3) of the lemma. Then it is easy to see that γ_1 is the only root in $\gamma_1 + \mathbb{R}(\alpha - \beta)$. On the other hand, if there exist some real numbers t_1 and t_2, such that $\gamma_3 = \gamma_2 + t_1\gamma_1 + t_2(\alpha - \beta)$ is a root of $\mathfrak{g}$ other than γ_2, then we have $t_2 \in \{-1, 0, 1\}$. If $t_2 = 0$, then $\gamma_3 = \gamma_2 + t_1\gamma_1$, with $t_1 \neq 0$, is a root of $\mathfrak{g}_2$. This is impossible, Since $\gamma_1 \pm \gamma_2$ are not roots of $\mathfrak{g}_2$. If $t_2 = \pm 1$ then $\pm\beta = t_1\gamma_1 + \gamma_2$ is a root of $\mathfrak{g}_2$ other than γ_2. Similarly we can get a contradiction. This implies that the pair of roots γ_1 and γ_2 satisfy the conditions (1)–(4) of Lemma 3.10, which is a contradiction. □

Let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}_2$ generated by $\mathfrak{g}_{\pm\beta}$ and $\mathfrak{h} \cap \mathfrak{g}_2$. It has the same rank as $\mathfrak{g}_2$ and can be decomposed as a direct sum $\mathfrak{k} = A_1 \oplus (\mathfrak{h} \cap \mathfrak{g}_2)$, in which the A_1-factor is generated by $\mathfrak{g}_{\pm\beta}$. By Lemma 6.1, the pair $(\mathfrak{g}_2, \mathfrak{k})$ satisfies Condition (A) in [16]. Then by Proposition 6.1 of [16], the pair $(\mathfrak{g}_2, \mathfrak{k})$ must be one of the following:

$$(A_1, A_1), (A_2, A_1 \oplus \mathbb{R}) \text{ or } (C_n, A_1 \oplus C_{n-1}).$$

Correspondingly, the pair $(\mathfrak{g}, \mathfrak{h})$ must be one of the following:

$$(A_1 \oplus A_1, A_1), (A_1 \oplus A_2, A_1 \oplus \mathbb{R}) \text{ or } (A_1 \oplus C_n, A_1 \oplus C_{n-1}),$$

in which the A_1-factor in $\mathfrak{h}$ is the diagonal subalgebra. Thus the corresponding homogeneous Finsler space is equivalent to the symmetric homogeneous sphere $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$, or the Wilking's space $\mathrm{SU}(3) \times \mathrm{SO}(3)/\mathrm{U}(2)$ (which coincides with the Aloff–Wallach's space $S_{1,1}$, see [1] and [17]), or the homogeneous sphere $S^{4n-1} = \mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n-1)\mathrm{Sp}(1)$.

To summarize, we have the following theorem, which gives a complete classification of odd-dimensional positively curved reversible homogeneous Finsler spaces in Case II.

Theorem 6.2 *Let $(G/H, F)$ be an odd-dimensional positively curved reversibly homogeneous Finsler space of Case II, i.e., with respect to a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and a fundamental Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, there are roots α and β of $\mathfrak{g}$ from different simple factors such that $\mathrm{pr}_{\mathfrak{h}}(\alpha) = \mathrm{pr}_{\mathfrak{h}}(\beta) = \alpha'$ is a root of $\mathfrak{h}$. Then $(G/H, F)$ is equivalent to one of the following homogeneous Finsler spaces:*

(1) *The symmetric homogeneous sphere* $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$*;*
(2) *The homogeneous spheres* $\mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n-1)\mathrm{Sp}(1)$*;*
(3) *The Wilking's space* $\mathrm{SU}(3) \times \mathrm{SO}(3)/\mathrm{U}(2)$.

6.2 The case I: The proof for Theorem 2

Let $(G/H, F)$ be an odd-dimensional positively curved reversible homogeneous Finsler space in Case I, i.e., with respect to the chosen bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and a fundamental Cartan subalgebra $\mathfrak{t}$, each root plane of $\mathfrak{h}$ is also a root plane of $\mathfrak{g}$. Keep all the relevant notation as before. The root system of $\mathfrak{h}$ is then a subset of the root system of $\mathfrak{g}$, that is, $\Delta_{\mathfrak{h}} \subset \Delta_{\mathfrak{g}} \cap \mathfrak{h}$. For each root α of $\mathfrak{g}$, we have either $\mathfrak{g}_{\pm\alpha} = \mathfrak{h}_{\pm\alpha} \subset \mathfrak{h}$ or $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{m}$.

Suppose $\mathfrak{g}$ has the following direct sum decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \tag{6.32}$$

where $\mathfrak{g}_0$ is abelian and each $\mathfrak{g}_i$, $1 \leq i \leq n$, is a simple ideal. Given a nonzero vector w in $\mathfrak{t} \cap \mathfrak{m}$, let $w = w_0 + \cdots + w_n$ be the decomposition of w with respect to (6.32). Then it follows from Lemma 3.9 that $\mathfrak{g}_i$ is contained in $\mathfrak{h}$ if and only if $w_i = 0$, for any nonzero $w \in \mathfrak{t} \cap \mathfrak{m}$. Now we have the following cases:

Case 1. There exists $w \in \mathfrak{t} \cap \mathfrak{m}$ such that $w_0 \neq 0$.

We first assert that if α and β are roots of $\mathfrak{g}$, with $\alpha \neq \pm\beta$, such that none of them is a root of $\mathfrak{h}$, then at least one of $\alpha \pm \beta$ is a root of $\mathfrak{g}$. In fact, otherwise the pair of roots α, β will satisfy the conditions (1)–(4) of Lemma 3.10, which is a contradiction. Now let $\mathfrak{k}$ be the subalgebra generated by $\mathfrak{h}$ and $\mathfrak{t}$. Then we have $\mathfrak{k} = \mathfrak{h} \oplus (\mathfrak{t} \cap \mathfrak{m})$. Let K be a closed subgroup of G with $\mathrm{Lie}(K) = \mathfrak{k}$. Then we have $\mathrm{rk}K = \mathrm{rk}G$. This implies that the pair $(\mathfrak{g}, \mathfrak{k})$ satisfies the Condition (A) in [16]. Thus we can suppose that in the decomposition (6.32) of $\mathfrak{g}$, the following equation holds:

$$\mathfrak{k} = \mathfrak{g}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n,$$

where

$$(\mathfrak{g}_1, \mathfrak{k}_1) = (A_n, A_{n-1} \oplus \mathbb{R}), (C_n, C_{n-1} \oplus \mathbb{R}), \text{ or } (A_2, \mathbb{R} \oplus \mathbb{R}).$$

Notice that in other spaces of Wallach's list, the abelian factor required for this situation does not appear. If $\mathfrak{g}_1 = A_2$, then by Lemma 3.9, no root of $\mathfrak{g}_1$ can be contained in $\mathfrak{t} \cap \mathfrak{h}$. Thus $(G/H, F)$ is equivalent to one of the following:

(1) The homogeneous sphere

$$S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1) \text{ or } S^{4n-1} = \mathrm{Sp}(n)\mathrm{U}(1)/\mathrm{Sp}(n-1)\mathrm{U}(1) \text{ for } n > 1;$$

(2) The U(3)-homogeneous presentations of Aloff–Wallach's spaces $S_{k,l} = \mathrm{U}(3)/T^2$, in which T^2 is a two-dimensional torus of diagonal matrices which does not contain the centre of U(3) and

$$T^2 \cap \mathrm{SU}(3) = U_{k,l} = \{\mathrm{diag}(z^k, z^l, z^{-k-l}) | z \in \mathbb{C}, |z| = 1\},$$

where k and l are integers satisfying $kl(k+l) \neq 0$.

Notice that the SU(3)-homogeneous space $S_{k,l}$ have infinitely many different presentation as U(2)-homogeneous spaces; see [1].

Case 2. There exists $w \in \mathfrak{t} \cap \mathfrak{m}$ with decomposition $w = w_1 + w_2$, where both w_1 and w_2 are nonzero.

Up to equivalence, we can assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

We first assert that there does not exist a root α of $\mathfrak{g}_1$, and a root β of $\mathfrak{g}_2$ such that $\alpha \notin \mathbb{R}w_1$, $\beta \notin \mathbb{R}w_2$, and none of them is a root of $\mathfrak{h}$. In fact, otherwise the pair of roots α and

β will satisfy (1)–(4) of Lemma 3.10, which is a contradiction. Without loss of generality, we can assume that all the roots of $\mathfrak{g}_1$ outside $\mathbb{R}w_1$ are roots of $\mathfrak{h}$, i.e., they are contained in $\mathfrak{t}\cap\mathfrak{h}\cap\mathfrak{g}_1$ which is bi-invariant orthogonal to w_1. By the simpleness of $\mathfrak{g}_1$, we must have $\mathfrak{g}_1 = A_1$, and the only roots in $\mathfrak{t}\cap\mathfrak{g}_1 = \mathbb{R}w_1$ are $\pm\alpha$. There are two subcases:

Subcase 1 There exists a root β of $\mathfrak{g}_2$ contained in $\mathbb{R}w_2$.

Obviously neither α nor β is a root of $\mathfrak{h}$, i.e., their root planes are contained in $\mathfrak{m}$. Let $\mathfrak{t}'$ be the bi-invariant orthogonal complement of w_2 in $\mathfrak{g}_2$ and T' be the corresponding torus in H. Using Lemma 3.1 for T', we get a positively curved reversible homogeneous Finsler space $\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)$ in Case I, in which U(1) is not contained in any of the simple factors. To prove the reversible homogeneous space G/H can not be positively curved in this subcase, we only need to consider the situation that $\mathfrak{g}_2 = A_1$, and the only roots are $\pm\beta$. By suitably reordering the two simple factors, we can assume that $\alpha + c\beta \in \mathfrak{t}\cap\mathfrak{m}$ with $|c|\geq 1$. Denote $\alpha' = \mathrm{pr}_{\mathfrak{h}}(\alpha)$ and $\beta' = \mathrm{pr}_{\mathfrak{h}}(\beta)$. Then the above assumption implies that β' is not an even multiple of α'.

Let $\{u, u'\}$ be a bi-invariant orthonormal basis of $\mathfrak{g}_{\pm\alpha}$, and v a nonzero vector in $\mathfrak{g}_{\pm\beta}$ such that $\langle u', v\rangle_u^F = 0$. Obviously u and v are linearly independent and commutative. By the $\mathrm{Ad}(H)$-invariance, the Minkowski norm $F|_{\mathfrak{g}_{\pm\alpha}}$ coincides with the bi-invariant inner product up to scalar changes. Thus

$$\langle u', u\rangle_u^F = \langle[\mathfrak{t}, u], u\rangle_u^F = 0.$$

By the assumption that β' is not an even multiple of α' and Lemma 3.8, we have

$$\langle\mathfrak{t}\cap\mathfrak{m}, u\rangle_u^F = \langle\mathfrak{t}\cap\mathfrak{m}, v\rangle_u^F = 0. \tag{6.33}$$

Then a direct calculation shows that

$$[u,\mathfrak{m}]_{\mathfrak{m}} = \mathfrak{t}\cap\mathfrak{m} + [\mathfrak{t}, u]. \tag{6.34}$$

So by (6.33), we get

$$\langle[u,\mathfrak{m}]_{\mathfrak{m}}, u\rangle_u^F = \langle\mathfrak{t}\cap\mathfrak{m}, u\rangle_u^F + \langle\mathbb{R}u', u\rangle_u^F = 0, \tag{6.35}$$

and

$$\langle[u,\mathfrak{m}]_{\mathfrak{m}}, v\rangle_u^F = \langle\mathfrak{t}\cap\mathfrak{m}, v\rangle_u^F + \langle\mathbb{R}u', v\rangle_u^F = 0. \tag{6.36}$$

Now a direct calculation shows that

$$[v,\mathfrak{m}]_{\mathfrak{m}} = \mathfrak{t}\cap\mathfrak{m} + [\mathfrak{t}\cap\mathfrak{m}, v] = \mathfrak{t}\cap\mathfrak{m} + [\mathfrak{t}\cap\mathfrak{h}, v]. \tag{6.37}$$

For any $w'\in\mathfrak{t}\cap\mathfrak{h}$, we have, by Theorem 3.1 of [8],

$$\langle[w', v], u\rangle_u^F = -\langle v, [w', u]\rangle_u^F - 2C_u^F([w', u], v, u) = 0.$$

So by Lemma 3.8 and (6.37), we have

$$\langle[v,\mathfrak{m}]_{\mathfrak{m}}, u\rangle_u^F = \langle[v,\mathfrak{t}\cap\mathfrak{h}], u\rangle_u^F = 0. \tag{6.38}$$

Taking the summation of (6.35), (6.36) and (6.38), we get $U(u, v) = 0$. Hence by Theorem 3.5, $K^F(o, u, u\wedge v) = 0$, which is a contradiction. Hence the corresponding coset space does not admit any invariant Finsler metric with positive flag curvature.

Subcase 2 There does not exist any root of $\mathfrak{g}_2$ in $\mathbb{R}w_2$.

Then by the simpleness of $\mathfrak{g}_2$, there is a root β of $\mathfrak{g}_2$ which is not bi-invariant orthogonal to w_2. Let u and v be any nonzero vectors in $\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_{\pm\beta}$, respectively. Then they are linearly independent and commutative. The subalgebra $\mathfrak{t}' = \mathfrak{t}\cap\mathfrak{h}\cap\mathfrak{g}_2$ coincides with $w_2^\perp\cap\mathfrak{t}\cap\mathfrak{g}_2$, the bi-invariant orthogonal complement of w_2 in $\mathfrak{t}\cap\mathfrak{g}_2$. Denote T' the corresponding torus in H. Since the inner product $\langle\cdot,\cdot\rangle_u^F$ is $\mathrm{Ad}(T')$-invariant, by Lemma 3.7, $\mathfrak{m}$ can be g_u^F-orthogonally decomposed as the sum of $\mathfrak{m}' = \hat{\hat{\mathfrak{m}}}_0 = \mathfrak{t}\cap\mathfrak{m}+\mathfrak{g}_{\pm\alpha}$ for the trivial irreducible T'-representation and $\mathfrak{m}'' \subset \mathfrak{g}_2$ for nontrivial irreducible T'-representations. Notice that $\mathfrak{m}''$ is the sum of some root planes in $\mathfrak{g}_2$, and u and v are contained in $\mathfrak{m}'$ and $\mathfrak{m}''$, respectively.

Now a direct calculation shows that

$$[u,\mathfrak{m}]_\mathfrak{m} = \mathfrak{t}\cap\mathfrak{m} + [\mathfrak{t},u] \subset \mathfrak{m}' \text{ and } [v,\mathfrak{m}]_\mathfrak{m} \subset \mathfrak{t}\cap\mathfrak{m}+\mathfrak{m}''. \tag{6.39}$$

Moreover, the $\mathrm{Ad}(T_H)$ invariance of $F|_{\mathfrak{g}_{\pm\alpha}}$ implies that $F|_{\mathfrak{g}_{\pm\alpha}}$ coincides with the restriction of a bi-invariant inner product up to scalar changes. Thus we have

$$\langle[\mathfrak{t},u],u\rangle_u^F = \langle[\mathfrak{t}\cap\mathfrak{h},u],u\rangle_u^F = 0. \tag{6.40}$$

By Lemma 3.8,

$$\langle\mathfrak{t}\cap\mathfrak{m},u\rangle_u^F = 0. \tag{6.41}$$

Taking the summation of (6.39), (6.40) and (6.41), we get

$$\langle[u,\mathfrak{m}]_\mathfrak{m},u\rangle_u^F = \langle[u,\mathfrak{m}]_\mathfrak{m},v\rangle_u^F = \langle[v,\mathfrak{m}]_\mathfrak{m},u\rangle_u^F = 0.$$

Therefore, $U(u,v)=0$. Now by Theorem 3.5, $K^F(o,u,u\wedge v)=0$. This is a contradiction. Hence the corresponding coset space does not admit any invariant Finsler metric with positive flag curvature.

Case 3. There exists $w\in\mathfrak{t}\cap\mathfrak{m}$ such that $w = w_1+\cdots+w_m$, where $m>2$ and $w_i\neq 0, \forall 1\leq i\leq m$.

If there is a root $\alpha\notin\mathbb{R}w_1$ of $\mathfrak{g}_1$, and a root $\beta\notin\mathbb{R}w_2$ of $\mathfrak{g}_2$ such that they are not roots of $\mathfrak{h}$, then they satisfy the conditions (1)–(4) of Lemma 3.10, which is a contradiction. Similarly to the previous case, we can assume that $\mathfrak{g}_1 = A_1$. Let $\pm\alpha$ be the only roots of $\mathfrak{g}_1$, then we have $\mathfrak{g}_{\pm\alpha}\subset\mathfrak{m}$. We can also find a root β of $\mathfrak{g}_2$ which is not bi-invariant orthogonal to w_2, then $\mathfrak{g}_{\pm\beta}\subset\mathfrak{m}$. Let u and v be any nonzero vectors in $\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_{\pm\beta}$, respectively. Notice there does not exist any root which is contained in $\mathbb{R}(w_2+\cdots+w_m)$, thus a similar argument as for Subcase 2 of Case 2 can be applied to prove $K^F(o,u,u\wedge v)=0$, which is a contradiction. Hence the corresponding coset space does not admit any invariant reversible Finsler metric with positive flag curvature.

The only case left is that w belongs to a simple factor; in this case G/H is equivalent to an odd-dimensional positively curved reversible homogeneous Finsler space G'/H' such that G' is a compact simple Lie group and H' is regular in G'. Summarizing the above argument in this subsection, we have proven Theorem 2.

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