

Spaces of Goldberg type on certain measured metric spaces

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Abstract In this paper, we define a space $\mathfrak{h}^1(M)$ of Hardy–Goldberg type on a measured metric space satisfying some mild conditions. We prove that the dual of $\mathfrak{h}^1(M)$ may be identified with $\mathfrak{bmo}(M)$, a space of functions with “local” bounded mean oscillation, and that if p is in $(1, 2)$, then $L^p(M)$ is a complex interpolation space between $\mathfrak{h}^1(M)$ and $L^2(M)$. This extends previous results of Strichartz, Carbonaro, Mauceri and Meda, and Taylor. Applications to singular integral operators on Riemannian manifolds are given.

Keywords Local Hardy space · Atom · Goldberg-type space · Riesz transform

Mathematics Subject Classification Primary 42B30 · Secondary 42B25

1 Introduction

This paper focuses on the study of spaces of Hardy–Goldberg type on certain measured metric spaces and is based on previous results of the second-named author [28]. Our goal is twofold: On the one hand, we aim at extending previous work on the subject by Strichartz [25], Carbonaro et al. [7, 8], and Taylor [27]. On the other hand, our results pave the way to further developments concerning Riesz transforms on a certain class of noncompact Riemannian manifolds that will appear in a forthcoming paper.

Strichartz worked on compact Lie groups; some of his far-reaching ideas have been subsequently developed by Taylor to successfully extend Strichartz’s results to the setting of

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Riemannian manifolds with strongly bounded geometry. A comparison between the results contained in [7] and [27] may help understanding our motivations and contributions. Before coming to this, we mention that in [29,30], the authors established a quite complete theory of local Hardy spaces on RD-spaces, i.e., on spaces of homogenous type in the sense of Coifman and Weiss where a reverse doubling property holds. See also [16] for results concerning Triebel–Lizorkin spaces on RD-spaces and their relationships with local Hardy spaces. In particular, note that if M is an RD-space, then our space $\mathfrak{h}^1(M)$ reduces to the space $H_\ell^{1,2}(M)$ of [29]. However, we emphasise the fact that though our theory works also on spaces of homogeneous type in the sense of Coifman and Weiss (without assuming the RD-property), our main goal is to develop a theory that works on a certain class of manifolds with exponential growth, where the doubling property fails.

Coming back to the comparison between the results in [7] and [27], observe that in [7] the authors consider a metric measured space (M, μ, d) satisfying three conditions: the approximate midpoint property (AMP), the local doubling condition (LDC), and Cheeger's isoperimetric property (IP) (see Sect. 2 for the definitions). The AMP is a very mild assumption, very often satisfied, the LDC is a very natural assumption for the applications we have in mind to Riemannian manifolds, whereas the IP is a comparatively restrictive assumption, for it implies that the volume growth of M be at least exponential [18, Proposition 3.1 (i)]. In this setting, the authors introduce an atomic Hardy space $H^1(M)$, identify the dual space of $H^1(M)$ with $BMO(M)$ (suitably defined), and prove that if p is in $(2, \infty)$, then $L^p(M)$ is an interpolation space between $L^2(M)$ and $BMO(M)$. Also, applications to spectral multipliers and Riesz transforms are given. It is important to keep in mind that atoms are functions in $L^2(M)$, with support contained in balls of radius at most 1, say, satisfying the standard size estimate and cancellation property (the same as those satisfied by atoms in the classical Hardy space $H^1(\mathbb{R}^n)$).

In [27], Taylor works on a Riemannian manifold M of bounded geometry in a very strong sense, which requires a uniform local control of all derivatives of the metric tensor in exponential co-ordinates around each point, but a mild control on the volume growth of the manifold. He defines a local Hardy space $\mathfrak{h}^1(M)$, which is a direct generalisation of the classical local Hardy space $\mathfrak{h}^1(\mathbb{R}^n)$, introduced by Goldberg [13], and of the extension thereof to compact Lie groups by Strichartz. Taylor defines $\mathfrak{h}^1(M)$ via a suitable grand maximal function, identifies the dual space of $\mathfrak{h}^1(M)$ with $\mathfrak{bmo}(M)$ (suitably defined), and proves that if p is in $(2, \infty)$, then $L^p(M)$ is an interpolation space between $L^2(M)$ and $\mathfrak{bmo}(M)$. Applications to a wide class of pseudo-differential operators are provided. Taylor also proves that $\mathfrak{h}^1(M)$ has an atomic decomposition, whose atoms are either atoms in $H^1(M)$ (in the sense of [7]), or functions in $L^2(M)$, supported in a ball of radius exactly equal to 1 and satisfying the standard size condition, but possibly not the cancellation condition. One of the limitations of this approach is that the geometric assumptions on the Riemannian manifolds are, as mentioned above, quite stringent. One of its advantages is that it reduces any estimate involving $\mathfrak{h}^1(M)$ to corresponding local estimates for $\mathfrak{h}^1(\mathbb{R}^n)$.

It is worth observing that each of the spaces $H^1(M)$ and $\mathfrak{h}^1(M)$ has its own advantages and range of applications. Clearly $\mathfrak{h}^1(M)$ is a flexible space that is preserved by the action of suitable classes of pseudo-differential operators [13,25,27]. However, it is not apt to obtain endpoint estimates for certain singular integral operators like, for instance, the purely imaginary powers of the translated Ornstein–Uhlenbeck operator [9], where $H^1(M)$ functions perfectly.

As mentioned above, one of the motivations of our work is to extend considerably the range of applicability of the approach of Strichartz and Taylor. Our ambient space is a measured metric space possessing AMP and LDP. It is well known that the assumptions above are

satisfied whenever M is a Riemannian manifold with Ricci curvature bounded from below (without assuming that M has positive injectivity radius), a condition that does not require any control on the derivatives of the metric tensor. Note that such manifolds may have exponential volume growth, so that they may not be homogeneous spaces in the sense of Coifman–Weiss. Note that we do not assume that M possesses the so-called uniform ball size condition, i.e., it may happen that $\inf \{\mu(B) : r_B = r\} = 0$ and $\sup \{\mu(B) : r_B = r\} = +\infty$ for each $r > 0$.

We emphasise the fact that our methods are quite different from those of Taylor, for we cannot reduce the analysis to that of Goldberg on Euclidean spaces. We give an atomic definition of $\mathfrak{h}^1(M)$: When M is a manifold of strongly bounded geometry, $\mathfrak{h}^1(M)$ agrees with the space defined by Taylor. We prove that the topological dual of $\mathfrak{h}^1(M)$ may be identified with a local space $\mathfrak{bmo}(M)$ of functions of bounded mean oscillation in an appropriate sense (see Sects. 5 and 6), and that if $p \in (1, 2)$, then $L^p(M)$ is a complex interpolation space between $\mathfrak{h}^1(M)$ and $L^2(M)$ (see Sect. 8). Applications to the study of the translated Riesz transform and of spectral multipliers of the Laplace–Beltrami operator on manifolds with Ricci curvature bounded from below will be given in Sect. 10.

Finally, a few words concerning our second goal. A basic question concerning the Riesz transform $\mathcal{R} = \nabla \mathcal{L}^{-1/2}$ (here \mathcal{L} denotes the Laplace–Beltrami operator on M) is to characterise the space $H^1_{\mathcal{R}}(M)$ of all functions f in $L^1(M)$ such that $|\mathcal{R}f|$ is in $L^1(M)$. In many cases, for instance in \mathbb{R}^n , such space is just the Hardy space $H^1(\mathbb{R}^n)$. Recent results of Mauzeri et al. [21] show that if \mathbb{D} denotes the hyperbolic disc, then $H^1_{\mathcal{R}}(\mathbb{D})$ is not $H^1(\mathbb{D})$. The analysis of $\mathfrak{h}^1(M)$ performed in this paper will be the key to provide a characterisation of $H^1_{\mathcal{R}}(M)$ for a comparatively large class of Riemannian manifolds.

We will use the “variable constant convention” and denote by C , possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

For each p in $[1, \infty]$, we denote by p' the index conjugate to p , i.e., $p' = p/(p - 1)$.

2 Notation, terminology, and geometric assumptions

Suppose that (M, d, μ) is a measured metric space, and denote by \mathcal{B} the family of all balls on M . We assume that $\mu(M) > 0$ and that every ball has finite measure. For each B in \mathcal{B} , we denote by c_B and r_B the centre and the radius of B , respectively. Furthermore, we denote by kB the ball with centre c_B and radius kr_B . For each s in \mathbb{R}^+ , we denote by \mathcal{B}_s the family of all balls B in \mathcal{B} such that $r_B \leq s$.

We say that M possesses the *local doubling property* (LDP) if for every s in \mathbb{R}^+ , there exists a constant D_s such that

$$\mu(2B) \leq D_s \mu(B) \quad \forall B \in \mathcal{B}_s.$$

Remark 1 The LDP implies that for each $\tau \geq 1$ and for each s in \mathbb{R}^+ , there exists a constant C such that

$$\mu(B') \leq C \mu(B) \tag{1}$$

for each pair of balls B and B' , with $B \subset B'$, B in \mathcal{B}_s , and $r_{B'} \leq \tau r_B$. We shall denote by $D_{\tau,s}$ the smallest constant for which (1) holds. In particular, if (1) holds (with the same constant) for all balls B in \mathcal{B} , then μ is doubling and we shall denote by $D_{\tau,\infty}$ the smallest constant for which (1) holds.

We say that M possesses the *approximate midpoint property* (AMP) if there exist R_0 in $[0, \infty)$ and β in $[1/2, 1)$ such that for every pair of points x and y in M with $d(x, y) > R_0$, there exists a point z in M such that $d(x, z) < \beta d(x, y)$ and $d(y, z) < \beta d(x, y)$. This is clearly equivalent to the requirement that there exists a ball B containing x and y such that $r_B < \beta d(x, y)$.

If M is a measured metric space for which $R_0 = 0$ and each segment has a midpoint, then we say that M possesses the midpoint property (MP). Typically graphs enjoy the AMP, but quite often a “segment” in a graph has not a midpoint. On the other hand, every connected Riemannian manifold possesses the MP, and the constant R_0 is equal to 0.

All the results in this paper hold under the assumption that M possesses the local doubling property LDP and the approximate midpoint property AMP. However, for the sake of simplicity, **hereafter we assume that M possesses the local doubling property LDP and the midpoint property MP** (with $R_0 = 0$). This leads to cleaner statements and allows us to avoid certain annoying technicalities, which makes the reading more difficult. The interested reader may easily fill the additional details and come to prove our results under the assumption that M satisfies the AMP only. To this end, [7] may serve as a guide, for the details of proofs therein are done under the assumption that M possesses the AMP only.

Given a positive number η , a set \mathfrak{M} of points in M is a η -discretisation of M if it is maximal with respect to the following property:

$$\min\{d(z, w) : z, w \in \mathfrak{M}, z \neq w\} > \eta \quad \text{and} \quad d(\mathfrak{M}, x) \leq \eta \quad \forall x \in M.$$

It is straightforward to show that η -discretisations exist for every η . For each subset E of M , we set

$$\mathfrak{M}_E := \{z \in \mathfrak{M} : B_{2\eta}(z) \cap E \neq \emptyset\},$$

and denote by $\#\mathfrak{M}_E$ its cardinality. If x is a point in M , we write \mathfrak{M}_x instead of $\mathfrak{M}_{\{x\}}$, for simplicity. Note that $\#\mathfrak{M}_x$ is the number of balls of the covering $\{B_{2\eta}(z) : z \in \mathfrak{M}\}$ that contain x .

Lemma 1 *Suppose that M possesses the LDP and the MP (with $R_0 = 0$, see, however, the remark before the definition of discretisations). Assume that c is a positive number and that \mathfrak{M} is a $c/2$ -discretisation. The following hold:*

- (i) *the family $\{B_c(z) : z \in \mathfrak{M}\}$ is a locally uniformly finite covering of M , and there exists a constant C , depending on c , such that $\sup_{x \in M} \#\mathfrak{M}_x \leq C$;*
- (ii) *for every $b > c$ there exists a constant C , which depends on b and c , such that $\#\mathfrak{M}_B \leq C$ for every ball B of radius b .*

Proof First we prove (i). Since \mathfrak{M} is a $c/2$ -discretisation, $d(\mathfrak{M}, x) \leq c/2$ for every x in M , so that $\{B_c(z) : z \in \mathfrak{M}\}$ is a covering of M . Observe that if z is in \mathfrak{M}_x , then $B_c(z) \subset B_{2c}(x) \subset B_{3c}(z)$. This and the LDP (1) imply that

$$\mu(B_{2c}(x)) \leq \mu(B_{3c}(z)) \leq D_{12,c/4} \mu(B_{c/4}(z)).$$

Since $B_{c/4}(z) \subset B_{2c}(x)$ and the balls of the family $\{B_{c/4}(z) : z \in \mathfrak{M}\}$ are pairwise disjoint,

$$\begin{aligned} \mu(B_{2c}(x)) &\geq \mu\left(\bigcup_{z \in \mathfrak{M}_x} B_{c/4}(z)\right) \\ &= \sum_{z \in \mathfrak{M}_x} \mu(B_{c/4}(z)) \end{aligned}$$

$$\geq \frac{\#\mathfrak{M}_x}{D_{12,c/4}} \mu(B_{2c}(x)),$$

whence $\#\mathfrak{M}_x \leq D_{12,c/4}$, as required.

Now we prove (ii). Denote by B' the ball with centre c_B and radius $b + 2c$. Observe that if z is in \mathfrak{M}_B , and x belongs to $B_c(z)$, then $d(x, c_B) < b + 2c$. Therefore, x is in B' . This and (i) imply that

$$\sum_{z \in \mathfrak{M}_B} \mathbf{1}_{B_c(z)} \leq D_{12,c/4} \mathbf{1}_{B'}.$$

By integrating both sides of this inequality, we see that

$$\sum_{z \in \mathfrak{M}_B} \mu(B_{c/4}(z)) \leq \sum_{z \in \mathfrak{M}_B} \mu(B_c(z)) \leq D_{12,c/4} \mu(B').$$

Recall that the balls $B_{c/4}(z)$, $z \in \mathfrak{M}_B$, are pairwise disjoint, and that $\mu(B_{c/4}(z)) \geq D_{4(b/c)+8}^{-1} \mu(B')$ by the LDP, so that

$$\#\mathfrak{M}_B D_{4(b/c)+8}^{-1} \mu(B') \leq D_{12,c/4} \mu(B'),$$

from which the required estimate follows directly. □

Remark 2 A careful examination of the proof of Lemma 1 reveals that, in fact, we have proved the following: $\sup_{x \in M} \#\mathfrak{M}_x \leq D_{12,c/4}$ and $\#\mathfrak{M}_B \leq D_{4(b/c)+8} D_{12,c/4}$ (see Remark 1 for the definition of $D_{\tau,s}$). We have made here the choice not to keep track of the precise dependence of the constants appearing in the statement from the various parameters. We shall do the same in all the subsequent sections.

3 The local Hardy space $\mathfrak{h}^1(M)$

Definition 1 Suppose that p is in $(1, \infty]$ and let p' denote the index conjugate to p . Suppose that b is a positive number. A *standard p -atom at scale b* is a function a in $L^1(M)$ supported in a ball B in \mathcal{B}_b satisfying the following conditions:

(i) *size condition:*

$$\|a\|_\infty \leq \mu(B)^{-1} \text{ if } p = \infty \text{ and } \|a\|_p \leq \mu(B)^{-1/p'} \text{ if } p \in (1, \infty);$$

(ii) *cancellation condition:*

$$\int_B a \, d\mu = 0.$$

A *global p -atom at scale b* is a function a in $L^1(M)$ supported in a ball B of radius *exactly equal to b* satisfying the size condition above (but possibly not the cancellation condition). Standard and global p -atoms will be referred to simply as *p -atoms*.

Definition 2 Let b be a positive number. The *local atomic Hardy space $\mathfrak{h}_b^{1,p}(M)$* is the space of all functions f in $L^1(M)$ that admit a decomposition of the form

$$f = \sum_{j=1}^\infty \lambda_j a_j, \tag{2}$$

where the a_j 's are p -atoms at scale b and $\sum_{j=1}^\infty |\lambda_j| < \infty$. The norm $\|f\|_{\mathfrak{h}_b^{1,p}}$ of f is the infimum of $\sum_{j=1}^\infty |\lambda_j|$ over all decompositions (2) of f .

We shall prove that $\mathfrak{h}_b^{1,p}(M)$ is independent of p and b , and later the space $\mathfrak{h}_1^{1,p}(M)$ will be denoted simply by $\mathfrak{h}^1(M)$.

The following lemma produces an economical decomposition of atoms supported in ‘‘big’’ balls as finite linear combinations of atoms supported in smaller balls. This result extends to global atoms the economical decomposition for standard atoms proved in [20, Lemma 6.1]; see also [7, Prop. 4.3 (i)] for a ‘‘less economical’’ decomposition. It is worth observing that our proof does not require the uniform ball size condition, which, instead, is used in [20, Lemma 6.1]. Furthermore, the proof of the following lemma is somewhat simpler than the proof of [20, Lemma 6.1], for we can decompose atoms supported in ‘‘big’’ balls as finite linear combinations of *global* atoms supported in smaller balls, so that we need not care about cancellations.

Lemma 2 *Suppose that p is in $(1, \infty]$ and that $b > c > 0$. Then, each p -atom a at scale b may be written as a finite linear combination of global p -atoms at scale c , and there exists a constant C , independent of the atom a , such that $\|a\|_{\mathfrak{h}_c^{1,p}} \leq C$.*

Proof Suppose that a is a p -atom at scale b (either standard or global), supported in the ball B , and denote by \mathfrak{M} a $c/2$ -discretisation of M . We denote by B_1, \dots, B_N the balls with centre at points in \mathfrak{M}_B and radius c , and define

$$\psi_j := \frac{\mathbf{1}_{B_j}}{\sum_{k=1}^N \mathbf{1}_{B_k}}.$$

Clearly $\sum_{k=1}^N \psi_j$ is equal to 1 on B . Set $\lambda_j := \|a\psi_j\|_p \mu(B_j)^{1/p'}$, $b_j := a\psi_j \lambda_j^{-1}$, and write $a = \sum_{j=1}^N \psi_j a = \sum_{j=1}^N \lambda_j b_j$. Clearly b_j is a global p -atom at scale c , whence

$$\begin{aligned} \|a\|_{\mathfrak{h}_c^{1,p}} &\leq \sum_{j=1}^N |\lambda_j| \\ &= \sum_{j=1}^N \|a\psi_j\|_p \mu(B_j)^{1/p'} \\ &\leq \left[\sum_{j=1}^N \|a\psi_j\|_p^p \right]^{1/p} \left[\sum_{j=1}^N \mu(B_j) \right]^{1/p'}; \end{aligned}$$

we have used Hölder’s inequality with exponents p and p' in the last inequality. Observe that the balls B_j are contained in the ball with centre c_B and radius $b + 2c$. Since, by Lemma 1 (i) each point in $B_{b+2c}(c_B)$ is covered by at most C balls B_j , with C depending only on c ,

$$\sum_{j=1}^N \mu(B_j) \leq C \mu(B_{b+2c}(c_B)).$$

Similarly,

$$\sum_{j=1}^N \|a\psi_j\|_p^p = \int_M \sum_{j=1}^N \psi_j^p |a|^p \, d\mu$$

$$\begin{aligned} &\leq \|a\|_p^p \\ &\leq \mu(B)^{-p/p'}; \end{aligned}$$

we have used the fact that $0 \leq \psi_j \leq 1$, that $p > 1$, and that $\sum_{k=1}^N \psi_j = 1$ on B in the first inequality above and the size condition of the p -atom a in the second. By combining the preceding estimates, we obtain that there exists a constant C , depending on c and on p , such that

$$\|a\|_{\mathfrak{h}_c^{1,p}} \leq C \mu(B)^{-1/p'} \mu(B_{b+2c}(c_B))^{1/p'}.$$

Since B and $B_{b+2c}(c_B)$ have the same centre,

$$\mu(B_{b+2c}(c_B)) \leq D_{1+2(c/b),b} \mu(B),$$

whence $\|a\|_{\mathfrak{h}_c^{1,p}} \leq C D_{1+2(c/b),b}^{1/p'}$, as required. □

Proposition 1 *Suppose that p is in $(1, \infty]$ and that $b > c > 0$. A function f is in $\mathfrak{h}_c^{1,p}(M)$ if and only if f is in $\mathfrak{h}_b^{1,p}(M)$. Furthermore, there exist positive constants C_1 and C_2 , depending on b, c and p , such that*

$$C_1 \|f\|_{\mathfrak{h}_b^{1,p}} \leq \|f\|_{\mathfrak{h}_c^{1,p}} \leq C_2 \|f\|_{\mathfrak{h}_b^{1,p}} \quad \forall f \in \mathfrak{h}_c^{1,p}(M).$$

Proof We begin by showing that $\mathfrak{h}_c^{1,p}(M) \subset \mathfrak{h}_b^{1,p}(M)$, and that the left-hand inequality holds.

If a is a p -atom at scale c with support contained in B , then $a \left[\frac{\mu(B)}{\mu((b/c)B)} \right]^{1/p'}$ is a p -atom at scale b , and

$$\begin{aligned} \|a\|_{\mathfrak{h}_b^{1,p}} &\leq \left[\frac{\mu((b/c)B)}{\mu(B)} \right]^{1/p'} \\ &\leq D_{b/c,c}^{1/p'}. \end{aligned}$$

This implies that if f belongs to $\mathfrak{h}_c^{1,p}(M)$, then f is in $\mathfrak{h}_b^{1,p}(M)$ and $\|f\|_{\mathfrak{h}_b^{1,p}} \leq D_{b/c,c}^{1/p'} \|f\|_{\mathfrak{h}_c^{1,p}}$.

The reverse inclusion follows directly from Lemma 2. □

Remark 3 Suppose that p is in $(1, \infty]$. Then, for every b and c such that $b > c > 0$ the spaces $\mathfrak{h}_b^{1,p}(M)$ and $\mathfrak{h}_c^{1,p}(M)$ are isomorphic (in fact, they contain the same functions) by Proposition 1. Hereafter, we denote the space $\mathfrak{h}_1^{1,p}(M)$, endowed with any of the equivalent norms defined above, simply by $\mathfrak{h}^{1,p}(M)$.

In Sect. 6, we shall prove that $\mathfrak{h}^{1,p}(M)$ does not depend on the parameter p in $(1, \infty)$, and then we shall denote all the spaces $\mathfrak{h}^{1,p}(M)$ simply by $\mathfrak{h}^1(M)$.

4 The local ionic space $\mathfrak{h}_I^1(M)$

In this section, we show that $\mathfrak{h}^1(M)$ admits a ‘‘ionic decomposition’’. Specifically, we shall define a ‘‘ionic’’ Hardy space $\mathfrak{h}_I^1(M)$. The space $\mathfrak{h}_I^1(M)$ is defined much as $\mathfrak{h}^1(M)$, but with ions in place of atoms. It will be clear from the definition that every atom is an ion, but

not conversely. In fact, we shall consider a one-parameter family of different types of ions. When this parameter is equal to one, and M is a Riemannian manifold with strongly bounded geometry, then $\mathfrak{h}_I^1(M)$ is the local Hardy space introduced by Taylor in [27].

Definition 3 Suppose that p is in $(1, \infty]$ and that α is a positive real number. A (p, α) -ion is a function g in $L^1(M)$ supported in a ball B with the following properties:

- (i) $\|g\|_\infty \leq \mu(B)^{-1}$ if $p = \infty$ and $\|g\|_p \leq \mu(B)^{-1/p'}$ if $p \in (1, \infty)$;
- (ii) $\left| \iint_B g \, d\mu \right| \leq r_B^\alpha$.

A $(p, 1)$ -ion will be simply called a p -ion.

Note that Taylor considered ∞ -ions only.

Definition 4 Suppose that b and α are positive real numbers. The local ionic Hardy space $\mathfrak{h}_{I,b}^{1,p,\alpha}(M)$ is the space of all functions f in $L^1(M)$ that admit a decomposition of the form

$$f = \sum_{j=1}^{\infty} \mu_j g_j, \tag{3}$$

where the g_j 's are (p, α) -ions supported in balls of radius at most b and $\sum_{j=1}^{\infty} |\mu_j| < \infty$. The norm $\|f\|_{\mathfrak{h}_{I,b}^{1,p,\alpha}}$ of f is the infimum of $\sum_{j=1}^{\infty} |\mu_j|$ over all decompositions (3) of f .

If $\alpha = 1$, then we denote $\mathfrak{h}_{I,b}^{1,p,\alpha}(M)$ simply by $\mathfrak{h}_{I,b}^{1,p}(M)$.

We shall prove that the spaces $\mathfrak{h}_{I,b}^{1,p,\alpha}(M)$ do not depend on α . Indeed, we shall show that all these spaces coincide with the atomic spaces $\mathfrak{h}_b^{1,p}(M)$ and that the corresponding norms are equivalent. We shall make use of the following remark.

Remark 4 If $\alpha \geq 1$, then it is easy to show that $\mathfrak{h}_{I,b}^{1,p,\alpha}(M) \subset \mathfrak{h}_{I,b}^{1,p}(M)$ and $\|f\|_{\mathfrak{h}_{I,b}^{1,p}} \leq \|f\|_{\mathfrak{h}_{I,b}^{1,p,\alpha}}$ for every f in $\mathfrak{h}_{I,b}^{1,p,\alpha}(M)$.

Indeed, consider a (p, α) -ion g supported in a ball B . If $r_B \geq 1$, then the size condition implies that

$$\left| \iint_B g \, d\mu \right| \leq \|g\|_p \mu(B)^{1/p'} \leq 1 \leq r_B.$$

If $r_B < 1$, then

$$\left| \iint_B g \, d\mu \right| \leq r_B^\alpha \leq r_B.$$

Hence, g is a p -ion. The inclusion $\mathfrak{h}_{I,b}^{1,p,\alpha}(M) \subset \mathfrak{h}_{I,b}^{1,p}(M)$ and the desired norm inequality follow.

Theorem 1 Suppose that $p \in (1, \infty]$, $\alpha > 0$ and $b > 0$. The spaces $\mathfrak{h}_{I,b}^{1,p,\alpha}(M)$ and $\mathfrak{h}_b^{1,p}(M)$ coincide. Furthermore, there exist a constant C_1 , depending on b and α , and a constant C_2 , depending on b, α and p , such that

$$C_1 \|f\|_{\mathfrak{h}_{I,b}^{1,p,\alpha}} \leq \|f\|_{\mathfrak{h}_b^{1,p}} \leq C_2 \|f\|_{\mathfrak{h}_{I,b}^{1,p,\alpha}} \quad \forall f \in \mathfrak{h}_b^{1,p}(M).$$

Proof First we prove that $\mathfrak{h}_b^{1,p}(M) \subset \mathfrak{h}_{I,b}^{1,p,\alpha}(M)$, by showing that each p -atom at scale b is a multiple of a (p, α) -ion supported in the same ball. Indeed, clearly each standard p -atom is a (p, α) -ion. Now, suppose that a is a global p -atom supported in a ball B of radius b . Then, the size condition implies that

$$\left| \int_B a \, d\mu \right| \leq \|a\|_p \mu(B)^{1/p'} \leq 1. \tag{4}$$

If $b \geq 1$, then $\left| \int_B a \, d\mu \right| \leq b^\alpha$ and a is a (p, α) -ion at scale b . If $b < 1$, then it is clear that $b^\alpha a$ is a (p, α) -ion at scale b . Therefore, $\|a\|_{\mathfrak{h}_{I,b}^{1,p,\alpha}} \leq 1/b^\alpha$. Thus, $\mathfrak{h}_b^{1,p}(M) \subset \mathfrak{h}_{I,b}^{1,p,\alpha}(M)$ and

$$\|f\|_{\mathfrak{h}_{I,b}^{1,p,\alpha}} \leq \max(1, b^{-\alpha}) \|f\|_{\mathfrak{h}_b^{1,p}} \quad \forall f \in \mathfrak{h}_b^{1,p}(M).$$

To prove the reverse inclusion, let g be a (p, α) -ion with support contained in B , with $r_B \leq b$. We write $g = a + h$, where

$$a = g - \frac{\chi_B}{\mu(B)} \int_B g \, d\mu \quad \text{and} \quad h = \frac{\chi_B}{\mu(B)} \int_B g \, d\mu.$$

Observe that a is a multiple of a standard p -atom at scale b . Indeed, $\int_B a \, d\mu = 0$ and

$$\begin{aligned} \|a\|_p &\leq \|g\|_p + \left| \int_B g \, d\mu \right| \frac{\|\chi_B\|_p}{\mu(B)} \\ &\leq \mu(B)^{-1/p'} + r_B^\alpha \mu(B)^{-1/p'} \\ &\leq (1 + b^\alpha) \mu(B)^{-1/p'}, \end{aligned}$$

so that $\|a\|_{\mathfrak{h}_b^{1,p}} \leq 1 + b^\alpha$. Now, if $r_B = b$, then

$$\begin{aligned} \|h\|_p &= \mu(B)^{-1/p'} \left| \int_B g \, d\mu \right| \\ &\leq \mu(B)^{-1/p'} b^\alpha, \end{aligned}$$

so that $b^{-\alpha} h$ is a global p -atom at scale b , whence $\|h\|_{\mathfrak{h}_b^{1,p}} \leq b^\alpha$, and $\|g\|_{\mathfrak{h}_b^{1,p}} \leq 1 + 2b^\alpha$.

If, instead, $r_B < b$, then we decompose h as a finite combination of $\mathfrak{h}_b^{1,p}$ -atoms as follows. Set $N := \lceil \log_2(b/r_B) \rceil$ and write $h = \sum_{i=1}^{N+2} h_i$, where

$$h_i = \left[\frac{\chi_{2^{i-1}B}}{\mu(2^{i-1}B)} - \frac{\chi_{2^i B}}{\mu(2^i B)} \right] \left(\int_B g \, d\mu \right) \quad i = 1, \dots, N + 1$$

and $h_{N+2} = \frac{\chi_{2^{N+1}B}}{\mu(2^{N+1}B)} \left(\int_B g \, d\mu \right)$. A straightforward computation shows that for all $i = 1, \dots, N + 1$,

$$\begin{aligned} \int_M \left| \frac{\chi_{2^{i-1}B}}{\mu(2^{i-1}B)} - \frac{\chi_{2^i B}}{\mu(2^i B)} \right|^p d\mu &\leq \frac{\mu(2^i B \setminus 2^{i-1}B)}{\mu(2^i B)^p} + \frac{\mu(2^{i-1}B)}{\mu(2^{i-1}B)^p} \\ &\leq \mu(2^i B)^{1-p} + \mu(2^{i-1}B)^{1-p} \\ &\leq 2 \mu(2^{i-1}B)^{1-p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|h_i\|_p &\leq r_B^\alpha 2^{1/p} \mu(2^{i-1}B)^{-1/p'} \\ &\leq r_B^\alpha 2^{1/p} D_{2,b/2}^{1/p'} \mu(2^i B)^{-1/p'}; \end{aligned}$$

the last inequality follows from the estimate $\mu(2^i B) \leq D_{2,b/2} \mu(2^{i-1} B)$. Since $2^i B \in \mathcal{B}_b$ for all $i = 1, \dots, N$, $h_i/[2^{1/p} D_{2,b/2}^{1/p'} r_B^\alpha]$ is a standard p -atom, so that $\|h_i\|_{\mathfrak{h}_b^{1,p}} \leq 2^{1/p} D_{2,b/2}^{1/p'} r_B^\alpha$.

Furthermore, the functions h_{N+1} and h_{N+2} are supported in the ball $2^{N+1}B$, which has radius $\leq 2b$. Denote by B' the ball with the same centre as B and radius $2b$. Then,

$$\begin{aligned} \|h_{N+1}\|_p &\leq 2^{1/p} r_B^\alpha \mu(2^{N+1}B)^{-1/p'} \\ &\leq 2^{1/p} r_B^\alpha \frac{\mu(B')^{1/p'}}{\mu(2^{N+1}B)^{1/p'}} \mu(B')^{-1/p'} \\ &\leq D_{2,2b}^{1/p'} 2^{1/p} r_B^\alpha \mu(B')^{-1/p'}, \end{aligned}$$

and, similarly, $\|h_{N+2}\|_p \leq D_{2,2b}^{1/p'} r_B^\alpha \mu(B')^{-1/p'}$. Thus,

$$\|h_{N+1} + h_{N+2}\|_{\mathfrak{h}_{2b}^{1,p}} \leq D_{2,2b}^{1/p'} (2^{1/p} + 1) r_B^\alpha.$$

Then, by Proposition 1, there exists a constant C , depending on b and p , such that

$$\|h_{N+1} + h_{N+2}\|_{\mathfrak{h}_b^{1,p}} \leq C r_B^\alpha.$$

By combining these estimates, we see that there exists a constant C , which depends on b and p such that

$$\|h\|_{\mathfrak{h}_b^{1,p}} \leq \sum_{i=1}^{N+2} \|h_i\|_{\mathfrak{h}_b^{1,p}} \leq C r_B^\alpha [N + 1].$$

Now, observe that

$$r_B^\alpha N \leq r_B^\alpha \log_2(b/r_B) \leq r_B^\alpha [\log_2 b - \log_2 r_B] \leq r_B^\alpha \log_2 b \leq b^\alpha \log_2 b.$$

Hence, $\|h\|_{\mathfrak{h}_b^{1,p}} \leq C$, so that each (α, p) -ion g is in $\mathfrak{h}_b^{1,p}(M)$ and $\|g\|_{\mathfrak{h}_b^{1,p}} \leq C$, where the constant C depends only on b, α and p , as required. \square

We have already mentioned that the spaces $\mathfrak{h}_b^{1,p}(M)$ will be proved to be independent of the parameters p and b . Then, by Theorem 1, for p in $(1, \infty)$, $b > 0$ and α in \mathbb{R}^+ , the spaces $\mathfrak{h}_{I,b}^{1,p,\alpha}(M)$ coincide with equivalence of the norms.

Remark 5 We shall denote by $\mathfrak{h}_I^1(M)$ all the spaces $\mathfrak{h}_{I,1}^{1,p,\alpha}(M)$, endowed with any of the equivalent norms defined above.

5 The space $\mathfrak{bmo}(M)$

Suppose that q is in $[1, \infty)$ and b is in \mathbb{R}^+ . For each locally integrable function f , define the local sharp maximal function $f_b^{\sharp,q}$ by

$$f_b^{\sharp,q}(x) = \sup_{B \in \mathcal{B}_b(x)} \left(\frac{1}{\mu(B)} \int_B |f - f_B|^q d\mu \right)^{1/q} \quad \forall x \in M,$$

where f_B denotes the average of f over B and $\mathcal{B}_b(x)$ denotes the family of all balls in \mathcal{B}_b centred at the point x . Define also the modified local sharp maximal function $N_b^q(f)$ by

$$N_b^q(f)(x) := f_b^{\sharp,q}(x) + \left[\frac{1}{\mu(B_b(x))} \int_{B_b(x)} |f|^q d\mu \right]^{1/q} \quad \forall x \in M,$$

where $B_b(x)$ denotes the ball with centre x and radius b . Denote by $\mathfrak{bmo}_b^q(M)$ the space of all locally integrable functions f such that $N_b^q(f)$ is in $L^\infty(M)$, endowed with the norm

$$\|f\|_{\mathfrak{bmo}_b^q} = \|N_b^q(f)\|_\infty.$$

The space $\mathfrak{bmo}_b^q(M)$ is related to the space $BMO_b^q(M)$, introduced in [7]. The latter is the Banach space of all locally integrable functions f (modulo constants) such that

$$\|f\|_{BMO_b^q} = \|f_b^{\sharp,q}\|_\infty < \infty.$$

As shown in [7], the spaces $BMO_b^q(M)$ do not depend on the parameters q and b and we denote them all by $BMO(M)$.

Remark 6 Given f in $\mathfrak{bmo}_b^q(M)$, we have

$$\|f_b^{\sharp,q}\|_\infty \leq \|N_b^q(f)\|_\infty = \|f\|_{\mathfrak{bmo}_b^q}.$$

Denote by $[f]$ the equivalence class in $BMO_b^q(M)$ which contains f . By the estimate above, the linear map $\iota : \mathfrak{bmo}_b^q(M) \rightarrow BMO_b^q(M)$, defined by $\iota(f) = [f]$, is continuous, i.e.,

$$\|\iota(f)\|_{BMO_b^q} \leq \|f\|_{\mathfrak{bmo}_b^q} \quad \forall f \in \mathfrak{bmo}_b^q(M). \tag{5}$$

In the following proposition, we show that the space $\mathfrak{bmo}_b^q(M)$ does not depend on the parameters b and q in the appropriate ranges.

Proposition 2 *Suppose that q is in $[1, \infty)$ and that $b > c > 0$. The following hold:*

- (i) $\mathfrak{bmo}_b^q(M)$ and $\mathfrak{bmo}_c^q(M)$ coincide and their norms are equivalent;
- (ii) $\mathfrak{bmo}_1^q(M)$ and $\mathfrak{bmo}_1^1(M)$ coincide and their norms are equivalent.

Proof First we prove (i). Suppose that f is in $\mathfrak{bmo}_b^q(M)$. Since $c < b$, $f_c^{\sharp,q}(x) \leq f_b^{\sharp,q}(x)$. Moreover, for each $x \in M$

$$\begin{aligned} \frac{1}{\mu(B_c(x))} \int_{B_c(x)} |f|^q d\mu &\leq \frac{1}{\mu(B_c(x))} \int_{B_b(x)} |f|^q d\mu \\ &= \frac{\mu(B_b(x))}{\mu(B_c(x))} \frac{1}{\mu(B_b(x))} \int_{B_b(x)} |f|^q d\mu \\ &\leq D_{b/c,c} \frac{1}{\mu(B_b(x))} \int_{B_b(x)} |f|^q d\mu, \end{aligned}$$

(see (1)). Therefore, $N_c^q(f)(x) \leq D_{b/c,c}^{1/q} N_b^q(f)(x)$. Thus, f is in $\mathfrak{bmo}_c^q(M)$ and $\|f\|_{\mathfrak{bmo}_c^q} \leq D_{b/c,c}^{1/q} \|f\|_{\mathfrak{bmo}_b^q}$.

To prove the reverse inequality, observe that, by [7, Prop. 5.1], there exists a constant C_1 , depending only on b, c and M , such that

$$\|f_b^{\sharp,q}\|_\infty \leq C_1 \|f_c^{\sharp,q}\|_\infty \quad \forall f \in \mathfrak{bmo}_c^q(M).$$

Now suppose that B_b is a ball of radius b . Then,

$$\left[\frac{1}{\mu(B_b)} \int_{B_b} |f|^q \, d\mu \right]^{1/q} = \frac{1}{\mu(B_b)^{1/q}} \sup_{\|\phi\|_{L^{q'}(B_b)} \leq 1} \left| \int_{B_b} f \phi \, d\mu \right|,$$

where q' is the exponent conjugate to q . If ϕ is a function in $L^{q'}(B_b)$ satisfying $\|\phi\|_{L^{q'}(B_b)} \leq 1$, then $\mu(B_b)^{-1/q} \phi$ is a q' -global atom at scale b . Therefore, by Lemma 2, there exist q' -global atoms a_1, \dots, a_N at scale c supported in balls B_j such that $\mu(B_b)^{-1/q} \phi = \sum_{j=1}^N \lambda_j a_j$, with

$$\sum_j |\lambda_j| \leq C,$$

where C depends only on b, c and p . Thus, by Hölder’s inequality,

$$\begin{aligned} \frac{1}{\mu(B_b)^{1/q}} \left| \int_{B_b} f \phi \, d\mu \right| &= \left| \sum_{j=1}^N \lambda_j \int_{B_j} f a_j \, d\mu \right| \\ &\leq \sum_{j=1}^N |\lambda_j| \left[\int_{B_j} |f|^q \, d\mu \right]^{1/q} \|a_j\|_{q'} \\ &\leq \sum_{j=1}^N |\lambda_j| \left[\frac{1}{\mu(B_j)} \int_{B_j} |f|^q \, d\mu \right]^{1/q} \\ &\leq C \|f\|_{\mathfrak{bmo}_c^q}. \end{aligned}$$

The above estimates imply that $\|f\|_{\mathfrak{bmo}_b^q} \leq (C_1 + CN) \|f\|_{\mathfrak{bmo}_c^q}$, as required to conclude the proof of (i).

Next we prove (ii). Recall that the spaces $BMO^1(M)$ and $BMO^q(M)$ agree (with equivalence of norms) for all q in $(1, \infty)$ [7, Corollary 5.5]. Therefore, there exists a constant C such that

$$\|\iota(f)\|_{BMO^q} \leq C \|\iota(f)\|_{BMO^1} \leq C \|f\|_{\mathfrak{bmo}^1} \quad \forall f \in \mathfrak{bmo}^1(M),$$

where the last inequality follows from (5). Thus,

$$\left[\frac{1}{\mu(B)} \int_B |f - f_B|^q \, d\mu \right]^{1/q} \leq C \|f\|_{\mathfrak{bmo}^1} \quad \forall f \in \mathfrak{bmo}^1(M) \quad \forall B \in \mathcal{B}_1.$$

Now suppose that B_1 is a ball of radius 1. By the triangle inequality

$$\begin{aligned} \left[\frac{1}{\mu(B_1)} \int_{B_1} |f|^q \, d\mu \right]^{1/q} &\leq \left[\frac{1}{\mu(B_1)} \int_{B_1} |f - f_{B_1}|^q \, d\mu \right]^{1/q} + |f_{B_1}| \\ &\leq C \|f\|_{\mathfrak{bmo}^1} + \frac{1}{\mu(B_1)} \int_{B_1} |f| \, d\mu \\ &\leq (C + 1) \|f\|_{\mathfrak{bmo}^1}. \end{aligned}$$

These estimates imply that

$$\|f\|_{\mathfrak{bmo}^q} \leq (2C + 1)\|f\|_{\mathfrak{bmo}^1} \quad \forall f \in \mathfrak{bmo}^1(M),$$

whence $\mathfrak{bmo}^1(M) \subseteq \mathfrak{bmo}^q(M)$.

To prove the reverse containment, observe that, by Hölder’s inequality,

$$N_1^1(f)(x) \leq N_1^q(f)(x) \quad \forall x \in M,$$

so that $\|f\|_{\mathfrak{bmo}^1} \leq \|f\|_{\mathfrak{bmo}^q}$, and $\mathfrak{bmo}^q(M) \subseteq \mathfrak{bmo}^1(M)$.

The proof of (ii) is complete. □

Remark 7 In view of the observation above, all the spaces $\mathfrak{bmo}_b^q(M)$, $b > 0$, q in $[1, \infty)$, coincide. We shall denote them simply by $\mathfrak{bmo}(M)$, endowed with any of the equivalent norms $\|\cdot\|_{\mathfrak{bmo}_b^q}$. This remark will be important in the proof of the duality between $\mathfrak{h}^1(M)$ and $\mathfrak{bmo}(M)$.

6 Duality

In this section, we shall prove that the topological dual of $\mathfrak{h}^{1,p}(M)$ is isomorphic to $\mathfrak{bmo}^{p'}(M)$, where p' denotes the index conjugate to p . In view of Remark 7, we consider $\mathfrak{bmo}^{p'}(M)$ endowed with the norm $\|N_1^{p'}(\cdot)\|_\infty$. Similarly, in view of Remark 3, we may, and shall, consider $\mathfrak{h}^{1,p}(M)$, endowed with the $\mathfrak{h}_1^{1,p}(M)$ -norm.

We need more notation and some preliminary observations. Suppose that p is in $[1, \infty)$. For each closed ball B in M , we denote by $L^p(B)$ the space of all functions in $L^p(M)$ which are supported in B . The union of all spaces $L^p(B)$ as B varies over all balls coincides with the space $L_c^p(M)$ of all functions in $L^p(M)$ with compact support. Fix a reference point o in M and for each positive integer k denote by B_k the ball centred at o with radius k . A convenient way of topologising $L_c^p(M)$ is to interpret $L_c^p(M)$ as the strict inductive limit of the spaces $L_c^p(B_k)$ (see [2, II, p. 33] for the definition of the strict inductive limit topology). We denote by X^p the space $L_c^p(M)$ with this topology and write X_k^p for $L_c^p(B_k)$.

We denote by $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ the subspace of $\mathfrak{h}^{1,p}(M)$ consisting of all finite linear combinations of p -atoms. Clearly, $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ is dense in $\mathfrak{h}^{1,p}(M)$ with respect to the norm of $\mathfrak{h}^{1,p}(M)$. A natural norm on $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ is defined as follows:

$$\|f\|_{\mathfrak{h}_{\text{fin}}^{1,p}} = \inf \left\{ \sum_{j=1}^N |c_j| : f = \sum_{j=1}^N c_j a_j, \ a_j \text{ is a } p\text{-atom}, \ N \in \mathbb{N}^+ \right\}. \tag{6}$$

Note that the infimum is taken over *finite* linear combinations of atoms. Obviously,

$$\|f\|_{\mathfrak{h}^{1,p}} \leq \|f\|_{\mathfrak{h}_{\text{fin}}^{1,p}} \quad \forall f \in \mathfrak{h}_{\text{fin}}^{1,p}(M). \tag{7}$$

Remark 8 Observe also that $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ and $L_c^p(M)$ agree as vector spaces. Indeed, on the one hand, each function in $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ has finite L^p -norm and is compactly supported; hence, it belongs to $L_c^p(M)$. On the other hand, suppose that g is in $L_c^p(M)$ and denote by B a ball of radius ≥ 1 that contains the support of g . Then, $a := \|g\|_p^{-1} \mu(B)^{-1/p'} g$ is a global p -atom at scale r_B , which, by Lemma 2, may be written as a finite linear combination of global p -atoms at scale 1. Therefore, a is in $\mathfrak{h}_{\text{fin}}^{1,p}(M)$, whence so is g .

Define

$$f^{s,q}(x) := \sup_{B \in \mathcal{B}_1(x)} \inf_{c \in \mathbb{C}} \left[\frac{1}{\mu(B)} \int_B |f - c|^q d\mu \right]^{1/q} \quad \forall x \in M.$$

It is straightforward to check that $f^{s,q}(x) \leq f^{\sharp,q}(x) \leq 2 f^{s,q}(x)$ for all x in M . Thus,

$$\|f^{s,q}\|_\infty + \sup_{x \in M} \left[\frac{1}{\mu(B_1(x))} \int_{B_1(x)} |f|^q d\mu \right]^{1/q}$$

is an equivalent norm on $\text{bmo}^q(M)$. We shall write f^s , instead of $f^{s,1}$.

Lemma 3 *If $f \in \text{bmo}^q(M)$, then $|f| \in \text{bmo}^q(M)$ and $\||f|\|_{\text{bmo}^q} \leq 2 \|f\|_{\text{bmo}^q}$.*

Proof Indeed,

$$\begin{aligned} |f|^{\sharp,q}(x) &\leq 2 |f|^{s,q}(x) \\ &\leq 2 \sup_{B \in \mathcal{B}_1(x)} \left[\frac{1}{\mu(B)} \int_B \left| |f| - |f_B| \right|^q d\mu \right]^{1/q} \\ &\leq 2 \sup_{B \in \mathcal{B}_1(x)} \left[\frac{1}{\mu(B)} \int_B |f - f_B|^q d\mu \right]^{1/q} \\ &= 2 f^{\sharp,q}(x), \end{aligned}$$

whence

$$\begin{aligned} N^q(|f|)(x) &= |f|^{\sharp,q}(x) + \left[\frac{1}{\mu(B_1(x))} \int_{B_1(x)} |f|^q d\mu \right]^{1/q} \\ &\leq 2 f^{\sharp,q}(x) + \left[\frac{1}{\mu(B_1(x))} \int_{B_1(x)} |f|^q d\mu \right]^{1/q} \\ &\leq 2 N^q(f)(x), \end{aligned}$$

as required. □

Next we identify the dual of $\mathfrak{h}^1(M)$ with $\text{bmo}(M)$. The proof follows the lines of the classical result of Coifman and Weiss [10] in the case of spaces of homogeneous type, and of [7].

Theorem 2 *Suppose that p is in $(1, \infty)$ and let p' be the index conjugate to p . The following hold:*

(i) *for every g in $\text{bmo}^{p'}(M)$ the functional F , initially defined on $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ by the rule*

$$F(f) = \int_M f g d\mu,$$

has a unique bounded extension to $\mathfrak{h}^{1,p}(M)$. Furthermore,

$$\|F\| \leq 4 \|g\|_{\text{bmo}^{p'}},$$

where $\|F\|$ denotes the norm of F as a continuous linear functional on $\mathfrak{h}^{1,p}(M)$.

(ii) *for every continuous linear functional F on $\mathfrak{h}^{1,p}(M)$, there exists a function g_F in $\text{bmo}^{p'}(M)$ such that $\|g_F\|_{\text{bmo}^{p'}} \leq 3 \|F\|$ and*

$$F(f) = \int_M f g_F d\mu \quad \forall f \in \mathfrak{h}_{\text{fin}}^{1,p}(M).$$

Proof The proof of (i) is a straightforward adaptation of the original proof of Coifman and Weiss in the case of spaces of homogeneous type. The argument makes use of Lemma 3 above. We omit the details.

Next we prove (ii). Since F is a continuous linear functional on $\mathfrak{h}^{1,p}(M)$, for every p -atom a

$$|Fa| \leq \|F\| \|a\|_{\mathfrak{h}^{1,p}} \leq \|F\|,$$

because each p -atom has $\mathfrak{h}^{1,p}(M)$ -norm at most 1. Thus,

$$\sup \{|Fa| : a \text{ is a } \mathfrak{h}^{1,p}\text{-atom}\} \leq \|F\|.$$

If f is in $L^p(B)$, and $r_B \geq 1$, then $\|f\|_p^{-1} \mu(B)^{-1/p'} f$ is a global p -atom at scale r_B . Then, by Lemma 2, there exists a constant C , independent of f , such that

$$\|f\|_{\mathfrak{h}^{1,p}} \leq C \mu(B)^{1/p'} \|f\|_p,$$

whence

$$|Ff| \leq C \|F\| \mu(B)^{1/p'} \|f\|_p.$$

Hence, the restriction of F to X_k^p is a bounded linear functional on X_k^p for each k . Therefore, F is a continuous linear functional on X^p . Since the dual of X^p is the space $L_{loc}^{p'}(M)$, there exists a function g_F in $L_{loc}^{p'}(M)$ such that

$$Ff = \int_M f g_F \, d\mu \quad \forall f \in X^p. \tag{8}$$

In particular, this holds whenever f is a p -atom.

To conclude the proof, it suffices to prove that g_F belongs to $\mathfrak{bmo}^{p'}(M)$ and that

$$\|g_F\|_{\mathfrak{bmo}^{p'}} \leq 3 \|F\|. \tag{9}$$

Recall that we consider $\mathfrak{h}^{1,p}(M)$ endowed with the $\mathfrak{h}_1^{1,p}(M)$ norm (see the beginning of this section). Thus, we need to consider only atoms with support in balls of radius ≤ 1 . Suppose that B is a ball of radius at most 1, and observe that

$$\left[\int_B |g_F - (g_F)_B|^{p'} \, d\mu \right]^{1/p'} = \sup_{\|\varphi\|_{L^p(B)}=1} \left| \int_B \varphi (g_F - (g_F)_B) \, d\mu \right|.$$

But

$$\begin{aligned} \int_B \varphi (g_F - (g_F)_B) \, d\mu &= \int_B (\varphi - \varphi_B) (g_F - (g_F)_B) \, d\mu \\ &= \int_B (\varphi - \varphi_B) g_F \, d\mu, \end{aligned}$$

and since $\|\varphi\|_{L^p(B)} = 1$

$$|\varphi_B| \leq \left[\frac{1}{\mu(B)} \int_B |\varphi|^p \, d\mu \right]^{1/p} \leq \mu(B)^{-1/p}.$$

Moreover,

$$\begin{aligned} \|\varphi - \varphi_B\|_{L^p(B)} &\leq \|\varphi\|_{L^p(B)} + |\varphi_B| \mu(B)^{1/p} \\ &\leq 2, \end{aligned}$$

so that the function $(\varphi - \varphi_B)/(2\mu(B)^{1/p'})$ is a standard p -atom. Therefore,

$$\left| \int_B (\varphi - \varphi_B) g_F d\mu \right| \leq 2 \|F\| \mu(B)^{1/p'}.$$

By combining the estimates above, we conclude that for every ball B of radius at most 1

$$\left[\frac{1}{\mu(B)} \int_B |g_F - (g_F)_B|^{p'} d\mu \right]^{1/p'} \leq 2 \|F\|,$$

Now take a ball B of radius exactly equal to 1. We have

$$\left[\int_B |g_F|^{p'} d\mu \right]^{1/p'} = \sup_{\|\varphi\|_{L^p(B)=1}} \left| \int_B \varphi g_F d\mu \right|.$$

The function $\varphi/\mu(B)^{1/p'}$ is a global p -atom at scale 1, thus

$$\left| \int_B \varphi g_F d\mu \right| \leq \|F\| \mu(B)^{1/p'}.$$

Therefore, for every ball B of radius 1

$$\left[\frac{1}{\mu(B)} \int_B |g_F|^{p'} d\mu \right]^{1/p'} \leq \|F\|.$$

Combining these estimates, (9) follows. This concludes the proof of (ii) and of the theorem. □

In view of the last result, we are now able to prove that all the spaces $\mathfrak{h}^{1,p}(M)$, with p in $(1, \infty)$, coincide. Indeed, suppose that $1 < r < p < \infty$. Then, $\mathfrak{h}^{1,r}(M)^* = \mathfrak{h}^{1,p}(M)^*$, since $\mathfrak{bmo}^{r'}(M) = \mathfrak{bmo}^{p'}(M)$. Moreover, the identity is a continuous injection of $\mathfrak{h}^{1,p}(M)$ into $\mathfrak{h}^{1,r}(M)$ and $\mathfrak{h}^{1,p}(M)$ is a dense subspace of $\mathfrak{h}^{1,r}(M)$; therefore, the Hahn–Banach theorem implies that $\mathfrak{h}^{1,r}(M) = \mathfrak{h}^{1,p}(M)$.

7 Estimates for the operator N

The purpose of this section is to establish a basic $L^p(M)$ estimate for the operator N , which acts on a locally integrable function f by

$$Nf(x) = f^\sharp(x) + N_0f(x) \quad \forall x \in M,$$

where f^\sharp is the *local centred sharp maximal function* given by the formula

$$f^\sharp(x) = \sup_{B \in \mathcal{B}_1(x)} \frac{1}{\mu(B)} \int_B |f - f_B| d\mu \tag{10}$$

and

$$N_0f(x) = \frac{1}{\mu(B_1(x))} \int_{B_1(x)} |f| d\mu.$$

Note that $f^\sharp = f_1^{\sharp,1}$ in the notation of Sect. 5. The main result of this section, Theorem 3 below, will be the key to prove a basic interpolation results for $\mathfrak{h}^1(M)$ in the next section.

For each locally integrable function f , define the *local centred Hardy–Littlewood maximal function* $\mathcal{M}f$ as

$$\mathcal{M}f(x) = \sup_{0 < r \leq 1} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, d\mu.$$

The operator \mathcal{M} is bounded on $L^p(M)$ for every $p \in (1, \infty]$ and of weak type 1 (for the weak-type estimate, just follows the lines of the proof of the maximal inequality in [23]). Clearly $Nf(x) \leq 3 \mathcal{M}f(x)$, so that the L^p -boundedness of \mathcal{M} implies that for $1 < p < \infty$

$$\|f\|_p \geq C \|Nf\|_p \quad \forall f \in L^p(M).$$

In the next theorem, we prove a reverse inequality.

Theorem 3 *Suppose that p is in $(1, \infty)$. Then, there exists a constant C such that*

$$\|f\|_p \leq C \|Nf\|_p$$

for every $f \in L^1_{loc}(M)$ such that $Nf \in L^p(M)$.

We recall [7, Thm. 7.3] that if M possesses the isoperimetric property IP, then for each p in $(1, \infty)$ there exists a constant C such that

$$\|f\|_p \leq C \|f^\sharp\|_p \quad \forall f \in L^p(M). \tag{11}$$

Observe that this estimate may fail if M does not possess the isoperimetric property. For instance, (11) is false for $M = \mathbb{R}^n$, as shown in [17]. The inequality in Theorem 3 is weaker than (11), but it does not require the IP.

The proof of Theorem 3, which occupies the rest of this section, will make use of the so-called dyadic cubes introduced by David and Christ [6, 12] on spaces of homogeneous type. In fact, Christ’s construction requires only the local doubling property, as remarked in [7]. For the reader’s convenience, we recall the main properties of dyadic cubes.

Theorem 4 ([7, Thm. 3.2]) *There exist constants δ in $(0, 1)$, a_0, a_1 in \mathbb{R}^+ and a collection $\mathcal{Q} := \{Q_\alpha^k : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets of M such that*

- (i) for each k in \mathbb{Z} , the set $\bigcup_\alpha Q_\alpha^k$ is of full measure in M ;
- (ii) if $\ell \geq k$, then either $Q_\beta^\ell \subset Q_\alpha^k$ or $Q_\beta^\ell \cap Q_\alpha^k = \emptyset$;
- (iii) for each (k, α) and each $\ell < k$, there is a unique β such that $Q_\alpha^k \subset Q_\beta^\ell$;
- (iv) $\text{diam}(Q_\alpha^k) \leq a_1^k$;
- (v) there exists a point z_α^k in Q such that

$$B_{a_0 \delta^k} \left(z_\alpha^k \right) \subset Q_\alpha^k \subset B_{a_1 \delta^k} \left(z_\alpha^k \right).$$

We shall denote by \mathcal{Q}^k the class of all dyadic cubes of “resolution” k , i.e., the family of cubes $\{Q_\alpha^k : \alpha \in I_k\}$. We shall need the following additional properties of dyadic cubes.

Proposition 3 ([7, Prop. 3.4]) *Suppose that $b \in \mathbb{R}^+$, $\nu \in \mathbb{Z}$, and let δ, a_0 and a_1 be as in Theorem 4. The following hold:*

- (i) suppose that Q is in \mathcal{Q}^k for some $k \geq \nu$, and that B is a ball such that $c_B \in Q$. If $r_B \geq a_1 \delta^k$, then

$$\mu(B \cap Q) = \mu(Q); \tag{12}$$

if $r_B < a_1 \delta^k$, then

$$\mu(B \cap Q) \geq D_{a_1/(a_0 \delta), \delta^\nu}^{-1} \mu(B); \tag{13}$$

(ii) suppose that τ is in $[2, \infty)$. For each Q in \mathcal{Q} , the space $(Q, d|_Q, \mu|_Q)$ is of homogeneous type. Denote by $D_{\tau, \infty}^Q$ its doubling constant (see Remark 1 for the definition). Then,

$$\sup \left\{ D_{\tau, \infty}^Q : Q \in \bigcup_{k=v}^{\infty} \mathcal{Q}^k \right\} \leq D_{\tau, a_1 \delta^v} D_{a_1/(a_0 \delta), \delta^v};$$

(iii) for each ball B in \mathcal{B}_b , let k be the integer such that $\delta^k \leq r_B < \delta^{k-1}$, and let \tilde{B} denote the ball with centre c_B and radius $(1 + a_1) r_B$. Then, \tilde{B} contains all dyadic cubes in \mathcal{Q}^k that intersect B and

$$\mu(\tilde{B}) \leq D_{1+a_1, b} \mu(B);$$

(iv) suppose that B is in \mathcal{B}_b , and that k is an integer such that $\delta^k \leq r_B < \delta^{k-1}$. Then, there are at most $D_{(1+a_1)/(a_0 \delta), b}$ dyadic cubes in \mathcal{Q}^k that intersect B .

In particular, property (ii) states that, for fixed k , all the cubes in \mathcal{Q}^k are spaces of homogeneous type with doubling constants uniformly bounded from above. More precisely, for each cube Q in \mathcal{Q}^k

$$D_{\tau, \infty}^Q \leq C_{\tau, k} \quad \text{where} \quad C_{\tau, k} := D_{\tau, a_1 \delta^k} D_{a_1/(a_0 \delta), \delta^k} \tag{14}$$

For each locally integrable function f and each dyadic cube Q , the *noncentred Hardy–Littlewood maximal function* $\mathcal{M}^Q f$ is defined by

$$\mathcal{M}^Q f(x) = \sup_{B: B \cap Q \ni x} \frac{1}{\mu(B \cap Q)} \int_{B \cap Q} |f| \, d\mu \quad \forall x \in Q,$$

where each B is a ball in \mathcal{B} whose centre belongs to Q . The operator \mathcal{M}^Q is bounded on $L^p(Q)$ for every p in $(1, \infty]$ and of weak type 1. Furthermore, there exists a constant C_0 , depending only on the doubling constant of $(Q, d|_Q, \mu|_Q)$, such that for all Q in $\bigcup_{k \geq v} \mathcal{Q}^k$

$$\mu(\{x \in Q : \mathcal{M}^Q f(x) > \lambda\}) \leq \frac{C_0}{\lambda} \|f\|_{L^1(Q)}. \tag{15}$$

For each locally integrable function f and each dyadic cube Q , we define the *noncentred sharp maximal function* $f^{\sharp, Q}$ by

$$f^{\sharp, Q}(x) = \sup_{B: B \cap Q \ni x} \frac{1}{\mu(B \cap Q)} \int_{B \cap Q} |f - f_{B \cap Q}| \, d\mu \quad \forall x \in Q, \tag{16}$$

where B is a ball in \mathcal{B} whose centre belongs to Q and

$$f_{B \cap Q} = \frac{1}{\mu(B \cap Q)} \int_{B \cap Q} f \, d\mu.$$

We split the proof of Theorem 3 into a series of lemmas. For each $\lambda > 0$, we define

$$E_\lambda := \left\{ x \in Q : \mathcal{M}^Q f(x) > \lambda \right\}, \quad F_\lambda := \left\{ x \in Q : f^{\sharp, Q}(x) \leq \lambda \right\}$$

and $G_\lambda^{\beta, \gamma} := E_{\beta \lambda} \cap F_{\gamma \lambda}$.

Lemma 4 *Suppose that k is in \mathbb{Z} . Then, there exists a constant A such that for every $\beta > 2C_{2, k}$, $\gamma > 0$, f in $L^1_{loc}(M)$, and Q in \mathcal{Q}^k*

$$\mu(E_{\beta \lambda} \cap F_{\gamma \lambda}) \leq A \frac{\gamma}{\beta} \mu(E_\lambda)$$

for every $\lambda > \frac{C_0}{\mu(Q)} \|f\|_{L^1(Q)}$, where C_0 is as in (15).

We observe that the constant A in the statement above may very well depend on the resolution k . This will be no problem, for in the sequel we shall mainly work with cubes with a fixed resolution.

Proof Set $\lambda_0 := \frac{C_0}{\mu(Q)} \|f\|_{L^1(Q)}$. Since $\lambda > \lambda_0$, $\mu(E_\lambda) < \mu(Q)$, so that E_λ is a proper subset in Q . Since E_λ is open and Q is a space of homogeneous type, we can apply a Whitney-type covering lemma [10, Thm. 3.2] (with 1 in place of C and K therein), and obtain a sequence $\{B_i \cap Q\}$ of balls in Q , where $B_i \in \mathcal{B}$, such that:

- (i) $E_\lambda = \bigcup_i (B_i \cap Q)$;
- (ii) there exists a constant $K_0 = K_0(k)$ such that no point of E_λ belongs to more than K_0 balls $B_i \cap Q$;
- (iii) $(3B_i \cap Q) \cap ((E_\lambda)^c \cap Q) \neq \emptyset$.

Note that K_0 does not depend on the particular cube Q in \mathcal{Q}^k because K_0 depends only on the doubling constant of the space of homogeneous type, and for cubes of the same resolution, the doubling constants are uniformly bounded from above (see (14) above).

By assumption, $\beta > 2C_{2,k} > 2$. Then, $G_\lambda^{\beta,\gamma} \subset E_{\beta\lambda} \subset E_\lambda$, so that

$$\begin{aligned} \mu(G_\lambda^{\beta,\gamma}) &= \mu \left[G_\lambda^{\beta,\gamma} \cap \left(\bigcup_i (B_i \cap Q) \right) \right] \\ &= \mu \left[\bigcup_i (G_\lambda^{\beta,\gamma} \cap B_i) \right] \\ &\leq \sum_i \mu(G_\lambda^{\beta,\gamma} \cap B_i). \end{aligned}$$

If $G_\lambda^{\beta,\gamma} \cap B_i = \emptyset$ for some index i , we simply ignore the ball B_i ; otherwise, there exists at least a point $y_i \in G_\lambda^{\beta,\gamma} \cap B_i$, whence $f^\sharp, Q(y_i) \leq \gamma\lambda$.

We claim that

$$E_{\beta\lambda} \cap B_i \subseteq \left\{ x \in Q : \mathcal{M}^Q(f \chi_{5B_i})(x) > \frac{\beta\lambda}{C_{2,k}} \right\} \quad \forall \beta \geq C_{2,k}.$$

The claim will imply that

$$\mu \left(G_\lambda^{\beta,\gamma} \cap B_i \right) \leq \mu \left(E_{\beta\lambda} \cap B_i \right) \leq \mu \left(\left\{ \mathcal{M}^Q(f \chi_{5B_i}) > \frac{\beta\lambda}{C_{2,k}} \right\} \right).$$

To prove the claim, we consider the centred Hardy–Littlewood maximal function on the cube Q defined by

$$\widetilde{\mathcal{M}}^Q f(x) = \sup_{r>0} \frac{1}{\mu(B_r(x) \cap Q)} \int_{B_r(x) \cap Q} |f| \, d\mu \quad \forall x \in Q.$$

Since the restriction of μ to each cube Q is a doubling measure with doubling constant bounded above by $C_{2,k}$,

$$\mathcal{M}^Q f(x) \leq C_{2,k} \widetilde{\mathcal{M}}^Q f(x) \quad \forall x \in Q.$$

Suppose that $x \in E_{\beta\lambda} \cap B_i$ and $\beta \geq C_{2,k}$. We need to prove that

$$\mathcal{M}^Q (f \chi_{5B_i})(x) > \frac{\beta\lambda}{C_{2,k}}.$$

Clearly, $\widetilde{\mathcal{M}}^Q f(x) > \beta\lambda/C_{2,k}$, so that there exists a ball $B_r(x)$ such that

$$\frac{1}{\mu(B_r(x) \cap Q)} \int_{B_r(x) \cap Q} |f| d\mu > \frac{\beta\lambda}{C_{2,k}}.$$

Condition (iii) above implies that there exists a point x_i in $3B_i \cap Q$ such that

$$\mathcal{M}^Q f(x_i) \leq \lambda. \tag{17}$$

Since we have assumed that $\beta \geq C_{2,k}$, $x_i \notin B_r(x)$, for otherwise

$$\mathcal{M}^Q f(x_i) \geq \frac{1}{\mu(B_r(x) \cap Q)} \int_{B_r(x) \cap Q} |f| d\mu > \frac{\beta\lambda}{C_{2,k}} \geq \lambda.$$

Since x_i is in $3B_i \setminus B_r(x)$, $r < 4r_{B_i}$. Hence, $B_r(x) \subset 5B_i$ and

$$\frac{\beta\lambda}{C_{2,k}} < \frac{1}{\mu(B_r(x) \cap Q)} \int_{B_r(x) \cap Q} |f| \chi_{5B_i} d\mu \leq \mathcal{M}^Q(f \chi_{5B_i})(x).$$

This concludes the proof of the claim.

Now we observe that

$$\mathcal{M}^Q(f \chi_{5B_i})(x) \leq \mathcal{M}^Q((f - f_{5B_i \cap Q}) \chi_{5B_i})(x) + |f_{5B_i \cap Q}|.$$

Since x_i is in $3B_i \cap Q$ and $\mathcal{M}^Q f(x_i) \leq \lambda$ by (17),

$$|f_{5B_i \cap Q}| \leq \frac{1}{\mu(5B_i \cap Q)} \int_{5B_i \cap Q} |f| d\mu \leq \mathcal{M}^Q f(x_i) \leq \lambda.$$

Therefore, if $\beta > 2C_{2,k}$, then $|f_{5B_i \cap Q}| < \frac{\beta}{2C_{2,k}}\lambda$. This estimate, together with the weak type 1 inequality for \mathcal{M}^Q and the assumption that $f^{\#,Q}(y_i) \leq \gamma\lambda$, implies that, if $\beta > 2C_{2,k}$, then

$$\begin{aligned} \mu\left(\left\{\mathcal{M}^Q(f \chi_{5B_i}) > \frac{\beta\lambda}{C_{2,k}}\right\}\right) &\leq \mu\left(\left\{\mathcal{M}^Q((f - f_{5B_i \cap Q}) \chi_{5B_i}) > \frac{\beta\lambda}{2C_{2,k}}\right\}\right) \\ &\leq C_0 \frac{2C_{2,k}}{\beta\lambda} \int_Q |f - f_{5B_i \cap Q}| \chi_{5B_i} d\mu \\ &\leq C_0 \frac{2C_{2,k}}{\beta\lambda} \mu(5B_i \cap Q) f^{\#,Q}(y_i) \\ &\leq C_0 2C_{2,k} \frac{\gamma}{\beta} \mu(5B_i \cap Q). \end{aligned}$$

Thus, we have proved that

$$\mu\left(G_\lambda^{\beta,\gamma} \cap B_i\right) \leq C_0 2C_{2,k} \frac{\gamma}{\beta} \mu(5B_i \cap Q),$$

which, together with the doubling property on Q and condition (ii) above, implies that

$$\begin{aligned} \mu\left(G_\lambda^{\beta,\gamma}\right) &\leq 2C_{2,k} C_0 \frac{\gamma}{\beta} \sum_i \mu(5B_i \cap Q) \\ &\leq 2C_{2,k} C_0 C_{5,k} \frac{\gamma}{\beta} \sum_i \mu(B_i \cap Q) \\ &\leq 2C_{2,k} C_0 C_{5,k} K_0 \frac{\gamma}{\beta} \mu(E_\lambda), \end{aligned}$$

as required (with $A = 2C_{2,k} C_0 C_{5,k} K_0$).

□

Lemma 5 *For each integer k , there exists a constant $C = C(k)$ such that for each cube Q in \mathcal{Q}^k and every locally integrable function f*

$$\|f\|_{L^p(Q)}^p \leq C \left(\|f^{\sharp, Q}\|_{L^p(Q)}^p + \mu(Q)^{1-p} \|f\|_{L^1(Q)}^p \right).$$

Proof Since $\mathcal{M}^Q f \geq |f|$ almost everywhere, it suffices to show that

$$\|\mathcal{M}^Q f\|_{L^p(Q)}^p \leq C \left(\|f^{\sharp, Q}\|_{L^p(Q)}^p + \mu(Q)^{1-p} \|f\|_{L^1(Q)}^p \right),$$

We set $E_\lambda = \{x \in Q : \mathcal{M}^Q f(x) > \lambda\}$ and $\lambda_0 = \frac{C_0}{\mu(Q)} \|f\|_{L^1(Q)}$, as in Lemma 4. Note that for each $\beta > 0$

$$\begin{aligned} \|\mathcal{M}^Q f\|_{L^p(Q)}^p &= p \int_0^\infty \lambda^{p-1} \mu(E_\lambda) \, d\lambda \\ &= p \beta^p \int_0^\infty \lambda^{p-1} \mu(E_{\beta\lambda}) \, d\lambda \\ &= p \beta^p \int_0^{\lambda_0} \lambda^{p-1} \mu(E_{\beta\lambda}) \, d\lambda + p \beta^p \int_{\lambda_0}^{+\infty} \lambda^{p-1} \mu(E_{\beta\lambda}) \, d\lambda. \end{aligned}$$

Denote by I_1 and I_2 the first and the second integral in the last line above, respectively. Since the maximal operator \mathcal{M}^Q is of weak type 1,

$$\begin{aligned} I_1 &\leq C_0 \beta^{-1} \|f\|_{L^1(Q)} \int_0^{\lambda_0} \lambda^{p-2} \, d\lambda \\ &= \frac{C_0^p}{p-1} \beta^{-1} \mu(Q)^{1-p} \|f\|_{L^1(Q)}^p. \end{aligned}$$

Now, we choose $\beta > 2C_{2,k}$. Given $\gamma > 0$, we write I_2 as

$$\int_{\lambda_0}^{+\infty} \lambda^{p-1} \mu(E_{\beta\lambda} \cap \{f^{\sharp, Q} \leq \gamma\lambda\}) \, d\lambda + \int_{\lambda_0}^{+\infty} \lambda^{p-1} \mu(E_{\beta\lambda} \cap \{f^{\sharp, Q} > \gamma\lambda\}) \, d\lambda.$$

Then, by Lemma 4,

$$\begin{aligned} I_2 &\leq \frac{A\gamma}{\beta} \int_{\lambda_0}^{+\infty} \lambda^{p-1} \mu(E_\lambda) \, d\lambda + \int_{\lambda_0}^{+\infty} \lambda^{p-1} \mu(\{f^{\sharp, Q} > \gamma\lambda\}) \, d\lambda \\ &= \frac{A\gamma}{\beta} \int_{\lambda_0}^{+\infty} \lambda^{p-1} \mu(E_\lambda) \, d\lambda + \frac{1}{\gamma^p} \int_{\gamma\lambda_0}^{+\infty} \lambda^{p-1} \mu(\{f^{\sharp, Q} > \lambda\}) \, d\lambda \\ &\leq \frac{A\gamma}{p\beta} \|\mathcal{M}^Q f\|_{L^p(Q)}^p + \frac{1}{p\gamma^p} \|f^{\sharp, Q}\|_{L^p(Q)}^p. \end{aligned}$$

By combining the estimates above, we see that

$$(1 - A\beta^{p-1}\gamma) \|\mathcal{M}^Q f\|_{L^p(Q)}^p \leq \frac{C_0^p}{p-1} \beta^{p-1} \mu(Q)^{1-p} \|f\|_{L^1(Q)}^p + \frac{\beta^p}{\gamma^p} \|f^{\sharp, Q}\|_{L^p(Q)}^p.$$

Now, we choose $\gamma = 1/(2A\beta^{p-1})$ and obtain

$$\begin{aligned} \|\mathcal{M}^Q f\|_{L^p(Q)}^p &\leq \frac{2C_0^p}{p-1} \beta^{p-1} \mu(Q)^{1-p} \|f\|_{L^1(Q)}^p + 2^{p+1} A^p \beta^{p^2} \|f^{\sharp, Q}\|_{L^p(Q)}^p \\ &\leq C \left(\|f^{\sharp, Q}\|_{L^p(Q)}^p + \mu(Q)^{1-p} \|f\|_{L^1(Q)}^p \right), \end{aligned}$$

where $C = \max \left(2C_0^p \frac{p}{p-1} \beta^{p-1}, 2^{p+1} A^p \beta^{p^2} \right)$, as required. □

Lemma 6 For all integers k large enough and for each cube Q in \mathcal{Q}^k

$$\|f\|_{L^1(Q)} \leq D_{(1+a_1\delta^k)/a_0\delta^k, a_0\delta^k} \|N_0f\|_{L^1(Q)}.$$

Proof For the sake of definiteness, suppose that Q is the dyadic cube Q_α^k . Then, Q is contained in $B_{a_1\delta^k}(z_\alpha^k)$ by Theorem 4 (v). Denote by \tilde{B} the ball $B_{1+a_1\delta^k}(z_\alpha^k)$. Then, $B_1(x) \subset \tilde{B}$ for each x in Q . Furthermore, if k is large enough, then $B_1(x) \supset Q$ for every x in Q . so that $\mu(B_1(x)) \leq \mu(\tilde{B})$. Then, by Tonelli’s theorem,

$$\begin{aligned} \|N_0f\|_{L^1(Q)} &= \int_Q \frac{d\mu(x)}{\mu(B_1(x))} \int_{B_1(x)} |f(y)| d\mu(y) \\ &\geq \frac{1}{\mu(\tilde{B})} \int_Q d\mu(x) \int_Q |f(y)| d\mu(y) \\ &\geq \frac{\mu(Q)}{\mu(\tilde{B})} \int_Q |f(y)| d\mu(y). \end{aligned}$$

Recall that Q contains $B_{a_0\delta^k}(z_\alpha^k)$, so that

$$\frac{\mu(Q)}{\mu(\tilde{B})} \geq D_{(1+a_1\delta^k)/a_0\delta^k, a_0\delta^k}^{-1}$$

and the required estimate follows. □

Lemma 7 Suppose that k is an integer $> \lceil \log_\delta(1/(2a_1)) \rceil$, where δ and a_1 are as in Theorem 4. Then, there exists a constant C , depending on k , such that for each cube Q in \mathcal{Q}^k

$$f^{\sharp, Q}(x) \leq C f^\sharp(x) \quad \forall x \in Q$$

(see (16) and (10) for the definitions of $f^{\sharp, Q}$ and f^\sharp , respectively).

Proof For each $b > 0$, we define the noncentred sharp function \tilde{f}_b^\sharp of a locally integrable function f as

$$\tilde{f}_b^\sharp(x) = \sup_B \frac{1}{\mu(B)} \int_B |f - f_B| d\mu \quad \forall x \in M,$$

where the supremum is taken over all balls in \mathcal{B}_b that contain x .

We first show that there exists a constant C , depending on k , such that $f^{\sharp, Q}(x) \leq C \tilde{f}_{a_1\delta^k}^\sharp(x)$ for each cube Q in \mathcal{Q}^k and for any x in Q . (see Theorem 4 for the definition of a_1 and δ).

Choose Q in \mathcal{Q}^k . Take x in Q and suppose that B is a ball whose centre belongs to Q and such that $x \in B \cap Q$. We consider the cases where $r_B < a_1\delta^k$ and $r_B \geq a_1\delta^k$ separately. If $r_B < a_1\delta^k$, the triangle inequality gives

$$\begin{aligned} \frac{1}{\mu(B \cap Q)} \int_{B \cap Q} |f - f_{B \cap Q}| d\mu &\leq \frac{1}{\mu(B \cap Q)} \int_{B \cap Q} |f - f_B| d\mu + |f_B - f_{B \cap Q}| \\ &\leq \frac{2}{\mu(B \cap Q)} \int_{B \cap Q} |f - f_B| d\mu. \end{aligned}$$

By Proposition 3 (i), we have that $\mu(B \cap Q) \geq D_{a_1/(a_0\delta), \delta^k}^{-1} \mu(B)$, so that

$$\frac{1}{\mu(B \cap Q)} \int_{B \cap Q} |f - f_{B \cap Q}| d\mu \leq \frac{2 D_{a_1/(a_0\delta), \delta^k}}{\mu(B)} \int_B |f - f_B| d\mu.$$

Since the ball B belongs to $\mathcal{B}_{a_1\delta^k}$, the right-hand side of the formula above is majorised by $2 D_{a_1/(a_0\delta),\delta^k} \tilde{f}_{a_1\delta^k}^\sharp(x)$.

Now assume that $r_B \geq a_1\delta^k$. For the sake of definiteness, suppose that Q is the dyadic cube Q_α^k . Recall that $\text{diam}(Q) \leq a_1\delta^k$, by Theorem 4 (iv), whence $Q \cap B = Q$. Moreover, $B_{a_0\delta^k}(z_\alpha^k) \subset Q \subset B_{a_1\delta^k}(z_\alpha^k)$, by Theorem 4 (v). Denote by \bar{B} the ball $B_{a_1\delta^k}(z_\alpha^k)$. Then, by the triangle inequality,

$$\begin{aligned} \frac{1}{\mu(B \cap Q)} \int_{B \cap Q} |f - f_{B \cap Q}| \, d\mu &\leq \frac{1}{\mu(B \cap Q)} \int_{B \cap Q} |f - f_{\bar{B}}| \, d\mu + |f_{\bar{B}} - f_{B \cap Q}| \\ &\leq \frac{2}{\mu(B \cap Q)} \int_{B \cap Q} |f - f_{\bar{B}}| \, d\mu \\ &\leq \frac{2}{\mu(B_{a_0\delta^k}(z_\alpha^k))} \int_{\bar{B}} |f - f_{\bar{B}}| \, d\mu. \end{aligned}$$

Now, the local doubling property implies that

$$\mu(\bar{B}) \leq D_{a_1/a_0,a_0\delta^k} \mu(B_{a_0\delta^k}(z_\alpha^k));$$

hence, the right-hand side can be estimated from above by

$$\frac{2 D_{a_1/a_0,a_0\delta^k}}{\mu(\bar{B})} \int_{\bar{B}} |f - f_{\bar{B}}| \, d\mu,$$

which, in turn, may be majorised by $2 D_{a_1/a_0,a_0\delta^k} \tilde{f}_{a_1\delta^k}^\sharp(x)$, for the ball \bar{B} has radius $a_1\delta^k$. By taking the supremum over all balls B containing x and whose centre belongs to Q , we get

$$f^{\sharp,Q}(x) \leq 2 D_{a_1/(a_0\delta),\max(1,a_0)\delta^k} \tilde{f}_{a_1\delta^k}^\sharp(x) \quad \forall x \in Q.$$

The local doubling property ensures that for each b in \mathbb{R}^+ , there exists a constant C such that $\tilde{f}_b^\sharp \leq C f_{2b}^\sharp$. Therefore,

$$f^{\sharp,Q}(x) \leq C f_{2a_1\delta^k}^\sharp(x) \quad \forall x \in Q.$$

Now, if the integer k large enough so that $2a_1\delta^k \leq 1$, i.e., $k > \lceil \log_\delta(1/2a_1) \rceil$, we get $f_{2a_1\delta^k}^\sharp \leq f^\sharp$, which gives the desired conclusion. \square

Now we are ready to prove the main result of this section.

Proof of Theorem 3 Fix an integer k so large that Lemmas 6 and 7 hold. In particular, k must be $> \lceil \log_\delta(1/2a_1) \rceil$. The cubes in \mathcal{Q}^k are pairwise disjoint, and their union is a set of full measure in M , so that

$$\begin{aligned} \|f\|_{L^p(M)}^p &= \sum_{Q \in \mathcal{Q}^k} \|f\|_{L^p(Q)}^p \\ &\leq C \sum_{Q \in \mathcal{Q}^k} \left[\|f^{\sharp,Q}\|_{L^p(Q)}^p + \mu(Q)^{1-p} \|f\|_{L^1(Q)}^p \right] \\ &\leq C \sum_{Q \in \mathcal{Q}^k} \left[\|f^\sharp\|_{L^p(Q)}^p + \mu(Q)^{1-p} \|N_0 f\|_{L^1(Q)}^p \right]; \end{aligned}$$

the first inequality above follows from Lemma 5, and the second is a consequence of Lemmas 6 and 7. Furthermore, by Hölder’s inequality,

$$\begin{aligned} \frac{1}{\mu(Q)^{1/p'}} \int_Q |N_0 f| \, d\mu &\leq \frac{1}{\mu(Q)^{1/p'}} \left[\int_Q |N_0 f|^p \, d\mu \right]^{1/p} \mu(Q)^{1/p'} \\ &= \|N_0 f\|_{L^p(Q)}. \end{aligned}$$

Thus,

$$\begin{aligned} \|f\|_{L^p(M)}^p &\leq C \left[\|f^\sharp\|_{L^p(M)}^p + \|N_0 f\|_{L^p(M)}^p \right] \\ &\leq C \|Nf\|_{L^p(M)}^p, \end{aligned}$$

as required. □

8 Interpolation

Suppose that X and Y are Banach spaces, and that θ is in $(0, 1)$. We denote by S the strip $\{z \in \mathbb{C} : \operatorname{Re} z \in (0, 1)\}$, and by \bar{S} its closure. We consider the class $\mathcal{F}(X, Y)$ of all functions $F : \bar{S} \rightarrow X + Y$ with the following properties:

1. F is continuous and bounded in \bar{S} and analytic in S ;
2. the functions $t \mapsto F(it)$ and $t \mapsto F(1 + it)$ are continuous from \mathbb{R} into X and Y , respectively;
3. $\lim_{|t| \rightarrow +\infty} \|F(it)\|_X = 0$ and $\lim_{|t| \rightarrow +\infty} \|F(1 + it)\|_Y = 0$.

We endow $\mathcal{F}(X, Y)$ with the norm

$$\|F\|_{\mathcal{F}(X,Y)} = \sup\{\max(\|F(it)\|_X, \|F(1 + it)\|_Y) : t \in \mathbb{R}\}.$$

We define the complex interpolation space $(X, Y)_{[\theta]}$ by

$$(X, Y)_{[\theta]} = \{F(\theta) : F \in \mathcal{F}(X, Y)\},$$

endowed with the norm

$$\|f\|_{(X,Y)_{[\theta]}} = \inf\{\|F\|_{\mathcal{F}(X,Y)} : F \in \mathcal{F}(X, Y) \text{ and } F(\theta) = f\}.$$

For more on the complex interpolation method, see, for instance, [1].

Theorem 5 *Suppose that θ is in $(0, 1)$. The following hold:*

- (i) if p_θ is $2/(1 - \theta)$, then $(L^2(M), \mathfrak{bmo}(M))_{[\theta]} = L^{p_\theta}(M)$;
- (ii) if p_θ is $2/(2 - \theta)$, then $(\mathfrak{h}^1(M), L^2(M))_{[\theta]} = L^{p_\theta}(M)$.

Proof First we prove (i). Observe that

$$L^{p_\theta}(M) = (L^2(M), L^\infty(M))_{[\theta]} \subseteq (L^2(M), \mathfrak{bmo}(M))_{[\theta]};$$

the containment above follows from the fact that $L^\infty(M) \subset \mathfrak{bmo}(M)$.

In order to prove the reverse inclusion, suppose that f is in the interpolation space $(L^2(M), \mathfrak{bmo}(M))_{[\theta]}$. Then, given $\epsilon > 0$ there exists a function F in $\mathcal{F}(L^2(M), \mathfrak{bmo}(M))$ such that $F(\theta) = f$ and

$$\|F\|_{\mathcal{F}(L^2, \mathfrak{bmo})} \leq \|f\|_{(L^2, \mathfrak{bmo})_{[\theta]}} + \epsilon.$$

Let ϕ be any measurable function which associates to any point x in M a ball $\phi(x)$ in $\mathcal{B}_1(x)$. Furthermore, let $\eta : M \times M \rightarrow \mathbb{C}$ be any measurable function with $|\eta| = 1$. We consider the linear operators $S^{\phi,\eta}$ and T^η which act on a function f in $L^2(M)$ as follows:

$$S^{\phi,\eta} f(x) = \frac{1}{\mu(\phi(x))} \int_{\phi(x)} [f - f_{\phi(x)}] \eta(x, \cdot) \, d\mu \quad \forall x \in M$$

and

$$T^\eta f(x) = \frac{1}{\mu(B_1(x))} \int_{B_1(x)} f \eta(x, \cdot) \, d\mu \quad \forall x \in M.$$

Then,

$$\sup_{\phi,\eta} |S^{\phi,\eta} f| = f^\sharp \quad \text{and} \quad \sup_{\eta} |T^\eta f| = N_0 f. \tag{18}$$

For each ϕ and η as before, consider the functions $S^{\phi,\eta} F$ and $T^\eta F$, where F is in the space $\mathcal{F}(L^2(M), \text{bmo}(M))$.

We claim that $S^{\phi,\eta} F$ and $T^\eta F$ belong to the class $\mathcal{F}(L^2(M), L^\infty(M))$,

$$\|S^{\phi,\eta} F\|_{\mathcal{F}(L^2, L^\infty)} \leq C \|F\|_{\mathcal{F}(L^2, \text{bmo})}$$

and

$$\|T^\eta F\|_{\mathcal{F}(L^2, L^\infty)} \leq C \|F\|_{\mathcal{F}(L^2, \text{bmo})}.$$

Indeed, recall that $g^\sharp \leq 2 \mathcal{M} g$ and that \mathcal{M} is bounded on $L^2(M)$. Thus,

$$\|S^{\phi,\eta} F(it)\|_2 \leq \|F(it)^\sharp\|_2 \leq 2 \|\mathcal{M} F(it)\|_2 \leq C \|F(it)\|_2.$$

Note that the constant C in the above inequality does not depend on ϕ and η . Moreover,

$$\|S^{\phi,\eta} F(1 + it)\|_\infty \leq \|F(1 + it)^\sharp\|_\infty \leq \|F(1 + it)\|_{\text{bmo}}.$$

Similarly,

$$\|T^\eta F(it)\|_2 \leq \|\mathcal{M} F(it)\|_2 \leq C \|F(it)\|_2;$$

and

$$\|T^\eta F(1 + it)\|_\infty \leq \|N_0 F(1 + it)\|_\infty \leq \|F(1 + it)\|_{\text{bmo}},$$

where C is independent of η . Hence,

$$\begin{aligned} \|S^{\phi,\eta} f\|_{p_\theta} &= \|S^{\phi,\eta} F(\theta)\|_{(L^2, L^\infty)_{[\theta]}} \\ &\leq \|S^{\phi,\eta} F\|_{\mathcal{F}(L^2, L^\infty)} \\ &\leq C \|F\|_{\mathcal{F}(L^2, \text{bmo})} \\ &\leq C (\|f\|_{(L^2, \text{bmo})_{[\theta]}} + \epsilon). \end{aligned}$$

By taking the infimum over all $\epsilon > 0$, we get

$$\|S^{\phi,\eta} f\|_{p_\theta} \leq C \|f\|_{(L^2, \text{bmo})_{[\theta]}}.$$

Now, by taking the supremum over all ϕ and η , we obtain the estimate

$$\|f^\sharp\|_{p_\theta} \leq C \|f\|_{(L^2, \text{bmo})_{[\theta]}}. \tag{19}$$

Similarly, we get

$$\|T^\eta f\|_{p_\theta} \leq C \|f\|_{(L^2, \mathfrak{bmo})_{[\theta]}}$$

and taking the supremum over all functions η , we have

$$\|N_0 f\|_{p_\theta} \leq C \|f\|_{(L^2, \mathfrak{bmo})_{[\theta]}}. \tag{20}$$

Now, applying Theorem 3 and combining (19) and (20), we may conclude that

$$\begin{aligned} \|f\|_{p_\theta} &\leq C \|Nf\|_{p_\theta} \\ &\leq C (\|f^\sharp\|_{p_\theta} + \|N_0 f\|_{p_\theta}) \\ &\leq C \|f\|_{(L^2, \mathfrak{bmo})_{[\theta]}} \quad \forall f \in (L^2(M), \mathfrak{bmo}(M))_{[\theta]} \end{aligned}$$

and the required inclusion $(L^2(M), \mathfrak{bmo}(M))_{[\theta]} \subset L^{p_\theta}(M)$ follows.

To prove (ii), we may apply a duality argument [1, Corollary 4.5.2]. We omit the details. \square

9 On the $\mathfrak{h}^1 - L^1$ boundedness of operators

One of the reasons which make $\mathfrak{h}^1(M)$ useful is that to prove that a linear operator T maps $\mathfrak{h}^1(M)$ to a Banach space X it suffices to prove that T is uniformly bounded on atoms. This extends to the space $\mathfrak{h}^1(M)$ the analogous result for $H^1(M)$ (see [22]).

We need more notation. Suppose that p is in $[1, \infty)$. For each closed ball B in M , we denote by $L^p(B)$ the space of all functions in $L^p(M)$ which are supported in B . The union of all spaces $L^p(B)$ as B varies over all balls coincides with the space $L^p_c(M)$ of all functions in $L^p(M)$ with compact support. Fix a reference point o in M , and for each positive integer k , denote by B_k the ball centred at o with radius k . A convenient way of topologising $L^p_c(M)$ is to interpret $L^p_c(M)$ as the strict inductive limit of the spaces $L^p_c(B_k)$ (see [2, II, p. 33] for the definition of the strict inductive limit topology). We denote by X^p the space $L^p_c(M)$ with this topology, and write X^p_k for $L^p_c(B_k)$. It is well known that the topological dual of X^p is $L^{p'}_{loc}(M)$, where p' denotes the index conjugate to p .

Note that the spaces X^p_k and X^p differ from the spaces, denotes exactly in the same way, considered in [22], for functions in our version of X^p_k and X^p need not have vanishing integral.

Theorem 6 *Suppose that p is in $(1, \infty)$ and that T is a $L^1(M)$ -valued linear operator defined on $\mathfrak{h}^{1,p}_{fin}(M)$ with the property that*

$$A := \sup \{\|Ta\|_1 : a \text{ is a } p\text{-atom}\} < \infty.$$

Then, there exists a unique bounded operator \tilde{T} from $\mathfrak{h}^1(M)$ to $L^1(M)$ which extends T .

Proof Suppose that B is a ball of radius $r_B \geq 1$. For each $f \in L^p(B)$ such that $\|f\|_p = 1$ set $a = \mu(B)^{-1/p'} f$, where p' denotes the index conjugate to p . Then, a is a p -atom at scale r_B and by Lemma 2 there exist global p -atoms at scale 1, a_1, \dots, a_N such that $a = \sum_{j=1}^N c_j a_j$,

with $|c_j| \leq C$, where C and N are constants, which depend only on r_B and M . Thus, we get

$$\begin{aligned} \|Tf\|_1 &= \|T\left(\mu(B)^{1/p'} a\right)\|_1 \\ &\leq \mu(B)^{1/p'} \sum_{j=1}^N |c_j| \|T a_j\|_1 \\ &\leq C N A \mu(B)^{1/p'} \end{aligned}$$

for every $f \in L^p(B)$ such that $\|f\|_p = 1$.

In particular, the restriction of T to X_k^p is bounded from X_k^p to $L^1(M)$ for each k . Therefore, T is bounded from X^p to $L^1(M)$. It follows that the transpose operator T^* is bounded from $L^\infty(M)$ to the dual of X^p , which can be identified with $L_{loc}^{p'}(M)$. Therefore, for every f in $L^\infty(M)$ and every p -atom a we have

$$\langle Ta, f \rangle = \langle a, T^* f \rangle = \int_M a T^* f \, d\mu,$$

so that

$$\left| \int_M a T^* f \, d\mu \right| = |\langle Ta, f \rangle| \leq \|Ta\|_1 \|f\|_\infty \leq A \|f\|_\infty. \tag{21}$$

Now we show that $T^* f$ belongs to $\text{bmo}(M)$ and that

$$\|T^* f\|_{\text{bmo}} \leq 3 A \|f\|_\infty \quad \forall f \in L^\infty(M).$$

Suppose that B is a ball of radius at most 1; we have

$$\left[\int_B |T^* f - (T^* f)_B|^{p'} \, d\mu \right]^{1/p'} = \sup_{\|\varphi\|_{L^p(B)}=1} \left| \int_B \varphi (T^* f - (T^* f)_B) \, d\mu \right|. \tag{22}$$

Observe that

$$\begin{aligned} \int_B \varphi (T^* f - (T^* f)_B) \, d\mu &= \int_B (\varphi - \varphi_B) (T^* f - (T^* f)_B) \, d\mu \\ &= \int_B (\varphi - \varphi_B) T^* f \, d\mu. \end{aligned} \tag{23}$$

Since $\|\varphi\|_{L^p(B)} = 1$,

$$|\varphi_B| \leq \left[\frac{1}{\mu(B)} \int_B |\varphi|^p \, d\mu \right]^{1/p} \leq \mu(B)^{-1/p}.$$

Then,

$$\begin{aligned} \|\varphi - \varphi_B\|_{L^p(B)} &\leq \|\varphi\|_{L^p(B)} + |\varphi_B| \mu(B)^{1/p} \\ &\leq 2, \end{aligned}$$

so that $(\varphi - \varphi_B) \chi_B / (2 \mu(B)^{1/p'})$ is a standard p -atom, and (21) implies that

$$\left| \int_B (\varphi - \varphi_B) T^* f \, d\mu \right| \leq 2 A \|f\|_\infty \mu(B)^{1/p'}.$$

Thus, by (23) and (22), we may conclude that for every ball B of radius at most 1

$$\left[\frac{1}{\mu(B)} \int_B |T^* f - (T^* f)_B|^{p'} \, d\mu \right]^{1/p'} \leq 2 A \|f\|_\infty.$$

Now take a ball B of radius exactly equal to 1. We have

$$\left[\int_B |T^* f|^{p'} d\mu \right]^{1/p'} = \sup_{\|\varphi\|_{L^p(B)}=1} \left| \int_B \varphi T^* f d\mu \right|.$$

The function $\varphi/\mu(B)^{1/p'}$ is a global p -atom, and, by (21),

$$\left| \int_B \varphi T^* f d\mu \right| \leq A \|f\|_\infty \mu(B)^{1/p'}.$$

Therefore, for every ball B of radius 1

$$\left[\frac{1}{\mu(B)} \int_B |T^* f|^{p'} d\mu \right]^{1/p'} \leq A \|f\|_\infty.$$

Combining the above estimates, we get

$$\|T^* f\|_{\text{bmo}} \leq \|T^* f\|_{\text{bmo}^{p'}} \leq 3 A \|f\|_\infty \quad \forall f \in L^\infty(M),$$

as required.

Now we prove that T extends to a bounded operator from $\mathfrak{h}^1(M)$ to $L^1(M)$. Observe that X^p and $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ coincide as vector spaces. For every g in $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ and every f in $L^\infty(M)$

$$\begin{aligned} |\langle Tg, f \rangle| &= |\langle g, T^* f \rangle| \\ &\leq C \|g\|_{\mathfrak{h}^1} \|T^* f\|_{\text{bmo}} \\ &\leq 3 C A \|g\|_{\mathfrak{h}^1} \|f\|_\infty. \end{aligned}$$

By taking the supremum of both sides over all functions f in $L^\infty(M)$ with $\|f\|_\infty = 1$, we obtain that

$$\|Tg\|_1 \leq 3 C A \|g\|_{\mathfrak{h}^1} \quad \forall g \in \mathfrak{h}_{\text{fin}}^{1,p}(M).$$

Since $\mathfrak{h}_{\text{fin}}^{1,p}(M)$ is dense in $\mathfrak{h}^1(M)$ with respect to the norm of $\mathfrak{h}^1(M)$, the required conclusion follows by a density argument. □

Suppose that T is a bounded linear operator on $L^2(M)$. Then, T is automatically defined on $\mathfrak{h}_{\text{fin}}^{1,2}(M)$. If we assume that

$$A := \sup \{ \|Ta\|_1 : a \text{ is a 2-atom} \} < \infty,$$

then, by the previous theorem, the restriction of T to $\mathfrak{h}_{\text{fin}}^{1,2}(M)$ has a unique bounded extension to an operator \tilde{T} from $\mathfrak{h}^1(M)$ to $L^1(M)$. We wonder if the operators T and \tilde{T} are consistent, i.e., if they agree on the intersection $\mathfrak{h}^1(M) \cap L^2(M)$ of their domains. As in the case of the same problem on the space $H^1(M)$ (see [22, Prop. 4.2]), the answer is in the affirmative, as shown in the next proposition.

Proposition 4 *Suppose that T is a bounded linear operator on $L^2(M)$ and that*

$$A := \sup \{ \|Ta\|_1 : a \text{ is a 2-atom} \} < \infty.$$

Denote by \tilde{T} the unique bounded extension of the restriction of T to $\mathfrak{h}_{\text{fin}}^{1,2}(M)$ to an operator from $\mathfrak{h}^1(M)$ to $L^1(M)$. Then, the operators T and \tilde{T} coincide on $\mathfrak{h}^1(M) \cap L^2(M)$.

Proof Assume that f is in $L^2(M) \cap L^\infty(M)$ and that g is in $L_c^2(M)$. Denote by T^* the transpose operator of T (as an operator on $L^2(M)$). Then,

$$\int_M g T^* f \, d\mu = \int_M Tg f \, d\mu.$$

Since g is in $\mathfrak{h}_{\text{fin}}^{1,2}(M)$ and the operators T and \tilde{T} agree on $\mathfrak{h}_{\text{fin}}^{1,2}(M)$, we get

$$\int_M Tg f \, d\mu = \int_M \tilde{T}g f \, d\mu.$$

Denote by $(\tilde{T})^*$ the transpose of \tilde{T} as an operator from $\mathfrak{h}^1(M)$ to $L^1(M)$. Then,

$$\int_M \tilde{T}g f \, d\mu = \langle g, (\tilde{T})^* f \rangle.$$

Since $(\tilde{T})^* f$ is in $\mathfrak{bmo}(M)$ and g is in $\mathfrak{h}_{\text{fin}}^{1,2}(M)$, we can write the last scalar product $\langle g, (\tilde{T})^* f \rangle$ (with respect to the duality between $\mathfrak{h}^1(M)$ and $\mathfrak{bmo}(M)$) as

$$\langle g, (\tilde{T})^* f \rangle = \int_M g (\tilde{T})^* f \, d\mu.$$

Thus, combining the above equalities, we obtain that

$$\int_M g [T^* f - (\tilde{T})^* f] \, d\mu = 0 \quad \forall g \in L_c^2(M),$$

i.e., for all g in X^2 . This implies that $T^* f - (\tilde{T})^* f = 0$ is in the dual space of X^2 , i.e., in $L_{\text{loc}}^2(M)$. Thus, $T^* f = (\tilde{T})^* f$ almost everywhere.

Now, suppose that f is in $L^2(M) \cap L^\infty(M)$ and that g is in $\mathfrak{h}^1(M) \cap L^2(M)$. Then,

$$\begin{aligned} \int_M Tg f \, d\mu &= \int_M g T^* f \, d\mu \\ &= \int_M g (\tilde{T})^* f \, d\mu \\ &= \int_M \tilde{T}g f \, d\mu. \end{aligned}$$

Thus, we have obtained that

$$\int_M [Tg - \tilde{T}g] f \, d\mu = 0$$

for an arbitrary f in $L^2(M) \cap L^\infty(M)$. This implies that $Tg = \tilde{T}g$ for all g in $\mathfrak{h}^1(M) \cap L^2(M)$. □

10 Applications to SIO

The purpose of this section is to show that the Hardy space $\mathfrak{h}^1(M)$ may be used to obtain endpoint estimates for interesting singular integral operators on Riemannian manifolds.

Hereafter in this section, we assume that M is a complete connected noncompact n -dimensional Riemannian manifold with *bounded geometry*, that is with Ricci curvature bounded from below and positive injectivity radius. We view M as a measured metric space with respect to the Riemannian distance and measure. Clearly the MP property holds (with $R_0 = 0$). Furthermore, it is well known that manifolds with bounded geometry possess the

LDP, as a consequence of the Bishop–Gromov comparison theorem (see, for instance, [14], [5, Thm III.4.5]). Thus, the theory of local Hardy spaces $h^1(M)$ developed in the previous chapters applies to this setting. Denote by $-\mathcal{L}$ the Laplace–Beltrami operator on M : \mathcal{L} is a symmetric operator on $C_c^\infty(M)$, and its closure is a self-adjoint operator on $L^2(M)$ which we still denote by \mathcal{L} .

We consider the (translated) Riesz transforms $R_a := \nabla(a\mathcal{I} + \mathcal{L})^{-1/2}$, where ∇ denotes the Riemannian gradient, and a is a positive number, and spectral multipliers of \mathcal{L} satisfying a Mihlin-type condition at infinity.

The latter operators are treated in [27] and in [19]. A comparison between the results obtained therein and our result is in order. We extend the result in [27] by relaxing significantly the assumptions on the geometry of M , as already illustrated in Introduction. In [19], the Riemannian manifold M is assumed to have bounded geometry in the same sense as here, but an additional hypothesis is made, i.e., that the bottom b of the L^2 spectrum of \mathcal{L} is strictly positive. This assumption rules out, for instance, all Riemannian manifolds of polynomial volume growth [3]. The reason for this additional assumption is that the local Hardy space $H_1^1(M)$ used in [19] is known to interpolate with $L^2(M)$ to give $L^p(M)$, $1 < p < 2$, only when $b > 0$.

The problem of establishing endpoint estimates for R_a when $p = 1$ in the setting of noncompact Riemannian manifolds has been widely studied. In particular, Coulhon and Doung [4] proved that if M is locally doubling, of exponential growth, and supports an L^2 -scaled Poincaré inequality, then R_a is of weak type 1. Russ [24] complemented this result by showing that, for a large enough, R_a is bounded from the atomic Hardy space $H_1^1(M)$ to $L^1(M)$. Note, however, that Russ’ result is known to interpolate with $L^2(M)$ to give $L^p(M)$ estimates only when M has bounded geometry and spectral gap (see [7] and the remarks above).

Here we prove, under the assumption that M has bounded geometry, that if a is suitably large, then R_a is bounded from $h^1(M)$ to $L^1(M)$. This result complements the analogous result in [7].

10.1 Spectral multipliers

First we define the class of symbols which will be needed in the statement of Theorem 7.

Definition 5 Suppose that J is a positive integer and that W is in \mathbb{R}^+ . Denote by \mathbf{S}_W the strip $\{\zeta \in \mathbb{C} : \text{Im}(\zeta) \in (-W, W)\}$ and by $H^\infty(\mathbf{S}_W; J)$ the vector space of all bounded *even* holomorphic functions f in \mathbf{S}_W for which there exists a positive constant C such that

$$|D^j f(\zeta)| \leq C (1 + |\zeta|)^{-j} \quad \forall \zeta \in \mathbf{S}_W \quad \forall j \in \{0, 1, \dots, J\}. \tag{24}$$

We denote by $\|f\|_{\mathbf{S}_W; J}$ the infimum of all constants C for which (24) holds. If $\|f\|_{\mathbf{S}_W; J} < \infty$, we say that f satisfies a *Mihlin condition of order J at infinity on \mathbf{S}_W* .

Denote by ω an even function in $C_c^\infty(\mathbb{R})$ which is supported in $[-3/4, 3/4]$, is equal to 1 in $[-1/4, 1/4]$, and satisfies

$$\sum_{j \in \mathbb{Z}} \omega(t - j) = 1 \quad \forall t \in \mathbb{R}.$$

Denote by \mathcal{D} the operator $\sqrt{\mathcal{L} - b}$, where b denotes the bottom of the L^2 spectrum of \mathcal{L} . Clearly spectral multipliers of \mathcal{L} may equivalently be expressed as spectral multipliers of

\mathcal{D} (with a different multiplier). Recall that the heat semigroup is the one-parameter family $\{\mathcal{H}_t\}_{t \geq 0}$ defined, at least on $L^2(M)$, by

$$\mathcal{H}_t f := e^{-t\mathcal{L}} f \quad \forall f \in L^2(M).$$

It is well known that \mathcal{H}_t extends to a contraction semigroup on $L^p(M)$ for all $p \in [1, \infty]$. Furthermore, since M has Ricci curvature bounded from below, the heat semigroup $\{\mathcal{H}_t\}$ satisfies the following ultracontractivity estimate [14, Section 7.5]

$$\|\mathcal{H}_t\|_{1;2} \leq C e^{-bt} t^{-n/4} (1+t)^{n/4-\delta/2} \quad \forall t \in \mathbb{R}^+ \tag{25}$$

for some δ in $[0, \infty)$. Recall also that a lower bound for the Ricci curvature implies also an upper bound of the volume growth of M (see (26)). Indeed, there are positive constants α, β and C such that

$$\mu(B(p, r)) \leq C r^\alpha e^{2\beta r} \quad \forall r \in [1, \infty) \quad \forall p \in M. \tag{26}$$

The following result should be compared with [27, Proposition B.5]. It provides an end-point result to the multiplier theorem [26, Thm 1.1].

Theorem 7 *Assume that α and β are as in (26), and δ as in (25). Denote by N the integer $\lfloor n/2 + 1 \rfloor + 1$. Suppose that J is an integer $> \max(N + 2 + \alpha/2 - \delta, N + 1/2)$. Then, there exists a constant C such that*

$$\|m(\mathcal{D})\|_{\mathfrak{h}^1} \leq C \|m\|_{\mathfrak{S}_{\beta;J}} \quad \forall m \in H^\infty(\mathfrak{S}_{\beta;J}).$$

Proof We claim that it suffices to prove that for each 2-atom a at scale 1, the function $m(\mathcal{D})a$ may be written as the sum of 2-atoms supported in balls of \mathcal{B}_1 , with ℓ^1 norm of the coefficients controlled by $C \|m\|_{\mathfrak{S}_{\beta;J}}$.

Indeed, suppose that f is a function in $\mathfrak{h}^1(M)$ and that $f = \sum_j \lambda_j a_j$ is an atomic decomposition of f with $\|f\|_{\mathfrak{h}^1} \geq \sum_j |\lambda_j| - \varepsilon$. Since for each 2-atom a , we have $\|m(\mathcal{D})a\|_1 \leq C \|m\|_{\mathfrak{S}_{\beta;J}}$, by Theorem 6 $m(\mathcal{D})$ extends to a bounded operator from $\mathfrak{h}^1(M)$ to $L^1(M)$. Then, $m(\mathcal{D})f = \sum_j \lambda_j m(\mathcal{D})a_j$, where the series is convergent in $L^1(M)$. But the partial sums of the series $\sum_j \lambda_j m(\mathcal{D})a_j$ is a Cauchy sequence in $\mathfrak{h}^1(M)$; hence, the series is convergent in $\mathfrak{h}^1(M)$, and the sum must be the function $m(\mathcal{D})f$. Therefore,

$$\begin{aligned} \|m(\mathcal{D})f\|_{\mathfrak{h}^1} &\leq \sum_j |\lambda_j| \|m(\mathcal{D})a_j\|_{\mathfrak{h}^1} \\ &\leq C \|m\|_{\mathfrak{S}_{\beta;J}} \sum_j |\lambda_j| \\ &\leq C \|m\|_{\mathfrak{S}_{\beta;J}} (\|f\|_{\mathfrak{h}^1} + \varepsilon), \end{aligned}$$

and the required conclusion follows by taking the infimum of both sides with respect to all admissible decompositions of f .

It has already been shown in the proof of [19, Thm 3.4] that the claim holds for standard atoms. Therefore, it suffices to prove it for global atoms.

As in the proof of [19, Thm 3.4], we split the operator $m(\mathcal{D})$ into the sum of two operators and analyse them separately. The functions $\widehat{\omega} * m$ and $m - \widehat{\omega} * m$ (ω is the cut-off function defined above) are bounded. Define the operators \mathcal{S} and \mathcal{T} spectrally by

$$\mathcal{S} = (\widehat{\omega} * m)(\mathcal{D}) \quad \text{and} \quad \mathcal{T} = (m - \widehat{\omega} * m)(\mathcal{D}).$$

Thus $m(\mathcal{D}) = \mathcal{S} + \mathcal{T}$.

Suppose that a is a global 2-atom supported in $B_1(p)$ for some p in M . Observe that the function $\widehat{w} * m$ is bounded and

$$\|\widehat{w} * m\|_\infty \leq C \|m\|_\infty \leq C \|m\|_{\mathbf{S}_\beta; J}. \tag{27}$$

Therefore, $(\widehat{w} * m)(\mathcal{D})$ is bounded on $L^2(M)$ by the spectral theorem, and

$$\|(\widehat{w} * m)(\mathcal{D})\|_2 \leq \|\widehat{w} * m\|_\infty \leq C \|m\|_{\mathbf{S}_\beta; J}.$$

We have used (27) in the second inequality above. Observe that the support of the kernel of the operator $(\widehat{w} * m)(\mathcal{D})$ is contained in $\{(x, y) : d(x, y) \leq 1\}$, for \mathcal{L} possesses the finite propagation speed property; hence, the function $(\widehat{w} * m)(\mathcal{D})a$ is supported in the ball with centre p and radius 2. Moreover,

$$\begin{aligned} \|(\widehat{w} * m)(\mathcal{D})a\|_2 &\leq C \|(\widehat{w} * m)(\mathcal{D})\|_2 \|a\|_2 \\ &\leq C \|m\|_{\mathbf{S}_\beta; J} \mu(B_1(p))^{-1/2} \\ &\leq C \|m\|_{\mathbf{S}_\beta; J} \mu(B_2(p))^{-1/2}. \end{aligned}$$

We have used the LDP in the last inequality. Thus, $(\widehat{w} * m)(\mathcal{D})a$ is a constant multiple of a global atom at scale 2 and, by Lemma 2,

$$\|(\widehat{w} * m)(\mathcal{D})a\|_{\mathfrak{h}^1} \leq C \|m\|_{\mathbf{S}_\beta; J}.$$

Now we analyse $\mathcal{T}a$. In the proof of [19, Thm 3.4], it is shown that if b is a standard 2-atom, then $\mathcal{T}b$ may be decomposed as

$$\mathcal{T}b = \sum_{j=1}^\infty \lambda_j b_j,$$

where b_j is an atom at scale $j + 2$, and

$$|\lambda_j| \leq C \|m\|_{\mathbf{S}_\beta; J} j^{N+\alpha/2-J-\delta} \quad \forall j = 1, 2, 3, \dots \tag{28}$$

A close examination of the proof reveals that the cancellation property of b is used to show that the atoms b_j also have this property, but it is not required in the proof of (28). Thus, by arguing as in Step IV in the proof of [19, Thm 3.4], we may conclude that $\mathcal{T}a$ may be written as

$$\mathcal{T}a = \sum_{j=1}^\infty \lambda_j a_j,$$

where a_j is a global 2-atom at scale $j + 2$ and λ_j satisfies estimate (28). Now Lemma 2 (and its version for Riemannian manifolds [19, Lemma 5.7]) implies that there exists a constant C such that

$$\|a_j\|_{\mathfrak{h}^1} \leq C j \quad j = 1, 2, 3, \dots$$

Therefore,

$$\|\mathcal{T}a\|_{\mathfrak{h}^1} \leq C \|m\|_{\mathbf{S}_\beta; J} \sum_{j=1}^\infty j^{1+N+\alpha/2-J-\delta} \leq C \|m\|_{\mathbf{S}_\beta; J},$$

where C is independent of a . Hence, \mathcal{T} extends to a bounded operator from $\mathfrak{h}^1(M)$ to $\mathfrak{h}^1(M)$.

So far, we have proved that there exists a constant C such that for every global atom a

$$\|\mathcal{S}a\|_{\mathfrak{h}^1} + \|\mathcal{T}a\|_{\mathfrak{h}^1} \leq C \|m\|_{\mathbf{S}_{\beta;J}}.$$

Hence,

$$\|m(\mathcal{D})a\|_{\mathfrak{h}^1} \leq C \|m\|_{\mathbf{S}_{\beta;J}}.$$

The required conclusion follows from the claim at the beginning of the proof. □

10.2 The translated Riesz transform

We shall need the following local estimate for the space derivative of the heat kernel.

Lemma 8 *There exists $\eta > 0$ such that for all $y \in M, t > 0$*

$$\int_{d(x,y) \geq \sqrt{t}} |\nabla_x h_s(x, y)| d\mu(x) \leq \begin{cases} C e^{-\eta t/s} s^{-1/2} & \forall s \in (0, 1] \\ C e^{-\eta t/s} e^{cs} s^{-1/2} & \forall s \in (1, \infty) \end{cases}.$$

This result is stated in [4], though its proof is given in full detail only in the case where M is globally doubling. However, it is not hard to modify the argument to produce a proof of Lemma 8. The proof hinges on upper estimates for the heat kernel and its time derivatives (see [11, 14, 15]) and on weighted estimates for the space derivative of the heat kernel ([4]).

Theorem 8 *There exists $a > 0$ such that the translated Riesz transform $\nabla(a\mathcal{S} + \mathcal{L})^{-1/2}$ is bounded from $\mathfrak{h}^1(M)$ to $L^1(M)$.*

Proof We know that if a is large enough, then $\nabla(a\mathcal{S} + \mathcal{L})^{-1/2}$ is bounded from $H^1(M)$ to $L^1(M)$ by [24]. Therefore, it suffices to show that the kernel k of $\nabla(a\mathcal{S} + \mathcal{L})^{-1/2}$ satisfies the condition

$$\sup_{y \in M} \int_{B_2(y)^c} |k(x, y)| d\mu(x) < \infty \tag{29}$$

and then apply [9, Prop 4.5]. The kernel is given, off the diagonal, by

$$k(x, y) = \int_0^{+\infty} \frac{e^{-as}}{\sqrt{s}} \nabla_x h_s(x, y) ds.$$

By Fubini’s theorem, we obtain

$$\begin{aligned} \int_{B_2(y)^c} |k(x, y)| d\mu(x) &= \int_{B(y)^c} \left| \int_0^{+\infty} \frac{e^{-as}}{\sqrt{s}} \nabla_x h_s(x, y) ds \right| d\mu(x) \\ &\leq \int_0^{+\infty} \frac{e^{-as}}{\sqrt{s}} \int_{d(x,y) \geq 2} |\nabla_x h_s(x, y)| d\mu(x) ds \\ &:= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_0^1 \frac{e^{-as}}{\sqrt{s}} \int_{d(x,y) \geq 2} |\nabla_x h_s(x, y)| d\mu(x) ds$$

and

$$I_2 = \int_1^{+\infty} \frac{e^{-as}}{\sqrt{s}} \int_{d(x,y) \geq 2} |\nabla_x h_s(x, y)| d\mu(x) ds.$$

Now we apply Lemma 8 to estimate the inner integrals. We get

$$I_1 \leq C \int_0^1 \frac{e^{-as-4\eta/s}}{s} ds \leq C \int_0^1 \frac{e^{-4\eta/s}}{s^2} ds = C \frac{e^{-4\eta}}{4\eta},$$

and

$$I_2 \leq C \int_1^{+\infty} \frac{e^{-(a-c)s-4\eta/s}}{s} ds \leq C \int_1^{+\infty} e^{-(a-c)s} ds.$$

Note that the last integral converges only when $a > c$. Therefore, (29) holds if $a > c$, and for such a , the operator $\nabla(a\mathcal{J} + \mathcal{L})^{-1/2}$ extends to a bounded operator from $\mathfrak{h}^1(M)$ to $L^1(M)$, as required. \square

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