

A note on a fixed point theorem on topological cylinders

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Abstract We present a fixed point theorem on topological cylinders in normed linear spaces for maps satisfying a property of stretching a space along paths. This result is a generalization of a similar theorem obtained by D. Papini and F. Zanolin. In view of the main result, we discuss the existence of fixed points for maps defined on different types of domains and we propose alternative proofs for classical fixed point theorems, as Brouwer, Schauder and Krasnosel'skiĭ ones.

Keywords Fixed point theorems · Fixed point index · Brouwer fixed point theorem · Schauder fixed point theorem · Krasnosel'skiĭ fixed point theorem in cones

Mathematics Subject Classification 47H10 · 37C25 · 47H11 · 54H25

1 Introduction

Fixed point theorems in finite and infinite dimensional spaces have recently received a lot of attention and frequently classical results has been reinterpreted in a new perspective, even with the purpose to obtain new useful tools for the study of nonlinear differential and integral equations. As an example, in [12, 13] the author dealt with Krasnosel'skiĭ cone fixed point theorem, and he discussed more general results in cones and cylinders. Or still, the paper [15] is devoted to an application of a general version of the classical Poincaré–Miranda zeros theorem, which is equivalent to Brouwer fixed point theorem, as well known. As a last example, we mention [18], where the author proposed elementary proofs of generalization in \mathbb{R}^n of classical fixed point theorems for expansive-compressive mappings.

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The present paper deals with the search and the localization of fixed points for compact maps defined on “cylindrical” domains of a normed linear space. The starting point and motivation for the present paper is the interesting work [21] by Papini and Zanolin. The two authors have studied fixed point theorems for maps satisfying a property of stretching a space along paths connecting two of its own disjoint subsets. In [21], they have considered cylinders of the form

$$[a, b] \times B[0, R],$$

so with “base” a ball in \mathbb{R}^n , while our goal is to work with domains of the form

$$[a, b] \times A,$$

where A is an *absolute retract*. Afterwards, we prove that the existence of fixed points remains valid under homeomorphic transformation of the cylinder.

The plan of the paper is the following. In Sect. 2, we present our main results. First of all, we generalize [21, Theorem 6] by considering a cylinder with an absolute retract as “base” and a compact map defined over it. Under the assumption of “stretching along the paths”, we obtain an existence result for fixed points in a particular region of the cylinder, using the fixed point index and in the same spirit of [21]. Subsequently, we discuss further possible generalization of the theorem and we verify that the property of “stretching along the paths” is preserved under homeomorphisms.

In Sect. 3, we review some classical fixed point theorems, as Brouwer and Schauder theorems, from the perspective of the main result. Using simple tricks, we show that these results are direct consequences of our main theorem.

Section 4 is devoted to the study of maps defined on domains of different type. In more detail, we consider particular region of cones in normed linear spaces, as in Krasnosel’skii’s cone fixed point theorem. As we shall see, the hypotheses of our main result are clearly of expansive type, hence it allows us to analyse only the expansive form of Krasnosel’skii’s theorem. For this reason, in that section we introduce a corollary useful to demonstrate Krasnosel’skii’s compressive form and other compressive-type fixed point results. In addition, we explore other fixed point theorems related to the main result.

We conclude our work with an appendix dedicated to the fixed point index. We recall the main properties and we enunciate the Leray–Schauder continuation principle. This tools are used during the proof of the main theorem in Sect. 2.

Now we introduce some symbols and some notations. We denote by \mathbb{Q} and \mathbb{R} the usual numerical sets and let $\mathbb{R}^+ := [0, +\infty[$.

Given a topological space (X, τ) and a subset $A \subseteq X$, we denote by \bar{A} its closure, with $\text{int}(A)$ its inner part and with ∂A its boundary. If $A, B \subseteq X$, with $A \setminus B$ we mean the relative complement of B in A .

What is dealt in the article takes place in the class of metrizable spaces. Usually, we denote by $(X, \|\cdot\|)$ or simply by X a normed linear space, where $\|\cdot\|$ is its norm. The symbols $B(x_0, r)$ and $B[x_0, r]$, where $x_0 \in X$ and $r > 0$, represent the open and closed ball of centre x_0 and radius r , respectively, i.e.

$$B(x_0, r) := \{x \in X : \|x - x_0\| < r\}, \quad B[x_0, r] := \{x \in X : \|x - x_0\| \leq r\}.$$

We indicate with

$$\text{dist}(A, B) := \inf_{x \in A, y \in B} \|x - y\|$$

the usual distance between two sets $A, B \subseteq X$.

We denote by Id or Id_X the identity map on the space X . Given a function f , $f|_D$ represents the restriction of the function f in D , where D is a subset of the domain of f .

2 Main results

In this section we present our main result (Theorem 1) which deals with cylinders having a topological retract as “base”. Then we prove the validity of the same theorem for other type of domains.

Let $(X, \| \cdot \|)$ be a normed linear space and let D be a subset of $\mathbb{R} \times X$. Consider a map

$$\phi = (\phi_1, \phi_2) : D \rightarrow \mathbb{R} \times X.$$

Our main goal is to provide an existence result for fixed points of ϕ . In other words, we want to prove the existence of elements $\tilde{z} = (\tilde{t}, \tilde{x}) \in D$ such that

$$\begin{cases} \tilde{t} = \phi_1(\tilde{t}, \tilde{x}) \\ \tilde{x} = \phi_2(\tilde{t}, \tilde{x}). \end{cases}$$

Now we introduce some preliminary definitions and notations. First of all we recall that a nonempty subset A of a topological space Y is a *retract* of Y if there exists a continuous map $r : Y \rightarrow A$ (called *retraction*), such that $r|_A = Id_A$.

Let A be a retract of X and fix $a, b \in \mathbb{R}$ with $a < b$. We denote by

$$C := [a, b] \times A$$

the *cylinder* with “base” the retract A and “height” the interval $[a, b]$. We also set

$$C_l := \{a\} \times A \quad \text{and} \quad C_r := \{b\} \times A$$

the cylinder’s left base and the cylinder’s right base, respectively.

In the following, we will put our attention on *paths* contained in C or in other subsets of $\mathbb{R} \times X$. With the term *path* we mean an element $(\sigma, \bar{\sigma})$, where σ is a continuous map defined on a compact interval $I \subseteq \mathbb{R}$ (usually we take $I = [0, 1]$) and with values on the normed linear space $\mathbb{R} \times X$, and $\bar{\sigma} := \sigma(I) \subseteq \mathbb{R} \times X$ is the *support* (image) of σ . Furthermore we say that $(\gamma, \bar{\gamma})$ is a *sub-path* of the path $(\sigma, \bar{\sigma})$ (defined on I) if $\gamma : [s_0, s_1] (\subseteq I) \rightarrow \mathbb{R} \times X$ and $\gamma = \sigma|_{[s_0, s_1]}$. In order to avoid too heavy notation, we simply employ σ to denote the path $(\sigma, \bar{\sigma})$. In addition, we often use the symbol σ meaning its support; for example, if $Z \subseteq \mathbb{R} \times X$, we write $\sigma \subseteq Z$ instead of $\bar{\sigma} \subseteq Z$, $\sigma \cap Z \neq \emptyset$ in place of $\bar{\sigma} \cap Z \neq \emptyset$ and so on. Moreover if σ and γ are two paths, the notation $\gamma \subseteq \sigma$ means that γ is a sub-path of σ .

As usual, a map $f : X \rightarrow Y$ between two metric spaces is *compact* if it is continuous and the closure of $f(X)$ is a compact subset of Y .

We now state and prove the main result (see also Fig. 1).

Theorem 1 *Let $(X, \| \cdot \|)$, D , A , C and ϕ be as above. Suppose that there exists a closed set $W \subseteq D \cap C$ such that*

- (i) ϕ is compact on W ;
- (ii) for every path $\sigma \subseteq C$ with $\sigma \cap C_l \neq \emptyset$ and $\sigma \cap C_r \neq \emptyset$ there exists a sub-path $\gamma \subseteq \sigma \cap W$ with $\phi(\gamma) \subseteq C$, $\phi(\gamma) \cap C_l \neq \emptyset$ and $\phi(\gamma) \cap C_r \neq \emptyset$.

Then there exists $\tilde{z} \in W$ such that $\phi(\tilde{z}) = \tilde{z}$.

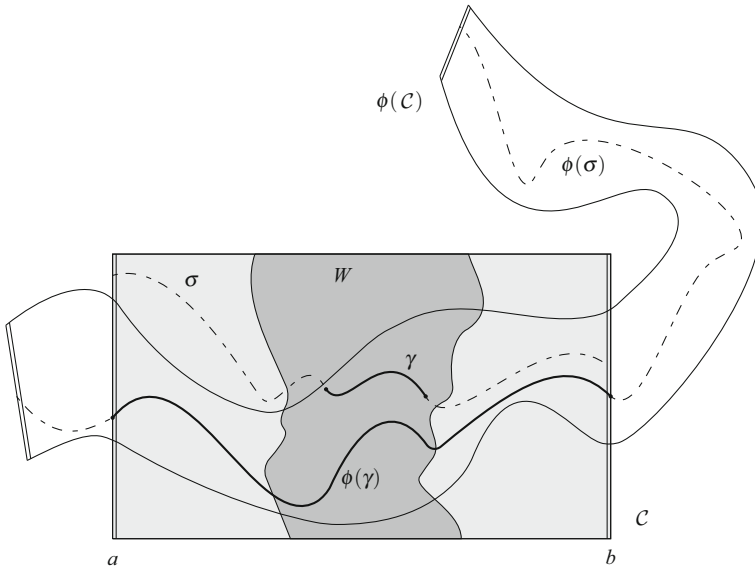


Fig. 1 Figure shows a possible situation satisfying conditions (i) and (ii) of Theorem 1. In the picture, it is assumed that $D = C$

Proof Consider the compact operator

$$\phi: W \rightarrow \mathbb{R} \times X.$$

As a first step, we extend ϕ to $\mathbb{R} \times X$ via a compact map

$$\tilde{\phi}: \mathbb{R} \times X \rightarrow \mathbb{R} \times X,$$

i.e. $\tilde{\phi}|_W = \phi|_W$. The existence of $\tilde{\phi}$ compact is guaranteed by an application of Urysohn embedding theorem (see, for instance, [6, Theorem B.11, p. 597]) and Dugundji extension theorem (cf. [3] and [6, Theorem 7.4, pp. 163–164]) together with the results in [6, § 6.2, pp. 116–118].

Let $r: X \rightarrow A$ be a retraction. We introduce the operator

$$\psi = (\psi_1, \psi_2): \mathbb{R} \times X \rightarrow \mathbb{R} \times A,$$

defined as follows

$$\psi_1(t, x) := \tilde{\phi}_1(t, x), \quad \psi_2(t, x) := r(\tilde{\phi}_2(t, x)).$$

We observe that ψ is a compact map (by the compactness of $\tilde{\phi}$ and the continuity of r). We define

$$\hat{A} := \overline{\psi_2(\mathbb{R} \times X)}.$$

The set \hat{A} is nonempty, compact and contained in A (since A is closed, by [8, Theorem 5.1, p. 18]).

We also note that if $\tilde{z} = (\tilde{t}, \tilde{x})$ is a fixed point of ψ with $\tilde{z} \in W$ and $\phi_2(\tilde{z}) \in A$, then

$$\tilde{t} = \psi_1(\tilde{z}) = \tilde{\phi}_1(\tilde{z}) = \phi_1(\tilde{z})$$

and

$$\tilde{x} = \psi_2(\tilde{z}) = r(\tilde{\phi}_2(\tilde{z})) = r(\phi_2(\tilde{z})) = \phi_2(\tilde{z}),$$

hence $\tilde{z} \in W$ is a fixed point of ϕ .

Now we study the fixed point problem

$$x = \psi_2(t, x), \quad x \in X,$$

where t is treated as a parameter in $[a, b]$. From the definition of \hat{A} , it is obvious that $\psi_2(t, x) \in \hat{A} \subseteq A$ for all $(t, x) \in [a, b] \times X$. We fix $r > 0$ such that the open ball $B(0, r)$ contains the closed and bounded set \hat{A} . It follows that, for all $t \in [a, b]$,

$$x - \psi_2(t, x) \neq 0, \quad \forall x \in \partial B(0, r).$$

Consequently, the fixed point index $i_X(\psi_2(t, \cdot), B(0, r))$ is well defined and constant, for $t \in [a, b]$. Consider now the compact homotopy $h: [0, 1] \times B[0, r] \rightarrow X, h_\lambda(x) := \lambda\psi_2(t, x)$. For every $t \in [a, b]$ we have that

$$x - \lambda\psi_2(t, x) \neq 0, \quad \forall \lambda \in [0, 1], \forall x \in \partial B(0, r),$$

since $\lambda\psi_2(t, x) \in \lambda\hat{A} \subseteq B(0, r)$ for every $\lambda \in [0, 1]$ and $x \in X$.

By the homotopy invariance property, the integer $i_X(h_\lambda, B(0, r))$ is constant with respect to $\lambda \in [0, 1]$. In particular, for every $t \in [a, b]$,

$$i_X(\psi_2(t, \cdot), B(0, r)) = i_X(0, B(0, r)) = 1.$$

Therefore we can apply the *Leray–Schauder continuation principle* (Theorem 11), which ensures that the nonempty set

$$\Sigma := \{(t, x) \in [a, b] \times B(0, r) : x = \psi_2(t, x)\}$$

contains a compact and connected (*continuum*) set S such that

$$S \cap C_l \neq \emptyset \quad \text{and} \quad S \cap C_r \neq \emptyset.$$

As consequence, $p_1(S) = [a, b]$, where $p_1: \mathbb{R} \times X \rightarrow \mathbb{R}, p_1(t, x) = t$, and for every open \tilde{A} containing \hat{A}

$$p_2(S) \subseteq \hat{A} \subseteq \tilde{A},$$

where $p_2: \mathbb{R} \times X \rightarrow X, p_2(t, x) = x$.

Fix an open and bounded set \tilde{A} such that $\hat{A} \subseteq \tilde{A}$ (for example $\tilde{A} = B(0, r)$, as before) and fix $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < \text{dist}(\hat{A}, X \setminus \tilde{A}) < +\infty$. By the continuity of the retraction r , for every $x \in X$ we can find a positive number $\delta_x = \delta_x(\varepsilon)$, with $0 < \delta_x < \varepsilon$, such that for every $y \in X$ with $\|y - x\| < \delta_x$ it holds that $\|r(y) - r(x)\| < \varepsilon$.

Consider the open covering of S defined as

$$\{]t - \varepsilon, t + \varepsilon[\times B(x, \delta_x) : (t, x) \in S \}.$$

Since S is compact, we can extract a finite sub-covering

$$\{]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, \delta_{x_i}) : i = 1, \dots, m \},$$

where $(t_i, x_i) \in S$, so $t_i \in [a, b]$ and $x_i \in \hat{A} \subseteq A$, for all $i \in \{1, \dots, m\}$.

Now we set

$$\mathcal{U}_\varepsilon := \bigcup_{i=1}^m]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, \delta_{x_i}),$$

which is contained in $]a - \varepsilon, b + \varepsilon[\times \tilde{A}$, because $0 < \delta_{x_i} < \varepsilon < \text{dist}(\hat{A}, X \setminus \tilde{A})$, for all $i = 1, \dots, m$. The set \mathcal{U}_ε is open and connected (because $S \subseteq \mathcal{U}_\varepsilon$ and $(t_i, x_i) \in S \cap (]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, \delta_{x_i})) \neq \emptyset$ for all $i \in \{1, \dots, m\}$). Hence \mathcal{U}_ε is also arcwise connected. Consequently there exists a continuous map $\vartheta: [0, 1] \rightarrow \mathcal{U}_\varepsilon$ such that $\vartheta(0) \in \mathcal{C}_l$ and $\vartheta(1) \in \mathcal{C}_r$. Passing, if necessary, to a sub-path inside the cylinder and reparameterizing the curve, we can also assume that, for all $s \in [0, 1]$,

$$\vartheta_1(s) := p_1(\vartheta(s)) \in [a, b].$$

Then we define a second curve $\zeta(s) := (\zeta_1(s), \zeta_2(s))$, $s \in [0, 1]$, as follows

$$\zeta_1(s) := p_1(\vartheta(s)) = \vartheta_1(s), \quad \zeta_2(s) := r(p_2(\vartheta(s))) = r(\vartheta_2(s)).$$

We claim that the curve ζ satisfies

- $\zeta(s) \in \mathcal{V}_\varepsilon \cap \mathcal{C}$, for all $s \in [0, 1]$;
- $\zeta(0) \in \mathcal{C}_l$ and $\zeta(1) \in \mathcal{C}_r$;

where

$$\mathcal{V}_\varepsilon := \bigcup_{i=1}^m]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, \varepsilon).$$

From the definition, we derive immediately that $\zeta(s) \in \mathcal{C}$ and $\zeta_1(s) = \vartheta_1(s) \in [a, b] \subseteq p_1(\mathcal{V}_\varepsilon)$, for every $s \in [0, 1]$. Consequently, to prove the first property it is sufficient to verify that there exists $j \in \{1, \dots, m\}$ such that $\zeta_2(s) \in B(x_j, \varepsilon)$. Fix $s \in [0, 1]$. Let $k \in \{1, \dots, m\}$ be such that $\vartheta_2(s) \in B(x_k, \delta_{x_k})$. We distinguish two cases. If $\vartheta_2(s) \in A$, then $\zeta_2(s) = \vartheta_2(s) \in B(x_k, \delta_{x_k}) \subseteq B(x_k, \varepsilon)$. If $\vartheta_2(s) \notin A$, then

$$\|\zeta_2(s) - x_k\| = \|r(\vartheta_2(s)) - r(x_k)\| < \varepsilon,$$

by the choice of $\delta_x = \delta_x(\varepsilon)$ and by the fact that $x_k \in A$ (hence $r(x_k) = x_k$). In each case, we conclude that $\zeta(s) \in \mathcal{V}_\varepsilon$ for all $s \in [0, 1]$. The second property is obvious from the definition of ζ . Thus our claim is proved.

From the two properties it follows that the path ζ is contained in the cylinder \mathcal{C} and it has a nonempty intersection with the left and the right bases. Thus, taking $\sigma := \zeta$, hypothesis (ii) implies that there must be a sub-path γ of σ such that $\gamma \subseteq W$ with $\phi(\gamma) \subseteq \mathcal{C}$, $\phi(\gamma) \cap \mathcal{C}_l \neq \emptyset$ and $\phi(\gamma) \cap \mathcal{C}_r \neq \emptyset$. Let

$$\xi = (\xi_1, \xi_2): [0, 1] \rightarrow \mathbb{R} \times X$$

be a continuous map such that the path $(\xi, \bar{\xi})$ is the reparametrization to $[0, 1]$ of the sub-path $(\gamma, \bar{\gamma})$. By the assumptions, it follows that

- $\xi(s) \in \mathcal{V}_\varepsilon \cap W$, for all $s \in [0, 1]$;
- $\phi(\xi(s)) \in \mathcal{C}$, for all $s \in [0, 1]$;
- $\phi(\xi(0)) \in \mathcal{C}_l$ and $\phi(\xi(1)) \in \mathcal{C}_r$.

We consider now the continuous map $g: [0, 1] \rightarrow \mathbb{R} \times X$ defined by $g(s) = \xi_1(s) - \phi_1(\xi(s))$. Since $\xi(s) \in \gamma \subseteq \sigma \subseteq \mathcal{C}$ for all $s \in [0, 1]$, we obtain that $\xi_1(0) \geq a$, so $g(0) \geq a - a = 0$. Similarly $g(1) \leq 0$. By Bolzano's theorem, we derive that there exists $\hat{s} = \hat{s}_\varepsilon \in [0, 1]$ such that $g(\hat{s}) = \xi_1(\hat{s}) - \phi_1(\xi(\hat{s})) = 0$. Setting

$$\hat{t} = \hat{t}_\varepsilon := \xi_1(\hat{s}), \quad \hat{x} = \hat{x}_\varepsilon := \xi_2(\hat{s}), \quad \hat{z} = \hat{z}_\varepsilon := (\hat{t}, \hat{x}),$$

it follows that

$$\hat{z} = \xi(\hat{s}) \in \mathcal{V}_\varepsilon \cap W, \quad \hat{t} = \xi_1(\hat{s}) = \phi_1(\hat{z}), \quad \phi_2(\hat{z}) = \phi_2(\xi(\hat{s})) \in A.$$

From the definition of ψ , we obtain that

$$\hat{z} \in \mathcal{V}_\varepsilon \cap W, \quad \hat{t} = \psi_1(\hat{z}), \quad \phi_2(\hat{z}) \in A. \tag{1}$$

Furthermore, since $\hat{z} \in \mathcal{V}_\varepsilon$, there is an index $i \in \{1, \dots, m\}$ such that $\|\hat{z} - (t_i, x_i)\| < \varepsilon$. Hence we conclude that, for every $0 < \varepsilon < \text{dist}(\hat{A}, X \setminus \hat{A})$, there exists a point $\hat{z} = \hat{z}_\varepsilon$ satisfying (2) and, moreover, there exists $\hat{y}_\varepsilon \in S$ such that $\|\hat{z}_\varepsilon - \hat{y}_\varepsilon\| < \varepsilon$.

Setting $\varepsilon = \varepsilon_n = 1/n$, we derive that, for $n > 1/\text{dist}(\hat{A}, X \setminus \hat{A})$, there exists an element $\hat{z}_n \in \mathcal{V}_{1/n} \cap W$ with the characteristics listed above. In addition, for each \hat{z}_n there exists a $\hat{y}_n \in S$ such that $\|\hat{z}_n - \hat{y}_n\| < 1/n$. Since S is compact, possibly passing to a subsequence, the sequence $(\hat{y}_n)_n$ converges to an element $\tilde{z} = (\tilde{t}, \tilde{x}) \in S$. Passing to the corresponding subsequence of $(\hat{z}_n)_n$, we also obtain that $(\hat{z}_n)_n$ converges to $\tilde{z} \in S$. By the continuity of ϕ and ψ and using the fact that W and A are closed, we have

$$\tilde{z} \in S \cap W, \quad \tilde{t} = \psi_1(\tilde{z}), \quad \phi_2(\tilde{z}) \in A.$$

Being S contained in the set Σ , we find that

$$\tilde{x} = \psi_2(\tilde{t}, \tilde{x})$$

and consequently $\tilde{z} = (\tilde{t}, \tilde{x}) \in W$ is a fixed point of ψ . As shown at the beginning of the proof and since $\phi_2(\tilde{z}) \in A$, we get that $\tilde{z} \in W$ is a fixed point of ϕ . The theorem follows. \square

Remark 1 We observe that in Theorem 1 it is not restrictive to suppose that $\phi(D \cap C) \subseteq \mathbb{R} \times A$. In fact, if $\phi(D \cap C) \subseteq \mathbb{R} \times X$, we define

$$\hat{D} := \{z \in D \cap C : \phi_2(z) \in A\} \subseteq C$$

and we consider the restriction $\hat{\phi} := \phi|_{\hat{D}} : \hat{D} \rightarrow \mathbb{R} \times A$. Now we prove that $\hat{\phi}$ satisfies the two conditions of the theorem with respect to a suitable closed set $\hat{W} \subseteq \hat{D}$. Setting

$$\hat{W} := \{z \in W : \phi(z) \in C\} \subseteq \hat{D},$$

clearly condition (i) is satisfied (as $\hat{W} \subseteq W$). We prove (ii). Suppose to have a path $\sigma \subseteq C$ with $\sigma \cap C_l \neq \emptyset$ and $\sigma \cap C_r \neq \emptyset$. We know that there exists a sub-path $\gamma \subseteq \sigma \cap W$ with $\phi(\gamma) \subseteq C$, $\phi(\gamma) \cap C_l \neq \emptyset$ and $\phi(\gamma) \cap C_r \neq \emptyset$. Obviously $\gamma \subseteq \hat{W}$. Hence (ii) follows.

Remark 2 We note that in condition (ii) of Theorem 1 we can replace the term *path* with the concept of *continuum* (i.e. compact and connected set). Following the proof of [21, Theorem 7], we obtain the same thesis in the case of retracts.

Our next aim is to give an existence result with respect to a cylinder having an absolute retract as a “base”. For reader’s convenience, we recall that a space A is an *absolute retract* (or simply an *AR*) if A is metrizable and for any metrizable Y and every closed $M \subseteq Y$ each continuous function $f : M \rightarrow A$ is continuously extendable over Y . We also recall that a space A is an *absolute neighbourhood retract* (or simply an *ANR*) if A is metrizable and for any metrizable Y , every closed $M \subseteq Y$ and each continuous function $f : M \rightarrow A$ there exists a neighbourhood U of M and a continuous extension of f over U . Clearly an AR is an ANR. For properties and characterization of ARs and ANRs, we refer to [6, 8].

The next corollary deals with the case of a cylinder with base an AR.

Corollary 1 *Let A be an AR and $a, b \in \mathbb{R}$ with $a < b$. We denote by*

$$C := [a, b] \times A$$

the cylinder with “base” the absolute retract A and “height” the interval $[a, b]$. We indicate with

$$C_l := \{a\} \times A \quad \text{and} \quad C_r := \{b\} \times A$$

the cylinder’s left base and the cylinder’s right base, respectively.

Let $D \subseteq C$. Consider the operator

$$\phi = (\phi_1, \phi_2): D \rightarrow \mathbb{R} \times A$$

and suppose that there exists a closed set $W \subseteq D$ such that

- (i) ϕ is compact on W ;
- (ii) for every path $\sigma \subseteq C$ with $\sigma \cap C_l \neq \emptyset$ and $\sigma \cap C_r \neq \emptyset$ there exists a sub-path $\gamma \subseteq \sigma \cap W$ with $\phi(\gamma) \subseteq C$, $\phi(\gamma) \cap C_l \neq \emptyset$ and $\phi(\gamma) \cap C_r \neq \emptyset$.

Then there exists $\tilde{z} \in W$ such that $\phi(\tilde{z}) = \tilde{z}$.

Proof Given the absolute retract A , we can fix a normed linear space X where A is a retract (cf. [6, Theorem 7.6, p. 164]). Finally we use Theorem 1 to conclude.

The following example shows that completely continuity is not sufficient for the claim of Theorem 1. We recall that a map $f: X \rightarrow Y$ between normed linear spaces is *completely continuous*, if f is continuous and the closure of $f(B)$ is a compact subset of Y , for each bounded subset $B \subseteq X$.

Example 1 Consider the map $\phi: [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$, $\phi(t, x) = (t, x + 1)$, and set $W = [0, 1] \times \mathbb{R}$. Clearly \mathbb{R} is a retract (of \mathbb{R}) and W is closed. We note that the map ϕ is completely continuous, but it is not compact on W , since $\phi(W) = [0, 1] \times \mathbb{R}$. Moreover condition (ii) is satisfied, but ϕ has not fixed points.

With the following example we point out that, if we have an ANR at the place of the AR, the assertion under the same hypotheses is not true.

Example 2 Let $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere in \mathbb{R}^n . It is known that S^{n-1} is an ANR but not an AR (see [6, Theorem 1.5, p. 280] and [8, pp. 20–21]). Fix $\alpha \in]0, 2\pi[$ ($\alpha/2\pi \notin \mathbb{Q}$) and consider the map $\phi: [0, 1] \times S^1 \rightarrow [0, 1] \times S^1$, $\phi(t, x) = (t, xe^{i\alpha t})$, which represents the rotation around zero of angle α . Clearly ϕ is continuous (hence compact), but it has not fixed points (or periodic points).

In almost all the applications to differential equations we cannot work with a cylinder. Thus our new goal is to present an existence result for fixed points of maps defined in more general domains.

We recall that a topological space X satisfies the *fixed point property* if any continuous function $f: X \rightarrow X$ has a fixed point. From classical results, we know that an homeomorphism preserves the fixed point property. In the following result, we show that also our property is preserved under homeomorphisms.

Corollary 2 *Let Y be a metric space and $M \subseteq Y$ a closed subset homeomorphic to the cylinder $C := [a, b] \times A$, where A is an AR and $a, b \in \mathbb{R}$ with $a < b$. Call*

$$h: M \rightarrow [a, b] \times A$$

the homeomorphism onto its image and

$$M_l := h^{-1}(\{a\} \times A) \quad \text{and} \quad M_r := h^{-1}(\{b\} \times A)$$

the “left base” and the “right base” of M , respectively.

Let $N \subseteq Y$. Consider the operator $\phi: N \rightarrow Y$ and suppose that there exists a closed set $V \subseteq M \cap N$ such that

- (i) ϕ is compact on V ;
- (ii) for every path $\sigma \subseteq M$ with $\sigma \cap M_l \neq \emptyset$ and $\sigma \cap M_r \neq \emptyset$ there exists a sub-path $\gamma \subseteq \sigma \cap V$ with $\phi(\gamma) \subseteq M$, $\phi(\gamma) \cap M_l \neq \emptyset$ and $\phi(\gamma) \cap M_r \neq \emptyset$.

Then there exists $\tilde{z} \in V$ such that $\phi(\tilde{z}) = \tilde{z}$.

Proof Define

$$H := \{z \in V : \phi(z) \in M\}$$

and observe that $H = \phi^{-1}(M) \cap V$ is closed (since M and V are closed), $\hat{\phi} := \phi|_H: H \rightarrow Y$ is compact and $\phi(H) \subseteq M$. Set $D := h(H)$, which is closed and contained in $h(M) = C$.

Consider this diagram

$$\begin{array}{ccc}
 H(\subseteq V) & \xrightarrow{\hat{\phi}} & M \\
 h^{-1} \uparrow & & \downarrow h \\
 h(H) = D & \xrightarrow{\hat{\psi}} & h(M) = C
 \end{array}$$

where

$$\hat{\psi} := h \circ \hat{\phi} \circ h^{-1}: D \rightarrow C = [a, b] \times A.$$

We claim that $\hat{\psi}$ is a compact map. Suppose to have a sequence $(y_n)_n$ contained in $\hat{\psi}(D)$, i.e. $y_n = \hat{\psi}(x_n) = h(\hat{\phi}(h^{-1}(x_n)))$ with $x_n \in D$. By the compactness of $\hat{\phi}$, the closure of $\hat{\phi}(h^{-1}(D)) = \hat{\phi}(H)$ is compact. Then there exists a subsequence (x_{k_n}) such that $\hat{\phi}(h^{-1}(x_{k_n})) \rightarrow \bar{x} \in M$ (since M closed). By the continuity of h , we conclude that $y_{k_n} = \hat{\psi}(x_{k_n}) = h(\hat{\phi}(h^{-1}(x_{k_n}))) \rightarrow h(\bar{x}) \in h(M) = C$. Hence the compactness of $\hat{\psi}$.

Consider a path $\hat{\sigma} \subseteq C$ with $\hat{\sigma} \cap C_l \neq \emptyset$ and $\hat{\sigma} \cap C_r \neq \emptyset$. Consequently $\sigma := h^{-1}(\hat{\sigma}) \subseteq M$, $\sigma \cap M_l \neq \emptyset$ and $\sigma \cap M_r \neq \emptyset$. Then there exists a sub-path $\gamma \subseteq \sigma \cap V$ with $\phi(\gamma) \subseteq M$, $\phi(\gamma) \cap M_l \neq \emptyset$ and $\phi(\gamma) \cap M_r \neq \emptyset$. Hence $\hat{\gamma} := h(\gamma)$ is a sub-path of $\hat{\sigma}$ such that $\hat{\gamma} \subseteq \hat{\sigma} \cap D$, $\hat{\psi}(\hat{\gamma}) \subseteq C$, $\hat{\psi}(\hat{\gamma}) \cap C_l \neq \emptyset$ and $\hat{\psi}(\hat{\gamma}) \cap C_r \neq \emptyset$.

Setting $W = D$, every hypothesis of Corollary 1 is satisfied. Then there exists $\tilde{z} \in D = h(H)$ such that $\hat{\psi}(\tilde{z}) = \tilde{z}$. We conclude that the point $\tilde{z} = h^{-1}(\tilde{z}) \in V$ is a fixed point of ϕ . □

3 Some classical theorems

In this section we present three classical results of fixed point theory revised in view of Theorem 1. In other words, we check that our main theorem is a generalization of these results.

Certainly *Brouwer fixed point theorem* is one of the most famous. In the proposed proof, we note that a simple trick allows us to put ourselves in the hypotheses (i) and (ii) of Theorem 1.

Theorem 2 (Brouwer fixed point theorem) *Let $B = B[0, 1] \subseteq \mathbb{R}^n$ be the unit closed ball. Any continuous function $f: B \rightarrow B$ has a fixed point in B .*

Proof First of all, we observe that B is a retract of \mathbb{R}^n and f is compact (by the finite dimension of the space). Consider the application $\phi = (\phi_1, \phi_2): [0, 1] \times B \rightarrow [0, 1] \times B$ defined as

$$\phi_1(t, x) := t, \quad \phi_2(t, x) := f(x).$$

By the hypothesis, ϕ is well defined and satisfies conditions (i) and (ii) of Theorem 1 (with $W = [0, 1] \times B$). Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in [0, 1] \times B$ such that $\phi(\tilde{z}) = \tilde{z}$, i.e. $(\tilde{t}, \tilde{x}) = (\tilde{t}, f(\tilde{x}))$. We conclude that $\tilde{x} \in B$ is a fixed point of f .

The second classical result we look over is *Schauder fixed point theorem*.

Theorem 3 (Schauder fixed point theorem) *Let C be a closed, convex and nonempty subset of a normed linear space X . Any compact map $\varphi: C \rightarrow C$ has a fixed point in C .*

Now we prove directly a more general theorem, of which one can find an alternative proof in [6, Theorem 7.9, p. 165].

Theorem 4 (Generalized Schauder theorem) *Let Z be an AR. Any compact map $\varphi: Z \rightarrow Z$ has a fixed point in Z .*

Proof Consider the application $\phi = (\phi_1, \phi_2): [0, 1] \times Z \rightarrow [0, 1] \times Z$ defined as

$$\phi_1(t, x) := t, \quad \phi_2(t, x) := \varphi(x).$$

By the hypothesis, ϕ is well defined and satisfies conditions (i) and (ii) of Corollary 1 (with $W = [0, 1] \times Z$). Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in [0, 1] \times Z$ such that $\phi(\tilde{z}) = \tilde{z}$, i.e. $(\tilde{t}, \tilde{x}) = (\tilde{t}, \varphi(\tilde{x}))$. We conclude that $\tilde{x} \in Z$ is a fixed point of φ . □

It is well known that Brouwer fixed point theorem is equivalent to the classical *Poincaré–Miranda zeros theorem*. Now we present a theorem that is the “fixed point version” of Poincaré–Miranda result, alternatively it can be viewed as a n -dimensional generalization of Bolzano’s theorem. Our aim is to provide a direct proof in the light of the main theorem, without using previous classical results.

The theorem we state below has been proposed in some recent works, in this or in other versions. As examples, see [4,9,12,13,15,17,18,22].

Theorem 5 (Poincaré–Miranda) *Let $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$ be a n -dimensional rectangle. We denote by $I_i^- = \{x \in \mathcal{R}: x_i = a_i\}$ and $I_i^+ = \{x \in \mathcal{R}: x_i = b_i\}$ its i -faces. Let $g = (g_1, \dots, g_n): \mathcal{R} \rightarrow \mathbb{R}^n$ be a continuous vector field such that for each $i \in \{1, \dots, n\}$ one of the two following possibilities is true:*

- (e_{*i*}) $g_i(x) \leq a_i, \forall x \in I_i^-, \quad g_i(x) \geq b_i, \forall x \in I_i^+;$
- (c_{*i*}) $g_i(x) \geq a_i, \forall x \in I_i^-, \quad g_i(x) \leq b_i, \forall x \in I_i^+.$

Then there exists $z \in \mathcal{R}$ such that $g(z) = z$.

Proof First of all, we introduce a useful notation. Let $I := \{1, \dots, n\}$, $I_e := \{i \in I: (e_i) \text{ is valid}\}$ and $I_c := \{i \in I: (c_i) \text{ is valid}\}$. By the hypotheses, $I = I_e \cup I_c$ and $I_e \cap I_c = \emptyset$.

We first assume that $\emptyset \neq I_e = \{k\}$. We define

$$\mathcal{R}_k := [a_1, b_1] \times \dots \times [a_{k-1}, b_{k-1}] \times [a_{k+1}, b_{k+1}] \times \dots \times [a_n, b_n]$$

and we observe that \mathcal{R}_k is a retract of \mathbb{R}^{n-1} , hence $[a_k, b_k] \times \mathcal{R}_k$ is a cylinder where we can apply Theorem 1. Without loss of generality and for simplicity, we assume $k = 1$. Consider the operator

$$\tilde{g}: [a_1, b_1] \times \mathcal{R}_1 = \mathcal{R} \rightarrow \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$$

defined as

$$\tilde{g}(x) := (g_1(x), p_2(g_2(x)), \dots, p_n(g_n(x))),$$

where $p_i(x) := \min\{b_i, \max\{x_i, a_i\}\}$ is the projection on $[a_i, b_i]$, for all $i \in I_c$. We claim that \tilde{g} and $W = \mathcal{R}$ satisfy conditions (i) and (ii) of Theorem 1. Clearly \tilde{g} is continuous (hence compact). Suppose that the path $(\sigma, \bar{\sigma})$, with $\sigma: [0, 1] \rightarrow \mathcal{R}$, is such that $\sigma(0) \in I_1^-$ and $\sigma(1) \in I_1^+$. Hypothesis (e₁) implies that $g_1(\sigma(0)) \leq a_1$ and $g_1(\sigma(1)) \geq b_1$. Then, by the continuity of g_1 and σ , there exists a subinterval $[s_0, s_1] \subseteq [0, 1]$ such that $g_1(\sigma([s_0, s_1])) = [a_1, b_1]$. In particular the sub-path $\gamma := \sigma|_{[s_0, s_1]}$ is such that $\tilde{g}(\gamma) \subseteq \mathcal{R}$, $\tilde{g}(\gamma) \cap I_1^- \neq \emptyset$ and $\tilde{g}(\gamma) \cap I_1^+ \neq \emptyset$. We deduce the validity of (ii).

By Theorem 1, there exists an element $z = (z_1, z_2, \dots, z_n) \in \mathcal{R}$ such that $\tilde{g}(z) = z$. We claim that $z \in \mathcal{R}$ is a fixed point of g . We have to prove that $g_i(z) \in [a_i, b_i]$ for all $i \in I_c$. Suppose that there is an index $i \in I_c$ such that $g_i(z) < a_i$ or $g_i(z) > b_i$. Consider the first case. Then $z_i = p_i(g_i(z)) = a_i$, so $g_i(z) \geq a_i$, by (c_i). This is a contradiction. Similarly one can prove that $g_i(z) > b_i$ does not occur. We have the assertion.

Now suppose that $\emptyset \neq I_e = \{1, \dots, j\}$. Define $\hat{g}: \mathcal{R} \rightarrow \mathbb{R}^n$ as follows

$$\hat{g}(x) := (g_1(x), 2x_2 - g_2(x), \dots, 2x_j - g_j(x), g_{j+1}(x), \dots, g_n(x)).$$

It is easy to see that \hat{g} satisfies (e₁) and also (c_i) for all $i \in \{2, \dots, n\}$. Consequently we can apply the first case and observe that a fixed point of \hat{g} is also a fixed point of g .

The last case is $I_e = \emptyset$. We proceed as in the second case, defining $\hat{g}: \mathcal{R} \rightarrow \mathbb{R}^n$ as

$$\hat{g}(x) := (2x_1 - g_1(x), g_2(x), \dots, g_n(x)).$$

The theorem is thus proved. □

4 Other consequences of the main theorem

We conclude our work by explore other fixed point theorems available in the literature. The final aim is to relate Krasnosel'skii theorems on cones to our main result.

First of all, we analyse still a fixed point theorem on cylinders: *Kwong's theorem*. We state and prove a more general version compared to [12, Theorem 3.2] and [13, Theorem 3].

Theorem 6 *Let A be an AR and C, C_l, C_r as in Corollary 1. Consider a compact map $T = (T_1, T_2): C \rightarrow \mathbb{R} \times A$.*

– (Expansive form) *T has at least a fixed point in C if*

$$(E_a) \quad T_1(z) \leq a, \forall z \in C_l;$$

$$(E_b) \quad T_1(z) \geq b, \forall z \in C_r.$$

– (Compressive form) *T has at least a fixed point in C if*

$$(C_a) \quad T_1(z) \geq a, \forall z \in C_l;$$

$$(C_b) \quad T_1(z) \leq b, \forall z \in C_r.$$

Proof Expansive form. We are going to prove that conditions (i) and (ii) of Corollary 1 hold, with respect to $W = \mathcal{C}$. By the compactness hypothesis, (i) follows. Now suppose that the path $(\sigma, \bar{\sigma})$, with $\sigma : [0, 1] \rightarrow \mathcal{C}$, is such that $\sigma(0) \in \mathcal{C}_l$ and $\sigma(1) \in \mathcal{C}_r$. By hypotheses (E_a) and (E_b) , $T_1(\sigma(0)) \leq a$ and $T_1(\sigma(1)) \geq b$. Then, by the continuity of T_1 and σ , there exists a subinterval $[s_0, s_1] \subseteq [0, 1]$ such that $T_1(\sigma([s_0, s_1])) = [a, b]$. In particular the sub-path $\gamma := \sigma|_{[s_0, s_1]}$ is such that $T(\gamma) \subseteq \mathcal{C}$, $T(\gamma) \cap \mathcal{C}_l \neq \emptyset$ and $T(\gamma) \cap \mathcal{C}_r \neq \emptyset$. We deduce the validity of (ii). By Corollary 1 we have the assertion.

Compressive form. We reduce this case to the expansive form. Define a new operator $S = (S_1, S_2) : \mathcal{C} \rightarrow \mathbb{R} \times A$ as follows

$$S_1(t, x) := 2t - T_1(t, x), \quad S_2(t, x) := T_2(t, x).$$

We observe that the operator S is compact, since T is compact. Moreover, if $t = a$, $T_1(t, x) \geq a$ by (C_a) , then $S_1(t, x) \leq 2a - a = a$. Similarly $S_2(t, x) \geq b$ if $t = b$, using (C_b) . Then S satisfies the expansive form conditions. Consequently there is $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{C}$ such that $S(\tilde{z}) = \tilde{z}$. Hence

$$\tilde{t} = S_1(\tilde{t}, \tilde{x}) = 2\tilde{t} - T_1(\tilde{t}, \tilde{x}), \quad \tilde{x} = S_2(\tilde{t}, \tilde{x}) = T_2(\tilde{t}, \tilde{x}).$$

We conclude that $\tilde{z} \in \mathcal{C}$ is also a fixed point of T . □

Remark 3 In the second part of the proof of Theorem 6, we bring back the compressive form to the expansive framework employing the same trick that Kwong employed in [12] to transform an expansive operator into a compressive one.

An alternative way to prove the compressive form is to use directly the generalized Schauder fixed point theorem that, as already noted, is a direct consequence of our main result.

Till now, we have always considered fixed point theorems of expansive type or we have reduced the problem to them. In view of Corollary 2, we present a result of compressive type that allows us to easily prove Krasnosel’skiĭ fixed point theorems.

Corollary 3 *Let Y be a metric space and $M' \subseteq Y$ a closed subset homeomorphic to the cylinder $\mathcal{C}' := [a', b'] \times A$, where A is an AR and $a', b' \in \mathbb{R}$ with $a' < b'$. Call $h : M' \rightarrow \mathcal{C}'$ the homeomorphism onto its image. Let $M \subseteq M'$ and suppose there exist $a, b \in \mathbb{R}$ with $a' < a < b < b'$ such that $h(M) = \mathcal{C} := [a, b] \times A$. We define*

$$M_l := h^{-1}(\{a\} \times A) \quad \text{and} \quad M_r := h^{-1}(\{b\} \times A)$$

the left base and the right base of M , respectively.

Consider the operator $\phi : M \rightarrow Y$ and suppose that

- (i) ϕ is compact on M ;
- (ii) $\phi(M_l) \subseteq h^{-1}([a, b'] \times A)$ and $\phi(M_r) \subseteq h^{-1}([a', b] \times A)$.

Then there exists $\tilde{z} \in M$ such that $\phi(\tilde{z}) = \tilde{z}$.

Proof First of all, we observe that condition (ii) is equivalent to require that $\psi(\{a\} \times A) \subseteq [a, b'] \times A$ and $\psi(\{b\} \times A) \subseteq [a', b] \times A$, where $\psi := h \circ \phi \circ h^{-1}$. Secondly, as in the proof of Corollary 2, we consider the homeomorphism $h|_M$ between M and \mathcal{C} . Finally, we apply the compressive form in Theorem 6 to $T = \psi$ and \mathcal{C} . Hence we obtain the assertion. □

Remark 4 As we shall see in the sequel, Corollary 3 is a useful tool to prove compressive forms. However we can also note that, unlike Corollary 2, we do not have a localization of fixed points (for example inside a closed subset of M).

Hereafter we deal with cones in normed linear spaces. Let $(X, \|\cdot\|)$ be a normed linear space. We recall that a subset $K \subseteq X$ is a *cone* if the following conditions are satisfied:

- K is closed;
- $\alpha u + \beta v \in K, \forall u, v \in K, \forall \alpha, \beta \in \mathbb{R}^+$;
- $K \cap (-K) = \{0\}$, i.e. if $u \in K$ and $-u \in K$ then $u = 0$.

In the following example, we analyse some subsets of a cone in a normed space. What we observe allows us to define a homeomorphism between a region inside the cone and a suitable cylinder.

Example 3 Let $(X, \|\cdot\|)$ be a normed linear space and $K \subseteq X$ a cone. By Dugundji extension theorem, K is an AR, since K is convex.

Consider the subset $K_a := \{x \in K : \|x\| = a\}$, with $a \in \mathbb{R}^+$. We claim that K_a is a retract of K ; therefore, it is an AR (cf. [6, Proposition 7.2, p. 162]). If $a = 0$, we consider the retraction $r_a \equiv 0$. If $a \neq 0$, we fix an element $y \in K \setminus \{0\}$ and we define the retraction $r_a : K \rightarrow K_a$ as follows

$$r_a(x) := a \frac{x + (a - \|x\|)^2 y}{\|x + (a - \|x\|)^2 y\|}, \quad x \in K.$$

The map r_a is well defined, because $x + (a - \|x\|)^2 y \in K \setminus \{0\}, \forall x \in K$. Moreover $r_a(x) \in K$ and $\|r_a(x)\| = a, \forall x \in K$. Clearly r_a is continuous and $r|_{K_a} = Id_{K_a}$.

Let $l : X \rightarrow \mathbb{R}$ be a continuous functional, positive (i.e. $l(x) \geq 0, \forall x \in K$) and positively homogeneous. Consider the set $K_a^l := \{x \in K : l(x) = a\}$, with $a \in \mathbb{R}^+$. If the functional l is also strictly positive (i.e. $l(x) > 0, \forall x \in K \setminus \{0\}$), using the previous retractions (where we replace $\|\cdot\|$ with $l(\cdot)$), we obtain that K_a^l is a retract of K , therefore an AR.

Suppose that l is not strictly positive and assume that l is linear and $l \not\equiv 0$ on K (otherwise K_a^l is the whole cone K or the empty set). Fix an element $y \in K$ with $l(y) \neq 0$. Fix $a \neq 0$. Define the retraction $r_a : K \rightarrow K_a^l$ as follows

$$r_a(x) := a \frac{x + (a - l(x))^2 y}{l(x + (a - l(x))^2 y)}, \quad x \in K.$$

The map r_a is well defined, because $l(x + (a - l(x))^2 y) = l(x) + (a - l(x))^2 l(y) > 0, \forall x \in K$. Moreover $r_a(x) \in K$ and $l(r_a(x)) = a, \forall x \in K$. Clearly r_a is continuous and $r_a|_{K_a^l} = Id_{K_a^l}$. If $a = 0$, K_a^l is closed, convex (by the linearity of l) and nonempty ($0 \in K_a^l$). By Dugundji extension theorem, we conclude that K_0^l is a retract of X , so it is an AR.

We note that we can alternatively suppose that $l : K \rightarrow \mathbb{R}^+$ is defined only on K , is positively homogeneous and is such that

$$l(\alpha x_1 + \beta x_2) \geq \alpha l(x_1) + \beta l(x_2), \quad \forall x_1, x_2 \in K, \forall \alpha, \beta \geq 0.$$

In the case of positively homogeneous functionals, we stress that this hypothesis is equivalent to assume that l is concave on K .

Furthermore we also remark that the assumption of positive homogeneity is necessary for our definition of the retractions r_a .

For definitions, properties and characterizations of functionals defined on cones, we refer to [7].

Now we have all the tools to enunciate and prove Krasnosel'skiĭ fixed point theorem. The original version and the original proof are located in [10, 11]. Alternative proofs can be found in [6, 7, 23].

Theorem 7 (Krasnosel’skiĭ) *Let $(X, \| \cdot \|)$ be a normed linear space and K a cone in X . Fix $a, b \in \mathbb{R}^+$ with $0 < a < b$. Define $K_{a,b} := \{x \in K : a \leq \|x\| \leq b\}$ and $K_d := \{x \in K : \|x\| = d\}$, for all $d \in \mathbb{R}^+$.*

Consider a compact map $T : K \rightarrow K$.

– (Expansive form) *T has at least a fixed point in $K_{a,b}$ if*

$$(E_a) \quad \|T(x)\| \leq \|x\|, \forall x \in K_a;$$

$$(E_b) \quad \|T(x)\| \geq \|x\|, \forall x \in K_b.$$

– (Compressive form) *T has at least a fixed point in $K_{a,b}$ if*

$$(C_a) \quad \|T(x)\| \geq \|x\|, \forall x \in K_a;$$

$$(C_b) \quad \|T(x)\| \leq \|x\|, \forall x \in K_b.$$

Proof Consider the cylinder $C = [a, b] \times K_1$. As observed in Example 3, the set K_1 is a retract of K (hence of X). Now we define an homeomorphism h between $K_{a,b}$ and C as follows

$$h(x) := \left(\|x\|, \frac{x}{\|x\|} \right), \quad x \in K_{a,b}.$$

Clearly h is a continuous bijection with continuous inverse $h^{-1} : (\alpha, z) \mapsto \alpha z$. Moreover we note that $K_a = h^{-1}(\{a\} \times K_1)$ and $K_b = h^{-1}(\{b\} \times K_1)$.

Expansive form. We use Corollary 2 with $Y = X, N = K$ and $M = V = K_{a,b}$. Condition (i) is obvious. We prove (ii). Suppose that the path $(\sigma, \bar{\sigma})$, with $\sigma : [0, 1] \rightarrow K_{a,b}$, is such that $\sigma(0) \in K_a$ and $\sigma(1) \in K_b$. From (E_a) and (E_b) we derive that $\|T(\sigma(0))\| \leq a$ and $\|T(\sigma(1))\| \geq b$. By the continuity of $\|T(\sigma(\cdot))\|$, there exists a subinterval $[s_0, s_1] \subseteq [0, 1]$ such that $\|T(\sigma([s_0, s_1]))\| = [a, b]$. In particular, the sub-path $\gamma := \sigma|_{[s_0, s_1]}$ is such that $T(\gamma) \subseteq K_{a,b}, T(\gamma) \cap K_a \neq \emptyset$ and $T(\gamma) \cap K_b \neq \emptyset$. We deduce the validity of (ii) and hence the claim.

Compressive form. Fix $\varepsilon > 0$ such that $a - \varepsilon > 0$. Define $a' = a - \varepsilon$ and $b' = b + \varepsilon$. Set $M' = K_{a',b'}$ and $\phi := r \circ T|_{M'}$, where $r : K \rightarrow K$ is defined as follows

$$r(z) = \begin{cases} r_{a'}(z), & \text{if } z \in K_{0,a'}; \\ z, & \text{if } z \in K_{a',b'}; \\ r_{b'}(z), & \text{if } z \in K \setminus K_{0,b'}; \end{cases}$$

and $r_{a'}, r_{b'}$ are the retractions defined in Example 3. Using Corollary 3 we obtain the thesis. □

From what has been discussed in Example 3, one can easily generalize Theorem 7 by the following result. For other versions of Krasnosel’skiĭ fixed point theorem, we refer to [7, 14, 23].

Theorem 8 *Let $(X, \| \cdot \|)$ be a normed linear space and K a cone in X . Let $l : X \rightarrow \mathbb{R}$ be a continuous functional, strictly positive and positively homogeneous. Fix $a, b \in \mathbb{R}^+$ with $a < b$. Define $K_{a,b}^l := \{x \in K : a \leq l(x) \leq b\}$ and $K_d^l := \{x \in K : l(x) = d\}$, for all $d \in \mathbb{R}^+$.*

Consider a compact map $T : K \rightarrow K$.

– (Expansive form) *T has at least a fixed point in $K_{a,b}^l$ if*

$$(E_a^l) \quad l(T(x)) \leq a, \forall x \in K_a^l;$$

$$(E'_b) \quad l(T(x)) \geq b, \forall x \in K^l_b.$$

– (Compressive form) T has at least a fixed point in $K^l_{a,b}$ if

$$(C'_a) \quad l(T(x)) \geq a, \forall x \in K^l_a;$$

$$(C'_b) \quad l(T(x)) \leq b, \forall x \in K^l_b.$$

Proof The proof is the same of Theorem 8, replacing $\| \cdot \|$ with $l(\cdot)$.

Remark 5 We note that usually Krasnosel’skiĭ fixed point theorems are set on Banach spaces, while our versions are also valid on normed spaces. Furthermore it is typically required that the neighbourhoods of the origin are bounded and that the operator is completely continuous (which is equivalent to the compactness, if the neighbourhoods are bounded). We remark that hypotheses of Theorem 7 and Theorem 8 do not require the boundedness of the set $\{x \in K : l(x) \leq b\}$, but we demand the compactness of the operator T .

We end this section by considering other forms for the domain of the operator T . We start with the following theorem, still presented in [12, 13] in the framework of a Banach space. It was originally stated and proved in [2] (using Lefschetz theory). See also [6, Corollary 6.3, p. 452] and [15, § 4].

Theorem 9 *Let $(X, \| \cdot \|)$ be a normed linear space and $(B[x_i, r_i])_i$ a collection of $2n$ closed balls pairwise disjoint contained in $B[0, R]$. Define $L := B[0, R] \setminus \bigcup_{i=1}^{2n} B(x_i, r_i)$ and consider a compact map $T : L \rightarrow B[0, R]$ such that $T(\partial B(x_i, r_i)) \subseteq B[x_i, r_i]$ for all $i \in \{1, \dots, n\}$. Then T has at least a fixed point in L .*

Using the fixed point index, the proof is quite straightforward: assuming that there are no fixed points on ∂L , it is proved that $i_X(T, \text{int}(L)) = 1 - 2n$, using the additivity property of the fixed point index.

From the computation of $i_X(T, \text{int}(L)) = 1 - 2n$, we immediately see that the assumptions of the theorem are very restrictive; in fact, to have a fixed point in L , it is sufficient to assume that the number of holes is different from 1. Moreover, if we suppose that this number is equal to 1, the constant map $T \equiv x_1$ defines a possible counterexample: clearly T satisfies every hypothesis, but it has no fixed points in L .

The following result shows a possible solution for the case of a ball with a single hole in an infinite dimensional normed space. Assuming in addition to the hypotheses of Theorem 9 that $\partial B(0, R)$ is invariant (i.e. $T(\partial B(0, R)) \subseteq \partial B(0, R)$), the assertion with a single hole is valid, as shown in the expansive form of this theorem.

Theorem 10 *Let $(X, \| \cdot \|)$ be an infinite dimensional normed linear space. Fix $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 < r_2$ and define the annulus*

$$A := \{x \in X : r_1 \leq \|x\| \leq r_2\}$$

and $A_d := \partial B(0, d) = \{x \in X : \|x\| = d\}$, for all $d \in \mathbb{R}^+$. Consider a compact map $T : A \rightarrow X$.

– (Expansive form) T has at least a fixed point in A if

$$(E_a) \quad \|T(x)\| \leq \|x\|, \forall x \in A_{r_1};$$

$$(E_b) \quad \|T(x)\| \geq \|x\|, \forall x \in A_{r_2}.$$

– (Compressive form) T has at least a fixed point in A if

- (C_a) $\|T(x)\| \geq \|x\|, \forall x \in A_{r_1};$
- (C_b) $\|T(x)\| \leq \|x\|, \forall x \in A_{r_2}.$

Proof The proof is the same of Theorem 7, observing that A_1 is an AR in an infinite dimensional normed linear space (cf. [3, Theorem 6.2] or [6, Theorem 7.7, pp. 164–165]). \square

The statement is false in finite dimension. For example, it is sufficient to consider a nontrivial rotation ϑ of centre 0 in \mathbb{R}^2 . Clearly ϑ is continuous (hence compact) and it has no fixed points.

Remark 6 We underline that Theorem 9 remains valid if we consider sets homeomorphic to the ball at the place of $B(0, R)$ and $B(x_i, r_i)$ and, moreover, we could also set Theorem 9 in the context of ARs, as in [2].

Remark 7 Concerning Theorem 9 set in infinite dimensional spaces, if we replace the assumption $T(\partial B(x_i, r_i)) \subseteq B[x_i, r_i]$, for all $i \in \{1, \dots, n\}$, with $T(\partial B(x_i, r_i)) \subseteq L$, for all $i \in \{1, \dots, n\}$, we obtain the same thesis by applying Generalized Schauder Theorem (notice that L is an AR, see [3, Theorem 6.2] or [6, Theorem 7.7, pp. 164–165]).

Appendix

In this final section, we present the fixed point index defined on sets contained in ANRs. In particular, we list the axioms and the main properties which are relevant for this paper. For more details and proofs, we refer to [1, 5, 6, 16, 19, 20] and the references therein.

Let X be an ANR and $U \subseteq X$ an open subset. Consider a continuous map $f : U \rightarrow X$ such that $\text{Fix}(f) := \{x \in U : f(x) = x\}$ is a compact set (possibly empty) and such that there exists an open neighbourhood V of $\text{Fix}(f)$ with $\bar{V} \subseteq U$ such that $f|_{\bar{V}}$ is compact. If all the previous assumptions are satisfied, the triplet (X, U, f) is said to be *admissible*.

To an admissible triplet (X, U, f) we associate an integer

$$i_X(f, U),$$

called the *fixed point index of f on U relatively to X* , satisfying the following properties.

(a) *Additivity.* If $U_1, U_2 \subseteq U$ are open and disjoint subsets and $\text{Fix}(f) \subseteq U_1 \cup U_2$, then

$$i_X(f, U) = i_X(f, U_1) + i_X(f, U_2).$$

(b) *Homotopy invariance.* Let $h : [0, 1] \times U \rightarrow X, h_\lambda(x) := h(\lambda, x)$, be a continuous homotopy such that $\Sigma := \bigcup_{\lambda \in [0, 1]} \{x \in U : x - h_\lambda(x) = 0\}$ is a compact set and there exists an open neighbourhood V of Σ such that $\bar{V} \subseteq U$ and $h|_{[0, 1] \times \bar{V}}$ is a compact map. Then $i_X(h_\lambda, U)$ is constant with respect to $\lambda \in [0, 1]$.

(c) *Weak normalization.* Let $f(x) = p, \forall x \in U$, then

$$i_X(f, U) := \begin{cases} 1, & \text{if } p \in U; \\ 0, & \text{if } p \notin U. \end{cases}$$

(d) *Strong normalization.* If $U = X$ and the map f is compact, then the Lefschetz number of f is defined and

$$i_X(f, U) = \Lambda(f).$$

(e) *Fixed point property.* If $i_X(f, U) \neq 0$, then $\text{Fix}(f) \neq \emptyset$, i.e. f has a fixed point.

(f) *Excision.* Let U_0 be an open subset of U such that $\text{Fix}(f) \subseteq U_0$. Then

$$i_X(f, U) = i_X(f, U_0).$$

(g) *Multiplicativity.* If the triplets $(X_1, U_1, f_1), (X_2, U_2, f_2)$ are admissible, then

$$i_{X_1 \times X_2}(f_1 \times f_2, U_1 \times U_2) = i_{X_1}(f_1, U_1) \cdot i_{X_2}(f_2, U_2),$$

where $f_1 \times f_2: U_1 \times U_2 \rightarrow X_1 \times X_2, (f_1 \times f_2)(u, v) := (f_1(u), f_2(v))$.

(h) *Commutativity.* Let U_1, U_2 be open subsets of the ANRs X_1, X_2 , respectively. Suppose that $f_1: U_1 \rightarrow X_2, f_2: U_2 \rightarrow X_1$ are continuous maps and f_1 is compact in a neighbourhood of $\{x \in U_1: f_2 f_1(x) = x\}$ (or f_2 is compact in a neighbourhood of $\{x \in U_2: f_1 f_2(x) = x\}$), where $f_2 f_1: f_1^{-1}(U_2) \rightarrow X_1$ and $f_1 f_2: f_2^{-1}(U_1) \rightarrow X_2$. If the triplets $(f_1^{-1}(U_2), U_1, f_2 f_1), (f_2^{-1}(U_1), U_2, f_1 f_2)$ are admissible, then

$$i_{X_1}(f_2 f_1, f_1^{-1}(U_2)) = i_{X_2}(f_1 f_2, f_2^{-1}(U_1)).$$

(i) *Contraction.* Let $Y \subseteq X$ be an ANR such that the inclusion $j: Y \rightarrow X$ is continuous and $f(U) \subseteq Y$. Then

$$i_X(f, U) = i_Y(f, U \cap Y).$$

(j) *Localization.* Let $U_1, U_2 \subseteq U$ be open and disjoint subsets and $\text{Fix}(f) \subseteq U_1 \cup U_2$. If $i_X(f, U) \neq 0$ and $i_X(f, U_1) = 0$, then $\text{Fix}(f|_{U_2}) \neq \emptyset$.

(k) *Multiplicity.* Let $U_1, U_2 \subseteq U$ be open and disjoint subsets and $\text{Fix}(f) \subseteq U_1 \cup U_2$. If $i_X(f, U) = 0$ and $i_X(f, U_1) \neq 0$, then $\text{Fix}(f|_{U_1}) \neq \emptyset$ and $\text{Fix}(f|_{U_2}) \neq \emptyset$.

Remark 8 It is obvious to observe that the axioms listed above are not independent. For example, properties (j) and (k) are direct consequence of axioms (a) and (e), or further (h) implies (i).

Remark 9 For the applications shown in this paper, it is sufficient to give the axioms of the fixed point index in a less abstract framework. If we just assume that U is an open subset of X and $f: \bar{U} \rightarrow X$ is a compact map which has no fixed points on ∂U , then the triplet (X, U, f) is admissible.

In that framework, the homotopy property can be written in this easier way. Let X be an ANR and $U \subseteq X$ an open subset. Let $h: [a, b] \times \bar{U} \rightarrow X$ be a compact map such that $\text{Fix}(h_\lambda) \cap \partial U = \emptyset$, for all $\lambda \in [a, b]$. Then $i_X(h_\lambda, U)$ is constant with respect to $\lambda \in [a, b]$. The remaining axioms remain almost unchanged.

We conclude the appendix with a theorem which is crucial in the proof of the main result. It is stated in a less general framework than the one in which we have defined the fixed point index.

Theorem 11 (Leray–Schauder continuation principle) *Let X be an ANR, U an open subset of $[a, b] \times X$, and let $\psi: \bar{U} \rightarrow X$ be a compact map such that*

- (i) *the sets ∂U and $\Sigma := \{(t, x) \in \bar{U}: \psi(t, x) = x\}$ are disjoint;*
- (ii) *$i_X(\psi(a, \cdot), U_a) \neq 0$, where $U_a = \{x \in X: (a, x) \in U\}$.*

Then there exists a continuum $S \subseteq \Sigma$ joining the sets

$$A = (X \times \{a\}) \cap \Sigma \quad \text{and} \quad B = (X \times \{b\}) \cap \Sigma.$$

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References

1. Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **18**, 620–709 (1976)
2. Bourgin, D.G.: Fixed points on neighborhood retracts. *Rev. Math. Pures Appl.* **2**, 371–374 (1957)
3. Dugundji, J.: An extension of Tietze's theorem. *Pac. J. Math.* **1**, 353–367 (1951)
4. Fonda, A., Gidoni, P.: Generalizing the Poincaré–Miranda theorem: the avoiding cones condition. *Ann. Matematica Pura Appl.* **195**, 1347–1371 (2016)
5. Granas, A.: The Leray–Schauder index and the fixed point theory for arbitrary ANRs. *Bull. Soc. Math. Fr.* **100**, 209–228 (1972)
6. Granas, A., Dugundji, J.: *Fixed Point Theory*. Springer Monographs in Mathematics. Springer, New York (2003)
7. Guo, D.J., Lakshmikantham, V.: *Nonlinear Problems in Abstract Cones*. Academic Press, Boston (1988)
8. Hu, S.-T.: *Theory of Retracts*. Wayne State University Press, Detroit (1965)
9. Kampen, J.: On fixed points of maps and iterated maps and applications. *Nonlinear Anal.* **42**, 509–532 (2000)
10. Krasnosel'skiĭ, M.A.: Fixed points of cone-compressing or cone-extending operators. *Sov. Math. Dokl.* **1**, 1285–1288 (1960)
11. Krasnosel'skiĭ, M.A.: *Positive Solutions of Operator Equations*. P. Noordhoff Ltd., Groningen (1964)
12. Kwong, M.K.: On Krasnoselskii's cone fixed point theorem. *Fixed Point Theory Appl.* Art. ID 164,537, 18 pp. (2008)
13. Kwong, M.K.: The topological nature of Krasnoselskii's cone fixed point theorem. *Nonlinear Anal.* **69**, 891–897 (2008)
14. Leggett, R.W., Williams, L.R.: Multiple positive fixed points of nonlinear operators on ordered Banach spaces. *Indiana Univ. Math. J.* **28**, 673–688 (1979)
15. Marsh, M.M.: Fixed point theorems for partially outward mappings. *Topol. Appl.* **153**, 3546–3554 (2006)
16. Mawhin, J.: Leray–Schauder degree: a half century of extensions and applications. *Topol. Methods Nonlinear Anal.* **14**, 195–228 (1999)
17. Mawhin, J.: Variations on Poincaré–Miranda's theorem. *Adv. Nonlinear Stud.* **13**, 209–217 (2013)
18. Mawhin, J.: Variations on some finite-dimensional fixed-point theorems. *Ukr. Math. J.* **65**, 294–301 (2013)
19. Nussbaum, R.D.: *The Fixed Point Index and Some Applications*, Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], vol. 94. Presses de l'Université de Montréal, Montreal (1985)
20. Nussbaum, R.D.: The fixed point index and fixed point theorems. In: *Topological Methods for Ordinary Differential Equations* (Montecatini Terme, 1991), Lecture Notes in Mathematics, vol. 1537, pp. 143–205. Springer, Berlin (1993)
21. Papini, D., Zanolin, F.: Some results on periodic points and chaotic dynamics arising from the study of the nonlinear Hill equations. *Rend. Semin. Mat. Univ. Politec. Torino* **65**, 115–157 (2007)
22. Pireddu, M., Zanolin, F.: Cutting surfaces and applications to periodic points and chaotic-like dynamics. *Topol. Methods Nonlinear Anal.* **30**, 279–319 (2007)
23. Schmitt, K.: Fixed point and coincidence theorems with applications to nonlinear differential and integral equations. Séminaires de Mathématique Appliquée et Mécanique, Université catholique de Louvain, Vander, Louvain-la-Neuve, Rapport No. 97 (1976)