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Boggio's formula for fractional polyharmonic Dirichlet problems

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Abstract Boggio's formula in balls is known for integer-polyharmonic Dirichlet problems and for fractional Dirichlet problems with fractional parameter less than 1. We give here a consistent formulation for fractional polyharmonic Dirichlet problems such that Boggio's formula in balls yields solutions also for the general fractional case.

Mathematics Subject Classification 35J40

1 Introduction

Goal of this paper is to extend Boggio's classical formula [4, p. 126] for solutions to Dirichlet problems in the unit ball $B := B_1(0)$ of $\mathbb{R}^n$ to "polyharmonic" operators of any fractional order $s > 0$. In this context, we give a consistent definition for $(-\Delta)^s$ when applied to functions which are merely in $H^s(\mathbb{R}^n) \cap H^{2s}(B)$. For operators of order $s \in \mathbb{N}$, the formula of Boggio is a classical tool to construct solutions of the s-polyharmonic equation with right-hand side equal to f and subject to homogeneous boundary data.

More precisely, Boggio's formula (see also [14, p. 51]) states that the polyharmonic Green function with $s \in \mathbb{N}$ in the unit ball $B = B_1(0) \subset \mathbb{R}^n$ is given by

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$$G_s(x,y) := k_{s,n}|x-y|^{2s-n} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} \left(v^2-1\right)^{s-1} v^{1-n}\, dv, \tag{1}$$

$$\text{with} \quad k_{s,n} := \frac{1}{n e_n 4^{s-1}\Gamma(s)^2}, \quad \text{and} \quad e_n := \frac{\pi^{n/2}}{\Gamma(1+n/2)} \tag{2}$$

for $x, y \in \overline{B}$, $x \neq y$. This formula showed up also in potential theory (see [20,24]) for $s \in (0,1)$. For a probabilistic point of view, see [3,19]. Recently, the Dirichlet problem for the fractional Laplacian attracted quite some attention (see, e.g., [5,6,9,28] and the references therein), and it was shown in Theorems 3.1 and 3.3 in [5] that Boggio's formula remains valid also for $s \in (0,1)$. It is a quite remarkable fact that such G_s has a rather explicit expression, in terms of the fundamental solution and a weighted integral containing a Kelvin transformation.

To sum up, this means that, defining

$$u(x) := \int_B G_s(x,y) f(y)\, dy,$$

one obtains the unique solution u of the s-polyharmonic Dirichlet problem in the unit ball B, provided that

- $s \in \mathbb{N}$, see [4];
- $s \in (0,1)$, see [5] and the references therein.

We shall show that Boggio's formula holds true for any $s \in (0,\infty)$ and $n \in \mathbb{N}$. To this end, we only need to consider $s > 1$. We write

$$s = m + \sigma, \quad \text{with} \quad m \in \mathbb{N} \quad \text{and} \quad \sigma \in (0,1), \tag{3}$$

and define for $u \in H^s(\mathbb{R}^n) \cap H^{2s}(B)$

$$(-\Delta)^s u := (-\Delta)^m (-\Delta)^\sigma u. \tag{4}$$

Here, $(-\Delta)^\sigma u$ is the so-called fractional Laplacian, for which we use the usual nonlocal definition given, for instance, in [5,6,9,10]:

$$(-\Delta)^\sigma u(x) := C(n,\sigma) \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}},$$
$$\text{where} \quad C(n,\sigma) := \frac{4^\sigma \Gamma\left(\frac{n}{2}+\sigma\right)}{-\Gamma(-\sigma)\,\pi^{n/2}}. \tag{5}$$

The operator $(-\Delta)^m$ has to be understood in the classical pointwise sense. In this regularity class, the order of applying $(-\Delta)^\sigma$ and $(-\Delta)^m$ in (4) is quite essential.

For functions u, which are even in $H^{2s}(\mathbb{R}^n)$, the operator $(-\Delta)^s$ may be equivalently defined by using the Fourier transform, that is, for any $\xi \in \mathbb{R}^n$,

$$\mathcal{F}\left((-\Delta)^s u\right)(\xi) = |\xi|^{2s} \hat{u}(\xi), \tag{6}$$

where

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, \mathrm{d}x$$

denotes the Fourier transform of u (see [9, Prop. 3.3] and also [26]).

In order to state our main result, we denote by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ a multi-index and by $|\alpha| = \alpha_1 + \cdots + \alpha_n$ the length of α.

We will show that the Green function introduced in (1) can be used to get a solution u to the s-polyharmonic Dirichlet problem in the following sense:

$$(-\Delta)^s u = f \text{ in } B, \qquad u \in H^s(\mathbb{R}^n).$$

More precisely, the main result that we prove in this paper is as follows:

Theorem 1 *Let $s \in (0, \infty)$ and $f \in C_0^\infty(B)$. Set*

$$u(x) := \begin{cases} \int_B G_s(x, y) f(y)\, dy & \text{for } x \in B, \\ 0 & \text{for } x \notin B. \end{cases} \tag{7}$$

Then, $u \in H^s(\mathbb{R}^n)$ is the unique solution to

$$\begin{cases} (-\Delta)^s u = f & \text{in } B, \\ \quad u = 0 & \text{in } \mathbb{R}^n \setminus B. \end{cases} \tag{8}$$

In addition, $u \in C^{m,\sigma}(\overline{B}) \cap C^\infty(B)$ and

$$u(x) = (1 - |x|^2)_+^s \tilde{u}(x) \tag{9}$$

with some $\tilde{u} \in C^\infty(\mathbb{R}^n)$.

Once Boggio's solution is proved to belong to a unique class, then it coincides with any solution obtained by variational or functional analytic methods and the abstract theory developed by [26,28] applies.

We emphasize that in order to interpret (8) in a strong sense, the order of differentiation as defined in (4) is essential, i.e., $(-\Delta)^m((-\Delta)^\sigma u) = f$ in B. Indeed, Theorem 1 shows the consistency of this definition: The fractional polyharmonic operator should be defined such that the function given by Boggio's formula yields the unique solution of the corresponding Dirichlet problem.

As a by-product of Theorem 1 and the strict positivity of Boggio's Green function, we have the following positivity preserving property:

Let f and u be as in Theorem 1. Then, $f \geq 0$, $f \not\equiv 0$ implies that $u > 0$ in B.

It is known that such a property fails in general domains for integer $s \geq 2$, see the discussion and the references in [14]. While the present paper was submitted and under review, Abatangelo et al. [1] sent us their preprint where precisely this question is studied for $s > 1$. Independently and (almost) simultaneously, they achieved among others the same result as in our Theorem 1, but their approach is quite different.

This type of fractional higher-order operators plays an important role in analysis, see [8, 12,16,18,22,27,29,30], in geometry, see [7,15,17], and in some applications, see [21,31]. See in particular [26], where Pohozhaev identities for higher-order fractional Laplacians have been obtained.

The Boggio-type formula obtained in Theorem 1 is in perfect agreement with the cases $s \in (0, 1) \cup \mathbb{N}$, which were already known, but the extension to the whole interval $s \in (0, +\infty)$ is based on appropriate series expansions and analytic continuation.

The paper is organized as follows. In Sect. 2, we show the covariance under Möbius transformations of the operator introduced in (4). In Sect. 3, we provide an expression for the fractional Laplacian of order $\sigma \in (0, 1)$ of the Green function as given by formula (1) with pole at the origin. This will imply that the function in (1) is the Green function for the operator $(-\Delta)^s$ with pole at the origin. Here, we strongly rely on fractional differentiation results due to Dyda [10], which were developed further by him, Kuznetsov, and Kwaśnicki in the recent work [11].

Then, by the covariance under Möbius transformations, we will obtain the Green function for the fractional Laplacian of any order $s \in (0, \infty)$ at any pole.

In Sect. 4, we complete the proof of Theorem 1.

2 Covariance under Möbius transformations

In this section, we show that the operator introduced in (4) is covariant under Möbius transformations. This property holds true for polyharmonic operators, see, e.g., Lemma 6.14 in [14], and for the fractional Laplacian, see, e.g., Lemma 2.2 and Corollary 2.3 in [13] and Proposition A.1 in [25].

The extension of these covariance properties to operators of any fractional powers was not available in the literature (to the best of our knowledge), and we obtain it by using complex analysis methods and unique continuation of analytic functions, given that the desired formulas hold in a nontrivial interval in the reals. To this aim, we use the notation $\mathcal{S}$ to denote the Schwartz space of smooth and rapidly decreasing functions, and we have the following result:

Lemma 2 *Fix $x \in \mathbb{R}^n$ and $v \in \mathcal{S}$. Then, the map*

$$\mathcal{T}_{v,x}(s) := (-\Delta)^s v(x)$$

is analytic for $s \in (0, +\infty)$.

Proof We use complex analysis, so we will take $s \in \mathbb{C}$, with $\Re s > 0$. We define

$$w(s) := \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, |\xi|^{2s}\, \hat{v}(\xi)\, d\xi. \tag{10}$$

We observe that, for any $z \in \mathbb{C}$,

$$\begin{aligned} |e^z - 1 - z| &= \left| \sum_{k=2}^{+\infty} \frac{z^k}{k!} \right| \le \sum_{k=2}^{+\infty} \frac{|z|^k}{k!} = |z|^2 \sum_{k=2}^{+\infty} \frac{|z|^{k-2}}{k!} \\ &\le |z|^2 \sum_{k=2}^{+\infty} \frac{|z|^{k-2}}{(k-2)!} = |z|^2 e^{|z|}. \end{aligned}$$

So, for any $h \in \mathbb{C}$, we use this formula with $z := 2h \log|\xi|$, and we obtain

$$\begin{aligned} \left| |\xi|^{2(s+h)} - |\xi|^{2s} - 2h|\xi|^{2s} \log|\xi| \right| &= |\xi|^{2\Re s} \left| |\xi|^{2h} - 1 - 2h \log|\xi| \right| \\ &= |\xi|^{2\Re s} \left| e^{2h \log|\xi|} - 1 - 2h \log|\xi| \right| \\ &\le 4|h|^2 |\xi|^{2\Re s} \log^2|\xi|\, e^{2|h|\,|\log|\xi||}. \end{aligned}$$

Now, we observe that

$$e^{|\log|\xi||} = \begin{cases} |\xi| & \text{if} \quad |\xi| \ge 1, \\ |\xi|^{-1} & \text{if} \quad |\xi| < 1, \end{cases}$$

and so, if $|h| < \Re s/2$,

$$e^{2|h|\,|\log|\xi||} \le |\xi|^{\Re s} + |\xi|^{-\Re s}.$$

Hence, we have found that

$$\left| |\xi|^{2(s+h)} - |\xi|^{2s} - 2h|\xi|^{2s} \log |\xi| \right| \leq 4|h|^2 \left(|\xi|^{3\Re s} + |\xi|^{\Re s} \right) \log^2 |\xi|.$$

In consequence of this, for any $h \in \mathbb{C}$ with $0 < |h| < \Re s/2$,

$$\begin{aligned} &\left| \frac{w(s+h) - w(s)}{h} - 2 \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, |\xi|^{2s} \, \log |\xi| \, \hat{v}(\xi) \, d\xi \right| \\ &\quad = \left| \frac{1}{h} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, \hat{v}(\xi) \left(|\xi|^{2(s+h)} - |\xi|^{2s} - 2h|\xi|^{2s} \, \log |\xi| \right) d\xi \right| \\ &\quad \leq 4|h| \int_{\mathbb{R}^n} |\hat{v}(\xi)| \left(|\xi|^{3\Re s} + |\xi|^{\Re s} \right) \log^2 |\xi| \, d\xi \leq C \, |h|, \end{aligned}$$

for some $C > 0$ (possibly depending on n, s, and v). By sending $h \to 0$, we then obtain that w is differentiable in the complex sense, and so analytic in $\{\Re s > 0\}$, which gives the desired result by comparing (6) and (10). □

With the aid of the latter result, we can now prove the desired covariance under Möbius transformations:

Lemma 3 *Let ϕ be a smooth Möbius transformation in $\mathbb{R}^n \setminus \{x_0\}$, and let J_ϕ be the modulus of the Jacobian determinant.*

Then, for any $s \in (0, \infty)$ and any $u \in C_0^\infty(\mathbb{R}^n \setminus \{x_0\})$,

$$(-\Delta)^s \left(J_\phi^{\frac{1}{2} - \frac{s}{n}} u \circ \phi \right) = J_\phi^{\frac{1}{2} + \frac{s}{n}} \left((-\Delta)^s u \right) \circ \phi. \tag{11}$$

Proof We recall that any Möbius transform ϕ can be seen as a finite combination of similarities and inversions. In particular, from Corollary 4 on page 39 of [23], we have that we can write $\phi = \phi_1 \circ j \circ \phi_2$, where j is an inversion, i.e., $j(x) = |x|^{-2}x$, and ϕ_1, ϕ_2 are similarities, i.e., $\phi_i(x) = a_i + c_i F_i x$, for $i = 1, 2$, with $c_i > 0$, $a_i \in \mathbb{R}^n$ and F_i an orthogonal matrix. Therefore,

it suffices to show (11) for translations, rotations and inversions. (12)

Notice that formula (11) is easily verified if ϕ is a translation or a rotation, since in these cases the Jacobian is equal to 1, and therefore, using the invariance under translations and rotations of polyharmonic operators and fractional Laplacians,

$$\begin{aligned} (-\Delta)^s \, (u \circ \phi) &= (-\Delta)^m \left[(-\Delta)^\sigma \, (u \circ \phi) \right] = (-\Delta)^m \left[\left((-\Delta)^\sigma u \right) \circ \phi \right] \\ &= \left[(-\Delta)^m (-\Delta)^\sigma u \right] \circ \phi \\ &= \left((-\Delta)^s u \right) \circ \phi, \end{aligned}$$

as desired.

Now, we focus on the case in which ϕ is the inversion with respect to the unit sphere. Namely, we consider the transformation that associates with any $x \in \mathbb{R}^n \setminus \{0\}$ the point $x^* := x/|x|^2 \in \mathbb{R}^n \setminus \{0\}$, and we observe that $J_\phi(x) = |x|^{-2n}$. Hence, we set

$$u^*(x) := J_\phi^{\frac{1}{2} - \frac{s}{n}}(x) \, u(x^*) = \frac{1}{|x|^{n-2s}} u(x^*),$$

and we have that, according to our assumption, $u^* \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. With this notation, we claim that

$$(-\Delta)^s u^*(x) = \frac{1}{|x|^{n+2s}} (-\Delta)^s u(x^*). \tag{13}$$

To prove this, use the notation introduced in Lemma 2 to say that (13) is equivalent to

$$F(s) := \mathcal{T}_{u^*,x}(s) - \frac{1}{|x|^{n+2s}}\, \mathcal{T}_{u,x^*}(s) = 0.$$

We remark first that, thanks to our smoothness and compact support assumptions,

$$\hat{v}(s,\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} |x|^{2s-n} u\left(\frac{x}{|x|^2}\right) dx$$

depends holomorphically on s in $\Re s > 0$. The smoothness and fast decay of $\xi \mapsto \hat{v}(s,\xi)$ is locally uniform with respect to s. Hence, F is analytic when $s \in (0,+\infty)$, thanks to Lemma 2. Also, $F(s) = 0$ for any $s \in (0,1]$, since (13) is known for this range of parameters s (see Lemma 2.2 and Corollary 2.3 in [13] and Proposition A.1 in [25]). Then, by analytic unique continuation, we conclude that F vanishes identically in $(0,+\infty)$, which proves (13). Then, the proof of Lemma 3 is also completed, thanks to (12). □

3 Green function with pole at the origin

In this section, we focus on the case in which the pole for the Green function is the origin. For this, we define, for $r \in (0,\infty)$,

$$\tilde{G}_s(r) := \begin{cases} r^{2s-n} \int_1^{1/r} (v^2-1)^{s-1} v^{1-n}\, dv & \text{if } \ r \in (0,1], \\ 0 & \text{if } \ r \in (1,\infty) \end{cases} \tag{14}$$

so that

$$G_s(0,y) = k_{s,n} \tilde{G}_s(|y|), \quad y \in \overline{B} \setminus \{0\}.$$

We will prove that $\tilde{G}_s$ (suitably renormalized) is the Green function for $(-\Delta)^s$ with pole at the origin. More precisely, we prove that:

Proposition 4 *For $r = |y| \in (0,1)$, we have that*

$$(-\Delta)^s \tilde{G}_s(|y|) = \big(k_{s,n}\big)^{-1} \delta_0(y), \tag{15}$$

where $k_{s,n}$ is given by formula (2).

Then, the general case (i.e., when the pole is not the origin) will follow from the covariance under Möbius transformation of the operator $(-\Delta)^s$, as stated in Sect. 2. Proposition 4 gives a precise statement of what is outlined in a recent work of Dyda et al., see [11, Remark 1]. One should, however, observe that the Green's functions mentioned there were obtained before only for $s \in (0,1)$.

In order to prove Proposition 4, we shall deduce a series expansion for $\tilde{G}_s$, where we can rely on calculations by Dyda [10] in order to compute $(-\Delta)^\sigma_y G_s(0,y)$. For this purpose, we recall the definition of the *Pochhammer* symbols:

$$(a)_k := \prod_{j=0}^{k-1} (a+j), \qquad a \in \mathbb{R}, \quad k \in \mathbb{N}_0. \tag{16}$$

In what follows, we perform calculations with hypergeometric series which are related to those by Bucur [5, Section 3] and by Dyda et al. [11]. We start by writing $\tilde{G}_s$ in a useful way for our computations.

Lemma 5 *For $r \in (0, \infty)$, we have that*

$$\tilde{G}_s(r) = \sum_{k=0}^{\infty} \frac{\left(\frac{n}{2}\right)_k}{2(s)_{k+1}} (1 - r^2)_+^{k+s}. \tag{17}$$

This claim could also be deduced from [11, Remark 1].

Proof of Lemma 5 Differentiating $\tilde{G}_s$ yields for $r \in (0, 1]$:

$$\begin{aligned}\tilde{G}_s'(r) &= (2s-n)r^{2s-n-1} \int_1^{1/r} (v^2-1)^{s-1} v^{1-n}\, dv - r^{2s-n-2} \left(\frac{1}{r^2} - 1\right)^{s-1} r^{n-1} \\ &= \frac{(2s-n)}{r} \tilde{G}_s(r) - \frac{1}{r}(1-r^2)^{s-1}.\end{aligned}$$

Hence, $\tilde{G}_s$ solves the following initial value problem:

$$\begin{cases} \tilde{G}_s'(r) - \frac{(2s-n)}{r} \tilde{G}_s(r) = -\frac{1}{r}(1-r^2)^{s-1}, & r \in (0, 1], \\ \tilde{G}_s(1) = 0. \end{cases} \tag{18}$$

A direct calculation shows that the unique solution of (18) is given by the right-hand side of (17), and this finishes the proof of Lemma 5. □

Now, we employ Lemma 5 and Theorem 1 in [10] to find an expression of $(-\Delta)_y^\sigma \tilde{G}_s(0, y)$ (extended by 0 outside B) in terms of an auxiliary function $G_s^\sharp$ (given in (19)).

Lemma 6 *For $r \in (0, 1)$, we have that*

$$(-\Delta)^\sigma \tilde{G}_s(r) = \frac{4^{\sigma-\frac{1}{2}} \Gamma(\frac{n}{2} + \sigma)\Gamma(s)}{\Gamma(\frac{n}{2}) \cdot m!} G_s^\sharp(r),$$

with

$$G_s^\sharp(r) = \sum_{k=0}^{\infty} \frac{\left(\frac{n}{2}\right)_k}{(m+1)_k} {}_2F_1\left(\frac{n}{2} + \sigma, -k-m; \frac{n}{2}; r^2\right). \tag{19}$$

Here, ${}_2F_1$ denotes Gauß's hypergeometric function.

Proof We recall Euler's beta-function $B(p, q) := \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. When applying Dyda's fractional differentiation result [10, Theorem 1], one should observe that he uses $\Delta^\sigma = -(-\Delta)^\sigma$. This yields immediately:

$$\begin{aligned}(-\Delta)^\sigma \tilde{G}_s(r) &= \frac{4^{\sigma-\frac12}\Gamma(\frac n2+\sigma)}{\Gamma(\frac n2)\,\Gamma(-\sigma)} \\ &\quad \cdot \sum_{k=0}^{\infty} \frac{\left(\frac n2\right)_k B(-\sigma, k+m+\sigma+1)}{(s)_{k+1}}\, {}_2F_1\left(\frac n2+\sigma, -k-m; \frac n2; r^2\right) \\ &= \frac{4^{\sigma-\frac12}\Gamma(\frac n2+\sigma)}{\Gamma(\frac n2)} \sum_{k=0}^{\infty} \frac{\left(\frac n2\right)_k \Gamma(k+m+\sigma+1)}{(s)_{k+1}\Gamma(k+m+1)}\, {}_2F_1\left(\frac n2+\sigma, -k-m; \frac n2; r^2\right) \\ &= \frac{4^{\sigma-\frac12}\Gamma(\frac n2+\sigma)}{\Gamma(\frac n2)} \sum_{k=0}^{\infty} \frac{\left(\frac n2\right)_k \Gamma(k+s+1)}{(s)_{k+1}(m+1)_k\Gamma(m+1)}\, {}_2F_1\left(\frac n2+\sigma, -k-m; \frac n2; r^2\right) \\ &= \frac{4^{\sigma-\frac12}\Gamma(\frac n2+\sigma)}{\Gamma(\frac n2)\cdot m!} \sum_{k=0}^{\infty} \frac{\left(\frac n2\right)_k (s)_{k+1}\Gamma(s)}{(s)_{k+1}(m+1)_k}\, {}_2F_1\left(\frac n2+\sigma, -k-m; \frac n2; r^2\right) \\ &= \frac{4^{\sigma-\frac12}\Gamma(\frac n2+\sigma)\Gamma(s)}{\Gamma(\frac n2)\cdot m!} \sum_{k=0}^{\infty} \frac{\left(\frac n2\right)_k}{(m+1)_k}\, {}_2F_1\left(\frac n2+\sigma, -k-m; \frac n2; r^2\right),\end{aligned}$$

which gives the desired result. □

Finally, we provide an expression for the function $G_s^\sharp$, also in terms of the Green function for $(-\Delta)^m$. Indeed, we have the following result:

Lemma 7 *There exist real numbers $a_0, \ldots, a_{m-1}$ such that, for $r \in (0,1)$, we have that*

$$G_s^\sharp(r) = \sum_{k=0}^{m-1} a_k(1-r^2)^k + 2m\frac{\Gamma(m+\sigma)\Gamma(\frac n2)}{\Gamma(\frac n2+\sigma)\Gamma(m)}\tilde{G}_m(r). \tag{20}$$

Proof If we directly insert the definition of ${}_2F_1\left(\frac n2+\sigma, -k-m; \frac n2; r^2\right)$ into (19), we would come up with polynomials with terms of alternating sign. The whole series would then not be absolutely convergent, and one could not rearrange the order of summation. In order to bypass this difficulty, we use [2, Corollary 2.3.3]. Here, it is quite important that σ is not an integer. We first notice that

$${}_2F_1\left(\frac n2+\sigma, -k-m; \frac n2; r^2\right) = \frac{(-\sigma)_{k+m}}{(\frac n2)_{k+m}}\, {}_2F_1\left(\frac n2+\sigma, -k-m; \sigma+1-k-m; 1-r^2\right),$$

where

$$\begin{aligned}{}_2F_1\left(\frac n2+\sigma, -k-m; \sigma+1-k-m; 1-r^2\right) &= \sum_{\ell=0}^{k+m} \frac{\left(\frac n2+\sigma\right)_\ell (-k-m)_\ell}{\ell!\,(\sigma+1-k-m)_\ell}(1-r^2)^\ell \\ = \sum_{\ell=0}^{k+m} \frac{\left(\frac n2+\sigma\right)_\ell (m+k)\cdot\ldots\cdot(m+k-\ell+1)}{\ell!\,(m+k-\sigma-1)\cdot\ldots\cdot(m+k-\sigma-\ell)}(1-r^2)^\ell \\ = \sum_{\ell=0}^{k+m} \frac{\left(\frac n2+\sigma\right)_\ell (m+k-\ell+1)_\ell}{\ell!\,(m+k-\sigma-\ell)_\ell}(1-r^2)^\ell.\end{aligned}$$

Here, one has that all summands but for $\ell = k+m$ are positive. This means that in what follows, we have absolute convergence and may rearrange the order of summation. In view of Lemma 6, we obtain that, for suitable real numbers a_ℓ,

$$
\begin{aligned}
G_s^\sharp(r) &= \sum_{k=0}^{\infty} \frac{(\frac{n}{2})_k\,(-\sigma)_{k+m}}{(m+1)_k\,(\frac{n}{2})_{k+m}} \sum_{\ell=0}^{k+m} \frac{(\frac{n}{2}+\sigma)_\ell\,(m+k-\ell+1)_\ell}{\ell!\,(m+k-\sigma-\ell)_\ell}(1-r^2)^\ell \\
&= \sum_{k=0}^{\infty} \frac{(-\sigma)_{k+m}}{(m+1)_k\,(\frac{n}{2}+k)_m} \sum_{\ell=0}^{k+m} \frac{(\frac{n}{2}+\sigma)_\ell\,(m+k-\ell+1)_\ell}{\ell!\,(m+k-\sigma-\ell)_\ell}(1-r^2)^\ell \\
&= \sum_{k=m}^{\infty} \frac{(-\sigma)_{k}}{(m+1)_{k-m}\,(\frac{n}{2}+k-m)_m} \sum_{\ell=0}^{k} \frac{(\frac{n}{2}+\sigma)_\ell\,(k+1-\ell)_\ell}{\ell!\,(k-\sigma-\ell)_\ell}(1-r^2)^\ell \\
&= \sum_{\ell=0}^{m-1} a_\ell\,(1-r^2)^\ell \\
&\quad + \sum_{\ell=m}^{\infty} \frac{(\frac{n}{2}+\sigma)_\ell}{\ell!}(1-r^2)^\ell \sum_{k=\ell}^{\infty} \frac{(-\sigma)_k\,(k+1-\ell)_\ell}{(m+1)_{k-m}\,(\frac{n}{2}+k-m)_m\,(k-\sigma-\ell)_\ell}.
\end{aligned}
$$

Interchanging k and ℓ leads us to

$$
\begin{aligned}
G_s^\sharp(r) &= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k \\
&\quad + \sum_{k=m}^{\infty} \frac{(\frac{n}{2}+\sigma)_k}{k!}(1-r^2)^k \sum_{\ell=k}^{\infty} \frac{(-\sigma)_\ell\,(\ell+1-k)_k}{(m+1)_{\ell-m}\,(\frac{n}{2}+\ell-m)_m\,(\ell-\sigma-k)_k} \\
&= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k \\
&\quad + \sum_{k=m}^{\infty} \frac{(\frac{n}{2}+\sigma)_k}{k!}(1-r^2)^k \sum_{\ell=0}^{\infty} \frac{(-\sigma)_{\ell+k}\,(\ell+1)_k}{(m+1)_{\ell+k-m}\,(\frac{n}{2}+\ell+k-m)_m\,(\ell-\sigma)_k} \\
&= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k \\
&\quad + \sum_{k=m}^{\infty} \frac{(\frac{n}{2}+\sigma)_k}{k!\,(m+1)_{k-m}}(1-r^2)^k \sum_{\ell=0}^{\infty} \frac{(-\sigma)_\ell\,(\ell+1)_k\,\cdot\,\ell!}{(k+1)_\ell\,(\frac{n}{2}+\ell+k-m)_m\,\cdot\,\ell!} \\
&= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k + \sum_{k=m}^{\infty} \frac{(\frac{n}{2}+\sigma)_k}{(m+1)_{k-m}}(1-r^2)^k \sum_{\ell=0}^{\infty} \frac{(-\sigma)_\ell}{\ell!(\frac{n}{2}+\ell+k-m)_m}.
\end{aligned}
\tag{21}
$$

We show now that for $k \geq m$

$$
\frac{(\frac{n}{2}+k-m)_\ell}{(\frac{n}{2}+k)_\ell} = \frac{(\frac{n}{2}+k-m)_m}{(\frac{n}{2}+k+\ell-m)_m}. \tag{22}
$$

Indeed, the claim is obvious for $\ell = m$, while for $\ell > m$ terms cancel as follows:

$$
\begin{aligned}
\frac{(\frac{n}{2}+k-m)_\ell}{(\frac{n}{2}+k)_\ell} &= \frac{(\frac{n}{2}+k-m)\cdot\ldots\cdot(\frac{n}{2}+k)\cdot\ldots\cdot(\frac{n}{2}+k+\ell-m-1)}{(\frac{n}{2}+k)\cdot\ldots\cdot(\frac{n}{2}+k+\ell-1)} \\
&= \frac{(\frac{n}{2}+k-m)_m}{(\frac{n}{2}+k+\ell-m)_m}.
\end{aligned}
$$

On the other hand, if $\ell < m$, we see that

$$\frac{(\frac{n}{2}+k-m)_m}{(\frac{n}{2}+k+\ell-m)_m} = \frac{(\frac{n}{2}+k-m)\cdot\ldots\cdot(\frac{n}{2}+k-1)}{(\frac{n}{2}+k+\ell-m)\cdot\ldots\cdot(\frac{n}{2}+k)\cdot\ldots\cdot(\frac{n}{2}+k+\ell-1)}.$$
$$= \frac{(\frac{n}{2}+k-m)_\ell}{(\frac{n}{2}+k)_\ell}.$$

This completes the proof of (22).

Combining (21) and (22), we conclude further that

$$\begin{aligned}
G_s^\sharp(r) &= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k \\
&\quad + \sum_{k=m}^{\infty} \frac{(\frac{n}{2}+\sigma)_k}{(m+1)_{k-m}\,(\frac{n}{2}+k-m)_m}(1-r^2)^k \sum_{\ell=0}^{\infty} \frac{(-\sigma)_\ell\,(\frac{n}{2}+k-m)_\ell}{\ell!\,(\frac{n}{2}+k)_\ell} \\
&= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k \\
&\quad + \sum_{k=m}^{\infty} \frac{(\frac{n}{2}+\sigma)_k}{(m+1)_{k-m}\,(\frac{n}{2}+k-m)_m}(1-r^2)^k\,{}_2F_1(-\sigma, \frac{n}{2}+k-m; \frac{n}{2}+k; 1).
\end{aligned}$$

We apply now Gauß's summation theorem, see [2, Theorem 2.2.2], and obtain further:

$$\begin{aligned}
G_s^\sharp(r) &= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k + \sum_{k=m}^{\infty} \frac{(\frac{n}{2}+\sigma)_k\Gamma(\frac{n}{2}+k)\,\Gamma(m+\sigma)}{(m+1)_{k-m}\,(\frac{n}{2}+k-m)_m\Gamma(\frac{n}{2}+k+\sigma)\,\Gamma(m)}(1-r^2)^k \\
&= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k + \frac{\Gamma(m+\sigma)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+\sigma)\Gamma(m)} \sum_{k=m}^{\infty} \frac{(\frac{n}{2})_k}{(m+1)_{k-m}\,(\frac{n}{2}+k-m)_m}(1-r^2)^k \\
&= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k + \frac{\Gamma(m+\sigma)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+\sigma)\Gamma(m)} \sum_{k=m}^{\infty} \frac{(\frac{n}{2})_{k-m}}{(m+1)_{k-m}}(1-r^2)^k \\
&= \sum_{k=0}^{m-1} a_k\,(1-r^2)^k + 2m\frac{\Gamma(m+\sigma)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+\sigma)\Gamma(m)} \sum_{k=0}^{\infty} \frac{(\frac{n}{2})_k}{2(m)_{k+1}}(1-r^2)^{k+m}.
\end{aligned}$$

Thus, we recall Lemma 5 and end up with

$$G_s^\sharp(r) = \sum_{k=0}^{m-1} a_k\,(1-r^2)^k + 2m\frac{\Gamma(m+\sigma)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+\sigma)\Gamma(m)}\tilde{G}_m(r).$$

This concludes the proof of Lemma 7. □

With all this in hand, we are able to complete the proof of Proposition 4.

Proof of Proposition 4 Recalling the definition of $(-\Delta)^s$ in (4) and combining Lemmata 6 and 7, we obtain by writing $r = |y|$ that

$$
\begin{aligned}
(-\Delta)^s \tilde{G}_s(r) &= (-\Delta)^m\,(-\Delta)^\sigma \tilde{G}_s(r)\\
&= \frac{4^{\sigma-\frac{1}{2}}\Gamma(\frac{n}{2}+\sigma)\Gamma(s)}{\Gamma(\frac{n}{2})\cdot m!}\,(-\Delta)^m G_s^\sharp(r)\\
&= \frac{4^{\sigma-\frac{1}{2}}\Gamma(\frac{n}{2}+\sigma)\Gamma(s)}{\Gamma(\frac{n}{2})\cdot m!}\cdot 2m\,\frac{\Gamma(m+\sigma)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+\sigma)\Gamma(m)}(-\Delta)^m \tilde{G}_m(r)\\
&= \frac{4^{\sigma-\frac{1}{2}}\Gamma(\frac{n}{2}+\sigma)\Gamma(s)}{\Gamma(\frac{n}{2})\cdot m!}\cdot 2m\,\frac{\Gamma(m+\sigma)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+\sigma)\Gamma(m)}\cdot\Big(ne_n 4^{m-1}\Gamma(m)^2\Big)\,\delta_0(y).
\end{aligned}
$$

Now, we recall that $s=m+\sigma$ and that $\Gamma(m)=(m-1)!$, therefore

$$(-\Delta)^s \tilde{G}_s(|y|) = ne_n 4^{s-1}\Gamma(s)^2\delta_0(y),$$

which gives the desired result, thanks to (2). □

4 Proof of Theorem 1

In this section, we complete the proof of Theorem 1, with the aid of the results on the Green function obtained in Sects. 2 and 3. To this aim, in the following result, we show, by straightening the boundary of the unit ball, that $x\mapsto(1-|x|^2)_+^\sigma\in H^\sigma(\mathbb{R}^n)$ (from this, in Corollary 9, we will obtain that $x\mapsto(1-|x|^2)_+^s\in H^s(\mathbb{R}^n)$).

Lemma 8 *For any $\sigma\in(0,1)$, the function $\mathbb{R}^n\ni x\mapsto(1-|x|^2)_+^\sigma$ belongs to $H^\sigma(\mathbb{R}^n)$.*

Proof With a covering argument, we may focus on proving that $u(x):=(1-|x|^2)_+^\sigma$ belongs to $H^\sigma(B_{1/10}(-e_n))$. We use the notation $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ and, for any $|x|\le 1$, we define

$$y=(y',y_n)=\Phi(x):=\Big(x',\ x_n+\sqrt{1-|x'|^2}\Big).$$

Notice that this map sends the unit ball B into $\mathbb{R}^{n-1}\times(0,+\infty)$, and we can consider the inverse map

$$x=\Psi(y):=\Big(y',\ y_n-\sqrt{1-|y'|^2}\Big).$$

Notice that Φ is smooth in $B_{1/10}(-e_n)$; hence, Ψ is also smooth on the image of $B_{1/10}(-e_n)$, which is in turn contained in $D:=B^{n-1}_{1/10}\times\big[-\frac{11}{10},\ \frac{11}{10}\big]$, where the index $n-1$ indicates that this is an $(n-1)$-dimensional ball. Hence, we define

$$v(y):=u(\Psi(y))=(2y_n\sqrt{1-|y'|^2}-y_n^2)_+^\sigma,$$

and we aim at showing that $v\in H^\sigma(D)$. Notice also that, if $y\in D$,

$$\sqrt{1-|y'|^2}\in\left[\sqrt{\frac{99}{100}},\ 1\right],$$

and so

$$2\sqrt{1-|y'|^2}-y_n\ge 2\sqrt{\frac{99}{100}}-\frac{11}{10}>0.$$

Consequently, $v(y) = 0$ for $y \in D$ if and only if $y_n < 0$, and so we can write

$$v(y) = (2y_n\sqrt{1-|y'|^2} - y_n^2)_+^\sigma = (y_n)_+^\sigma\,(2\sqrt{1-|y'|^2} - y_n)^\sigma = (y_n)_+^\sigma\, w(y),$$

with w smooth in D.

As a consequence, we only need to show that $\zeta(y) := (y_n)_+^\sigma$ belongs to $H^\sigma(D)$. This follows from the computation below:

$$\begin{aligned}
&\int_{B^{n-1}_{1/10}} dx' \int_{-11/10}^{11/10} dx_n \int_{B^{n-1}_{1/10}} dy' \int_{-11/10}^{11/10} dy_n \,\frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^{n+2\sigma}}\\
&= 2\int_{B^{n-1}_{1/10}} dx' \int_{-11/10}^{11/10} dx_n \int_{B^{n-1}_{1/10}} dy' \int_{-11/10}^{x_n} dy_n \,\frac{|(x_n)_+^\sigma-(y_n)_+^\sigma|^2}{|x-y|^{n+2\sigma}}\\
&= 2\int_{B^{n-1}_{1/10}} dx' \int_{-11/10}^{11/10} dx_n \int_{B^{n-1}_{1/10}} dy' \int_{-11/10}^{x_n} dy_n \,\frac{|(x_n)_+^\sigma-(y_n)_+^\sigma|^2}{|x_n-y_n|^{n+2\sigma}\left(1+\frac{|x'-y'|^2}{|x_n-y_n|^2}\right)^{\frac{n+2\sigma}{2}}}\\
&\le 2\int_{B^{n-1}_{1/10}} dx' \int_{-11/10}^{11/10} dx_n \int_{\mathbb{R}^{n-1}} d\mu' \int_{-11/10}^{x_n} dy_n \,\frac{|(x_n)_+^\sigma-(y_n)_+^\sigma|^2}{|x_n-y_n|^{1+2\sigma}\left(1+|\mu'|^2\right)^{\frac{n+2\sigma}{2}}}\\
&\le C\int_{-11/10}^{11/10} dx_n \int_{-11/10}^{x_n} dy_n \,\frac{|(x_n)_+^\sigma-(y_n)_+^\sigma|^2}{|x_n-y_n|^{1+2\sigma}}\\
&= C\int_0^{11/10} dx_n \int_{-11/10}^{0} dy_n \,\frac{x_n^{2\sigma}}{|x_n-y_n|^{1+2\sigma}} + C\int_0^{11/10} dx_n \int_0^{x_n} dy_n \,\frac{|x_n^\sigma-y_n^\sigma|^2}{|x_n-y_n|^{1+2\sigma}}\\
&= C + C\int_0^{11/10} dx_n \int_0^1 d\tau\, x_n \frac{x_n^{2\sigma}(1-\tau^\sigma)^2}{x_n^{1+2\sigma}(1-\tau)^{1+2\sigma}}\\
&= C,
\end{aligned}$$

for some $C > 0$, possibly varying from line to line. □

As a consequence of Lemma 8, we obtain

Corollary 9 *For any $m \in \mathbb{N}$ and $\sigma \in (0,1)$, the function $\mathbb{R}^n \ni x \mapsto (1-|x|^2)_+^{m+\sigma}$ belongs to $H^{m+\sigma}(\mathbb{R}^n)$.*

Proof Let $u(x) := (1-|x|^2)_+^{m+\sigma}$ and $\alpha \in \mathbb{N}^n$ with $\alpha_1+\cdots+\alpha_n = m$. By iterating Lemma 8, we obtain that $D^\alpha u \in H^\sigma(\mathbb{R}^n)$. That is, using the equivalent norm in Fourier space,

$$+\infty > \int_{\mathbb{R}^n} |\xi|^{2\sigma} \left|\mathcal{F}(D^\alpha u)(\xi)\right|^2 d\xi = (2\pi)^m \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\xi_1|^{2\alpha_1} \ldots |\xi_n|^{2\alpha_n} |\hat{u}(\xi)|^2\, d\xi.$$

Choosing $\alpha = me_j$, with $j \in \{1,\ldots,N\}$, we find that

$$+\infty > \sum_{j=1}^n \int_{\mathbb{R}^n} |\xi|^{2\sigma}|\xi_j|^{2m}\, |\hat{u}(\xi)|^2\, d\xi \ge \frac{1}{n^m}\int_{\mathbb{R}^n} |\xi|^{2\sigma}|\xi|^{2m}\, |\hat{u}(\xi)|^2\, d\xi,$$

that is the desired result. □

With this, we are now in the position of completing the proof of Theorem 1.

Proof of Theorem 1 We observe that

$$(-\Delta)^s_y G_s(x,y) = \delta_x(y), \quad (-\Delta)^s_x G_s(x,y) = \delta_y(x), \qquad \text{for any } x, y \in B, \tag{23}$$

in the distributional sense. To prove this, one can focus on the proof of the first claim, since the second is equivalent to that, due to the symmetry of G_s under the exchange of x and y. Also, the case $x = 0$ has already been considered in Proposition 4, so we can suppose that $x \neq 0$. Hence, we can consider the Möbius transformation

$$\phi_x : \overline{B} \to \overline{B}, \quad \phi_x(y) := \frac{1}{|x|^2}\left(x + (1-|x|^2)\frac{y - \frac{x}{|x|^2}}{\left|y - \frac{x}{|x|^2}\right|^2}\right),$$

which is an automorphism of $\overline{B}$ and satisfies $\phi_x(0) = x$, $\phi_x(x) = 0$ and $\phi_x \circ \phi_x = id_{\overline{B}}$. Let also $\eta \in C_0^\infty(B)$ and

$$\tilde{\eta}(y) := J_{\phi_x}^{\frac{1}{2}-\frac{s}{n}}(y)\, \eta(\phi_x(y)),$$

where

$$J_{\phi_x}(y) = (1-|x|^2)^n \left||x|y - \frac{x}{|x|}\right|^{-2n}$$

is the modulus of the Jacobian determinant of ϕ_x. Hence,

$$\tilde{\eta}(y) = (1-|x|^2)^{(n-2s)/2} \left||x|y - \frac{x}{|x|}\right|^{2s-n} \eta(\phi_x(y)).$$

We point out that the center of inversion $x/|x|^2$ is outside $\overline{B}$, hence $\phi_x \in C^\infty(\overline{B})$, and therefore $\tilde{\eta} \in C_0^\infty(B)$. Proposition 4 shows that

$$\tilde{\eta}(0) = \int_B G_s(0,y)(-\Delta)^s \tilde{\eta}(y)\, dy.$$

Therefore, using Lemma 3 and the fact that $J_{\phi_x}(\phi_x(y)) = \frac{1}{J_{\phi_x}(y)}$, we find that

$$\begin{aligned}
(1-|x|^2)^{(n-2s)/2}\eta(x) &= \tilde{\eta}(0) \\
&= \int_B G_s(0,y)\, J_{\phi_x}^{\frac{n+2s}{2n}}(y) \left((-\Delta)^s\eta\right)(\phi_x(y))\, dy \\
&= \int_B G_s(0,\phi_x(y))\, J_{\phi_x}^{\frac{n+2s}{2n}}(\phi_x(y))\, J_{\phi_x}(y) \left((-\Delta)^s\eta\right)(y)\, dy \\
&= \int_B G_s(0,\phi_x(y))\, J_{\phi_x}^{\frac{n-2s}{2n}}(y) \left((-\Delta)^s\eta\right)(y)\, dy \\
&= (1-|x|^2)^{(n-2s)/2} \int_B \left||x|y - \frac{x}{|x|}\right|^{2s-n} \\
&\qquad \times G_s(0,\phi_x(y)) \left((-\Delta)^s\eta\right)(y)\, dy,
\end{aligned}$$

hence

$$\eta(x) = \int_B \left||x|y - \frac{x}{|x|}\right|^{2s-n} G_s(0,\phi_x(y)) \left((-\Delta)^s\eta\right)(y)\, dy. \tag{24}$$

Now, using that

$$|\phi_x(y)| = \frac{|x-y|}{\left||x|y - \frac{x}{|x|}\right|}$$

and

$$G_s(0, \phi_x(y)) = k_{s,n} |\phi_x(y)|^{2s-n} \int_1^{1/|\phi_x(y)|} (v^2-1)^{s-1} v^{1-n}\, dv,$$

we find that

$$\begin{aligned}\left||x|y - \frac{x}{|x|}\right|^{2s-n} G_s(0, \phi_x(y)) &= k_{s,n}|x-y|^{2s-n} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (v^2-1)^{s-1} v^{1-n}\, dv \\ &= G_s(x, y).\end{aligned}$$

This, together with (24), implies that, for any $\eta \in C_0^\infty(B)$ and any $x \in B$,

$$\eta(x) = \int_B G_s(x, y)\ \left((-\Delta)^s \eta\right)(y)\, dy.$$

This completes the proof of (23).

Now, we take $f \in C_0^\infty(B)$ and u as in (7). We show first that

$$u \text{ is a weak solution to} (8). \tag{25}$$

To this end, let $\varphi \in C_0^\infty(B)$ and observe that

$$(-\Delta)^s \varphi = (-\Delta)^m (-\Delta)^\sigma \varphi = (-\Delta)^\sigma (-\Delta)^m \varphi.$$

With the help of this and (23), we conclude that

$$\begin{aligned}\int_B u(x)(-\Delta)^s \varphi(x)\, dx &= \int_B \int_B G_s(x, y) f(y) (-\Delta)_x^\sigma (-\Delta)_x^m \varphi(x)\, dy\, dx \\ &= \int_B \left(\int_B (-\Delta)_x^\sigma G_s(x, y)\, (-\Delta)_x^m \varphi(x)\, dx \right) f(y)\, dy \\ &= \int_B \left(\int_B \delta_y(x) \varphi(x)\, dx \right) f(y)\, dy \\ &= \int_B f(y) \varphi(y)\, dy,\end{aligned}$$

which proves (25).

We next show that u, extended outside B by 0, satisfies

$$u \in H^s(\mathbb{R}^n) \cap C^{m,\sigma}(\mathbb{R}^n) \cap C^\infty(B) \tag{26}$$

and so, in particular, u vanishes of order m on ∂B.

To prove (26), we observe that Boggio's formula can be written as

$$G_s(x, y) = k_{s,n}\, |x-y|^{2s-n} \cdot (h \circ g)(x, y) \tag{27}$$

with

$$g(x, y) := \frac{\left||x|y - \frac{x}{|x|}\right|}{|x-y|} = \sqrt{1 + \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}} \tag{28}$$

and

$$h(t) := \int_0^t (v^2 - 1)_+^{s-1} v^{1-n} \, dv. \tag{29}$$

Notice that

$$h(t) = \frac{1}{2}(t^2 - 1)_+^s \int_0^1 \tau^{s-1} \left(1 + (t^2 - 1)\tau\right)^{-n/2} \, d\tau$$

where

$$(0, \infty) \ni t \mapsto \frac{1}{2} \int_0^1 \tau^{s-1} \left(1 + (t^2 - 1)\tau\right)^{-n/2} \, d\tau$$

is smooth and can be modified on $[0, 1)$ to a C^∞-smooth function which is identically 0 for t close to 0. Hence, we may write

$$h(t) = (t^2 - 1)_+^s \cdot \tilde{h}(t), \quad \text{with } \tilde{h} \in C^\infty([0, \infty)) \quad \text{and} \quad \tilde{h}|_{[0,1/2]} = 0. \tag{30}$$

Furthermore, since we assume that $f \in C_0^\infty(B)$, we may find some $\delta > 0$ such that $f \in C_0^\infty(B_{1-\delta}(0))$. Hence, from now on, we take

$$y \in B_{1-\delta}(0).$$

As in [14, Lemma 4.4], we consider the sets

$$\begin{aligned}
&A := \overline{B} \times B_{1-\delta}(0), \quad C := \overline{B} \times \overline{B}, \quad D := \{(x, x) : x \in \overline{B}\},\\
&F := C \cap \left\{ \left| |x|y - \frac{x}{|x|} \right| \le 3|x - y| \right\}, \quad J := C \cap \left\{ \left| |x|y - \frac{x}{|x|} \right| \ge 2|x - y| \right\},
\end{aligned}$$

and we define $d(x) := \mathrm{dist}(x, \partial B)$.

We take into account first the case in which $(x, y) \in A \cap F$. According to [14, Lemma 4.4], we have that

$$d(x) = 1 - |x| \le c|x - y| \quad \text{and} \quad d(y) \le c|x - y|,$$

for some $c \ge 1$. Since $d(y) \ge \delta$, it follows here that

$$|x - y| \ge \frac{\delta}{c}. \tag{31}$$

Accordingly, by (28), we see that

$$g \in C^\infty(A \cap F), \tag{32}$$

and all its derivatives are bounded in $A \cap F$. Analogously,

$$\text{the function } A \cap F \ni (x, y) \mapsto |x - y|^{2s-n} \text{ belongs to } C^\infty(A \cap F). \tag{33}$$

Using this and (30), we can write in $A \cap F$

$$\begin{aligned}
G_s(x, y) &= k_{s,n} \, |x - y|^{2s-n} \, h(g(x, y))\\
&= k_{s,n} \, |x - y|^{2s-n} \left((1 - |x|^2)_+^s (1 - |y|^2)^s |x - y|^{-2s} \cdot \tilde{h}(g(x, y)) \right)\\
&= (1 - |x|^2)_+^s \, g^\sharp(x, y),
\end{aligned} \tag{34}$$

with some $g^\sharp \in C^\infty(A \cap F)$.

We consider next the case in which $(x, y) \in A \cap (J \setminus D)$. Here, we have that $\frac{1}{c} d(y) \leq d(x) \leq c d(y)$, for some $c \geq 1$, so that in particular

$$d(x) \geq \frac{\delta}{c} > 0 \tag{35}$$

and $\left| |x| y - \frac{x}{|x|} \right| \geq 1 - |x|\,|y| \geq \frac{1}{c}$. As a consequence, in this case we may rewrite Boggio's formula as follows:

$$\begin{aligned} G_s(x, y) &= k_{s,n} |x-y|^{2s-n} \int_1^2 \left(v^2-1\right)^{s-1} v^{1-n}\, dv \\ &\quad + k_{s,n} |x-y|^{2s-n} \int_2^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} \left(v^2-1\right)^{s-1} v^{1-n}\, dv \\ &= c_1 |x-y|^{2s-n} + k_{s,n}|x-y|^{2s-n} \int_2^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} v^{2s-1-n} \left(1 - \frac{1}{v^2}\right)^{s-1} dv \\ &= c_1 |x-y|^{2s-n} + k_{s,n}|x-y|^{2s-n} \sum_{k=0}^{\infty} \int_2^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (-1)^k \\ &\qquad \times \binom{s-1}{k} v^{2s-2k-1-n}\, dv, \end{aligned}$$

for some $c_1 > 0$. Observe that on $A \cap J$, one has local uniform convergence of the second summand and all its derivatives, and this justifies the exchange in the order of integration and summation that we have performed here above. Hence, we can integrate the powers of v in the previous formula and end up with

$$\begin{aligned} G_s(x, y) &= c_2 |x-y|^{2s-n} \\ &\quad + c_3 \sum_{k=0}^{\infty} (-1)^k \binom{s-1}{k} \frac{1}{2s-n-2k} \left| |x|y - \frac{x}{|x|} \right|^{2s-2k-n} |x-y|^{2k}, \end{aligned}$$

for some $c_2, c_3 \in \mathbb{R}$. Only in case that $2s-2k-n=0$ (which may occur only if s is a multiple of $\frac{1}{2}$), the corresponding summand has to be replaced by a multiple of $\log\left(\frac{\left||x|y-\frac{x}{|x|}\right|}{|x-y|}\right)|x-y|^{2k}$. In consequence of this, we conclude that in this case, $G_s(x, y)$ is the sum of $c_2|x-y|^{2s-n}$ and an analytic function that we denote by $H^\star(x, y)$, provided that $s - \frac{n}{2} \notin \mathbb{N}_0$. Notice that if $s - \frac{n}{2} \in \mathbb{N}_0$, we have a logarithmic singularity instead, which is treated analogously.

That is, we can write

$$G_s(x, y) = c_2 |x-y|^{2s-n} + H^\star(x, y), \tag{36}$$

with $H^\star(x, y) \in C^\infty(A \cap J)$.

We also set

$$\begin{aligned} \mathcal{D}_{1,x} &:= \{y \in B_{1-\delta}(0) \text{ s.t. } (x, y) \in A \cap F\} \\ \text{and } \mathcal{D}_{2,x} &:= \{y \in B_{1-\delta}(0) \text{ s.t. } (x, y) \notin A \cap F\}. \end{aligned}$$

With the help of (35), we see that $\mathcal{D}_{2,x} = \emptyset$ for $d(x)$ close to 0.

Using this notation, we have that

$$\begin{aligned}
&\int_{\mathcal{D}_{2,x}} |x-y|^{2s-n}\, f(y)\,dy \\
&= \int_{B_{1-\delta}(0)} |x-y|^{2s-n}\, f(y)\,dy - \int_{B_{1-\delta}(0)\setminus\mathcal{D}_{2,x}} |x-y|^{2s-n}\, f(y)\,dy \\
&= \int_{\mathbb{R}^n} |x-y|^{2s-n}\, f(y)\,dy + Z(x) \\
&= \int_{\mathbb{R}^n} |y|^{2s-n}\, f(x-y)\,dy + Z(x) \\
&= Z^\star(x),
\end{aligned}$$

for suitable $Z, Z^\star \in C^\infty(\mathbb{R}^n)$, thanks to (33).

From this, (34) and (36), we find that

$$\begin{aligned}
u(x) &= \int_{\mathcal{D}_{1,x}} G_s(x,y)\, f(y)\,dy + \int_{\mathcal{D}_{2,x}} G_s(x,y)\, f(y)\,dy \\
&= (1-|x|^2)_+^s \int_{\mathcal{D}_{1,x}} g^\sharp(x,y)\, f(y)\,dy \\
&\quad + c_2 \int_{\mathcal{D}_{2,x}} |x-y|^{2s-n}\, f(y)\,dy + \int_{\mathcal{D}_{2,x}} H^\star(x,y)\, f(y)\,dy \\
&= (1-|x|^2)_+^s\, \tilde{u}(x),
\end{aligned}$$

with some suitable $\tilde{u} \in C^\infty(\overline{B})$. This shows that $u \in C^{m,\sigma}(\overline{B}) \cap C^\infty(B)$ and also, recalling Corollary 9, that $u \in H^s(\mathbb{R}^n)$, which completes the proof of (26), and in turn of Theorem 1. □

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