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Analytic stacks and hyperbolicity

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Abstract In this article we give two notions of hyperbolicity for groupoids on the analytic site of complex spaces, which we call Kobayashi and Brody hyperbolicity. In the special case the groupoid is a complex analytic space, these notions of hyperbolicity give the classical ones due to Kobayashi and Brody. We prove that such notions are equivalent if the groupoid is a *compact* Deligne–Mumford analytic stack (in analogy with the Brody theorem). Moreover, under the same assumptions, such notions of hyperbolicity are completely detected by the coarse moduli space of the stack. We finally show that stack hyperbolicity, as we defined it, is expected to impose a peculiar behavior to the stack itself, much like hyperbolicity for complex spaces. For instance, a stronger notion of it (hyperbolicity of the coarse moduli space) implies a "strong asymmetry" on the stack in the compact case, namely that its automorphism 2-group has only finitely many isomorphism classes.

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1 Introduction

A classical notion, introduced by Kobayashi, is the one of a pseudodistance associated with any complex space which behaves as a contraction with respect to holomorphic maps [11]. In particular, this pseudodistance is a biholomorphic invariant. Hyperbolic complex spaces are precisely those for which this pseudodistance is, in fact, a distance. In this manuscript, a *Kobayashi hyperbolic space* is what we referred to as a hyperbolic complex space, whereas a *Brody hyperbolic space* is a complex space not admitting nonconstant holomorphic maps from $\mathbb{C}$; the latter notion is inspired by Brody's Theorem [3]. For the basic results in hyperbolicity, implications and conjectures in complex geometry we refer to [11] and [4].

At the core of the paper lies the notion of hyperbolicity for analytic stacks following this Brody semantics. Such notions ought at least to generalize the known ones for complex spaces and be an invariant of the analytic stack thus independent of the choice of the groupoid presentation, or atlas, of an analytic stack.

The exposition of the manuscript begins in the greater generality of $\mathcal{S}$-groupoids (groupoids for short), where $\mathcal{S}$ is the category of (Hausdorff and reduced) complex spaces and coverings induced by the strong topology. In what follows we then progressively impose further assumptions on the studied objects, when necessary, and restrict to Deligne–Mumford analytic stacks for the proof of the main theorems.

In the first two sections we list the necessary homotopical results for providing insight to the hyperbolicity definitions. Some of the notions appearing in the paper [1] in terms of simplicial sheaves can be effectively expressed in the category of groupoids by using the results appeared in [5] and [10]. The simplicial presheaf viewpoint enables us to adapt to groupoids the concept of Brody hyperbolicity already introduced in the paper [1], whose motivation we briefly recall here. The Brody hyperbolicity condition for a complex space Y, summarized in the bijectivity of

$$p^* : \mathsf{Hom}_{\mathrm{holo}}(X, Y) \to \mathsf{Hom}_{\mathrm{holo}}(\mathbb{C} \times X, Y)$$

for all complex spaces X where $p : \mathbb{C} \times X \to X$, can be extended to groupoids $\mathcal{G}$ in two ways. One by requiring the bijectivity of

$$p^* : \mathsf{Hom}_{\mathrm{Grp}/\mathcal{S}}(\mathcal{X}, \mathcal{G}) \to \mathsf{Hom}_{\mathrm{Grp}/\mathcal{S}}(\mathbb{C} \times \mathcal{X}, \mathcal{G})$$

for all complex spaces $\mathcal{X}$ and the other for all groupoids $\mathcal{X}$. In analogy with the classical definition we choose the latter (see Definition 4.2). With some more work, we rephrased this definition in terms of *presheaves of holotopy groups* of the groupoid (see Definition 4.4). In this form it is clear that groupoid Brody hyperbolicity extends the same classical property for complex spaces and that it is categorical equivalence invariant. Lack of Brody hyperbolicity of an analytic stack implies the existence of $\mathbb{C}$ parametrized families of objects in the stack (the first two chapters of [15] describe the connection between stacks and moduli problems).

In order to introduce Kobayashi hyperbolicity for groupoids $\mathcal{G}$, several definitions have been considered; the one we decided to use (see Definition 4.6) involves the holotopy presheaves π_0^{simpl}, π_1^{simpl}. It is based on the notion of *relative analytic disc* and *relative analytic chain* (see 4.2.1) joining two sections in $\pi_0^{\mathrm{simpl}}(\mathcal{G}, g)(U)$ or $\pi_1^{\mathrm{simpl}}(\mathcal{G}, g)(U)$, U being a complex space. It preserves the metric "flavour" of the classical notion, bearing a difference: two points of a connected complex space are always joined by an analytic chain, whereas for the holotopy presheaves this happens only for particular sections called *admissible*. For a pair of admissible sections of $\pi_0^{\mathrm{simpl}}(\mathcal{G}, g)(U)$ or $\pi_1^{\mathrm{simpl}}(\mathcal{G}, g)(U)$ a *Kobayashi pseudodistance* $\mathrm{d_{Kob}}$ is defined (see 10). If this is strictly positive for any pairs of admissible sections of $\pi_0^{\mathrm{simpl}}(\mathcal{G}, g)(U)$ and $\pi_1^{\mathrm{simpl}}(\mathcal{G}, g)(U)$ over any complex space U, the groupoid $\mathcal{G}$ is said to be *Kobayashi hyperbolic*.

The important Brody's Theorem, [3], states that a compact complex space is hyperbolic if the only holomorphic maps from $\mathbb{C}$ to it are constant. One of the main results of the paper is the generalization of Brody's theorem to compact Deligne–Mumford analytic stacks (compactness for stacks is discussed in Sect. 5).

The proof is given in Sects. 7 and 8 and extensively uses techniques from complex analytic geometry, applied to the presentation

$$[X_1 = X \times_{\mathcal{Y}} X \rightrightarrows X]$$

of the complex analytic stack $\mathcal{Y}$. This is possible since the hyperbolicity properties of the complex analytic stacks studied in the paper are independent of the groupoid presentation. A crucial role is played by the *coarse moduli space* $Q(\mathcal{Y})$ of a Deligne–Mumford analytic stack $X \to \mathcal{Y}$, whose points are equivalence classes of an equivalence relation on X. The geometrization of this set was proved, in a more general context, in [2] and is a key step to make all the parts of the proof patch together. In our case, by fixing a distance function on $Q(\mathcal{Y})$ induced by a length function and lifting it to X and X_1, we obtain distances such that the structure maps are local isometries. This allows us to define distances δ_U, $\delta_{\pi_0,U}$, $\delta_{\pi_1,U}$ on the sets $\mathsf{Dis}_{[X.]}(U)$, $\pi_0(\mathcal{Y}, y)(U)$ and $\pi_1(\mathcal{Y}, y)(U)$ (see Proposition 6.1) and show the following fundamental estimate (see Lemma 6.3): for every complex space U there exists a positive number $c(\mathcal{Y}; U) \leq +\infty$ such that if $\alpha_1, \alpha_2 \in \pi_i^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$, $i = 0, 1$, then

$$\mathrm{d_{Kob}}(\alpha_1, \alpha_2) \geq \frac{\delta_{\pi_i,U}(\alpha_1, \alpha_2)}{2\, c(\mathcal{Y}; U)}$$

Thus, Kobayashi hyperbolicity of a stack $\mathcal{Y}$ is reduced to the finiteness of $c(\mathcal{Y}; U)$ for any complex space U.

The statement "Kobayashi hyperbolicity implies Brody hyperbolicity" holds in general. This fact, which in the classical case is a simple consequence of non-Kobayashi hyperbolicity of $\mathbb{C}$ and that every holomorphic map is a contraction with respect to the Kobayashi presudodistance, in the context of stacks is not entirely obvious (see Theorem 7.1).

In Sect. 8 we prove that for a Deligne–Mumford analytic stack the converse is also true (see Theorem 8.1). We argue by contradiction assuming that $c(\mathcal{Y}; U) = +\infty$ for some complex space U; then, by the results of Sect. 6.1, there is a sequence $\{f^\nu\}_\nu$ of holomorphic maps $f^\nu : \mathbb{D} \to Q(\mathcal{Y})$ such that $\lim_{\nu\to+\infty} |\mathrm{d}f^\nu(0)| = +\infty$. By classical results, such as the "reparametrization Lemma" (cfr. [3]) and Ascoli-Arzelà Theorem, there exists a subsequence $\widetilde{f}^\mu$ uniformly convergent on compact sets to a holomorphic map $f : \mathbb{C} \to Q(\mathcal{Y})$, which is not constant since $|\mathrm{d}f(0)| = 1$. The hard part of the proof consists in showing that actually f lifts to a nonconstant morphism $\mathbb{C} \to \mathcal{Y}$.

The techniques previously developed allow to prove the second main theorem of the paper, namely that for a compact Deligne–Mumford analytic stack $\mathcal{Y}$, hyperbolicity follows from the classic hyperbolicity of the complex space $Q(\mathcal{Y})$ (see Corollary 8.5).

Finally, in Sect. 9 we give some applications of the methods and results contained in the previous sections. We show that hyperbolicity for compact Deligne–Mumford stacks is not equivalent to the hyperbolicity of its coarse moduli space, by providing an explicit example of an hyperbolic Deligne–Mumford stack with a torus as moduli space. Furthermore, we prove that the hyperbolicity of the coarse moduli space implies that a compact Deligne–Mumford stack has only few automorphisms; more precisely, the automorphism 2-group $\mathsf{Aut}(\mathcal{Y})$ has only a finite number of isomorphism classes (Theorem 9.2).

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2 Preliminaries

2.1 Notation and definitions

- $\mathcal{S}_T$ is the analytic site: the category $\mathcal{S}$ whose objects are complex spaces and coverings are those induced by the strong topology.
- Grp is the category of (set-theoretic) groupoids and by $\mathsf{Psh}(\mathcal{S}; \mathrm{Grp})$ we will denote the category of presheaves on $\mathcal{S}$ of set-theoretic groupoids.
- $\mathrm{Grp}/\mathcal{S}$ is the category of $\mathcal{S}$-groupoids whose objects are categories fibered in groupoids. A stack is an object of $\mathrm{Grp}/\mathcal{S}$ which is a sheaf on $\mathcal{S}_T$.
- $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J$ is the category of simplicial presheaves of sets on the site $\mathcal{S}_T$ with a topology T and endowed by *local, injective, simplicial model structure* on $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ (cfr. Joyal's model structure, [10, Sect. 5.1]). $\mathcal{H}_s$ (respectively $\mathcal{H}_{s_\bullet}$) is the homotopy category (respectively the pointed homotopy category) associated.
- Let $\mathcal{X}$ be a groupoid. Then $c_\mathcal{X} : \mathcal{X} \to \mathcal{C}\mathcal{X}$ is its stackification (cfr. [12, Lemma 3.2 and Observation 3.2.1 (3)]).
- The morphisms ∂_0 and ∂_1 will denote the face morphisms of a simplicial presheaf $\mathcal{X}.$ or, more frequently, of groupoids $[X.]$ and $\mathcal{C}[X.]$ from the presheaf in degree 1 to presheaf.

2.2 Simplicial presheaves and groupoids

Before being able to state what we think of as a (differently flavored) hyperbolic groupoid, we will expose the connection between simplicial presheaves and groupoids, by stating the relevant results contained in the manuscript [5]. The concept of Brody hyperbolicity, in particular, is directly transposed from simplicial presheaves. While the concept of groupoids in terms of categories fibered in set-theoretic groupoids (S-groupoids) probably goes back to ideas of Grothendieck, only recently these objects have been related to the homotopy theory of simplicial presheaves of sets.

In the paper [1] we dealt with simplicial presheaves of sets on the analytic site, i.e., the category $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ whose objects are complex spaces and coverings given by the strong topology on them. That category has been considered a model category by means of the Joyal injective *local* simplicial structure. Here the word "local," as opposed to "global," refers to the fact that the weak equivalences are morphisms inducing weak equivalences of simplicial sets on the stalks of the presheaves, as opposed to weak equivalences on simplicial sets of sections of the simplicial presheaves. After having localized this category with respect of such model structure, we defined Brody hyperbolic simplicial presheaves (cfr. [1, Sect. 3.1]). We want to relate these objects to groupoids in general and stacks in particular.

Such a relation is spread out in the papers [10] and [5]. We list here their results which are relevant to this manuscript. The starting point is Corollary 4.3 in [10] which claims the existence of an adjunction between the categories $\mathsf{Prsh}(\mathrm{Grp})$ and $\mathrm{Grp}/\mathcal{S}$. Endowing each of them with appropriate *global* model structures ([10, Proposition 4.1 and Theorem 4.2]) we furthermore have that the adjoint functors are a Quillen equivalence. Here "global" means that the weak equivalences are meant to be objectwise (respectively fiberwise) weak equivalences. These two categories are not directly related to $\mathsf{Prsh}_T(\mathcal{S})$, but a certain localization of them are. Keeping the same notation as the references, we let S to be the set of maps

$$S = \{p_U : \mathrm{hocolim}\, U. \to X : \ \amalg_i U_i = U \to X \text{ is a cover for the strong topology}\} \quad (1)$$

where hocolim is the homotopy colimit of the diagram $U. = \{\cdots U \times_X U \rightrightarrows U\}$. By using Bousfield localization theory, it can be shown that there are model structures on $\mathsf{Prsh}(\mathrm{Grp})$ and $\mathrm{Grp}/\mathcal{S}$ whose homotopy categories are the localization with respect to S ([10, Proposition 4.4]). These model structures are called *local* for a reason that will become clear later and to emphasize this model structure we will add L as a subscript. It follows that the aforementioned adjoint pair of functors, induces a Quillen equivalence $\mathsf{Prsh}(\mathrm{Grp})_L \leftrightarrows (\mathrm{Grp}/\mathcal{S})_L$ ([10, Corollary 4.5]). In the paper [5] the model structure on $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ is reinterpreted through the Bousfield localization with respect to the maps in S: by [5, Theorem 6.2], we know that its localized category is equivalent to the homotopy category $\mathcal{H}_s$. Thus, to relate $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J$ to $(\mathrm{Grp}/\mathcal{S})_L$, it suffices to relate $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_L$ to $\mathsf{Prsh}(\mathrm{Grp})_L$. Again, there is an adjunction (π_{oid}, N)

$$\pi_{oid} : \Delta^{op}\mathsf{Prsh}_T(\mathcal{S}) \leftrightarrows \mathsf{Prsh}(\mathrm{Grp}) : N$$

defined as follows: π_{oid} is the functor which sends a simplicial presheaf $F.$ to the $\mathcal{S}$-groupoid having $F_0(U)$ as objects, for each complex space U, and $\mathsf{Hom}(a, b)$ is the set of $\phi \in F_1(U)$ such that $\partial_0(\phi) = a$ and $\partial_1(\phi) = b$. To an $\mathcal{S}$-groupoid $\mathcal{G}$, the functor N associates the simplicial presheaf $N\mathcal{G}$ with $(N\mathcal{G})_0 = \mathsf{Ob}(\mathcal{G})$, $(N\mathcal{G})_1 = \mathsf{Mor}(\mathcal{G})$ and

$$(N\mathcal{G})_i = (N\mathcal{G})_1 \times_{(N\mathcal{G})_0} \overset{i}{\cdots} \times_{(N\mathcal{G})_0} (N\mathcal{G})_1$$

with the following structural face morphisms: $\partial_0, \partial_1 : (N\mathcal{G})_1 \to (N\mathcal{G})_0$ are the domain and codomain of the isomorphism, respectively; the three morphisms $(N\mathcal{G})_2 \to (N\mathcal{G}_1)$ send (f, g), respectively, in f, $g \circ f$ e g; in the general case an n-tuple of composable isomorphisms are sent to $(n-1)$-subtuples involving individual isomorphisms and compositions of them, when appropriate. The degenerations are induced by alternatively adding the identity morphism. (π_{oid}, N) is a Quillen pair and the following holds (cfr. [10, Theorem 5.4]):

Theorem 2.1 *The Quillen adjoint pair* (π_{iod}, N) *induces a Quillen equivalence between* $(\mathbb{S}^2)^{-1}\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_L$*, the* $\mathbb{S}^2$ *nullification of* $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_L$*, and* $\mathsf{Prsh}(\mathrm{Grp})_L$.

To trace this chain of equivalences back to the category $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J$ equipped with the Joyal simplicial model structure, we finally need to use the results in the [5] which imply that $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J$ and $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_L$ are Quillen equivalent. The previous considerations and this theorem prove

Corollary 2.2 *The adjoint pair* (π_{oid}, N) *induces a Quillen equivalence between the categories* $(\mathbb{S}^2)^{-1}\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J$ *and* $(\mathrm{Grp}/\mathcal{S})_L$.

In order to see how this relates to (analytic) stacks we need to invoke sharper results. Theorem 1.1 (see also Theorem 3.9) of [10] states that a groupoid $\mathcal{F}$, seen as a presheaf of groupoids, is a stack if and only if, for any covering $\amalg_i U_i = U \to X$, the canonical morphism

$$\mathcal{F}(X) \to \mathrm{holim}_n \Big\{ \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_{ij}) \Rrightarrow \prod \mathcal{F}(U_{ijk}) \cdots \Big\}$$

is an equivalence of categories for each complex space X, where $U_{i_1,\cdots,i_k}$ stands for $U_{i_1} \times \cdots \times U_{i_k}$. On the other hand, since the Bousfield $\mathcal{S}$-localizing structure on $\mathsf{Prsh}(\mathrm{Grp})$ relies on an underlying *global* (meaning weak equivalences and fibrations are objectwise) model structure, we deduce that the $\mathcal{S}$-fibrant objects are precisely those presheaves of groupoids $\mathcal{F}$ that are

- objectwise fibrant, i.e., all since a set-theoretic groupoid is simplicially fibrant if seen as simplcial set (by means of the functor N, precedently described) and
- such that the canonical morphism

$$\mathcal{F}(X) \to \mathrm{holim}_n \Big\{ \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_{ij}) \Rrightarrow \prod \mathcal{F}(U_{ijk}) \cdots \Big\}$$

  is a weak equivalence of set-theoretic groupoids.

Because of Theorem 2.1, the functor π_{oid} sends $\mathcal{S}$-fibrant presheaves of groupoids to $\mathcal{S}$-fibrant simplicial presheaves and by [5, Theorem 1.1] such simplicial presheaves are precisely those that are fibrant according to the Joyal simplicial model structure on $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$. This leads to

Theorem 2.3 *The chain of Quillen equivalences between* $\mathrm{Grp}/\mathcal{S}$ *and* $(\mathbb{S}^2)^{-1}\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J$ *induces an isomorphism between the (full) subcategories of stacks and fibrant simplicial presheaves.*

While for $\mathcal{S}$-groupoids there are two notions of equivalences, one *global* and one *local*, they coincide for stacks. Moreover, we may think of a stack as a simplicial presheaf of sets, a presheaf of groupoids, or a presheaf of categories where equivalences between them should be thought of as homotopy equivalences in the first two cases and as equivalences of categories in the latter.

2.3 Analytic stacks

A stack over the analytic site $\mathcal{S}_T$ is said to be an *analytic stack* if

1. the diagonal morphism $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable;
2. there exists a complex space X with a smooth and surjective morphism $p : X \to \mathcal{Y}$.

The morphism $p : X \to \mathcal{Y}$ is also called a *presentation* of $\mathcal{Y}$ and X an *atlas*.

An analytic stack $\mathcal{Y}$ with an étale presentation $p : X \to \mathcal{Y}$ is called a *Deligne–Mumford analytic stack*.

Let $\mathcal{P}$ be a presheaf of groupoids and $F : \mathcal{P} \to \mathcal{G}$ be a 1-morphism (functor) to a $\mathcal{S}$-groupoid $\mathcal{G}$. We build a groupoid out of it, denoted by $[\mathcal{P}.]$. Its objects over a complex space U are the sections in $\mathcal{P}(U)$ and $\mathrm{Hom}_{[P.](U)}(f, g)$ are the sections $\phi \in \mathcal{P}_1(U) := \mathcal{P}(U) \times_{\mathcal{Y}} \mathcal{P}(U)$ such that $\partial_0(\phi) = f$ and $\partial_1(\phi) = g$, where $\partial_i : \mathcal{P}_1 \to \mathcal{P}$, for $i = 0, 1$, are the projections on the factors, and the fiber product is taken in the category of groupoids. The remaining structure making $[X.]$ a groupoid is inherited by the one of $\mathcal{G}$ and it is explained in (2.4.3) and in Proposition 3.8 of [12].

Remark 2.1 1. If $X \to \mathcal{Y}$ is an analytic stack (see Sect. 2.1), then the objects of $[X.]$ over a complex space U are the holomorphic maps $U \to X$ and $\partial_i : X_1 \to X$ are holomorphic maps between complex spaces.

2. Our notation is slightly different from the one in [12, 2.4.3]: the groupoid $[X.]$ is denoted as $[X.]'$ there. Moreover, through this manuscript, we will identify the $\mathcal{S}$-*espace en groupoïdes* and its associated groupoid.

We recall the following general result

Proposition 2.4 *(cfr. [12, Prop. 3.8]) Let $F : \mathcal{P} \to \mathcal{Y}$ be a morphism (functor) between a presheaf and a stack. Then the canonical morphism (functor) $[\mathcal{P}.] \to \mathcal{Y}$ is a monomorphism and is epi if and only if F is.*

In the particular case $p : X \to \mathcal{Y}$ is an analytic stack, we get a simplicial complex space $[X.]$ such that

$$X \times_{\mathcal{Y}} \times \overset{i+1}{\cdots} \times_{\mathcal{Y}} X = X_i = X_1 \times_{\partial_0, X, \partial_1} \overset{i}{\cdots} \times_{\partial_0, X, \partial_1} X_1.$$

$[X.]$ is a *prestack*, as explained in the example [12, 3.4.3] and is precisely $N([\mathcal{P}.])$ with $\mathcal{P} = X$. To simplify the notation, we will drop the letter N and consider $[X.]$ indifferently as a $\mathcal{S}$-groupoid or a simplicial complex space, according to the needed properties. Notice that the stackification functor $[X.] \to \mathcal{C}[X.]$ corresponds to a fibrant resolution of simplicial presheaves. We conclude that

Proposition 2.5 *Let $p : X \to \mathcal{Y}$ be an analytic stack. Then p induces a groupoid equivalence $p : [X.] \to \mathcal{Y}$ and a stack equivalence $\mathcal{C}[X.] \to \mathcal{Y}$.*

An immediate consequence of this proposition is that, to work with a simplicial homotopy invariant, we can indifferently use any presentation and atlas of $\mathcal{Y}$:

Corollary 2.6 *Let $X, Z \to \mathcal{Y}$ be two presentations of an analytic stack. Then $[X.]$ and $[Z.]$ are equivalent groupoids and $\mathcal{C}[X.]$ and $\mathcal{C}[Z.]$ are equivalent stacks.*

In view of theorem 2.3, given an atlas $X \to \mathcal{Y}$, the stack $\mathcal{C}[X.]$ will simply be denoted as $[X \times_{\mathcal{Y}} X \rightrightarrows X]$ or $[X_1 \rightrightarrows X]$.

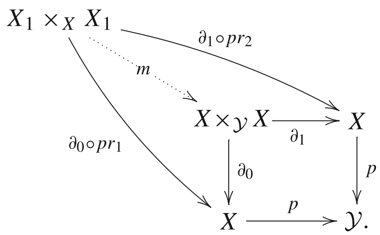
Digression 2.7 For future reference, we write in detail the product structure that $[X.]$ inherits from the presentation $p : X \to \mathcal{Y}$ of an algebraic stack. By definition of the fibered product in the category of groupoids, an element of the complex space $X_1 = X \times_{\mathcal{Y}} X$ is a triple $\underline{a} = (a_1, a_2, \alpha)$, where $a_1, a_2 \in X$ and $\alpha : p(a_1) \overset{\cong}{\to} p(a_2)$ is an isomorphism. Notice that there is a canonical holomorphic map $e : X \to X_1$, the one sending $a \in X$ to $(a, a, \mathrm{id}_a) \in X_1$. Let ∂_0 and ∂_1

$$X \times_{\mathcal{Y}} X \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} X \tag{2}$$

be the holomorphic maps involved in the definition of fiber products, i.e., making the following diagram commutative

$$\begin{array}{ccc} X \times_{\mathcal{Y}} X & \xrightarrow{\partial_0} & X \\ \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & \mathcal{Y}. \end{array} \tag{3}$$

The multiplication m is the holomorphic map associated to the pair $(\partial_0 \circ pr_1, \partial_1 \circ pr_2)$ and whose existence is a consequence of the universality of fiber products:



(4)

m is explicitely descibed as follows: if a is as before and $b = (b_1, b_2, \beta)$ is another point of X_1, then $m(a, b) = (a_1, b_2, \beta \circ \alpha)$. By the commutativity of the previous diagram, we have $(\partial_0 \circ m)(a, b) = a_1$ and $(\partial_1 \circ m)(a, b) = b_2$.

If $\mathcal{Y}$ is a Deligne–Mumford analytic stack then ∂_0 and ∂_1 are étale ; m also is étale because in the commutative diagram

$$\begin{array}{ccc} X_1 \times_X X_1 & \xrightarrow{pr_1} & X_1 \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle \partial_0} \\ X_1 & \xrightarrow{\partial_1} & X \end{array}$$

we have that pr_i are étale , hence $\partial_0 \circ pr_1$ is étale and from the commutativity of the diagram (4), we conclude that m is étale .

3 Simplicial parabolic holotopy presheaves $\pi_i^{\mathbf{simpl}}$

The notion of Brody hyperbolicity which we will introduce in Sect. 4 will be rephrased in terms of *simplicial parabolic holotopy presheaves*. It turns out that this is a very convenient way to work in practice with such seemingly abstract definitions. Those holotopy presheaves completely determine the *global* simplicial homotopy class of a groupoid, and since local simplicial weak equivalences between locally fibrant simplicial presheaves coincide with global simplicial weak equivalences, holotopy presheaves completely determine equivalences of stacks. In other words, a functor $F : \mathcal{X} \to \mathcal{Y}$ is an equivalence between stacks if and only if it induces isomorphisms between all holotopy presheaves of $\mathcal{X}$ and $\mathcal{Y}$.

We recall that the *parabolic $n-th$ dimensional circle* is the simplicial set $\mathsf{S}^1_s := \Delta^1/\partial\Delta^1$, where Δ^1 is the standard 1-dimensional simplex, seen as constant presheaf in the analytic site. As usual, in what follows, we let $\wedge$ be the monoidal structure in $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ and $\mathsf{S}^n_s := \mathsf{S}^1_s \wedge \overset{n}{\cdots} \wedge \mathsf{S}^1_s$.

Definition 3.1 1. For any complex space U and simplicial presheaf $\mathcal{X}$, we set

$$\pi_i^{\text{simpl}}(\mathcal{X}, x)(U) \overset{def}{=} \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^i_s \wedge U_+, (\mathcal{X}, x))$$

2. For any groupoid $\mathcal{G}$ and $U \in \mathcal{S}_T$ we set $\pi_i(\mathcal{G}, g)(U) = \pi_i^{\text{simpl}}(N\mathcal{G}, g)(U)$.

By definition, the presheaves π_i^{simpl} are functors from the category of simplicial presehaves to presheaves of sets and (abelian) groups and they will be called holotopy presheaves. They induce isomorphisms if applied to local and global weak equivalences or groupoid equivalences. We know already how to compute most of these presheaves for groupoids $\mathcal{G}$: because of Theorem 2.1, $\pi_i^{\text{simpl}}(\mathcal{G}, g)$ are isomorphic to the constant sheaf 0 for all i greater or equal to 2. In general, it is extremely hard to compute π_i^{simpl} of groupoids if $i = 0, 1$. The following result shows that, in the case the groupoid is a stack, these presheaves are related to some combinatorial data of the stack:

Proposition 3.2 *Let $\mathcal{Z}$ be a locally fibrant simplicial presheaf. Denote by ∂_I the composition $\partial_{i_1} \circ \cdots \circ \partial_{i_n}$ for a multiindex $I = (i_1, \cdots, i_n)$, where $\partial_{i_j} : \mathcal{Z}_j \to \mathcal{Z}_{j-1}$ are face morphisms. Then, $\pi_n^{simpl}(\mathcal{Z}, z)(U)$ is the quotient set $A_n(U)/\sim$ where*

1. *$A_n(U) = \{f : U \to \mathcal{Z}_n$ such that $\partial_I \circ f = \partial_J \circ f\ ,\ \forall I, J$ of lenght $n\}$*
2. *$\sim$ is the equivalence relation generated by $f \sim g$ if it exists an $H : U \to \mathcal{Z}_1$ such that $\partial_0 \circ H = \partial_I \circ f$ e $\partial_1 \circ H = \partial_I \circ g$.*

Proof One can get the given description of $A_n(U)$ by using the adjunction

$$\begin{aligned}\mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^n_s \wedge U_+, (\mathcal{P}, y)) &= \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^n_s, \mathsf{Map}_\bullet(U_+, (\mathcal{P}, y))) \\ &= \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^n_s, (\mathcal{P}(U), y)) \overset{def}{=} \pi_n^{\text{simpl}}(\mathcal{P}(U), y). \quad (5)\end{aligned}$$

and explicitely writing down the simplicial set S^n_s. Simplicial morphisms from S^n_s are determined by the image of the generating point in simplicial degree n. Since all the iterated face morphisms on that point are all constant to the same points, one at each degree, we get the condition describing the set $A_n(U)$. The fact that π_n^{simpl} is a quotient set of A_n follows by the fibrancy assumption. The equivalence relation translates the condition of simplicial homotopy between maps implied by the cylinder simplicial presheaf $\Delta^1 \wedge (\mathsf{S}^n_s \wedge U_+)$: two morphisms $f, g : \mathsf{S}^n_s \wedge U_+ \to \mathcal{Z}$ are simplicially homotopic if there exists $H : \Delta^1 \wedge (\mathsf{S}^n_s \wedge U_+) \to \mathcal{Z}$ such that when precomposed with the canonical embedding $\partial\Delta^1 \hookrightarrow \Delta^1$, yields $f \amalg g$. By explicitely writing down the simplicial set Δ^1, this last condition takes the shape of the equalities of 2). □

Remark 3.1 In particular, for an analytic stack $\mathcal{Z}$

$$\pi_0^{\text{simpl}}(\mathcal{Z}, z)(U) = \{f : U \to \mathcal{Z}_0\}/\sim$$

where $f \sim g$ if there exists a morphism $H : U \to \mathcal{Z}_1$ such that $\partial_0 \circ H = f$ and $\partial_1 \circ H = g$. Defining π_0^{simpl} this way for general groupoids would result in a non- (simplicial, local) homotopy invariant presheaf.

3.1 Descent data and simplicial parabolic holotopy presheaves

In the previous subsection we have observed that the only holotopy presheaves of a groupoid that are relevant are in degree zero and one. Moreover, if the groupoid is a stack $\mathcal{Y}$, the sections of these presheaves are expressible in terms of sections in the presheaves $\mathsf{Ob}(\mathcal{Y})$ and $\mathsf{Mor}(\mathcal{Y})$, or, more precisely, of the sections of $(N\mathcal{Y})_0$ and $(N\mathcal{Y})_1$ since by Definition 3.1 $\pi_i^{\text{simpl}}(\mathcal{Y}, x)(U) = \pi_i^{\text{simpl}}(N\mathcal{Y}, x)(U)$.

Proposition 3.3 *Let $X \to \mathcal{Y}$ be an analytic stack. Then*

$$\pi_i^{simpl}(\mathcal{Y}, y) \cong \pi_i^{simpl}([X.], x) \cong \pi_i^{simpl}(\mathcal{C}[X.], x)$$

Proof Use Proposition 2.5. □

In view of these isomorphisms, for any simplicially homotopy invariant considerations about $\mathcal{Y}$, we will replace $\mathcal{Y}$ with $\mathcal{C}[X.]$ for some appropriate choice of atlas X of $\mathcal{Y}$. Because of the relevance of the concept in the sequel, we explicitely recall

Definition 3.4 Let $\mathcal{G}$ be a groupoid, U a complex space and $\mathcal{U} = \{U_i\}_i$ a covering of U for the strong topology. Then,

1. A *descent datum* relative to $\mathcal{U}$ in $\mathcal{G}$ is a pair $((A_i), (h_{ij}))$, also denoted (A_i, h_{ij}) with: A_i objects of $\mathcal{G}(U_i)$ and $h_{ij} : A_i|_{U_{ij}} \to A_j|_{U_{ij}}$ isomorphisms, called *transition morphisms* satisying the *cocycle condition*: $h_{jk} \circ h_{ij} = h_{ik}$ on U_{ijk}. As always, U_{ij} and U_{ijk} stand for the double and triple intersections of the indixed complex spaces. The set of descent data will be denoted by $\mathsf{Dis}_{\mathcal{G}}(\mathcal{U})$.
2. A *descent data morphism* between (A_i, h_{ij}) and (B_i, g_{ij}) in $\mathcal{G}$ and relative to a covering $\mathcal{U}$ is a collection of isomorphisms $\{\phi_i : A_i \to B_i\}$ respecting the relation $g_{ij} \circ \phi_i = \phi_j \circ f_{ij}$ on U_{ij} for all i, j.

Remark 3.2 In a covering $\mathcal{U}$ associated to a descent datum we will possibly allow $U_i = U_j$ for $i \neq j$.

Given a complex space U, we will denote by $\mathrm{Cov}\, U$ the set of all the countable, locally finite open coverings such that $U_i \Subset U$ for all i. Any covering of U can be refined to one in $\mathrm{Cov}\, U$. This set is filtering with respect to the relation $\mathcal{U} \preceq \mathcal{U}'$ if $\mathcal{U}' = \{U_i\}_i$ is finer than $\mathcal{U}$ and $U'_i \Subset U_{\tau(i)}$, if $\tau : \mathbb{N} \to \mathbb{N}$ is the refining function.

Consider a refinement $\mathcal{U} \preceq \mathcal{U}'$ of two coverings of U. There is a correspondence $r_{\mathcal{U},\mathcal{U}'} : \mathsf{Dis}_{\mathcal{G}}(\mathcal{U}) \to \mathsf{Dis}_{\mathcal{G}}(\mathcal{U}')$ which associates to a descent datum (A_i, h_{ij}) the datum $(A_i|_{U'_i}, h'_{rs})$, where the new transition morphisms h'_{rs} on the double intersections U'_{rs} of complex spaces contained in U_i are defined as the identity: $A_i|_{U'_{rs}} \to A_i|_{U'_{rs}}$. The same can be said for a descent data morphism $\phi = \{\phi_i\}_i$ between a and b: it induces a morphism $r_{\mathcal{U},\mathcal{U}'}(a) \to r_{\mathcal{U},\mathcal{U}'}(b)$ in a unique way. Thus, we have the following

Lemma 3.5 *Let $X \to \mathcal{Y}$ be an analytic stack. The stack $\mathcal{C}[X.]$ associated to the groupoid $[X.]$ is explicitely described by the following:*

1. $\mathsf{Ob}(\mathcal{C}[X.])(U) = \varinjlim_{U \in Cov\, U} \mathsf{Dis}_{[X.]}(\mathcal{U})$*;*

2. *let* r *and* s *be two objects represented by descent data r and s, and defined on the same covering $\mathcal{U}$ (this is not restrictive). Then*

$$\mathsf{Hom}_{\mathcal{C}[X.]}(\mathrm{r}, \mathrm{s}) = \left\{\text{descent data (iso)morphisms } r \to s\right\}$$

(see Definition 3.4).

Remark 3.3 In [12, Lemma 3.2] a different description of the stack associated to a prestack is given. In particular the objects are simply descent data unidentified in the direct limit. We think this is not a useful definition as the following example shows: let $\mathcal{Y} = Y$ be a complex space and $X \to \mathcal{Y}$ be a presentation with $X = \amalg_i B_i$, B_i nontrivial open subspaces. Then the canonical morphism $\mathcal{C}[X.] \to \mathcal{Y}$ is a (presheaf) isomorphism, since the only isomorphisms in $\mathsf{Mor}\,\mathcal{C}[X.]$ are the identity morphisms. The objects of $\mathcal{C}[X.]$ over U are the sections $\mathcal{Y}(U)$; thus, they must be all the holomorphic maps $U \to Y$. We do have a canonical surjective correspondence from the descent data over U to holomorphic maps $U \to Y$, but this is not injective, factoring precisely through the relation defining the direct limit over the coverings of U.

The notion of descent data given in Definition 3.4 may be expressed in terms of holomorphic maps if the groupoid $\mathcal{G}$ in question is $\mathcal{C}[X.]$. Given a covering $\mathcal{U} \in \mathrm{Cov}\, U, \mathcal{U} = \{U_i\}_i$, a descent datum r (on U) in X relative to $\mathcal{U}$ is a pair $((r_i : U_i \to X)_i, (f_{ij} : U_{ij} \to X_1)_{ij})$ with r_i and f_{ij} holomorphic maps such that:

($\star$) $r_{i|U_{ij}} = \partial_0 \circ f_{ij}, \;\; r_{j|U_{ij}} = \partial_1 \circ f_{ij}$ on U_{ij};
($\star\star$) f_{ij}: $m(f_{ij} \times f_{jk}) = f_{ik}$ on U_{ijk} (cocycle relation)

and, like before,

$$\varinjlim_{\mathcal{U}\in \mathrm{Cov}\, U} \left\{\left((r_i : U_i \to X)_i, (f_{ij} : U_{ij} \to X_1)_{ij}\right)\right\} = \varinjlim_{\mathcal{U}\in \mathrm{Cov}\, U} \mathsf{Dis}_{[X.]}(\mathcal{U}) = \mathsf{Ob}(\mathcal{C}[X.])(U)$$

We are ready now to describe the zeroth and first holotopy presheaves of an analytic stack by means of the complex structure of any of its atlases:

Theorem 3.6 *Let $p : X \to \mathcal{Y}$ be an analytic stack and U a complex space. Then*

$$\pi_0(\mathcal{Y}, y)(U) \cong \varinjlim_{\mathcal{U}\in \mathrm{Cov}\, U} \left\{\left((r_i : U_i \to X)_i, (f_{ij} : U_{ij} \to X_1)_{ij}\right)\right\}/\!\sim_0$$

where $\sim_0$ is the equivalence relation generated by $(r_i, f_{ij}) \sim_0 (s_i, g_{ij})$ if and only if there exist holomorphic maps $\phi_i : U_i \to X_1$ such that

1. *$\partial_0 \circ \phi_i = s_i$ e $\partial_1 \circ \phi_i = r_i$ for all i;*
2. *$m(f_{ij} \times \phi_j) = m(\phi_i \times g_{ij})$.*

An isomorphism ϕ between two descent data $r = (r_i, f_{ij})$, $s = (s_i, g_{ij})$ is a collection of holomorphic maps $\phi_i : U_i \to X_1$ such that

(1′) $\partial_0 \circ \phi_i = r_i$ and $\partial_1 \circ \phi_i = s_i$ for all i;
(2′) $m(f_{ij}, \phi_j) = m(\phi_i, g_{ij})$ for all i, j.

Each collection $\{\phi_i\}_i$ determines a class in the filtered colimit, over the coverings $\mathcal{U}$ of U, of isomorphisms between the descent data r and s. Representatives of sections of $\pi_1^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$ are (classes of) automorphisms $(\phi)_i$ of r, for r ranging in $\varinjlim_{\mathcal{U}\in \mathrm{Cov}\, U} \mathsf{Dis}_{[X.]}(\mathcal{U})$.

Theorem 3.7 *Let $p : X \to \mathcal{Y}$ an analytic stack and U be a complex space. Then,*

$$\pi_1(\mathcal{Y}, y)(U) \cong \varinjlim_{\mathcal{U} \in \mathrm{Cov}\, U} (\phi_i)_i / \sim_1$$

where $\phi = (\phi_i)_i$ is an automorphism of a descent datum r in $[X.]$ relative to $\mathcal{U}$. If ϕ e ψ are automorphisms of descent data r and s, respectively, relative to $\mathcal{U} = \{U_i\}_i$, then $\sim_1$ is the equivalence relation generated by $\phi \sim_1 \psi$ if and only if there exists and isomorphism between $\partial_0 \circ \phi = s$ and $\partial_0 \circ \psi = r$, i.e., holomophic maps $H_i : U_i \to X \times_{\mathcal{Y}} X$ such that $\partial_0 \circ H_i = r_i$ and $\partial_1 \circ H_i = s_i$ satisfying the second of the conditions listed in the Theorem 3.6.

Proof (of Theorems 3.6 and 3.7) We use Proposition 3.2. By the Yoneda lemma the set A_0 coincides with the sections

$$\mathcal{C}[X.](U) = \mathsf{Ob}(\mathcal{C}[X.](U)) = \varinjlim_{\mathcal{U} \in \mathrm{Cov}\, U} \mathsf{Dis}_{[X.]}(\mathcal{U}).$$

Thus, $\pi_0^{\mathrm{simpl}}(\mathcal{C}[X.], x)(U)$ is the quotient of this set by the equivalence relation given by the simplicial homotopy that in turn is described in the part (2) of the above proposition. The same argument gives the statement for the groups $\pi_1^{\mathrm{simpl}}(\mathcal{C}[X.], x)(U)$. Finally, we identify the presheaves $\pi_0^{\mathrm{simpl}}(\mathcal{C}[X.], x)$ and $\pi_1^{\mathrm{simpl}}(\mathcal{C}[X.], x)$ with those of $\mathcal{Y}$ by means of the Proposition 3.3. □

Definition 3.8 Let $\mathcal{P}$ be a presheaf on $\mathcal{S}_T$. A section $\sigma \in \mathcal{P}(U)$ is *constant* if it lies in the image of the map $c^* : \mathcal{P}(\mathsf{pt}) \to \mathcal{P}(U)$, where $c : U \to \mathsf{pt}$.

4 Hyperbolicity

General assumption: *For the time being, given an analytic stack $X \to \mathcal{Y}$, we will assume that the two complex spaces X and $X \times_{\mathcal{Y}} X$ are reduced.*

The classical Brody's Theorem claims that two notions of hyperbolicity for complex spaces coincide. One is rooted in metric aspects of the complex space, the other is defined in terms of certain holomorphic maps. We are not aware of any possible candidates for the analogs of these notions about stacks in the literature. This section is devoted to providing ours.

4.1 Brody hyperbolicity

In the paper [1] we have given the following definition

Definition 4.1 A simplicial presheaf $\mathcal{P}$ is *Brody hyperbolic* if

1. is simplicially locally fibrant and
2. the projection $p_{\mathcal{X}} : \mathbb{C} \times \mathcal{X} \to \mathcal{X}$ induces set bijections

$$\mathsf{Hom}_{\mathcal{H}_s}(\mathcal{X}, \mathcal{P}) \stackrel{\cong}{\to} \mathsf{Hom}_{\mathcal{H}_s}(\mathbb{C} \times \mathcal{X}, \mathcal{P}) \tag{6}$$

for all $\mathcal{X} \in \mathsf{Prsh}_T(\mathcal{S})$.

Since a groupoid can be seen as a simplicial presheaf by means of the functor N, we will use the same definition.

Definition 4.2 A groupoid $\mathcal{G}$ is *Brody hyperbolic* if $N\mathcal{G}$ is a Brody hyperbolic simplicial presheaf.

Notice that a Brody hyperbolic groupoid is necessarily a stack. This definition can be rephrased in terms of holotopy presheaves:

Proposition 4.3 *Let $\mathcal{P}$ be a locally fibrant simplicial presheaf. The following conditions are equivalent:*

1. *$\mathcal{P}$ is Brody hyperbolic;*
2. *$p^*_{\mathcal{X}} : \mathsf{Map}(\mathcal{X}, \mathcal{P}) \to \mathsf{Map}(\mathbb{C} \times \mathcal{X}, \mathcal{P})$ is a weak equivalence of simplicial sets for any $\mathcal{X} \in \Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$, where Map is the simplicial mapping space;*
3. *the simplicial holotopy presheaves are Brody hyperbolic, i.e., the projection $p_U : \mathbb{C} \times U \to U$ induces isomorphisms $p^*_U : \pi_i^{simpl}(\mathcal{P}, y)(U) \overset{\cong}{\to} \pi_i^{simpl}(\mathcal{P}, y)(\mathbb{C} \times U)$ for each i and complex space U.*

Proof (2)$\Rightarrow$ (1). $\mathcal{X}$ is locally fibrant by assumption and

$$\mathsf{Hom}_{\mathcal{H}_s}(\mathcal{X}, \mathcal{P}) = \pi_0(\mathsf{Map}(\mathcal{X}, \mathcal{P})) \cong \pi_0(\mathsf{Map}(\mathbb{C} \times \mathcal{X}, \mathcal{P})) = \mathsf{Hom}_{\mathcal{H}_s}(\mathbb{C} \times \mathcal{X}, \mathcal{P}).$$

(1)$\Rightarrow$(2). We show that the condition (2) is equivalent to $\mathcal{P}(U) \cong \mathcal{P}(\mathbb{C} \times U)$ being a simplicial sets weak equivalence for all complex spaces U. By a small object argument there is a weak equivalence $\phi : \mathcal{Z}_n \to \mathcal{X}$, where $\mathcal{Z}_n$ are direct sums of simplicial sheaves of the kind $U \times \Delta^n$ for complex spaces U, because for any presheaf there is a surjection onto it from a direct sum of complex spaces. Moreover, for any simplicial presheaf $\mathcal{T}_.$, the canonical morphism hocolim $\mathcal{T}_n \to \mathcal{T}_.$ is a simplicial local weak equivalence. Since both domain and codomain are cofibrant objects (all objects are for the Joyal simplicial, injective, local model structure), by [9, Corollary 9.7.5.(2)], $\phi^* : \mathsf{Map}(\mathcal{X}, \mathcal{P}) \to \mathsf{Map}(\mathcal{Z}, \mathcal{P})$ is a simplicial weak equivalence. In turn this simplicial set is weakly equivalent to

$$\mathsf{Map}(\mathrm{hocolim}_n(\amalg_U U \times \Delta^n), \mathcal{P}) \cong \mathrm{holim}_n \mathsf{Map}(\amalg_U U \times \Delta^n, \mathcal{P}) = \mathrm{holim}_{n,U} \mathsf{Map}(U, \mathcal{P}).$$

Similarly,

$$\mathsf{Map}(\mathbb{C} \times \mathcal{X}, \mathcal{P}) \cong \mathrm{holim}_{n,U} \mathsf{Map}(\mathbb{C} \times U, \mathcal{P})$$

since finite limits commute with filtered colimits. Thus, if we knew that

$$\mathcal{P}(U) = \mathsf{Map}(U, \mathcal{P}) \cong \mathsf{Map}(\mathbb{C} \times U, \mathcal{P}) = \mathcal{P}(\mathbb{C} \times U)$$

for all complex spaces U, we would have that the homotopy limits are weakly equivalent and the condition (2) is verified. To prove that the condition (1) implies that the simplicial sets $\mathcal{P}(U)$ are weakly equivalent to $\mathcal{P}(\mathbb{C} \times U)$ for each U, we consider $\mathcal{X} = \mathsf{S}^n_s \wedge U_+$ and use the pointed version of the condition (1). We conclude that

$$\mathsf{Hom}_{\mathcal{H}_s}(\mathsf{S}^n_s \times U, \mathcal{P}) = \mathsf{Hom}_{\mathcal{H}_s}(\mathsf{S}^n_s \times \mathbb{C} \times U, \mathcal{P}).$$

On the other hand, by adjunction, we have

$$\begin{aligned} \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^n_s \wedge U_+, (\mathcal{P}, y)) &= \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^n_s, \mathsf{Map}_\bullet(U_+, (\mathcal{P}, y))) \\ &= \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^n_s, (\mathcal{P}(U), y)) \overset{def}{=} \pi_n^{\mathrm{simpl}}(\mathcal{P}(U), y). \end{aligned} \tag{7}$$

Similarly, we prove that

$$\mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^n_s \wedge U_+ \wedge \mathbb{C}_+, (\mathcal{P}, y)) = \pi_n^{\mathrm{simpl}}(\mathcal{P}(\mathbb{C} \times U), y),$$

hence the simplicial sets $\mathcal{P}(U)$ and $\mathcal{Y}(\mathbb{C} \times U)$ are weakly equivalent.

We have yet to prove that one of the conditions (1) or (2) is equivalent to the condition (3). For $\mathcal{P}$ this is equivalent to have the sets (groups, when applicable)

$$\begin{aligned}\pi_i^{\mathrm{simpl}}(\mathcal{P}, f)(U) &= \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^i_s \wedge U_+, (\mathcal{P}, y)) \\ &= \mathsf{Hom}_{\mathcal{H}_{s\bullet}}(\mathsf{S}^i_s, \mathsf{Map}_{\bullet}(U_+, (\mathcal{P}, y))) = \pi_i^{\mathrm{simpl}}(\mathcal{P}(U), y)\end{aligned} \tag{8}$$

isomorphic to $\pi_i^{\mathrm{simpl}}(\mathcal{P}(\mathbb{C} \times U), y)$, i.e., $p_U^* : \mathcal{P}(U) \to \mathcal{P}(\mathbb{C} \times U)$ is a weak homotopy equivalence, which is equivalent to condition (2) for what it has been previously proved. □

The notion of Brody hyperbolicity we will more frequently use is the (3) of the previous proposition.

Definition 4.4 1. A presheaf $\mathcal{P}$ is *Brody hyperbolic* if $p_U : \mathbb{C} \times U \to U$ induces bijections $p_U^* : \mathcal{P}(U) \to \mathcal{P}(\mathbb{C} \times U)$ for any complex space U.
2. A groupoid $\mathcal{G}$ is *Brody hyperbolic* if the holotopy presheaves $\pi_i^{\mathrm{simpl}}(\mathcal{G}, g)$ (Definition 3.1) are hyperbolic for all i, hence only for $i = 0, 1$, because of Theorem 2.1.

4.2 Kobayashi hyperbolicity

In the previous subsection we defined Brody hyperbolicity of a groupoid by first giving the same notion for a presheaf and then imposing that specification to the holotopy presheaves of the groupoid. The holotopy presheaves determine whether a groupoid is *Kobayashi hyperbolic*, as well. Classically (see [11]), Kobayashi hyperbolicity for complex spaces is a notion arising in the attempt to give complex spaces a biholomorphically invariant distance. In general the best that can be done is endowing complex spaces of a biholomorphic *pseudodistance*. When on a complex space X this happens to be a distance, X is said to be *Kobayashi hyperbolic*.

The notion of Kobayashi hyperbolicity for groupoids is based upon the concept of relative analytic disc.

4.2.1 Discs and analytic chains

Let $\mathbb{D}$ be the unitary open disc in $\mathbb{C}$. We recall that for a complex space U, we have denoted $\mathrm{Cov}\, U$ the set of countable, locally finite, open coverings $\mathcal{U} = \{U_i\}_i$ of U such that $U_i \Subset U$ for all i. If $\mathcal{U} \in \mathrm{Cov}\, U$ and $\mathcal{D} = \{D_a\}_a \in \mathrm{Cov}\, \mathbb{D}$ let $\{\mathcal{D} \times \mathcal{U}\}$ be the covering $\{D_a \times U_i\}_{ai}$ of $\mathbb{D} \times U$. The set $\mathrm{Cov}\, \mathbb{D} \times \mathrm{Cov}\, U$ is filtering in $\mathrm{Cov}\, \mathbb{D} \times U$.

Let $\mathcal{P}$ be a presheaf. A *relative analytic disc* of $\mathcal{P}$ on a complex space U is an object of $\mathsf{F} \in \mathcal{P}(\mathbb{D} \times U)$. For any $z \in \mathbb{D}$, the same letter will refer to the inclusion $\{z\} \times U \hookrightarrow \mathbb{D} \times U$. Let $r, s \in \mathcal{P}(U)$ be two sections and suppose there exists a relative analytic disc F and two points $z_1, z_2 \in \mathbb{D}$ such that $z_1^*\mathsf{F} = r$ and $z_2^*\mathsf{F} = s$. The sections r and s are then said to be *connected* by F. A *relative analytic chain* on U connecting r to s is the set $\mathsf{C}_{(r,s)}$ of the following data:

1. a collection $r_0 = r, \dots, r_k = s$ of sections;
2. $2k$ points $a_1, b_1, \dots, a_k, b_k$ in $\mathbb{D}$;

3. k relative analytic discs $\mathsf{F}_1, \ldots, \mathsf{F}_k$ such that the analytic disc F_i connects the sections r_{i-1} and r_i, i.e., $a_i^* \mathsf{F}_i = r_{i-1}$ and $b_i^* \mathsf{F}_i = r_i$ for all $1 \leq i \leq k$.

If a relative analytic chain $\mathsf{C}_{(r,s)}$ connects the section r with the section s, we call the pair (r, s) *admissible*. If $\mathcal{P} = Y$ is a complex space, admissibility of all section pairs in $\mathcal{P}(\mathsf{pt})$ is equivalent to the topological connectedness of Y.

Endow the unitary open disc $\mathbb{D}$ with the Poincaré metric

$$ds^2 = \frac{1}{(1-|z|^2)^2} dz \otimes d\bar{z}$$

and denote with $\varrho_{\mathbb{D}}(p, q)$ the associated distance function between two points a and b in $\mathbb{D}$. Then, for every chain $\mathsf{C}_{(r,s)}$ the nonnegative number

$$l(\mathsf{C}_{(r,s)}) = \varrho_{\mathbb{D}}(a_1, b_1) + \cdots + \varrho_{\mathbb{D}}(a_k, b_k) \tag{9}$$

is, by definition, the *(Kobayashi) length* of the relative analytic chain $\mathsf{C}_{(r,s)}$. If (r, s) is an admissible pair of sections, the nonnegative real number

$$\mathrm{d}^{\mathcal{P}}_{\mathrm{Kob}}(r, s) = \inf_{\mathsf{C}_{(r,s)}} l(\mathsf{C}_{(r,s)}) \tag{10}$$

defines a *pseudodistance* function on all the admissible pairs of sections in $\mathcal{P}(U)$ for all complex spaces U, called *Kobayashi pseudodistance* of $\mathcal{P}$.

Arguing as in the case of complex spaces it can be immediately seen that morphisms of presheaves decrease the Kobayashi pseudodistance.

Definition 4.5 A presheaf $\mathcal{P}$ is said to be *Kobayashi hyperbolic* if its Kobayashi pseudodistance is indeed a distance, hence if and only if $\mathrm{d}^{\mathcal{P}}_{Kob}(r, s) \neq 0$ for all admissible pairs $(r, s) \in \mathcal{P}(U)$ with $r \neq s$ and all complex spaces U.

The Kobayashi hyperbolicity for a groupoid is defined as follows:

Definition 4.6 A groupoid $\mathcal{G}$ is *Kobayashi hyperbolic* if the presheaves $\pi_0^{\mathrm{simpl}}(\mathcal{G}, g)$ and $\pi_1^{\mathrm{simpl}}(\mathcal{G}, g)$ are Kobayashi hyperbolic.

For general groupoids, we cannot go much further in expliciting what this amounts to; however, if $\mathcal{G}$ happens to be an analytic stack, the description of holotopy presheaves given in the Sect. 3 may be used to express the Kobayashi hyperbolicity in terms of holomorphic maps between complex spaces.

Thus, consider an analytic stack $p : X \to \mathcal{Y}$. Since Kobayashi hyperbolicity of $\mathcal{Y}$ is a prescription on the holotopy sheaves, we can replace $\mathcal{Y}$ with the stack $\mathcal{C}[X.]$, because of Proposition 3.3 and, by Theorem 3.6, this condition can be rephrased using descent data in X. In this case, a relative analytic disc is determined by a descent datum F on $\mathbb{D} \times U$ and therefore by an open covering $\mathcal{D} \times \mathcal{U}$ of $\mathbb{D} \times U$ and by holomorphic maps

$$F_{ai} : D_a \times U_i \to X, \;\; F_{ai,bj} : D_{ab} \times U_{ij} \to X_1 = X \times_{\mathcal{Y}} X,$$

where we wrote $D_{ab} \times U_{ij}$ for the set $(D_a \times U_i) \cap (D_b \times U_j)$. These maps satisfy the conditions $(\star)$ and $(\star\star)$ mentioned in the Sect. 3.1. Keeping the usual notation, this will be denoted as

$$\mathsf{F} = \mathsf{F}_{\mathcal{D}\times\mathcal{U}} = \big(F_{ai} : D_a \times U_i \to \mathbb{D} \times U, F_{aibj} : D_{ab} \times U_{ij} \to X_1\big) \tag{11}$$

with the coverings morphisms ranging among those of the kind $\amalg_{ai}(D_a \times U_i) \to \mathbb{D} \times U$.

The covering $\{D_a \times U_i\}_{ai}$ induces on $\{z\} \times U$ an open covering $\{U_{ai}\}_{ai}$ of $\{z\} \times U$ by setting $U_{ai} = (D_a \times U_i) \cap \{z\} \times U$ and the holomorphic maps F_{ai} and F_{aibj} restrict to holomorphic maps on U_{ai} and their intersections, satisfying ($\star$) and ($\star\star$). Notice that among the open subspaces $\{U_{ai}\}$ there may be some indexed by different pairs but are coincident as subspaces. A descent data $r = (r_i, f_{ij})$ on U relative to a covering $\{U_i\}_i$ determine a descent data r relative to the covering $\{U_{ai}\}_{ai}$ by letting $r_{ai} = r_i$ and $f_{aibj} = f_{ij}$.

Remark 4.1 1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of groupoids. Then for any complex space U and pair of admissible sections $r, s \in \pi_i^{\text{simpl}}(\mathcal{X}, x)(U)$, the sections $f_*(r)$ and $f_*(s)$ are admissible in $\pi_i^{\text{simpl}}(\mathcal{Y}, y)(U)$ and

$$\mathrm{d}^{\mathcal{Y}}_{\text{Kob}}(f_*(r), f_*(s)) \leq \mathrm{d}^{\mathcal{Y}}_{\text{Kob}}(r, s).$$

It follows that d_{Kob} is a simplicial, local weak equivalence invariant of stacks.

2. If $\mathcal{Y}$ is a complex space Y, then $\pi_i^{\text{simpl}} = 0$ for $i > 0$ and $\pi_0^{\text{simpl}}(\mathcal{Y}, y) \cong Y$ as (pre)sheaf. The definitions of Brody and Kobayashi hyperbolicity we have given involve all sections, holomorphic maps in this case, $U \to Y$ as opposed to just points $y \in Y$ as it is classically the case. It is easy to see that the relevant notions of hyperbolicity coincide.

(3) At the other extreme end, let $\mathcal{Y} = \mathcal{B}G$ be the classifying stack of a Lie group G. Then $X = \mathsf{pt}$ and G has to be finite, if we wish to restrict to Deligne–Mumford analytic stacks. Then there are no admissible pairs of sections in π_0^{simpl} and π_1^{simpl}, being similar to the case of a discrete complex space.

5 The coarse moduli space

Although the set $\mathcal{C}[X.](U) = \varinjlim_{\mathcal{U} \in \text{Cov}\, U} \mathsf{Dis}_{[X.]}(\mathcal{U})$ (see Lemma 3.5) is closely tied to the complex structure of an atlas X of an analytic stack $\mathcal{Y}$, it is unclear how to metrize it in a usable way and arguments employing complex variables theory, such as those necessary to prove Brody's theorem, seem not possible. Alternatively, it is possbile to link in a natural way this set (see the proof of Lemma 8.2) to a complex space, the coarse moduli space of $\mathcal{Y}$, which we will denote by $Q(\mathcal{Y})$.

5.1 Compactness

In the classical Brody theorem compactness is essential; similarly here we will need some finiteness condition on the stacks in order to proceed further.

Given an analytic stack with presentation $p_X : X \to \mathcal{Y}$, we will denote by j_X the holomorphic map

$$j_X = (\partial_0, \partial_1) : X_1 = X \times_{\mathcal{Y}} X \to X \times X.$$

Let $\mathcal{Y}$ be an analytic stack. An *open substack* $\mathcal{O} \subset \mathcal{Y}$ is a full category of $\mathcal{Y}$ such that

1. for any object $x \in \mathcal{O}$ all objects in $\mathcal{Y}$ isomorphic to x are also in $\mathcal{O}$;
2. the inclusion morphism $\mathcal{O} \subset \mathcal{Y}$ is represented by open immersions.

If every covering $\amalg_\alpha \mathcal{O}^\alpha \to \mathcal{Y}$ by open substacks admits a finite subcovering and for one presentation $W \to \mathcal{Y}$ the map j_W is proper, we will say $\mathcal{Y}$ is *compact by open coverings*.

The following three properties will serve to our purpose:

Proposition 5.1 *Let $X \to \mathcal{Y}$ be an analytic stack. The following are equivalent:*

1. *$\mathcal{Y}$ is compact by open coverings;*
2. *there exist two presentations $p_X : X = \amalg_{i=1}^{N} X^{(i)} \to \mathcal{Y}$, $p_Z : Z = \amalg_{i=1}^{M} Z^{(k)} \to \mathcal{Y}$ with $X^{(i)}$, $i = 1, \ldots, N$, $Z^{(k)}$, $k = 1, \ldots, M$ connected and a holomorphic map $\phi : X \to Z$ over $\mathcal{Y}$ such that $\phi|_{X^{(i)}}$ is an open embedding $X^{(i)} \hookrightarrow Z^{(k)}$ for some k with $\phi(X^{(i)}) \Subset Z^{(k)}$ and the map $j_Z : Z \times_{\mathcal{Y}} Z \to Z \times Z$ is proper.*

Under the equivalent conditions (1), (2) *we will say that $p_Z : Z \to \mathcal{Y}$ and $p_X : X \to \mathcal{Y}$ are* adapted *presentations.*

Proof Let $p_X : X = \amalg_{i=1}^{N} X^{(i)} \to \mathcal{Y}$, $p_Z : Z = \amalg_{i=1}^{N} Z^{(i)} \to \mathcal{Y}$ be adapted presentations $\mathcal{Y}$ and $\amalg_\alpha \mathcal{O}_\alpha \to \mathcal{Y}$ an open covering by substacks. Then $Z \times_{\mathcal{Y}} \mathcal{O}_\alpha \hookrightarrow Z$ is an open immersion of complex spaces for every α. It follows that $\amalg_\alpha (Z \times_{\mathcal{Y}} \mathcal{O}_\alpha)$ restricted to X is an open covering of X. If $\amalg_{j=1}^{n} (Z \times_{\mathcal{Y}} \mathcal{O}_{\alpha_j})$ covers $\overline{X}$, the closure of $\phi(X)$ in Z, then $\amalg_{j=1}^{n} \mathcal{O}_{\alpha_j} \to \mathcal{Y}$ is a covering of $\mathcal{Y}$.

Conversely, suppose that $\mathcal{Y}$ is compact by open coverings and let $p_W : \amalg_{i=1}^{+\infty} W^{(i)} \to \mathcal{Y}$ a presentation of $\mathcal{Y}$ with p_W open and $j_W : W \times_{\mathcal{Y}} W \to W \times W$ proper. Cover each $W^{(i)}$ by relatively compact balls and apply the shrinking lemma to obtain two presentations $p_{\hat{X}} : \hat{X} = \amalg_{i=1}^{+\infty} \hat{X}^{(i)} \to \mathcal{Y}$, $p_{\hat{Z}} : \hat{Z} = \amalg_{i=1}^{+\infty} \hat{Z}^{(i)} \to \mathcal{Y}$ with $\hat{X}^{(i)} \Subset \hat{Z}^{(i)}$, $i = 1, \ldots, N$. Consider the $\hat{X}_1 = \hat{X} \times_{\mathcal{Y}} \hat{X}$ saturations of the open spaces $\hat{X}^{(i)}$ (i.e., $\partial_1 \partial_0^{-1}(\hat{X}^{(i)})$) and call them $X^{(i)}$. Let $\mathcal{O}_i \subset \mathcal{Y}$ be the open substacks corresponding to them. Only finitely many of them, say $\{\mathcal{O}_{i_s} : j = 1, \ldots, N\}$ will be necessary to cover $\mathcal{Y}$, by assumption. Thus, $X := \amalg_{s=1}^{N} X^{(i_s)} \to \mathcal{Y}$ is an atlas. Doing the same with $\hat{Z}$ we produce an atlas $Z := \amalg_{k=1}^{M} Z^{i_k}$ of $\mathcal{Y}$. Since j_W is proper, $X^{(i_s)} \Subset Z^{(i_k)}$ for some k showing that X and Z are the sought for atlases.

This proves that conditions (1), (2) are equivalent. □

Definition 5.2 A stack satisfying either of the conditions of the proposition 5.1 will be called *compact*.

Proposition 5.3 *1. If a stack admits a compact atlas, the stack is compact;*
2. *compactness is invariant by equivalence of stacks;*
3. *if $\mathcal{Y}$ is a compact stack the diagonal morphism $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is proper.*

Proof Only the third statement requires a proof. Fix two adapted presentations $p_Z : Z \to \mathcal{Y}$ and $p_X : X \to \mathcal{Y}$ and consider the diagram

$$\begin{array}{ccc} S \times_{\mathcal{Y}\times\mathcal{Y}} \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow f & & \downarrow \Delta \\ S & \xrightarrow{(a,b)} & \mathcal{Y} \times \mathcal{Y} \end{array}$$

where S is a complex space. Since Δ is representable, showing that it is proper amounts to showing that f is a proper holomorphic morphism. Let $\{\xi_n = (s_n, y_n, \phi_n)\}$ be a sequence of points in $S \times_{\mathcal{Y}\times\mathcal{Y}} \mathcal{Y}$. Assume that $\{s_n\}$ converge to a point $\underline{s} \in S$. We are going to prove that there exists a subsequence of $\{\xi_n\}$ converging in $S \times_{\mathcal{Y}\times\mathcal{Y}} \mathcal{Y}$. We have a commutative diagram of complex spaces and holomorphic maps

$$\begin{array}{ccc} S \times_{\mathcal{Y}\times\mathcal{Y}} X & \xrightarrow{i} & S \times_{\mathcal{Y}\times\mathcal{Y}} Z \\ & \searrow^{id\times p_X} & \downarrow id\times p_Z \\ & & S \times_{\mathcal{Y}\times\mathcal{Y}} \mathcal{Y} \end{array}$$

by base changing over S the diagram of the adapted presentations. Both $id \times p_X$ and $id \times p_Z$ are surjective morphisms being base changes of essentially surjective functors. Lift $\{\xi_n\}$ to a sequence $\{u_n = (s_n, x_n, \phi_n)\} \in S \times_{\mathcal{Y}\times\mathcal{Y}} X$. Since $X \Subset Z$, there exists a subsequence $\{x_m\}$ converging to $\underline{x} \in Z$. The isomorphisms ϕ_n are isomorphisms between the object $(a(s_n), b(s_n))$ and (y_n, y_n) in $\mathcal{Y} \times \mathcal{Y}$, thus $\phi_n = (\phi'_n, \phi''_n)$ where ϕ'_n and ϕ''_n are isomorphisms of $\mathcal{Y}$. As such, they can naturally be seen as points of $X_1 = X \times_{\mathcal{Y}} X$ and of Z_1. Since $X \times X \Subset Z \times Z$, there exist subsequences of $\{j_X(\phi'_n)\}$ and $\{j_X(\phi''_n)\}$ converging in $Z \times Z$ ($j_X : X_1 = X \times_{\mathcal{Y}} X \to X \times X$ is the canonical map). Properness of j_Z, assumed by the compactness of $\mathcal{Y}$, implies that there exists a subsequence of ϕ_n converging to some point $\underline{\phi} \in Z_1$. Thus, $\{u_m\}$ converges to a point $\underline{u} = (\underline{s}, \underline{x}, \underline{\phi}) \in S \times_{\mathcal{Y}\times\mathcal{Y}} Z$. The point $id \times p_Z(\underline{u})$ is a limit point for ξ_n. □

5.2 Existence of the complex structure

In the paper [2] we have proved that to a flat analytic groupoid $\mathcal{X} = \{s, t : R \rightrightarrows X\}$ with $j = (s, t) : R \to X \times X$ finite (or equivalently j closed and $\mathcal{X}$ with finite stabilizer) is associated a GC quotient. This is a complex space $Q(\mathcal{X})$ with an holomorphic map $q : X \to X/R \to Q(\mathcal{X})$ satisfying a number of conditions. Here X/R is the quotient sheaf associated with the equivalence relation induced by R. That result applies to the analytic groupoid $\mathcal{X} = \left\{X \times_{\mathcal{Y}} \mathcal{X} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} X\right\}$ where $p_X : X = \amalg_{i=1}^{N} X^{(i)} \to \mathcal{Y}$ is a presentation of the compact Deligne–Mumford analytic stack $\mathcal{Y}$.

We can state another property of a stack equivalent to compactness:

Proposition 5.4 *Let $p : X \to \mathcal{Y}$ be an analytic stack with p open and admitting a coarse moduli space $Q(\mathcal{Y})$. Then $\mathcal{Y}$ is compact if and only if $Q(\mathcal{Y})$ is a compact complex space and j is proper for some presentation. In that case the diagonal morphism $\Delta_Q : \mathcal{Y} \to \mathcal{Y} \times_{Q(\mathcal{Y})} \mathcal{Y}$ is proper.*

Proof If $\mathcal{Y}$ is compact, $Q(\mathcal{Y})$ is compact as well. Conversely, let $W \to \mathcal{Y}$ be a presentation. As in the proof of proposition 5.1 we can always produce two atlases $A = \amalg_{i=1}^{\infty} A^{(i)}$ and $B = \amalg_{j=1}^{\infty} B^{(j)}$ satisfying all the conditions for the compactness of $\mathcal{Y}$ except the finiteness of the number of the connected components.

Let $q_A : A \to Q(\mathcal{Y})$ and $q_B : B \to Q(\mathcal{Y})$ be the canonical holomorphic maps and $\{U_k : k = 1, \ldots, N\}$ a finite subcover of $\{q_A(A^{(i)}), q_B(B^{(j)})\}_{i,j}$. Let $X^{(k)}$ be the A^{-1}-saturated of the open set $q_A^{-1} U_k$ and $Z^{(k)}$ be the B_1-saturated of $q_B^{-1} U_k$. Then $\amalg_{k=1}^{N} X^{(k)}$ and $\amalg_{k=1}^{N} Z^{(k)}$ are two atlases with the required properties. Indeed, each $X^{(k)}$ and $Z^{(k)}$ are the atlas of an open substack of $\mathcal{Y}$ and the union of them is surjective on $\mathcal{Y}$.

In order to prove that Δ_Q is proper set $Q = Q(\mathcal{Y})$ and consider the following diagram

$$\begin{array}{ccccc}
S \times_{(\mathcal{Y}\times_Q\mathcal{Y})} \mathcal{Y} & \longrightarrow & \mathcal{Y} & \xrightarrow{\text{id}} & \mathcal{Y} \\
\downarrow & & \downarrow{\scriptstyle \Delta_Q} & & \downarrow{\scriptstyle \Delta} \\
S & \longrightarrow & \mathcal{Y} \times_Q \mathcal{Y} & \longrightarrow & \mathcal{Y} \times \mathcal{Y}
\end{array}$$

where S is a complex space. Then one checks that the right side square is cartesian, i.e., $\mathcal{Y}$ is equivalent to the fiber product of $\mathcal{Y}$ and $\mathcal{Y} \times_Q \mathcal{Y}$ over $\mathcal{Y} \times \mathcal{Y}$; thus, properness of Δ_Q follows from that of Δ (see Remark 5.3). □

6 Topological and metric structures

6.1 Distances

For the time being, we will only consider compact, Deligne–Mumford analytic stacks. Let $\mathcal{Y}$ be such an analytic stack with atlases $X \Subset Z$, $X = \amalg_{i=1}^{N} X^{(i)}$, $Z = \amalg_{i=1}^{N} Z^{(i)}$ (see Sect. 5). We can assume X_i and Z_i are Stein.

Fix a distance $\mathrm{d} : Q(\mathcal{Y}) \times Q(\mathcal{Y}) \to \mathbb{R}_{\geq 0}$ on the quotient space $Q(\mathcal{Y})$ determined by a differentiable length function H. As we have previously introduced, $q = q_X$ is the projection $X \to Q(\mathcal{Y})$. Even if q^*H is only a pseudolength function, to it is associated a distance on any connected component of X, because q has locally proper and equifinite fibers. This is the same distance induced by the restriction of q_Z^*H to X. These distances on the connected components can be assembled together to a unique distance d_X on all X in the following way. Fix two points $x \neq y \in X$; a (piecewise differentiable) path γ through x and y is a set $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ of paths $[0, 1] \to X$ such that:

1. $\gamma_1(0) = x$ and $\gamma_m(1) = y$;
2. $q(\gamma_{i+1}(0)) = q(\gamma_i(1))$ for all $1 \leq i \leq m + 1$.

If $l(\gamma_i)$ denotes the length of γ_i with respect to the distance on the connected component of X in which the image of γ_i lies, the positive real number $l(\gamma) = \sum_{i=1}^{m} l(\gamma_i)$ is the length of γ by definition. We then set

$$\mathrm{d}_X(x, y) = \inf_{\gamma} l(\gamma).$$

We proceed likewise for the complex space $X_1 = X \times_{\mathcal{Y}} X$, considering on X_1 the length functions $(q \circ \partial_0)^*H$, $(q \circ \partial_1)^*H$ which give rise to the same distance making ∂_0 and ∂_1 into local isometries, since $q \circ \partial_0 = q \circ \partial_1$. A similar argument applies to $X_2 := X_1 \times_{\partial_0, X, \partial_1} X_1$ and to the multplication $m : X_2 \to X_1$ that becomes a local isometry, as well.

The distance just introduced allows to metrize in a natural way the sets $\mathsf{Dis}_{[X.]}(U)$, $\pi_0(\mathcal{Y}, y)(U)$ and $\pi_1(\mathcal{Y}, y)(U)$ as follows: let $r = (r_i, f_{ij}) \neq s = (s_i, g_{ij})$ two descent data in $\mathsf{Dis}_{[X.]}(\mathcal{U})$, with $\mathcal{U} \in \mathrm{Cov}\, U$; define

$$\delta_U(r, s) = \sup_{\substack{u \in U_i \\ i \in \mathbb{N}}} \big\{\mathrm{d}_X(r_i(u), s_i(u))\big\} + \sup_{\substack{u \in U_{ij} \\ i,j \in \mathbb{N}}} \big\{\mathrm{d}_{X_1}(f_{ij}(u), g_{ij}(u))\big\}. \tag{12}$$

The distance function $\delta_U(r, s)$ is invariant by restriction, that is $\delta_U(r, s) = \delta_U(r_{|\mathcal{U}'}, s_{|\mathcal{U}'})$ for any $\mathcal{U}' \succeq \mathcal{U}$, thus it is defined for pairs of objects in $\mathsf{Ob}(\mathcal{C}[X.](U))$. In turn it induces a distance function $\delta\pi_0, U$ on $\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$: for $\alpha, \beta \in \pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$

$$\delta_{\pi_0, U}(\alpha, \beta) = \inf_{r \in \alpha, s \in \beta} \delta_U(r, s). \tag{13}$$

For $\pi_1^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$ we proceed similarly. Given $\mathcal{U} = \{U_i\}_i \in \mathrm{Cov}\, U$, the elements of $\pi_1^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$ are represented by pairs $[r, \phi]$ of a descent datum r and $\phi = \{\phi_i\}_i$ is an automorphism of r (see Theorem 3.7). The distance δ between two such pairs (r, ϕ) and (s, ψ) is defined as

$$\delta'_U\big((r, \phi), (s, \psi)\big) = \delta(r, s) + \sup_{\substack{u \in U_i \\ i \in \mathbb{N}}} \big\{\mathrm{d}_{X_1}\big(\phi_i(u), \psi_i(u)\big)\big\} \tag{14}$$

and the distance between two classes $\alpha, \beta \in \pi_1^{\text{simpl}}(\mathcal{Y}, y)(U)$ is

$$\delta_{\pi_1,U}(\alpha, \beta) = \inf_{\substack{(r,\phi)\in\alpha \\ (s,\psi)\in\beta}} \delta'_U\big((r, \phi), (s, \psi)\big). \tag{15}$$

Proposition 6.1 *$\delta_{\pi_0,U}$ e $\delta_{\pi_1,U}$ are distances.*

Proof Let us introduce the following notation: $\widetilde{r} \neq \widetilde{s}$ are two classes in $\pi_0^{\text{simpl}}(\mathcal{Y}, y)(U)$ for a fixed complex space U and descent data $r = (r_i, f_{ij})$ and $s = (s_i, g_{ij})$ in X. It suffices to show that $\delta_{\pi_0,U}(\widetilde{r}, \widetilde{s}) > 0$, since the other axioms of being a distance are readily seen to be satisfied. Recall that, given a descent datum r, the composition $q \circ r$ denotes the well-defined holomorphic map $U \to Q(\mathcal{Y})$ defined locally by $q \circ r_i$ and gluing all these maps over U. We split the proof in two complementary cases: $q \circ r \neq q \circ s$, $q \circ r = q \circ s$.
$q \circ r \neq q \circ s$. Under this assumption we have that $\sup_{u\in U_i} \{\mathrm{d}_X (r_i(u), s_i(u))\} \geq N > 0$: let $V \subset U$ be an open set such that $\sup_{v\in V} \left\{\mathrm{d}_{Q(\mathcal{Y})}(q \circ r(v), q \circ s(v))\right\} = N > 0$. Then

$$\delta_U(r, s) \geq \sup_{\substack{u\in U_i \\ i\in\mathbb{N}}} \{\mathrm{d}_X (r_i(u), s_i(u))\} \geq N > 0$$

since the length function on X inducing d_X is q^*H, where H is a length function on $Q(\mathcal{Y})$, hence $\mathrm{d}_X(x_1, x_2) \geq \mathrm{d}_{Q(\mathcal{Y})}(q(x_1), q(x_2))$ for any $x_1, x_2 \in X$. To get the same inequality with $\delta_{\pi_0,U}(r, s)$ replacing $\delta_U(r, s)$, notice that $q \circ r' = q \circ r$ and $q \circ s' = q \circ s$, for any $r' \in \widetilde{r}$ and $s' \in \widetilde{s}$.
$q \circ r = q \circ s$. In this case, both of the summands in Eq. (12) will play a role. The first step is to show that r and s are strictly related: □

Lemma 6.2 *Given two descent data r and s relative to a covering $\mathcal{U}' = \{U'_i\}_i$ of a complex space U, such that $q \circ r = q \circ s$, there is a refinement $\mathcal{U} = \{U_i\}_i \succeq \mathcal{U}'$ and collection of holomorphic maps $\phi_i : U_i \to X_1 = X \times_{\mathcal{Y}} X$ related to r and s by the equations $\partial_0 \circ \phi_i = r_i$ and $\partial_1 \circ \phi_i = s_i$.*

Proof By definition of $Q(\mathcal{Y})$, there is a set-theoretic map $h_i : U'_i \to X_1$ such that $\partial_0 \circ h_i = r_i$ and $\partial_1 \circ h_i = s_i$. By possibly refining $\mathcal{U}'$ to $\mathcal{U}$, we may suppose that

1. $r_i(U_i)$ is contained in a ∂_0 uniformly covered open subspace $V_i \subset X$ for each i and
2. each U_i is simply connected.

Let $\partial_0^{-1}(V_i) = \amalg_{k=1}^m W_{ik}$. Then we can partition U_i in N subsets $S_k = h_i^{-1}(W_{ik})$ and there exists an a such that $\overline{S_a} \subset U_i$ has an nonempty interior. Define $\phi_i : U_i \to W_{i,a} \subset X_1$ to be the ∂_0 lifting of r_i. Such a holomorphic map satisfies the equations $\partial_0 \circ \phi_i = r_i$ on U_i, by construction, and $\partial_1 \circ \phi_i = s_i$ on $\overline{S_a}$, hence over all U_i, given the property of such a closed subspace and that the equality is between holomorphic maps. □

Notice that the collection $(\phi_i)_i$ does not need to be an isomorphism between the descent data r and s, the classifying stacks $\mathcal{B}G$ providing an example of this.

Remark 6.1 We will need a sharper version of the lemma just proved: by partitioning $U_i \cap U_j$, if such intersection is nonempty, we may assume that $\overline{S_a}$ is contained in $U_i \cap U_j$. Therefore, it is possible to lift simultaneously r_i to $\phi_i : U_i \to W_{ia}$ and r_j to $\phi_j : U_j \to W_{ia}$, both ϕ_i and ϕ_j satisfying the properties of the lemma.

We will assume the covering $\mathcal{U}$ associated to the descent data r and s enjoys the properties stated in the previous lemma, whose notation we are going to share. There are yet two distinct cases to be considered. Recall that $A \subset X$ is the subspace of q ramification points (see Sect. 5).

$V_i \cap A = \emptyset$. We may assume that $\partial_1(W_{ik_1}) \cap \partial_1(W_{ik_2}) = \emptyset$ if $k_1 \neq k_2$ (by possibly further refining $\mathcal{U}$). Denote by M the minimum of the distances between pairs of open subspaces $\partial_1(W_{ik})$, k ranging from 1 to m. We may assume $M > 0$, by shrinking V_i a little if necessary. Now, $r \nsim s$, so that in particular $r \neq s$, and we may assume $r_i \neq s_i$ as holomorphic maps $U_i \to X$. Since $q \circ r = q \circ s$ and because of Lemma 6.2, we conclude that

$$\delta_U(r, s) \geq \sup_{u \in U_i} \left\{ \mathrm{d}_X \left(r_i(u), s_i(u)\right) \right\} = \sup_{u \in U_i} \left\{ \mathrm{d}_X(\partial_1 \circ \phi_i(u), \partial_1 \circ \phi_i(u)) \right\} \geq M > 0$$

the same statement can be deduced for $\delta_{\pi_0, U}$ as the same argument applies for any $r' \sim r$ and $s' \sim s$ provided their associated open covering is $\mathcal{U}$ or finer.

$V_i \cap A \neq \emptyset$. In this case the number $\sup_{u \in U_i} \{\mathrm{d}_X (r_i(u), s_i(u))\}$ could possibly be arbitrarily small. However, the only way for $\sup_{u \in U_{ij}} \left\{\mathrm{d}_{X_1} \left(f_{ij}(u), g_{ij}(u)\right)\right\}$ to be arbitrarily small is if $f_{ij}(U_{ij})$ and $g_{ij}(U_{ij})$ both belong to W_{ih} for some $1 \leq h \leq m$, since otherwise that number would be greater or equal than $M' = \min_{h_1 \neq h_2} \{\mathrm{d}_{X_1}(W_{ih_1}, W_{ih_2})\}$ which can be assumed to be strictly greater than zero. Let now fix i, j so that $U_i \cap U_j \neq \emptyset$ and consider the worst case scenario, i.e., $f_{ij}(U_{ij})$ and $g_{ij}(U_{ij})$ lying in the same W_{ih}. We claim that the pair (ϕ_i, ϕ_j) forms an isomorphism between $r_{|U_i \cap U_j}$ and $s_{|U_i \cup U_j}$. That being the case, since $r \nsim s$, i.e., r is not isomorphic to s, there must be $U_{cd} \neq \emptyset$ such that $f_{cd}(U_{cd}) \subset W_{ih_1}$ and $g_{cd}(U_{cd}) \subset W_{ih_2}$ with $h_1 \neq h_2$. This implies that

$$\delta_U(r, s) \geq \sup_{\substack{u \in U_{ij} \\ i, j \in \mathbb{N}}} \left\{ \mathrm{d}_{X_1}(f_{ij}(u), g_{ij}(u)) \right\} \geq M' > 0$$

finishing the proof of the proposition.

In order to prove that the pair (ϕ_i, ϕ_j) is an isomorphism, it suffices to show that $m(f_{ij}, \phi_j) = m(\phi_i, g_{ij})$ seen as holomorphic maps defined on $U_i \cap U_j$ and with values in X_1. This statement follows from knowing that the images of $m(f_{ij}, \phi_j)$ and $m(\phi_i, g_{ij})$ lie in the same W_{ib}: indeed, this condition on the images implies that ∂_0 is injective when restricted these images and, since $\partial_0 \circ m(f_{ij}, \phi_j) = r_{i\,|U_i \cap U_j} = \partial_0 \circ m(\phi_i, g_{ij})$, we conclude that the map $m(f_{ij}, \phi_j)$ must necessarily be the same as $m(\phi_i, g_{ij})$. Showing that the images of $m(f_{ij}, \phi_j)$ and $m(\phi_i, g_{ij})$ belong to W_{ih} can be further reduced to the fact (see Remark 6.1) that the ones of ϕ_i and ϕ_j are both contained in W_{ia}. This follows by the continuity of m: we know that the images of f_{ij} and g_{ij} are in W_{ih} and, by the continuity of m we have that the maps $z \mapsto m(w_a, z)$ and $z \mapsto m(z, w_h)$, for $w_a \in W_{ia}$ and $w_h \in W_{ih}$, send W_{ih} and W_{ia}, respectively, to the same W_{ib}.

Consider now two classes $[(r, \phi)] \neq [(s, \psi)] \in \pi_1^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$. These two classes being different, in particular it implies that $\widetilde{r} \neq \widetilde{s} \in \pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$ (see Theorem 3.7). What already proved for $\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$, shows that the $\delta_U(r, s)$ summand in the formula (14) is going to be nonzero, hence $\delta_{\pi_1, U}\big([(r, \phi)], [(s, \psi)]\big) > 0$. □

6.2 The function $c(\mathcal{Y}, -)$

The function $c(\mathcal{Y}, -)$ associates to every complex space a positive number (possibly $+\infty$). It is defined as follows. Let U be a complex space and

$$\mathsf{F} = \mathsf{F}_{\mathcal{D} \times \mathcal{U}} = \left(F_{ai} : D_a \times U_i \to \mathbb{D} \times U,\ F_{aibj} : D_{ab} \times U_{ij} \to X_1\right)$$

be a relative analytic disc of $\mathcal{Y}$ on U. We associate a function $\mathbb{D} \to \mathbb{R}_{\geq 0}$ to F as follows. Take $z \in \mathbb{D}$ and a vector $v \in T\mathbb{D}$, the holomorphic tangent bundle on $\mathbb{D}$, and consider $D_a \times U_i$ with $z \in D_a$. Then, denoting with $\mathrm{d}F_{ai}$ the differential of the holomorphic map F_{ai}, $\mathrm{d}F_{ai}(z,\xi)v$ is a vector field tangent to X along the points of the image of $(F_{ai})_{z\times U_i}$, ξ ranging in U_i. Since the maps ∂_0, ∂_1 and m are local isometries (see Sect. 6.1), by differentiating with respect of the variable z the structural equations of F, $(\star)$ and $(\star\star)$ of Sect. 3.1, we notice that the real number

$$|\mathrm{d}F_{ai}(z,\xi)v| := q^*H(dF_{ai}(z,\xi)v)$$

only depends on ξ, z, v and not on the open subspaces $D_a \times U_i$ [1], so $(z,\xi,v) \mapsto |\mathrm{d}F_{ai}(z,\xi)v|$ is a well-defined real-valued continuous function which will be occasionally written as $|\mathrm{d}\mathsf{F}(z,\xi)v|$. Moreover, for any $(z,\xi) \in D_{ab} \times U_{ij}$ we have that

$$|\mathrm{d}F_{ai,bj}(z,\xi)v| = |\mathrm{d}F_{ai}(z,\xi)v|. \tag{16}$$

If $K \subset U$ is a compact subset and $z \in \mathbb{D}$, we define

$$|\mathrm{d}\mathsf{F}(z)|_K = \sup_{\xi\in K} \sup_{\substack{v\in T_z\mathbb{D}\\ v\neq 0}} \frac{|\mathrm{d}\mathsf{F}(z,\xi)v|}{|v|_{\mathrm{hyp}}}, \tag{17}$$

where $|v|_{\mathrm{hyp}}$ is the length induced by the Poincaré metric on $\mathbb{D}$, and

$$c_K(\mathcal{Y};\mathcal{U}) = \sup_{\substack{\mathsf{F}\in\mathcal{Y}(\mathcal{D}\times\mathcal{U})\\ \mathcal{D}\in\mathrm{Cov}\,\mathbb{D}}} \sup_{z\in\mathbb{D}} |\mathrm{d}\mathsf{F}(z)|_K. \tag{18}$$

Because of the transitivity of the action of $\mathrm{Aut}(\mathbb{D})$ and the invariance of the Poincaré metric we conclude that

$$c_K(\mathcal{Y};\mathcal{U}) = \sup_{\substack{\mathsf{F}\in\mathcal{Y}(\mathcal{D}\times\mathcal{U})\\ \mathcal{D}\in\mathrm{Cov}\,\mathbb{D}}} |\mathrm{d}\mathsf{F}(0)|_K. \tag{19}$$

We finally define

$$c(\mathcal{Y};U) = \sup_{\substack{K\Subset U\\ \mathcal{U}\in\mathrm{Cov}\,U}} c_K(\mathcal{Y};\mathcal{U}). \tag{20}$$

Since ∂_0 and ∂_1 are local isometries, if the relative analytic disc

$$\mathsf{G} = \mathsf{G}_{\mathcal{D}\times\mathcal{U}} = \left(G_{ai} : D_a \times U_i \to \mathbb{D}\times\mathbb{C},\, G_{aibj} : D_{ab}\times U_{ij} \to X_1\right) \tag{21}$$

is equivalent to F, i.e., they have the same class in $\pi_0^{\mathrm{simpl}}(\mathcal{Y},y)(\mathbb{D}\times U)$, the functions $|\mathrm{d}\mathsf{F}(z,\xi)v|$ and $|\mathrm{d}\mathsf{G}(z,\xi)v|$ coincide.

Given a pair $[\mathsf{F},\Phi]$ representing a class of $\pi_1^{\mathrm{simpl}}(\mathcal{Y},y)(\mathbb{D}\times U)$, where F is a relative analytic disc on U, and $\Phi = \{\Phi_{ai}\}_{ai}$ is an automorphism of F (see Sect. 3.1), for each compact set $K \subset U$ we have well-defined functions $|\mathrm{d}\mathsf{F}(z)|_K$ and $|\mathrm{d}\Phi(z)|_K$, as in Eq. (17). Keeping in mind the Theorem 3.7, we deduce that $|\mathrm{d}\mathsf{F}(z)|_K = |\mathrm{d}\Phi(z)|_K$. If $[\mathsf{G},\Psi]$ is a pair equivalent to $[\mathsf{F},\Phi]$, that is their images coincide in $\pi_1^{\mathrm{simpl}}(\mathcal{Y},y)(\mathbb{D}\times U)$, then

$$|\mathrm{d}\,\mathsf{F}(z)|_K = |\mathrm{d}\,\Phi(z)|_K = |\mathrm{d}\,\Psi(z)|_K = |\mathrm{d}\,\mathsf{G}(z)|_K. \tag{22}$$

Remark 6.2 Because of the last observations, the vanishing of the H-norm of the derivative of a descent data or of one of their isomorphisms is equivalent to their relevant classes in π_0^{simpl} or π_1^{simpl} being constant (see Definition 3.8).

Under the same notation as in Sect. 6.1, we prove the following fundamental

[1] We recall that H denotes the fixed length function on $Q(\mathcal{Y})$.

Lemma 6.3 *Let* $\alpha_1, \alpha_2 \in \pi_i^{simpl}(\mathcal{Y}, y)(U)$, *for* $i = 0, 1$ *and* $\mathsf{C}_{(alpha_1, \alpha_2)}$ *be an analytic chain through* α_1 *and* α_2. *Then*

$$\delta_{\pi_i, U}(\alpha_1, \alpha_2) \leq 2\, c(\mathcal{Y}; U)\, l(\mathsf{C}_{(\alpha_1, \alpha_2)}) \tag{23}$$

(see Eq. 9) for the definition of length of an analytic chain. In particular,

$$\mathrm{d}_{\mathrm{Kob}}(\alpha_1, \alpha_2) \geq \frac{\delta_{\pi_i, U}(\alpha_1, \alpha_2)}{2\, c(\mathcal{Y}; U)}. \tag{24}$$

for $i = 0, 1$ *where the left end side has been defined in Eq.* (10) *with* $\mathcal{P} = \mathcal{Y}$.

Proof It is sufficient to prove these statements for analytic discs instead of chains. Let $\alpha_1 \neq \alpha_2 \in \pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$ and $r = (r_i, f_{ij}) \neq s = (s_i, g_{ij})$ be two descent data in α_1, α_2, respectively. Let $\mathsf{F} = \left(F_{ai}, F_{aibj}\right)$ be an analytic disc such that $r = z_1^* \mathsf{F}$ and $s = z_2^* \mathsf{F}$ for z_1 and z_2 points which we initially suppose are in a D_a open of a covering of $\mathbb{D}$ associated to F. Using the notation introduced in Sect. 4.2.1, in particular we have that $F_{ai}(z_1, u) = r_{ai}(u)$ and $F_{ai}(z_2, u) = s_{ai}(u)$. Let $\gamma : [0, 1] \to D_a$ a geodesic arc in $\mathbb{D}$, endowed by the Poincaré metric, such that $\gamma(0) = z_1$, $\gamma(1) = z_2$ and, for $u \in U$ fixed, we call γ_u the path $t \to F_{ai}(\gamma(t), u)$. For each $u \in U_i$ we have

$$\begin{aligned} \mathrm{d}_X(r_{ai}(u), s_{ai}(u)) \leq \mathrm{length}(\gamma_u) &= \int_0^1 H\left(\mathrm{d}F_{ai}(\gamma(t), u) \cdot \gamma'(t)\right) \mathrm{d}t \\ &\leq c(\mathcal{Y}; U) \int_0^1 |\gamma'(t)|_{\mathrm{hyp}} \mathrm{d}t = c(\mathcal{Y}; U) \varrho_{\mathbb{D}}(z_1, z_2) \end{aligned} \tag{25}$$

The same argument applies to the transition functions f_{aibj} and g_{aibj} and, using the remark in Eq. (16), we get

$$\begin{aligned} \mathrm{d}_{X_1}(f_{aibj}(u), g_{aibj}(u)) \leq \mathrm{length}(\gamma_u) &= \int_0^1 H\left(\mathrm{d}F_{aibj}(\gamma(t), u)(\gamma'(t))\right) \mathrm{d}t \\ &\leq c(\mathcal{Y}; U) \int_0^1 |\gamma'(t)|_{\mathrm{hyp}} \mathrm{d}t = c(\mathcal{Y}; U) \varrho_{\mathbb{D}}(z_1, z_2). \end{aligned} \tag{26}$$

Taking the supremum with respect to u and the indices i, ai, we deduce from (25) and (26) that

$$\delta_U(r, s) \leq 2c(\mathcal{Y}; U) \varrho_{\mathbb{D}}(z_1, z_2)$$

which implies

$$\delta_{\pi_0, U}(\alpha_1, \alpha_2) = \inf_{\substack{r \in \alpha \\ s \in \beta}} \delta_U(r, s) \leq 2c(\mathcal{Y}; U) \varrho_{\mathbb{D}}(z_1, z_2).$$

In the general case, when z_1 and z_2 do not belong to the same open D_a, we partition $\gamma([0, 1]) \subset \mathbb{D}$ with points $z_1 = \zeta_1 = \gamma(t_1), \zeta_2 = \gamma(t_2), \ldots, \zeta_m = z_2 \gamma(1)$ so that the geodetic arcs $\gamma(\zeta_l, z_{l+1})$ are contained in $D_{a_0}, D_{a_1}, \ldots D_{a_{m-1}}$, respectively, for $0 \leq l \leq m - 1$. Then we argue as before on each arc. This proves th Eq. (23).

The statement regarding $\pi_1^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$ is proved in the same fashion and by using the equalities in (22). □

7 Kobayashi hyperbolicity implies Brody hyperbolicity

We can now proceed with the proof of the Brody theorem for stacks. Classically, that theorem refers to the implication "compactness and Brody hyperbolicity imply Kobayashi hyperbolicity", the other implication holding in general and simply a consequence of non- Kobayashi hyperbolicity of $\mathbb{C}$ and that every holomorphic map is a contraction with respect to the Kobayashi presudodistance. In the context of stacks, even this simple implication, though not using much of the theory developed in the last sections, is not entirely obvious and this section is devoted to its proof.

Theorem 7.1 *Let $\mathcal{Y}$ be a Kobayashi hyperbolic analytic stack. Then $\mathcal{Y}$ is Brody hyperbolic.*

Proof Suppose that $\pi_0^{\text{simpl}}(\mathcal{Y}, y)$ is not Brody hyperbolic; then there exists a section $r \in \pi_0^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C} \times U)$ not in the image of $p^* : \pi_0^{\text{simpl}}(\mathcal{Y}, y)(U) \to \pi_0^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C} \times U)$, p being the projection. Let us introduce the notation $r = (r_i(z, u), f_{ij}(z, u))$ over a covering $\{V\}_i$ of $\mathbb{C} \times U$; notice that we may take V_i to be of the kind $D_{k(i)} \times U_{\alpha(i)}$ for a disc $D_{k(i)}$ centered at some complex number $z \in \mathbb{C}$ and open set $U_{\alpha(i)}$ of U. We wish to construct two sections, or objects, $r_1 \ncong r_2 \in \text{Ob}(\mathcal{C}[X.](V))$ for some complex space V, whose Kobayashi pseudodistance is zero. Take $V = \mathbb{C} \times U$ and consider the two sections: $r_1 = s$ and $r_2 = p^*(i_0^*(s))$, where $i_0 : U \to \mathbb{C} \times U$ is the embedding in zero. By assumption, $r_1 \ncong r_2$. To show that the Kobayashi pseudodistance between r_1 and r_2 is zero, we construct relative analytic chains (see Sect. 4.2.1) s_n^λ between them for $n \in \mathbb{N}$, where $\lambda \in \mathbb{D}_2 = \{z \in \mathbb{C} : |z| < 2\}$. For each n, s_n^λ is, in fact, a relative analytic disc. We define first the descent data s_1^λ on $\mathbb{D}_2 \times \mathbb{C} \times U$; to describe the open covering we use its cross sections for λ =const as $t_{1/\lambda}(D_{k(i)}) \times U_{\alpha(i)}$, where $t_\omega : \mathbb{C} \to \mathbb{C}$ is the automorphism $z \to \omega z$. For $\lambda = 0$ the open covering of $\mathbb{C} \times U$ is $\{\mathbb{C} \times U_{\alpha_0}\}_{\alpha_0}$, where $\mathcal{W} = \{U_{\alpha_0}\}_{\alpha_0}$ is the covering of U given by the open sets appearing in the covering of $\{0\} \times U$ induced by the V_i. The holomorphic map is $(\lambda, z, u) \mapsto r_i(t_{1/\lambda}^{-1}(z), u) = (\lambda z, u)$ for $z \in t_{1/\lambda}(D_{k(i)})$ and $u \in U_{\alpha(i)}$. The transition functions are similarly defined as $(\lambda, z, u) \mapsto f_{ij}(\lambda z, u)$. Notice that $s^1 = r_1$ and $s^0 = r_2$. The general relative analytic disc s_n^λ is associated to the open covering having open sets $\{t_{\frac{1}{n\lambda}}(D_{k(i)}) \times U_{\alpha(i)} : \lambda \in \mathbb{D}_2\}$ and holomorphic maps $r_i(n\lambda z, u)$ and $f_{ij}(n\lambda z, u)$ for $\lambda \in \mathbb{D}_2$, $z \in t_{\frac{1}{n\lambda}}(D_{k(i)})$ and $u \in U_{\alpha(i)}$. For each n, we have $s_n^{1/n} = r_1$ and $s_n^0 = r_2$, thus (see Eq. 10)

$$\mathrm{d_{Kob}}(r_1, r_2) \le \rho_{\mathbb{D}_2}(1/n, 0) = \frac{1}{2} \log\left(\frac{2n+1}{2n-1}\right)$$

which tends to zero as n goes to infinity.

Consider now the case of $\pi_1^{\text{simpl}}(\mathcal{Y}, y)$ not being Brody hyperbolic, thus there exists a section $\phi \in \pi_1^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C} \times U)$ not in $p^*(\pi_1^{\text{simpl}}(\mathcal{Y}, y)(U))$. We recall (see Subection 3.1) that, for a complex space V, the sections in $\pi_1^{\text{simpl}}(\mathcal{Y}, y)(V)$ are represented by classes $[(\psi_i)_i]$ of automorphisms of some descent datum (r_i, f_{ij}). In particular, $\psi_i : V_i \to X \times_{\mathcal{Y}} X$ are holomorphic maps from open sets V_i of an open covering $\mathcal{V}$ of V, satisfying certain identities. From the section ϕ just mentioned, we produce two different sections θ_1 and θ_2 in $\pi_1^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C} \times U)$ as in the π_0^{simpl} case: $\theta_1 = \phi$ and $\theta_2 = p^*(i_0^*(\phi))$. Notice that θ_2 is an automorphism of a descent datum $r_2 \in \mathsf{Ob}(\mathcal{C}[X.](\mathbb{C} \times U))$ which lies in $p^*(\mathsf{Ob}(\mathcal{C}[X.](U)))$ and θ_1 is an automorphism of a descent data r_1 which may possibly be in the image of p^*, although θ_1 itself does not. However, it necessarily has to be $r_1 \neq r_2 \in \pi_0^{\text{simpl}}$ given the hypothesis on ϕ. Keeping this in mind, to denote the descent data r_1 and r_2 we will use the notation already introduced during the discussion of the π_0^{simpl} case: $r_1 = (r_i(z,u), f_{ij}(z,u))$ and $r_2 = (r_i'(z,u), f_{ij}'(z,u)), r_i'(z,u) = r(0,u)$ and $f_{ij}'(z,u) = f_{ij}(0,u)$. We wish to find an analytic disc Ω relative to $\mathbb{C}\times U$ between θ_1 and θ_2. Define $\Omega : \mathbb{D}_2 \times (\mathbb{C}\times U) \to \pi_1^{\mathrm{simpl}}(\mathcal{Y}, y)$ by

$$\Omega(\delta_U, z, u) = \{(\Omega_k)_k \text{ such that } \Omega_k(\delta, z, u_k) = \phi_k(\delta z, u_k)\},$$

where $\phi = (\phi_k)_k$ and $\phi_k : W_k \to X_1$ are holomorphic maps defining the extension of ϕ over the open covering $\mathcal{W}$ of U, precedently introduced. For any fixed $\delta \in \mathbb{D}_2, m(\phi_h, f_{hk}^{\delta})(z,u) = m(f_{hk}^{\delta}, \phi_k)(z,u)$, which shows that $\Omega_{|\{\delta\}\times\mathbb{C}\times U}$ is an automorphism of s^{δ} for any $\delta \in \mathbb{D}_2$ and Ω indeed induces an analytic disc in $\pi_1^{\mathrm{simpl}}(\mathcal{Y}, y)(\mathbb{C}\times U)$. Also, $\Omega_{|\{0\}\times\mathbb{C}\times U} = \theta_1$ and $\Omega_{|\{1\}\times\mathbb{C}\times U} = \theta_2$ so that it is an analytic disc between θ_1 and θ_2. The sequence of analytic discs Ω^n relative to $\mathbb{C}\times U$ is now introduced in analogy with the π_0^{simpl} earlier discussion and the vanishing of the Kobayashi pseudodistance between θ_1 and θ_2 follows at once. □

8 Compactness and Brody hyperbolicity imply Kobayashi hyperbolicity

Brody and Kobayashi hyperbolicity of an analytic stack $p : X \to \mathcal{Y}$ are statements concerning the holotopy presheaves $\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)$ and $\pi_1^{\mathrm{simpl}}(\mathcal{Y}, y)$, as conceived in definitions 4.4 and 4.6. Surprisingly, it turns out that for compact Deligne–Mumford analytic stacks, the Brody hyperbolicity content in the holotopy presheaf π_0^{simpl} absorbed the one of π_1^{simpl}, to the point of making the latter irrelevant.

Theorem 8.1 *Let $\mathcal{Y}$ be a compact, Deligne–Mumford analytic stack. If $\pi_0^{simpl}(\mathcal{Y}, y)$ is Brody hyperbolic, then*

(i) for any complex space U we have $c(\mathcal{Y}, U) < +\infty$;
(ii) $\pi_i^{simpl}(\mathcal{Y}, y)$ are Kobayashi hyperbolic, for $i = 0, 1$.

Proof The statement (i) implies statement (ii). Indeed, let $\alpha_1 \neq \alpha_2 \in \pi_i^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$ admissible sections. From (i) and Lemma 6.3, we have

$$\mathrm{d}_{\mathrm{Kob}}(\alpha_1, \alpha_2) \geq \frac{\delta_{\pi_i, U}(\alpha_1, \alpha_2)}{2c(\mathcal{Y}; U)} > 0$$

hence $\pi_i^{\mathrm{simpl}}(\mathcal{Y}, y)(U)$ are Kobayashi hyperbolic. The rest of the section will be devoted to the proof of the first assertion, which will be split in few steps.

Assume by contraddiction that $c(\mathcal{Y}, U) = +\infty$ for some complex space U. Then there exists a sequence $\{\mathcal{U}_\nu\}_\nu$ of coverings of U associated to a sequence of relative analytic discs $\{\mathsf{F}^\nu\}_\nu$, i.e., descent data, over $\mathbb{D}\times U$ and points $\{u_\nu\}_\nu \subset U$ all with the property that $\lim_{\nu\to+\infty} |\mathrm{d}\mathsf{F}^\nu(0, u_\nu)| = +\infty$ (see Eq. 19). The descent datum F^ν induces on $\mathbb{D}\times\{u_\nu\}$; hence on $\mathbb{D}$, a sequence of descent data which we will keep writing in the same way; this sequence will have the property that $\lim_{\nu\to+\infty} |\mathrm{d}\mathsf{F}^\nu| = +\infty$. This reduces the argument to $U = \{\mathrm{pt}\}$. □

Lemma 8.2 *If $c(\mathcal{Y}, U) = +\infty$, then there is a sequence $\{f^\nu\}_\nu$ of holomorphic maps $f^\nu : \mathbb{D} \to Q(\mathcal{Y})$ (see Sect. 5) such that* $\lim_{\nu\to+\infty} |\mathrm{d}f^\nu(0)| = +\infty$.

Proof We show that there exists a commutative diagram of sets

$$\begin{array}{ccc} \mathcal{C}[X.](U) & \longrightarrow & \pi_0(\mathcal{Y})(U) \\ & \phi \searrow \quad \swarrow \phi_Q & \\ & \mathrm{Hol}(U, Q(\mathcal{Y})) & \end{array} \tag{27}$$

where the application ϕ is defined by associating to a descent datum $r = (r_i, f_{ij})$ on $U = \cup_{i \in \mathbb{N}} U_i$ the holomorphic map $f_r : U \to Q(\mathcal{Y})$ defined for $u \in U_i$ as $f_r(u) = q(r_i(u))$, where $q : X \to Q(\mathcal{Y})$ denotes as usual the holomorphic projection. The map f^r is well defined since if $u \in U_i \cap U_j$, then $q(r_i(u)) = q(r_j(u))$, since there exists $w = f_{ij}(u) \in X \times_{\mathcal{Y}} X$ such that $\partial_0(w) = r_i(u)$ and $\partial_1(w) = r_j(u)$. If two descent data r and s represent the same class in $\pi_0^{\text{simpl}}(\mathcal{Y}, y)(U)$, then $\phi(r) = \phi(s)$, thus there is a well-defined application ϕ_Q making the diagram (27) commutative. In particular, to each relative analytic disc F on $\mathbb{D} \times U$ we have a holomorphic map $f := \phi(\mathsf{F}) : \mathbb{D} \times U \to Q(\mathcal{Y})$ and a real continuous function $|\mathrm{d} f(z)|$. From the very definition we have that $|\mathrm{d}\mathsf{F}(z)| = |\mathrm{d} f(z)|$; thus, the statement of the lemma follows at once by taking $f^\nu = \phi(\mathsf{F}^\nu)$. □

The sequence of the maps $f^\nu : \mathbb{D} \to Q(\mathcal{Y})$ may be reparametrized, by means of the "reparametrization Lemma" (cfr. [3]), to get a sequence of maps $\widetilde{f}^\nu : \mathbb{D}_\nu \to Q(\mathcal{Y})$, where $\mathbb{D}_\nu = \{|z| < \nu\}$ and $|\mathrm{d}\widetilde{f}^\nu(0)| = 1$ for all ν. By Ascoli-Arzelà Theorem, there exists a subsequence $\widetilde{f}^\mu$ uniformly convergent on compacts to a holomorphic map $f : \mathbb{C} \to Q(\mathcal{Y})$, which is not constant since $|\mathrm{d} f(0)| = 1$ (cfr. [3]). In the rest of the section, the reparametrized maps $\widetilde{f}^\nu$ will be denoted f^ν and the domain $\mathbb{D}_\nu$ will be specified to distinguish them from those defined on $\mathbb{D}$.

Recall (Sect. 5) that A is the ramification locus of $q : X \to Q(\mathcal{Y})$ and $B = q(A) \subset Q(\mathcal{Y})$ is the branch locus. One of the following three cases necessarily occurs

(Case 1) $f(\mathbb{C}) \cap B = \emptyset$;
(Case 2) $f^{-1}(B)$ is a discrete set of points of $\mathbb{C}$;
(Case 3) $f(\mathbb{C}) \subset B$.

We would like to "invert" the map ϕ, i.e., find an application ψ providing a descent datum in X associated to a holomorphic map $U \to Q(\mathcal{Y})$ such that $\phi \circ \psi = \mathsf{id}$. This property implies that if f is nonconstant, $\psi(f)$ is a nonconstant descent datum on $\mathbb{C}$ (see Definition 3.8) and $[\psi(f)] \in \pi_0^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C})$ is a nonconstant class (see Remark 6.2).

We are going to consider the existsence of ψ in the above three cases separately, starting with the first.

8.1 Case 1

Proposition 8.3 *Let $h : \mathbb{C} \to Q(\mathcal{Y}) \setminus B$ be a holomorphic map. Then there is a descent datum $\psi(h)$ in X associated to h, such that $\phi(\psi(h)) = h$.*

Proof Let U be any complex space; we will take $U = \mathbb{C}$ just at the end of the proof. Cover U with simply connected open subspaces U_i and lift the restrictions of h to continuous maps $h'_i : U_i \to X$. By [2], we know that $q : X \setminus A \to Q(\mathcal{Y}) \setminus B$ is a local biholomorphism, hence the h'_i are holomorphic maps. Let $U_i \cap U_j \neq \emptyset$; to relate h'_i and h'_j on $U_{ij} = U_i \cap U_j$ we argue as follows. Since $h_{|U_{ij}} = h_{|U_{ij}}$, there exists a set-theoretic map $\omega : U_{ij} \to X_1$ such that $\partial_0 \circ \omega = h'_{i\,|U_{ij}}$ and $\partial_1 \circ \omega = h'_{j\,|U_{ij}}$. Refining the covering $\{U_i\}_i$, if necessary, we can assume the U_i are relatively compact, U_{ij} are simply connected and $h'_i(U_{ij})$ are uniformly covered by ∂_0 so that $\partial_0^{-1} h'_i(U_{ij}) = \amalg_{k=1}^{l} V_k$ and $\partial_0 : V_k \to U_{ij}$ is a biholomorphism for any k. Consider first the case of a one-dimensional complex space U. As the codomain of ω has a finite number of connected components, there is an index $a \leq n$ such that $\omega^{-1}(V_a) = L$ contains uncountably many points; hence, there is an accumulation point among them. Call $f'_{ij} : U_{ij} \to V_a$ the ∂_0 lifting of $h'_{i|U_{ij}}$ to V_a; it coincides with ω on L since $\partial_{0|V_a}$ is injective. Thus, $\partial_0 \circ f'_{ij} = h'_{i|U_{ij}}$ on U_{ij}, $\partial_1 \circ f'_{ij} = h'_{j|U_{ij}}$ on L, hence on all U_{ij}. This shows that f'_{ij} is a transition function.

Let us assume inductively to be able to define the transition functions on each complex space U of dimension $n-1$. We think of U_{ij} to be embedded in $\mathbb{C}^N$ and consider the intersections $U_{ij} \cap H_\lambda$, H_λ ranging among all $(N-1)$ dimensional hyperplanes in $\mathbb{C}^N$. On each of these intersections we have transition functions with values in V_a where the index a may depend on λ, but being the indices (among which a ranges) finite, there are infinitely many μ such that $U_{ij} \cap H_\mu \neq \emptyset$ and whose transition functions have values in V_a for a fixed a. Call l_{ij} the holomorphic map on $L := \bigcup_\mu U_{ij} \cap H_\mu \to V_a$ defined by taking on each $U_{ij} \cap H_\mu$ the transition function on it; notice that l_{ij} is well defined since all the transition functions are ∂_0 liftings to V_a and $\partial_{0|V_a}$ is injective. Call now $f'_{ij} : U_{ij} \to V_a$ the ∂_0 lifting of $h'_{i|U_{ij}}$ to V_a. As expected, $\partial_0 \circ f'_{ij} = h'_{i|U_{ij}}$ and $\partial_1 \circ f'_{ij} = h'_{j|U_{ij}}$ coincide, as functions defined on U_{ij}, on an analytic subspace containing L, but the only such space is U_{ij} itself. A good candidate for the descent datum $\psi(h)$ could be (h'_i, f'_{ij}), which indeed looks very much like a descent datum on U, except that, in general, the f'_{ij} do not satisfy the cocyle condition $m(f'_{ij}, f'_{jk}) = f'_{ik}$ (see Definition 3.4), m being the multiplication of $[X.]$. The idea is the one of trying to replace f'_{ik} with $m(f'_{ij}, f'_{jk})$, the problem being that the latter is defined only on the triple intersection U_{ijk}. □

Lemma 8.4 *Given a nonconstant holomorphic map $h : U \to Q(\mathcal{Y})$, refer to $h'_i : U_i \to X_1$ and $f'_{ij} : U_{ij} \to X_1$ as the holomorphic maps obtained by means of the above constructions. Then there exists a refinement $\{B_\alpha\}_\alpha$ of $\{U_i\}_i$ and transition functions $\{g_{\alpha\beta}\}_{\alpha\beta}$ such that, if B_i, $i = 1, 2, 3$ are open sets with nonempty triple intersection, for any chosen transition functions g_{12} and g_{23}, there exists a holomorphic map $g_{13} : B_{13} \to X_1$ satisfying the cocycle condition $m(g_{12}, g_{23}) = g_{13}$ on B_{123}. Moreover, $\phi((h'_\alpha, g_{\alpha\beta})) = h$.*

Proof Denote the holomorphic map $m(f'_{12}, f'_{23}) : U_{123} \to X_1$ by l_{13}, choose points $u \in U_{123}$ and $p \in U_{13}$ and a path $\gamma : [0, 1] \to U_{13}$ connecting u to p. We would be done if $h'_1 \circ \gamma$ lifted to a $\tilde{\gamma}$ starting in $l_{13}(u) \in X_1^{13}$ where X_1^{13} is the connected component of X_1 containing the image of l_{13}, but this may necessarily not be the case since $h'_1(U_{13})$ might not be contained in $\partial_0(W_1^{13})$. The possibility of lifting such a map only depends on p, X and $\mathcal{Y}$; hence, using that $\partial_0 : X_1 \to X$ is a covering, we can refine $\{U_i\}_i$ to a covering $\{B_\alpha\}_{\alpha\in\mathbb{N}}$ of U with the following properties:

1. $\{B_\alpha\}_\alpha$ is a locally finite refinement of $\{U_i\}_i$;
2. the intersections $B_{\alpha\beta}$ are simply connected;
3. if we define $\tau(\alpha)$ to be an integer less than the minimum of the i such that $B_\alpha \subset U_i$, then for each triple intersection $B_{\alpha\beta\gamma} \neq \emptyset$ with $\alpha < \beta < \gamma$, we have $h'_{\tau(\alpha)}(B_{\alpha\beta})$, $h'_{\tau(\alpha)}(B_{\beta\gamma})$ and $h'_{\tau(\alpha)}(B_{\alpha\gamma})$ are contained in the same ∂_0 uniformly covered open subspace of X.

We now create a descent datum r on U using the covering $\{B_\alpha\}_\alpha$ and the holomorphic maps h'_i, f'_{ij}. For each α define $r_\alpha = h'_{\tau(\alpha)|B_\alpha}$ and let the transition functions $g_{\alpha\beta} : B_{\alpha\beta} \to X_1$ be

$$g_{\alpha\beta} = \begin{cases} \text{id}, & \text{if } B_\alpha \cup B_\beta \subset U_{\tau(\alpha)} \\ f'_{\tau(\alpha)\tau(\beta)|_{B_\alpha \cap B_\beta}}, & \text{otherwise.} \end{cases}$$

The third condition on the covering $\{B_\alpha\}_\alpha$ implies that, if $B_{\alpha\beta\gamma}$ is nonempty, $h'_{\alpha|B_{\alpha\beta}}$ lifts to a holomorphic map $\tilde{h}_\alpha$ with image in the same connected component of X_1 as the one of the image of $m(g_{\alpha\beta}, g_{\beta\gamma})$. Since $\partial_0 \circ \tilde{h}_\alpha = \partial_0 \circ m(g_{\alpha\beta}, g_{\beta\gamma})$, by the injectivity of ∂_0 on that connected component, we conclude that $\tilde{h}_\alpha$ is an extension of $m(g_{\alpha\beta}, g_{\beta\gamma})$ to all $B_{\alpha\gamma}$ and it can be taken to be the transition function $g_{\alpha\gamma}$ satisfying the cocyle condition. □

A recursive argument can extend this construction to the quadruple and higher intersections, the only possible indeterminacy being in how to choose the transition functions: for instance, g_{14} could be the extension of $m(g_{12}, g_{24})$ or $m(g_{13}, g_{34})$. However, using the cocycle relations

1. $m(g_{12}, g_{23}) = g_{13}$,
2. $m(g_{23}, g_{34}) = g_{24}$

and assuming to have defined g_{14} as the extension of $m(g_{12}, g_{24})$, we check that it verifies the relation $g_{14} = m(g_{13}, g_{34})$, as well: from the first equation, $m(g_{13}, g_{34}) = m(m(g_{12}, g_{23}), g_{34})$ and from the second relation and associativity we have that this is precisely $m(g_{12,} g_{24})$, which by definition is precisely g_{14}. □

To finish the proof of the proposition 8.3 we notice that we can cover $\mathbb{C}$ with open subspaces $\{B_i\}_i$ in such a way $B_j \cap \bigcup_{i<j} B_i$ is contractible; therefore, there cannot be conflicting conditions on the function g_{13} of the previous lemma for such a covering. □

This finishes the construction of the descent data $\psi(h)$ associated to a holomorphic map $h : U \to Q(\mathcal{Y})$, hence of the application ψ and the proof of Proposition 8.3 and Case 1 are settled.

Remark 8.1 Notice that if the holomorphic map h is nonconstant, then all the liftings h'_i are nonconstant. Using results proved in Sect. 6.2, we see that we can check whether a section in $[(h'_i, h'_{ij})] \in \pi_0^{\text{simpl}}(\mathcal{Y}, y)(\mathbb{C})$ is nonconstant or not, simply by showing that the real-valued function $z \mapsto |\mathrm{d}h'_i(z)|$ is nonzero for $z \in \mathbb{C}$. It follows that if h is nonconstant, so does $[(h'_i, h'_{ij})]$.

8.2 Case 2

Without loss of generality, we can assume that each connected component of the atlas $Z = \amalg_{i=1}^N Z^{(i)}$, containing X as a relatively compact open subspace, is Stein and that $X^{(i)} = \{\zeta \in Z^{(i)} : \phi_i(\zeta) < c\}$, where $\phi_i : Z^{(i)} \to \mathbb{R}$ is a strictly plurisubharmonic exhaustion function for $Z^{(i)}$.

By assumption $f^{-1}(B)$ is a discrete subset S of C. We will first show that for each $\zeta \in S$ there is a disc Δ_ζ centered in ζ having the property that $f_{|\Delta^*_\zeta}$ lifts to a holomorphic map $\tilde{f}_\zeta : \Delta^*_\zeta \to X$, where Δ^*_ζ is the punctured disc $\Delta_\zeta \setminus \{\zeta\}$. This follows if we prove that we can lift $f_{|\Delta^*_\zeta}$ to a continuous map in X which is equivalent to showing that

$$f_* \left(\pi_1(\Delta^*_\zeta, z)\right) \subset q_* \pi_1 \left(X \setminus A, x\right)) \subset \pi_1 \left(Q(\mathcal{Y}) \setminus B, q(x)\right) \tag{28}$$

where $q(x) = f(z)$. Let γ be a generator of $\pi_1(\mathbb{C} \setminus S, z)$, i.e., a simple closed path around one of the points of $\zeta \in S$. Since the holomorphic maps $f^\nu = \psi(\mathsf{F}^\nu)$ converge to f uniformly on compact sets and $Q(\mathcal{Y})$ is locally topologically contractible, there exists an index μ such that

1. $f^{\mu} \circ \gamma$ is homotopic to $f \circ \gamma$ and $(f^{\mu} \circ \gamma)([0, 1]) \cap B = \emptyset$;
2. $\gamma([0, 1]) \subset \mathbb{D}_{\mu}$.

The map f^{μ} can be lifted to X if restricted to an appropriate punctured open disc Δ^*_{ζ}, Δ_{ζ} being a contractible open neighborhood of ζ in $\mathbb{C}$: just take the holomorphic map F^{μ}_i part of the descent datum $\mathsf{F}^{\mu} = (F^{\mu}_i, F^{\mu}_{ij})$, which is defined on $\gamma([0, 1])$. We can assume that γ is close enough to ζ to have image contained on some open set $U_i \ni \zeta$ associated to F^{μ}. Let Δ_{ζ} be such that $\gamma([0, 1]) \subset \Delta_{\zeta} \subset U_i$. Now consider the closed path $F^{\nu}_i \circ \gamma$ in X: by construction and condition 8.2 on μ in particular it induces a homotopy class $[F^{\nu}_i \circ \gamma] \in \pi_1(X \setminus A, x)$; since $q \circ (F^{\nu}_i \circ \gamma) = f^{\nu} \circ \gamma$ we have

$$[f \circ \gamma] = [f^{\mu} \circ \gamma] = q_*(\overline{F^{\mu}_i \circ \gamma}) \in \pi_1(Q(\mathcal{Y}), y)$$

which means that $f_*([\gamma])$ lies in $q_*\pi_1(X \setminus A, x) \subset \pi_1(Q(\mathcal{Y}) \setminus B, f(z))$, and so does $f_*\pi_1(\mathbb{C} \setminus S, x)$. We can then lift $f_{\Delta^*_{\zeta}}$ to a holomorphic map $\widetilde{f_{\zeta}} : \Delta^*_{\zeta} \to X \setminus A$, since q is unramified on $X \setminus A$.

Take W to be an open subspace of $\mathbb{C} \setminus S$ such that $\{W, \Delta^*_{\zeta} : \zeta \in S\}$ is an open covering of U. Since $f(W) \subset Q(\mathcal{Y}) \setminus B$, by the Case 1 of the Theorem 8.3, $f_{|W}$ lifts to a descent datum $r = \psi(f_{|W})$. The datum r is extended to a descent datum on the whole $\mathbb{C} \setminus S$ by using the liftings $\widetilde{f_{\zeta}}$ as holomorphic maps on Δ^*_{ζ} and creating transition functions as shown in the construction of the application ψ in the unramified case. To further extend r to all of $\mathbb{C}$, we just have to extend the domain of the homolorphic maps $\widetilde{f_{\zeta}}$ from Δ^*_{ζ} to the whole Δ_{ζ}. Let $X^{(1)} \Subset Z^{(1)}$ be the connected components of the atlases X and Z containing the image of $\widetilde{f_{\zeta}}$. Embedding $Z^{(1)}$ as a closed analytic subspace in $\mathbb{C}^m$, the map $\widetilde{f_{\zeta}}$ is expressible in terms of m complex-valued holomorphic functions $f_1, f_2, \dots, f_m$ which are bounded given the compactness of $\overline{X}^{(1)}$, thus each of them extendable on Δ_{ζ} by Riemann's Extension Theorem. Call $\widehat{f_{\zeta}}$ the extension of $\widetilde{f_{\zeta}}$ to Δ_{ζ}. Suppose now that $\widehat{f_{\zeta}}(\zeta)$ is a point of the boundary of $X^{(1)}$ in $Z^{(1)}$, then the nonconstant plurisubharmonic function $\phi_1 \circ \hat{f_{\zeta}}$, where ϕ_1 is the exhaustion function introduced before, has a maximum in ζ, which is absurd. We conclude that the image of $\widehat{f_{\zeta}}$ is contained in $X^{(1)}$ and the collection of the functions $\widehat{f_{\zeta}}$, along with the rest of the data in s, yields the sought extension of s to $\mathbb{C}$. Because $|\mathrm{d}f(0)| = 1$, f is nonconstant, therefore as pointed out in Remark 8.1, the class of this descent datum in $\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)(\mathbb{C})$ is nonconstant, as well.

8.3 Case 3

We assume now that $f(\mathbb{C}) \subset B$. Moreover, since q is quasi finite and B is analytic ([2]), there exists a closed analytic subset B_1 of B such that $\dim_x(B_1) < \dim_x(B)$ for all $x \in B$ and q is a local biholomorphism from $A \setminus A_1$ to $B \setminus B_1$, with $A_1 = q^{-1}(B_1)$.

As before there are three possible cases: $f(\mathbb{C}) \cap B_1 = \emptyset$, $f^{-1}(B_1)$ is a discrete subset of $\mathbb{C}$ and $f(\mathbb{C}) \subset B_1$. If $f(\mathbb{C}) \cap B_1 = \emptyset$, then we lift f to the nonconstant descent datum $\psi(f)$ as in the Case 1 and prove that $\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)$ is not Brody hyperbolic. If $f(\mathbb{C}) \subset B_1$ then we repeat Case 3, but with all dimensions of the analytic spaces dropped, and recursively we will eventually end up with zero-dimensional ramification, which is treated exactly as in the general case below, except being simpler. Consider the case of $f^{-1}(B_1) =: S_1$ being a discrete subset. Two things can happen: there exists an index N such that for all $\nu > N$, $f^n(\mathbb{C}) \subset B$ which is treated exactly as in the Case 2. Otherwise, we proceed with few reduction steps. As usual we denote $F^{\nu}_{i(\nu)}$ a holomorphic map of the descent datum F^{ν} on some disc, such that $\phi(\mathsf{F}^{\nu}) = f^{\nu}$. By the compactness of $\mathcal{Y}$ we may assume X had only a finite number of connected components (see beginning of Sect. 5); hence, for a $\zeta \in S_1$, there is a punctured disc $\Delta^*_{\zeta} \subset \mathbb{C} \setminus S_1$ and an index k such that $F^{j}_{i(j)}(\Delta^*_{\zeta}) \subset X^{(k)}$ for infinitely many j. Considering only those indices j we may assume all the images of Δ^*_{ζ} through the sequence of functions $\{F^{\nu}_{i(\nu)}\}_n$ lie in the same connected component. Using again the compactness of $\mathcal{Y}$, we may suppose that the sequence of points $\{F^{\nu}_{i(n)}(\zeta)\}_n$ converges to a point $a_1 \in X^{(k)}$: otherwise it must necessarily converge to a point a_1 on the boundary of $X^{(k)}$ in $Z^{(k)}$ and $q^{-1}(q(a_1))$ has a point in the interior of another connected component $X^{(h)}$ of X, for which all the statements regarding $X^{(k)}$ hold as well. As far as our argument is concerned, we then replace $X^{(k)}$ with $X^{(h)}$. Let U_{a_1} be an open neighborhood of a_1 such that the closure $\overline{U_{a_1}} \subset X^{(i)}$. Then

there is a closed simple path γ in $\mathbb{C} \setminus S_1$ around ζ and close enough to it for

$$F^{\nu}_{i(\nu)}(\gamma([0, 1])) \subset U_{a_1}$$

for all $\nu \geq N \gg 0$.

Indeed, as in Sect. 6.1, let H be a distance on $Q(\mathcal{Y})$ and q^*H its pullback on X; set d to be a positive real number such that the set of points in $X^{(i)}$ whose d_{q^*H} distance from a_1 is less than d is contained in U_{a_1}. Choose a closed simple path γ as in the statement, in a way that $\mathrm{d}_H(f(\gamma([0, 1])), f(\zeta)) = d/2$. Then there exists N such that $\mathrm{d}_{q^*H}(F^{n}_{i(n)}(\gamma([0, 1])), a_1) \leq d$ for all $n \geq N$, since $q(a_1) = f(\zeta)$ and q is a contraction with respect to the length functions q^*H and H.

Replacing X with $\overline{U_{a_1}}$, we can consider the following simplified new situation: X is a connected, compact topological space, $Q(\mathcal{Y})$ compact, A, A_1, B and B_1 all compact subspaces.

Give X the structure of a CW complex with finitely many cells such that A and A_1 are sub CW complexes. Let $l = \mathrm{d}_H(f(\gamma([0, 1])), B_1)$; by possibly refining the cell structure we may take the q^*H-sizes of the cells to be of dimension ranging between $l/4$ and $l/2$. Call V_A the sub CW complex $A \subset V_A \subset X$ made of cells e_i having nonempty intersection with A.

1. Since the sizes $\mathrm{d}_{q^*H}(e_i) \geq l/4$ for all i, and $f(\gamma([0, 1])) \subset B$, there exists an N such that $F^{n}_{i(n)}(\gamma([0, 1])) \subset V_A$ for all $n \geq N$;
2. because $\mathrm{d}_{q^*H}(e_i) \leq l/2$, there exists $M \geq N$ such that if e_j are the cells of V_A with nonempty intersection with the image of $F^{\nu}_{i(\nu)} \circ \gamma$ for any of the $\nu \geq M$, then $e_j \cap A_1 = \emptyset$.

Finally, let $R \geq M$ with the property that $\mathrm{d}_{q^*H}(F^{\nu}_{i(\nu)}(\gamma([0, 1])), A) < l/8$. Then we can remove appropriately chosen points $p_1, \ldots, p_k$ from the interiors of some cells in V_A to obtain a topological space V'_A with a deformation retract $r : V'_A \to A$ such that $r(F^{R}_{i(R)}(\gamma([0, 1]))) \subset A \setminus A_1$.

Call γ' the closed path $r \circ F^{R}_{i(R)} \circ \gamma$. Then we have that

$$[f \circ \gamma] = q_*(F^{R}_{i(R)} \circ \gamma) = q_*(\gamma') \in \pi_1(A \setminus A_1, a)$$

thus f lifts to A on a small punctured disc Δ^*_{ζ}, being $[f \circ \gamma]$ in the image of

$$q_{|A \setminus A_1 *} : \pi_1(A \setminus A_1, a) \to \pi_1(B \setminus B_1, q(a))$$

and $q_{|A \setminus A_1}$ being a topological covering. As in the Case 2, we lift $f_{|\mathbb{C} \setminus S_1}$ to X (actually to A) and then extent the local lifting $\widetilde{f_{\zeta}}$ defined on the puntctured disc Δ^*_{ζ} to Δ_{ζ}. This argument produces a nonconstant descent datum in $\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)(\mathbb{C})$.

This concludes the proof of the part "$\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)$ Brody hyperbolic implies $\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)$ Kobayashi hyperbolic" part of the Theorem 8.1. It remains to be shown that if $\pi_0^{\mathrm{simpl}}(\mathcal{Y}, y)$

is Brody hyperbolic, then $\pi_1^{\text{simpl}}(\mathcal{Y}, y)$ is Kobayashi hyperbolic. However, by the proof so far written, we see that the Brody hyperbolicity implies that $c(\mathcal{Y}, y) < +\infty$; by Lemma 6.3, both of $\pi_0^{\text{simpl}}(\mathcal{Y}, y)$ and $\pi_1^{\text{simpl}}(\mathcal{Y}, y)$ are Kobayashi hyperbolic.

The proof of the Theorem 8.1 has highlighted the connection between hyperbolicity of $Q(\mathcal{Y})$ as a complex space and $\mathcal{Y}$ as Deligne–Mumford analytic stack.

Corollary 8.5 *Let $X \to \mathcal{Y}$ be a compact Deligne–Mumford analytic stack. Then*

1. *if $Q(\mathcal{Y})$ is hyperbolic, $\mathcal{Y}$ is hyperbolic;*
2. *$\mathcal{Y}$ is hyperbolic if and only if the presheaf $\pi_0^{simpl}(\mathcal{Y}, y)$ is hyperbolic if and only if* $\mathsf{Ob}(\mathcal{C}[X.])$ *is an hyperbolic presheaf.*

9 Hyperbolicity and the coarse moduli space

While hyperbolicity of the coarse moduli space implies hyperbolicity of the stack, the converse is generally not true. We give here an example.

9.1 An hyperbolic DM stack with nonhyperbolic coarse moduli space

Let $p : X \to \mathcal{Y}$ be an analytic stack with a one-dimensional torus $\mathbb{T}$ as coarse moduli space $Q(\mathcal{Y})$ and such that the ramification of $q : X \to Q(\mathcal{Y})$ is at least 4 in each of the points of the fiber over two branch points z_1 and $z_2 \in Q(\mathcal{Y})$. We show that such a stack $\mathcal{Y}$ is hyperbolic by proving that no nonconstant map $f : \mathbb{C} \to Q(\mathcal{Y})$ can be lifted to X. Indeed, let $\tilde{f} : \mathbb{C} \to \mathbb{C}$ be the lifting of f to the universal covering $h : \mathbb{C} \to Q(\mathcal{Y})$. Assume that f admits local liftings to X. Since f must be surjective, and h is unramified, any two points w_1 and w_2 in the h fiber of z_1 and z_2, respectively, will have the following property: $\tilde{f}^{-1}(w_1)$ and $\tilde{f}^{-1}(w_2)$ are two sets of points on which $\tilde{f}$ have ramification divisible by N_1 and N_2, respectively, both of them greater than 4. However, it is impossible for such an entire function to exist. Indeed, otherwise there would be two functions α and β such that $\alpha^{N_1} = \tilde{f} - w_1$ and $\beta^{N_2} = \tilde{f} - w_2$ and satisfying the equation $x^{N_1} - y^{N_2} = w_2 - w_1$. The function $z \to (\alpha(z), \beta(z))$ is defined over $\mathbb{C}$ and nonconstant with image in the affine part of a curve of genus greater than 2. This is impossible.

The following is an example of such a stack: consider a covering of a torus $\mathbb{T}$ by a curve $q : X \to \mathbb{T}$ of degree 4 and ramified in two points $x_1, x_2 \in X$ with ramification index 4, given by the composition $X \overset{q_2}{\to} X' \overset{q_1}{\to} \mathbb{T}$. Here both maps have degree 2 and are ramified in only two points. Their existence follows from

Lemma 9.1 *Given a complex curve C, and $B = \{c_1, c_2, \ldots, c_{2k}\} \subset C$, there exists a degree* 2 *covering $p : \widetilde{C} \to C$ with branch locus B.*

Proof It follows from the classification of the coverings of curves (see, for instance, [14, Prop. 4.9]). Coverings of C with branch locus contained in B are classified by homomorphisms $\pi_1(C \setminus B, x) \to S_2$ with transitive image, where S_2 is the symmetric group on two elements. The condition on the ramification corresponds to the image of loops γ_i around points of B going to the transpositon in S_2. $\pi_1(C \setminus B, x)$ can be presented as

$$\langle \gamma_i, \alpha_j, \beta_j : \prod_i \gamma_i \cdot \prod_j \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = 1 \rangle_{i=1,\ldots,2k,\ \ j=1,\ldots,\text{genus}(C)}$$

where α_j and β_j are the generators of $\pi_1(C, x)$. Therefore, such an homomorphism exists as B contains an even number of points. $\square$

The covering q is Galois, being the composition of two Galois extensions and because the monodromy group of a covering map is isomorphic to the Galois group of the rational functions field extension of the curves involved (cfr. [8], proposition at page 689, for instance). Therefore, $[X/\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(\mathbb{T}))]$ is a quotient stack with $\mathbb{T}$ as coarse moduli space satisfying the condition on the ramification stated above.

9.2 Automorphisms of a compact Deligne–Mumford analytic stack

Stack hyperbolicity is expected to impose a peculiar behavior to the stack itself, just like usual hyperbolicity does on a complex space. Here we prove that a compact Deligne–Mumford analytic stack with a hyperbolic coarse moduli space has few automorphisms:

Theorem 9.2 *Let $p : X \to \mathcal{Y}$ be a compact Deligne–Mumford analytic stack with hyperbolic coarse moduli space. Then there are finitely many isomorphism classes of the 2-group $\mathsf{Aut}(\mathcal{Y})$.*

Proof By assumption the coarse moduli space $Q(\mathcal{Y})$ is hyperbolic and compact, hence it has a finite number of biholomorphisms (cf. [11, Theorem 5.4.4]). We will prove that the kernel $\mathsf{Aut}_{Q(\mathcal{Y})}(\mathcal{Y})$ of the 2-homomorphism that associates to an automorphism $\mathcal{Y} \to \mathcal{Y}$ the induced biholomorphism $Q(\mathcal{Y}) \to Q(\mathcal{Y})$ has a finite number of equivalence classes. Let $q : X \to Q(\mathcal{Y})$ be the canonical projection.

Lemma 9.3 *There exist a dense open subspace $V \subset Q(\mathcal{Y})$ and a dense open substack $\mathcal{U} \subset \mathcal{Y}$ which is a gerbe over V.*

Proof An analytic stack $\mathcal{W}$ is a gerbe on a substack where the inertia stack morphism $\Delta' : \mathcal{I} \to \mathcal{Y}$ is flat. Consider the 2 commutative diagram

$$\begin{array}{ccccc} Z \subset W & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{Y} \\ \downarrow f & & \downarrow \Delta' & & \downarrow \Delta \\ f(Z) \subset X & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Y} \times \mathcal{Y} \end{array}$$

where the squares are cartesian. By a theorem of Frisch (cfr. [6]), f is flat outside an analytic subset $Z \subset W$. As f is proper (Δ is by assumption, see 5.3), $f(Z)$ is analytic in X. We conclude that $f : \mathcal{I} \times_{\mathcal{Y}} X \smallsetminus f(Z) \to X \smallsetminus f(Z)$ is flat. Now, $X \smallsetminus f(Z)$ is $X_1 = X \times_{\mathcal{Y}} X$ invariant, as it is maximal in the codomain onto which f is flat. Therefore, there exists an associated open dense substack $\mathcal{U} \subset \mathcal{Y}$ with the property that the basechanged morphism $\mathcal{I}_{\mathcal{U}} \to \mathcal{U}$ is flat, by faithfully flat descent. This shows that $\mathcal{U}$ is a gerbe onto $V := Q(\mathcal{Y}) \smallsetminus q(f(Z))$. □

We also have that $\pi_1(V)$ is finitely generated, since the following holds

Proposition 9.4 *Let C be a compact complex space and $B \subset C$ analytic; then $\pi_1(C \smallsetminus B)$ is finitely generated.*

Proof Embedding C in some $\mathbb{R}^N$ (cfr. [16]) and applying the triangularization theorem of Łojasiewicz [13, Theorem 3] we are reduced to the case where C is a finite simplicial complex K and B is a subcomplex $H < K$. Let $\gamma : S^1 \to |K| \smallsetminus |H|$ be a closed path; we will show that γ is homotopic to the topological realization of a simplicial map $\hat{\gamma} : \hat{S}^1 \to |D| \subset |K| \smallsetminus |H|$ for some simplicial complex $D < K$. This finishes the proof, since D is a finite subcomplex, thus $\pi_1(D)$ is finitely generated. Let γ be given. If the image of γ does not intersect the star of H, then we can let $\hat{\gamma}$ to be a simplicial approximation of γ in K. If $\gamma(S^1)$ intersects nontrivially the star of H, it is possible to continously deform γ inside each simplex of $\overline{|St(H)|}$ to get a γ' homotopic to γ with image not intersecting the star, but rather the link of H in K. In particular, the image of γ' will be contained in the simplicial complex $D = K \setminus St(H)$. Take now a simplicial approximation of γ' in D. □

We show that the restriction functor $\mathsf{End}_{Q(\mathcal{Y})}(\mathcal{Y}) \to \mathsf{End}_V(\mathcal{U})$ is injective on the associated isomorphism classes and then that the isomorphism classes of the codomain are finite. Assume that two morphisms $f, g : \mathcal{Y} \to \mathcal{Y}$ are isomorphic over $\mathcal{U}$ and let $h : \mathcal{I} \to \mathcal{Y}$ be the pullback of the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times_{Q(\mathcal{Y})} \mathcal{Y}$ through $(f, g) : \mathcal{Y} \to \mathcal{Y} \times_{Q(\mathcal{Y})} \mathcal{Y}$. Consider the diagram:

$$\begin{array}{ccccc} W & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Delta} \\ X & \longrightarrow & \mathcal{Y} & \xrightarrow{(f,g)} & \mathcal{Y} \times_{Q(\mathcal{Y})} \mathcal{Y} \end{array}$$

where the vertical morphisms are base changements of the diagonal Δ. Sections to h correspond to isomorphisms between f and g. The isomorphism between f and g over $\mathcal{U}$ yields a section $s : \mathcal{U} \to \mathcal{I}$ to h. Because of the compactness of $\mathcal{Y}$, the diagonal morphism is proper (see Proposition 5.4), h is proper. Moreover, $\mathcal{Y}$ being Deligne–Mumford, we conclude that its diagonal is étale, so h is as well. The same properties are inherited by the base changed holomorphic map $\phi : \mathcal{I} \times_{\mathcal{Y}} X =: W \to X$ of h. Let $a \in A = X \setminus U$ be a point and $a \in V_a \subset X$ be a simply connected open neighborhood. By assumption we have a section s to ϕ over $U \subset X$. As ϕ is a topological covering, s extends over V_a because it is simply connected and of the unicity of the lifting, once we fixed base points. Thus, s extends to a section of ϕ over the whole of X, so that we get a map $s' : X \to \mathcal{I}$ making the diagram commute. s' induces a stack morphism $[X_1 \rightrightarrows X] \to \mathcal{I}$ corresponding to a section $\mathcal{Y} \to \mathcal{I}$ through the canonical equivalence $p : [X_1 \rightrightarrows X] \to \mathcal{Y}$.

We now prove that $\mathsf{End}_V(\mathcal{U})$ has a finite number of isomorphism classes. $\mathsf{End}_V(\mathcal{U})$ is a finite disjoint union of gerbes: locally $\mathcal{U}$ is $V_i \times BG_i$ for some finite group G_i, and $\mathsf{Hom}(BG_i, BG_i) = [\mathsf{Hom}(G_i, G_i)/G_i]$, the quotient stack of the finite set $\mathsf{Hom}(G_i, G_i)$ by conjugation (cfr. [7, Proposition 5.3.5]), therefore a finite disjoint union of $B\mathrm{Stab}_{i_j}$, where Stab_{i_j} is the stabilizer of the j-th orbit of the G_i action on $\mathsf{Hom}(G_i, G_i)$. Each such gerbe has a global object over V, namely the identity morphism $\mathrm{id}_{\mathcal{U}}$ of $\mathcal{U}$. It follows that $\mathsf{End}_V(\mathcal{U})$ is equivalent to $B_V G$ where $G = \mathsf{Aut}(\mathrm{id}_{\mathcal{U}})$. G is a Lie group endowed by a holomorphic map $G \to V$ whose fibers are finite groups. Therefore, it suffices to show that there are a finite number of G_V-torsors on V modulo isomorphism. Because of the finiteness condition on $G \to V$, holomorphic G_V-torsors are the same as topological G-torsors; therefore, $B_V G(V)/\mathrm{iso} = \mathsf{Hom}(\pi_1(V), G)$ is finite since $\pi_1(V)$ is finite. □

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