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Hankel determinants, Padé approximations, and irrationality exponents for p-adic numbers

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Abstract The irrationality exponent of an irrational p-adic number ξ, which measures the approximation rate of ξ by rational numbers, is in general very difficult to compute explicitly. In this work, we shall show that the irrationality exponents of large classes of automatic p-adic numbers and Mahler p-adic numbers (which are transcendental) are exactly equal to 2. Our classes contain the Thue–Morse–Mahler p-adic numbers, the regular paperfolding p-adic numbers, the Stern p-adic numbers, among others.

Keywords Hankel determinant · Automatic sequence · Thue–Morse sequence · Periodicity · Regular paperfolding sequence · Stern sequence · Irrationality exponent

Mathematics Subject Classification 05A10 · 05A15 · 11B50 · 11B85 · 11J72 · 11J82

1 Introduction and results

By Roth's Theorem, the irrationality exponent of any irrational algebraic real number is equal to 2. However, it is in general a very difficult problem to determine the irrationality exponent of a given transcendental real number, unless its continued fraction expansion is known. Recently, a method based on Padé approximants was developed in [11] and generalized in several subsequent works; see [13] and the references given therein. In the present work, we shall show that this method can be also used to estimate the irrationality exponent of special classes of transcendental p-adic numbers.

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Fix p a prime number. We denote by $\mathbb{Q}_p$ the field of p-adic numbers, and by $|\cdot|_p$ the p-adic absolute value normalized such that $|p|_p = 1/p$ (for basic properties about p-adic numbers, see, for example, [32]). The irrationality exponent of a p-adic irrational number ξ, denoted by $\mu(\xi)$, is the supremum of the real numbers μ such that the inequality

$$\left|\xi - \frac{r}{s}\right|_p < \frac{1}{(H(r,s))^{\mu}}$$

has infinitely many solutions in integers r, s with $s > 0$, where we have set $H(r,s) = \max\{|r|, |s|\}$. It follows from a p-adic version of Dirichlet's theorem that the irrationality exponent of any irrational element of $\mathbb{Q}_p$ is at least equal to 2. By the p-adic analogue of Roth's theorem (see, for example, [26]), the irrationality exponent of any irrational algebraic p-adic number is equal to 2.

In the real case, one can easily construct, by means of the theory of continued fractions, explicit examples of real numbers with any given irrationality exponent ≥ 2. However, it seems that such constructions cannot be transposed to the p-adic setting. So, we have to use metric Diophantine approximation to prove that there exist p-adic numbers with any prescribed irrationality exponent ≥ 2; see Section 9.3 of [9]. When one looks for explicit examples, it is tempting to consider, for any real number $c \geq 2$, the p-adic number

$$\xi_c = \sum_{k \geq 1} p^{\lfloor c^k \rfloor},$$

where $\lfloor \cdot \rfloor$ denotes the integer part function. By truncating the expansion of ξ_c, one can easily get that $\mu(\xi_c) \geq c$. And triangle inequalities show that this inequality is indeed an equality if $c \geq (3+\sqrt{5})/2$. In the real case, a similar example has been considered in [10] (see also Section 7.6 in [12]), where continued fractions are used to prove that the irrationality exponent of $\sum_{k\geq 1} 2^{-\lfloor c^k \rfloor}$ (which can be viewed as the real analogue of ξ_c) is equal to c for any $c \geq 2$. As far as we are aware, the exact value of $\mu(\xi_c)$ is not yet known when c satisfies $2 \leq c < (3+\sqrt{5})/2$.

Actually, there are very few concrete examples of transcendental p-adic numbers whose irrationality exponent is known. One can mention Matala-Aho's paper [27]: The numbers

$$\mathrm{RR}(p) = \prod_{k=0}^{\infty} \frac{(1-p^{5k+2})(1-p^{5k+3})}{(1-p^{5k+1})(1-p^{5k+4})}$$

he considers have irrationality exponent equal to 2; however, they are not proved to be transcendental. For more examples on p-adic numbers with irrationality exponent equal to 2, the reader can consult the work [28] of Matala-Aho and Merilä.

The goal of the present paper is to present explicit examples of transcendental p-adic numbers whose irrationality exponent is equal to 2, with a special attention to automatic p-adic numbers.

A p-adic number ξ is automatic if there exist two integers $k, b \geq 2$ such that the b-adic expansion of ξ is k-automatic, where $b = p^w$ and w is a positive integer. This means that, if we write $\xi = \sum_{n\geq 0} a(n) b^n$ with $a(n) \in \mathbb{Z}$ $(n \geq 0)$ and $0 \leq a(n) < b$ $(n \geq 0)$, then the set of subsequences

$$\left\{ \left(a(k^r n + s)\right)_{n \geq 0} \;\middle|\; r \geq 0, \quad 0 \leq s < k^r \right\}$$

is finite (for more on automatic sequences, see, for example, Allouche [3] and also the book of Allouche and Shallit [4]). By following the proof of Theorem 6 in [1], one can show that an automatic p-adic number is either rational or transcendental. We have thus a large family of "simple" transcendental p-adic numbers, and one can then try to determine their irrationality exponents. In the present paper, we do not restrict our attention to automatic numbers and take a more general point of view.

Mahler's method [23–25] is a method in transcendence theory whereby one uses a function $F(z) \in \mathbb{Z}[[z]]$ that satisfies a functional equation of the form

$$\sum_{i=0}^{n} P_i(z) F(z^{d^i}) = 0,$$

for some integers $n \geq 1$ and $d \geq 2$, and polynomials $P_0(z), \ldots, P_n(z)$ in $\mathbb{Z}[x]$ with $P_0(z)P_n(z) \neq 0$, to give results about the nature of the p-adic numbers $F(b)$, where, as above, b is an integer power of p. We refer to such numbers $F(b)$ as *Mahler p-adic numbers*. It is well known that automatic p-adic numbers are special cases of Mahler p-adic numbers (see, for example [7, Theorem 1]).

From now on, we concentrate our attention on a special type of Mahler equation. Let $d \geq 2$ be an integer, $(c_m)_{m\geq 0}$ be an integer sequence, and set

$$f(z) = \sum_{m\geq 0} c_m z^m.$$

Suppose that there exist $A(z)$, $B(z)$, $C(z)$, $D(z) \in \mathbb{Z}[z]$ such that

$$f(z) = \frac{A(z)}{B(z)} + \frac{C(z)}{D(z)} f(z^d). \tag{1.1}$$

Under various assumptions on these polynomials, we shall show that, for every integer $w \geq 1$, the irrationality exponent of the p-adic number $f(p^w)$ is equal to 2. A precise statement is given in Theorem 3.1. We display several consequences of this result in Sect. 4. Among them is Theorem 1.1 below, which is the p-adic analogue of the main result of [11].

Recall here that the Thue–Morse sequence $(t_n)_{n\geq 0}$ on $\{0, 1\}$ is defined recursively by $t_0 = 0$, $t_{2n} = t_n$, and $t_{2n+1} = 1 - t_n$ for all integers $n \geq 0$. It is 2-automatic but not ultimately periodic (see, for example [4]); thus, all the p-adic numbers $\sum_{n\geq 0} t_n p^{wn}$ are transcendental.

Theorem 1.1 *Let $(t_n)_{n\geq 0}$ denote the Thue–Morse sequence over $\{0, 1\}$. Let p be a prime number and w a positive integer. Then, the irrationality exponent of the p-adic number $\sum_{n\geq 0} t_n p^{wn} = 2$.*

The proofs of our theorems essentially follow the same lines as the proofs of the corresponding statements in the real case (see [13]). However, there is an extra difficulty to overcome, since we do not know the exact degrees of the polynomials giving the Padé approximants to the power series f which satisfies (1.1).

2 Auxiliary results

The following result is the p-adic analogue of Lemma 4.1 in [2].

Lemma 2.1 *Let ξ be an element of $\mathbb{Q}_p$. Let λ, κ, and θ be real numbers such that $0 < \lambda \leq \kappa$ and $\theta \geq 1$. Suppose that there exists a sequence $(r_n/s_n)_{n\geq 1}$ of rational numbers and positive* *numerical constants c_0, c_1, c_2 such that, for all integers $n \geq 1$, we have*

$$H(r_n, s_n) < H(r_{n+1}, s_{n+1}) \leq c_0(H(r_n, s_n))^{\theta},$$
$$\frac{c_1}{(H(r_n, s_n))^{1+\kappa}} \leq \left|\xi - \frac{r_n}{s_n}\right|_p \leq \frac{c_2}{(H(r_n, s_n))^{1+\lambda}}.$$

Then, we have $\mu(\xi) \leq (1+\kappa)\theta/\lambda$.

Proof Let r/s be a rational number with $H(r, s)$ large enough. Then, there exists a unique integer $n = n(r, s) \geq 2$ such that

$$H(r_{n-1}, s_{n-1}) < (4c_2 H(r, s))^{1/\lambda} \leq H(r_n, s_n).$$

If $\frac{r}{s} \neq \frac{r_n}{s_n}$, then we obtain

$$\begin{aligned}\left|\xi - \frac{r}{s}\right|_p &\geq \left|\frac{r}{s} - \frac{r_n}{s_n}\right|_p - \left|\xi - \frac{r_n}{s_n}\right|_p \geq \frac{|rs_n - sr_n|_p}{|ss_n|_p} - \frac{c_2}{(H(r_n, s_n))^{1+\lambda}}\\ &\geq \frac{1}{|rs_n - sr_n|} - \frac{c_2}{(H(r_n, s_n))^{1+\lambda}}\\ &\geq \frac{1}{2H(r, s)H(r_n, s_n)} - \frac{1}{4H(r, s)H(r_n, s_n)}\\ &= \frac{1}{4H(r, s)H(r_n, s_n)},\end{aligned}$$

for we have $H(r_n, s_n) \geq (4c_2 H(r, s))^{1/\lambda}$. But

$$H(r_n, s_n) \leq c_0(H(r_{n-1}, s_{n-1}))^{\theta} < c_0(4c_2)^{\theta/\lambda}(H(r, s))^{\theta/\lambda}, \tag{2.1}$$

thus, by using that $0 < \lambda \leq \kappa$ and $\theta \geq 1$, we obtain

$$\left|\xi - \frac{r}{s}\right|_p \geq \frac{1}{c_3(H(r, s))^{1+\theta/\lambda}} \geq \frac{1}{c_3(H(r, s))^{(1+\kappa)\theta/\lambda}},$$

where we have put $c_3 = 4c_0(4c_2)^{\theta/\lambda}$.

If $\frac{r}{s} = \frac{r_n}{s_n}$, then we deduce from (2.1) that

$$\left|\xi - \frac{r}{s}\right|_p = \left|\xi - \frac{r_n}{s_n}\right|_p \geq \frac{c_1}{(H(r_n, s_n))^{1+\kappa}} \geq \frac{1}{c_4(H(r, s))^{(1+\kappa)\theta/\lambda}},$$

where we have set $c_4 = c_1^{-1} c_0^{1+\kappa}(4c_2)^{(1+\kappa)\theta/\lambda}$.

By the definition of $\mu(\xi)$, we obtain finally $\mu(\xi) \leq (1+\kappa)\theta/\lambda$. □

Below we summarize several basic facts on Padé approximation. For more details, we refer the reader to [6,8].

Let $\mathbb{F}$ be a field and z be an indeterminate over $\mathbb{F}$. For any sequence $\mathbf{c} = (c_m)_{m\geq 0}$ of elements in $\mathbb{F}$, we put $f = f(z) = \sum_{m\geq 0} c_m z^m$ and call it the generating function of $\mathbf{c}$. For all integers $n \geq 1$ and $k \geq 0$, the Hankel determinant of the power series f (or of the sequence $\mathbf{c}$) is defined by

$$\mathcal{H}_n^{(k)}(f) := \begin{vmatrix} c_k & c_{k+1} & \cdots & c_{k+n-1} \\ c_{k+1} & c_{k+2} & \cdots & c_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k+n-1} & c_{k+n} & \cdots & c_{k+2n-2} \end{vmatrix} \in \mathbb{F}.$$

By convention, we put $\mathcal{H}_0^{(k)}(f) = 1$, for all integers $k \geq 0$. For all integers $n \geq 0$, write $\mathcal{H}_n(f) := \mathcal{H}_n^{(0)}(f)$. The sequence $\mathcal{H}(f) := (\mathcal{H}_n(f))_{n\geq 0}$ is called the *sequence of the Hankel determinants* of f.

Let $r, s \geq 0$ be integers. By definition, the Padé approximant $[r/s]_f(z)$ to f is the rational fraction $P(z)/Q(z)$ in $\mathbb{F}[[z]]$ such that

$$\deg(P) \leq r, \ \deg(Q) \leq s, \ \text{and} \ f(z) - \frac{P(z)}{Q(z)} = \mathcal{O}(z^{r+s+1}).$$

The pair (P, Q) has no reason to be unique, but the fraction $P(z)/Q(z)$ is unique (see [8, p. 35]). Moreover, if we assume that P and Q are coprime, then we have $Q(0) \neq 0$.

If there exists an integer $k \geq 1$ such that $\mathcal{H}_k(f)$ is nonzero, then we know that the Padé approximant $[k-1/k]_f(z)$ exists and we have

$$f(z) - [k-1/k]_f(z) = \frac{\mathcal{H}_{k+1}(f)}{\mathcal{H}_k(f)} z^{2k} + \mathcal{O}(z^{2k+1}).$$

This formula is of little help if $\mathcal{H}_{k+1}(f) = 0$. But even in this case, we still have the following fundamental result (for the proof, see [13]).

Theorem 2.2 *With the same notation as above, suppose that there exist two integers ℓ, k such that $\ell > k \geq 1$ and $\mathcal{H}_\ell(f)\mathcal{H}_k(f) \neq 0$. Then the Padé approximant $[k-1/k]_f(z)$ exists, and there exist a nonzero element h_k in $\mathbb{F}$ and an integer k' such that $k \leq k' < \ell$ and*

$$f(z) - [k-1/k]_f(z) = h_k z^{k+k'} + \mathcal{O}(z^{k+k'+1}).$$

3 Irrationality exponent with Hankel determinants

In this section, we shall use Hankel determinants to bound from above the irrationality exponent of p-adic transcendental numbers, which are values at integer powers of p of power series satisfying a functional equation of a special type.

Theorem 3.1 *Let p be a prime number, w a positive integer, and $b = p^w$. Let $(c_m)_{m\geq 0}$ be an integer sequence and $f(z) = \sum_{m=0}^{+\infty} c_m z^m$. Suppose that there exist an integer $d \geq 2$ and $A(z), B(z), C(z), D(z) \in \mathbb{Z}[z]$ such that*

$$f(z) = \frac{A(z)}{B(z)} + \frac{C(z)}{D(z)} f(z^d) \tag{3.1}$$

and $B(b^{d^m})C(b^{d^m})D(b^{d^m}) \neq 0$, for all integers $m \geq 0$. If there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $\mathcal{H}_{n_i}(f) \neq 0$ for all integers $i \geq 0$, then, setting

$$\rho := \limsup_{i\to\infty} \frac{n_{i+1}}{n_i},$$

the p-adic number $f(b)$ is transcendental, and we have

$$\mu(f(b)) \leq (1+\rho)\rho^3 \min\{\rho^4, d\}.$$

In particular, the irrationality exponent of $f(b) = 2$ if $\rho = 1$.

Theorem 3.1 is the p-adic analogue of Theorem 4.1 in [13] and their proofs essentially follow the same lines. Observe, however, that the dependence on ρ in the upper bound for the irrationality exponent of $f(b)$ is much worse in Theorem 3.1 than in its real counterpart.

This is due to the fact that we have to control the degrees of the polynomials giving the Padé approximants to $f(z)$. Such a control was not needed in the real case. Anyway, in both the real and the p-adic settings, we have established that the irrationality exponent of $f(b)=2$ when $\rho=1$.

Proof Since there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $\mathcal{H}_{n_i}(f)\neq 0$ for all integers $i\geq 0$, we know by Kronecker's theorem (see [29, p. 5]) that $f(z)$ is not a rational function; thus, it is transcendental over $\mathbb{Q}(z)$ by Fatou's theorem (see [17]). But $f(z)$ has integer coefficients, so it is also transcendental over $\mathbb{C}_p(z)$ (see [31]), where $\mathbb{C}_p$ is the topological completion of a fixed algebraic closure of $\mathbb{Q}_p$. Then from Eq. (3.1) and the fact that $B(b^{d^m})C(b^{d^m})D(b^{d^m})\neq 0$ for all integers $m\geq 0$, we deduce immediately that $f(b)$ is transcendental (see, for example [34, p. 464]).

In the following, we shall only consider the case that $\rho<+\infty$.

By iteration of the formula (3.1), we have, for all integers $m\geq 1$,

$$f(z)=\frac{A_m(z)}{B_m(z)}+\frac{C_m(z)}{D_m(z)}f(z^{d^m}), \tag{3.2}$$

where we have set $C_m(z)=\prod_{j=0}^{m-1}C(z^{d^j})$, $D_m(z)=\prod_{j=0}^{m-1}D(z^{d^j})$, and

$$B_m(z)=D_{m-1}(z)\prod_{j=0}^{m-1}B(z^{d^j}),\ \ A_m(z)=\sum_{j=0}^{m-1}C_j(z)A(z^{d^j})\cdot\frac{B_m(z)}{D_j(z)B(z^{d^j})},$$

with $C_0(z)=D_0(z)=1$. Since B_m, B; C_m, C; and D_m, D share the same properties, we can always assume $d>\rho$ (otherwise everything which follows holds with d being replaced by d^k, where $k\geq 1$ is the smallest integer such that $d^k>\rho$).

Put $\alpha=\deg(A(z))$, $\beta=\deg(B(z))$, $\gamma=\deg(C(z))$, $\delta=\deg(D(z))$. Then, as proved in [13], we have:

$$\begin{aligned}\deg(A_m(z))&\leq(\alpha+\beta+\gamma+\delta)d^m,\\ \deg(B_m(z))&\leq(\delta+\beta)d^m,\\ \deg(C_m(z))&\leq\gamma d^m,\\ \deg(D_m(z))&\leq\delta d^m.\end{aligned}$$

Let $i\geq 0$ be an integer. By virtue of Theorem 2.2, we can find an integer n_i' $(n_i\leq n_i'<n_{i+1})$, $h_i\in\mathbb{Q}\setminus\{0\}$, and $P_i(z), Q_i(z)\in\mathbb{Z}[z]$ with $\deg(P_i(z))\leq n_i-1$, $\deg(Q_i(z))\leq n_i$, and $Q_i(0)\neq 0$ (see [13]) such that we have the following equality

$$f(z)-\frac{P_i(z)}{Q_i(z)}=h_iz^{n_i+n_i'}S_i(z), \tag{3.3}$$

where $S_i(z)=1+\sum_{j=1}^{+\infty}s_j^{(i)}z^j$, with $s_j^{(i)}\in\mathbb{Q}$ for all integers $j\geq 1$. Thus, for all integers $m\geq 1$, we have

$$f(z^{d^m})-\frac{P_i(z^{d^m})}{Q_i(z^{d^m})}=h_iz^{(n_i+n_i')d^m}S_i(z^{d^m}).$$

Combined with the formula (3.2), this gives

$$f(z)-\frac{A_m(z)}{B_m(z)}-\frac{C_m(z)}{D_m(z)}\cdot\frac{P_i(z^{d^m})}{Q_i(z^{d^m})}=h_iz^{(n_i+n_i')d^m}\frac{C_m(z)}{D_m(z)}S_i(z^{d^m}).$$

To simplify the notation, we define

$$P_{i,m}(z) = A_m(z)D_m(z)Q_i(z^{d^m}) + B_m(z)C_m(z)P_i(z^{d^m}),$$
$$Q_{i,m}(z) = B_m(z)D_m(z)Q_i(z^{d^m}).$$

Since $C(z)D(z) \neq 0$, then we can write

$$C(z) = c_\eta z^\eta(1 + z\tilde{C}(z)), \quad D(z) = d_\iota z^\iota(1 + z\tilde{D}(z))$$

with $\eta, \iota \geq 0$ integers, $c_\eta, d_\iota \in \mathbb{Z} \setminus \{0\}$, and $\tilde{C}(z), \tilde{D}(z) \in \mathbb{Q}[z]$. Note that $\tilde{C}(z), \tilde{D}(z)$ are bounded on the closed unit disk $\mathbb{Z}_p$; thus, there exists an integer $j_0 > 0$ such that for all integers $j \geq j_0$, we have $|b^{d^j}\tilde{C}(b^{d^j})|_p < 1$ and $|b^{d^j}\tilde{D}(b^{d^j})|_p < 1$; hence,

$$|1 + b^{d^j}\tilde{C}(b^{d^j})|_p = 1 \textit{ and } |1 + b^{d^j}\tilde{D}(b^{d^j})|_p = 1,$$

from which we obtain, for all integers $m > j_0$,

$$\sigma := \prod_{j=0}^{j_0} \left|1 + b^{d^j}\tilde{C}(b^{d^j})\right|_p = \left|\frac{C_m(b)}{c_\eta^m b^{\frac{\eta(d^m-1)}{d-1}}}\right|_p,$$
$$\tau := \prod_{j=0}^{j_0} \left|1 + b^{d^j}\tilde{D}(b^{d^j})\right|_p = \left|\frac{D_m(b)}{d_\iota^m b^{\frac{\iota(d^m-1)}{d-1}}}\right|_p.$$

Note that $Q_i(0) \neq 0$, so $Q_i(z)$ is different from zero in a neighborhood of zero in $\mathbb{Q}_p$, on which $S_i(z)$ converges by virtue of (3.3). Hence, we can find an integer $N_{0,i} > j_0$ (which depends only on i) such that for all integers $j \geq N_{0,i}$ we have $Q_i(b^{d^j}) \neq 0$ and

$$|S_i(b^{d^j}) - S_i(0)|_p < 1, \textit{for } z \mapsto S_i(z)$$

is continuous at the point $z = 0$. Thus, $|S_i(b^{d^j})|_p = |S_i(0)|_p = 1$. Note also that, by assumption, we have $B(b^{d^j})C(b^{d^j})D(b^{d^j}) \neq 0$ for all integers $j \geq 0$. Hence, $\sigma\tau \neq 0$, and for all integers $m > N_{0,i}$, we have

$$\left|f(b) - \frac{P_{i,m}(b)}{Q_{i,m}(b)}\right|_p = \left|h_i b^{(n_i+n_i')d^m}\frac{C_m(b)}{D_m(b)}S_i(b^{d^m})\right|_p$$
$$= \frac{\sigma}{\tau}|h_i|_p \left|\frac{c_\eta}{d_\iota}\right|_p^m |b|_p^{(n_i+n_i')d^m + \frac{(\eta-\iota)(d^m-1)}{d-1}}. \tag{3.4}$$

Unlike in the real case, we need control the degrees of $P_i(z)$ and $Q_i(z)$.

Note that, for all integers $i \geq 2$, we have

$$f(z) - \frac{P_{i-1}(z)}{Q_{i-1}(z)} = h_{i-1}z^{n_{i-1}+n_{i-1}'}(1 + \mathcal{O}(z)),$$
$$f(z) - \frac{P_i(z)}{Q_i(z)} = h_i z^{n_i+n_i'}(1 + \mathcal{O}(z)) = z^{n_{i-1}+n_{i-1}'}\mathcal{O}(z)$$

from which we deduce immediately that $\frac{P_{i-1}(z)}{Q_{i-1}(z)} \neq \frac{P_i(z)}{Q_i(z)}$. However, $\frac{P_{i-1}(z)}{Q_{i-1}(z)}$ is the Padé approximant $[n_{i-1} - 1/n_{i-1}]_f(z)$ to f. Then, by the unicity of Padé approximant (see [8,

p. 35]), we deduce that $\frac{P_i(z)}{Q_i(z)}$ is not the Padé approximant $[n_{i-1} - 1/n_{i-1}]_f(z)$ to f; thus, we have

$$\deg(P_i(z)) \geq n_{i-1} \quad \text{or} \quad \deg(Q_i(z)) > n_{i-1}. \tag{3.5}$$

This simple observation is crucial in the remaining part of the proof.

Recall that $Q_i(b^{d^j}) \neq 0$ for all integers $j \geq N_{0,i}$. Recall also that, by assumption, we have $B(b^{d^m})C(b^{d^m})D(b^{d^m}) \neq 0$ for all integers $m \geq 0$. Thus, we can find two constants $\alpha_{1,i}, \alpha_{2,i} > 0$ (which depend only on i) such that for all integers $m \geq 0$, we have

$$\begin{cases} |A(b^{d^m})| \leq \alpha_{2,i} b^{d^m \deg(A)}, \\ |P_i(b^{d^m})| \leq \alpha_{2,i} b^{d^m \deg(P_i)}, \\ \alpha_{1,i} b^{d^m \deg(B)} \leq |B(b^{d^m})| \leq \alpha_{2,i} b^{d^m \deg(B)}, \\ \alpha_{1,i} b^{d^m \deg(C)} \leq |C(b^{d^m})| \leq \alpha_{2,i} b^{d^m \deg(C)}, \\ \alpha_{1,i} b^{d^m \deg(D)} \leq |D(b^{d^m})| \leq \alpha_{2,i} b^{d^m \deg(D)}, \\ \alpha_{1,i} b^{d^m \deg(Q_i)} \leq |Q_i(b^{d^m})| \leq \alpha_{2,i} b^{d^m \deg(Q_i)}, \end{cases} \tag{3.6}$$

where the last one only holds for $m \geq N_{0,i}$. For all integers $i, m \geq 1$, put

$$q_{i,m} = |Q_{i,m}(b)|, \quad p_{i,m} = P_{i,m}(b)\operatorname{sgn}(Q_{i,m}(b)),$$
$$T_{i,m} = \max(\deg(Q_{i,m}), \deg(P_{i,m})) \quad \text{and} \quad t_{i,m} = \max(|p_{i,m}|, |q_{i,m}|, 1).$$

We note that $q_{i,m}$ and $p_{i,m}$ are integers. Fix $\varepsilon \in (0, \frac{1}{10})$ small enough such that $\frac{d}{\rho} > \frac{1+\varepsilon}{1-2\varepsilon}$. Since $\lim_{i\to+\infty} n_i = +\infty$, there exists an integer $N_1 \geq 1$ such that for all integers $i \geq N_1$, we have

$$\alpha + \beta + \gamma + \delta + \eta + \iota < \frac{\varepsilon n_i}{4}. \tag{3.7}$$

From the definition of $P_{i,m}$, $Q_{i,m}$ and the formula (3.7), we obtain that for all integers $i > N_1$ and $m \geq 1$, we have

$$T_{i,m} \leq \left(1 + \frac{\varepsilon}{2}\right) n_i d^m.$$

It then follows from the definition of $t_{i,m}$ and the formula (3.6) that there exists an integer $N_{1,i} \geq N_{0,i}$ such that for all integers $m \geq N_{1,i}$, we have

$$t_{i,m} \leq b^{(1+\varepsilon)n_i d^m}.$$

To get a lower bound for $t_{i,m}$ with $i > N_1$, we distinguish two cases:

Case I $\deg(Q_i(z)) \geq (1 - \frac{\varepsilon}{2})n_{i-1}$. Then, for all integers $m \geq 1$, we have

$$T_{i,m} \geq \deg(Q_{i,m}) \geq \deg(Q_i(z^{d^m})) \geq \left(1 - \frac{\varepsilon}{2}\right) n_{i-1} d^m.$$

It then follows from the definition of $t_{i,m}$ and the formula (3.6) that there exists an integer $N_{2,i} \geq N_{1,i}$ such that, for all integers $m > N_{2,i}$, we have

$$t_{i,m} \geq q_{i,m} \geq b^{(1-\varepsilon)n_{i-1} d^m}.$$

Case II $\deg(Q_i(z)) \leq (1-\frac{\varepsilon}{2})n_{i-1}$. Then, it follows from the formula (3.5) that $\deg(P_i(z)) \geq n_{i-1}$, and for all integers $m \geq 1$, we have

$$T_{i,m} \geq \deg(P_{i,m}) = \deg(B_m(z)C_m(z)P_i(z^{d^m})) \geq n_{i-1} d^m.$$

At the same time, from the definition of $t_{i,m}$ and the formula (3.6), we can find an integer $N_{3,i} \geq N_{1,i}$ such that for all integers $m > N_{3,i}$, we have

$$t_{i,m} \geq |p_{i,m}| \geq b^{(1-\varepsilon)n_{i-1}d^m}.$$

Hence, for all integers $i > N_1$ and $m > \max(N_{2,i}, N_{3,i})$, we always have

$$b^{(1-\varepsilon)n_{i-1}d^m} \leq t_{i,m} \leq b^{(1+\varepsilon)n_i d^m}. \tag{3.8}$$

Similarly by (3.4) and (3.7), there exists an integer $N_{4,i} > \max(N_{2,i}, N_{3,i})$ such that for all integers $i > N_1$ and $m \geq N_{4,i}$, we have

$$\frac{1}{b^{(n_i+n_i')(1+\varepsilon)d^m}} \leq \left| f(b) - \frac{p_{i,m}}{q_{i,m}} \right|_p \leq \frac{1}{b^{(n_i+n_i')(1-\varepsilon)d^m}}, \tag{3.9}$$

and by the formula (3.8), we obtain also

$$t_{i,m}^{-\frac{(n_i+n_i')(1+\varepsilon)}{n_{i-1}(1-\varepsilon)}} \leq \left| f(b) - \frac{p_{i,m}}{q_{i,m}} \right|_p \leq t_{i,m}^{-\frac{(n_i+n_i')(1-\varepsilon)}{n_i(1+\varepsilon)}}. \tag{3.10}$$

By the definition of ρ, there exists an integer $i_0 > N_1$ such that for all integers $i \geq i_0$, we have $\frac{n_{i+1}}{n_i} < \rho + \varepsilon \leq \rho(1+\varepsilon)$, and

$$\begin{cases} \frac{(n_i+n_i')(1+\varepsilon)}{n_{i-1}(1-\varepsilon)} \leq (1+\rho)\rho(1+3\varepsilon), \\ \frac{(n_i+n_i')(1-\varepsilon)}{n_i(1+\varepsilon)} \geq 2(1-3\varepsilon). \end{cases} \tag{3.11}$$

In particular, for all integers $i > i_0$ and for all integers $m \geq N_{4,i}$, we have

$$b^{n_i d^m (1-2\varepsilon)/\rho} \leq t_{i,m} \leq b^{n_i d^m (1+\varepsilon)}, \tag{3.12}$$

$$t_{i,m} < t_{i,m}^{\frac{d(1-2\varepsilon)}{\rho(1+\varepsilon)}} \leq t_{i,m+1} \leq t_{i,m}^{\frac{\rho d(1+\varepsilon)}{1-2\varepsilon}}, \tag{3.13}$$

$$t_{i,m}^{-(1+\rho)\rho(1+3\varepsilon)} \leq \left| f(b) - \frac{p_{i,m}}{q_{i,m}} \right|_p \leq t_{i,m}^{-2(1-3\varepsilon)}. \tag{3.14}$$

Applying Lemma 2.1 with (3.13) and (3.14), we obtain

$$\mu(f(b)) \leq \frac{(1+\rho)\rho(1+3\varepsilon)}{2(1-3\varepsilon)-1} \cdot \frac{\rho d(1+\varepsilon)}{1-2\varepsilon}.$$

Since ε is positive and can be chosen arbitrarily small, we get

$$\mu(f(b)) \leq d(1+\rho)\rho^2. \tag{3.15}$$

In the following, we assume $\rho < \sqrt[3]{d}$ and choose $\varepsilon > 0$ small enough such that $\rho < \sqrt[3]{d} - \varepsilon$. Fix $\ell > 1$ an integer such that $d^{\ell-1} > n_{i_0+1}$. Let $i_1 > i_0$ be the smallest integer such that $n_{i_1} \in [d^{\ell-1}, d^\ell - 1]$ (such an integer exists maybe not for all ℓ, but at least for infinitely many ℓ). Then, $n_{i_1-1} \leq d^{\ell-1} - 1$, and thus, we have

$$d^{\ell-1} \leq n_{i_1} < (\rho+\varepsilon)n_{i_1-1} \leq (\rho+\varepsilon)(d^{\ell-1}-1).$$

Since $n_{i_1+1} < (\rho+\varepsilon)n_{i_1}$, we can find an integer $i_2 > i_1$ such that

$$n_{i_2} < (\rho+\varepsilon)n_{i_1} \leq n_{i_2+1}.$$

So $n_{i_2+1} < (\rho+\varepsilon)n_{i_2} < (\rho+\varepsilon)^3(d^{\ell-1}-1) \le d^\ell - 1$. Set $i_3 = i_2 + 1$,

$$\mathcal{A}_\ell = \{n_{i_1}, n_{i_2}, n_{i_3}\} \cup \{n_j \in [d^{\ell-1}, d^\ell - 1] \mid j > i_3\},$$

and denote the elements of $\mathcal{A}_\ell$ as $n_{i_1} < n_{i_2} < \cdots < n_{i_\omega}$. Then, $\omega \ge 3$, and we have $d^\ell \le n_{i_\omega+1} < (\rho+\varepsilon)n_{i_\omega}$. Set

$$M_\ell = \max_{1\le i\le i_\omega} N_{4,i}.$$

We arrange the integers $t_{i_l,m}$ ($1 \le l \le \omega$ and $m \ge M_\ell$) as an increasing sequence, denoted by $(r_{\ell,j})_{j\ge 0}$.

Fix $j \ge 0$, and write $r_{\ell,j} = t_{i_l,m}$ with $1 \le l \le \omega$. By (3.12), we have

$$b^{n_{i_l} d^m (1-2\varepsilon)/\rho} \le t_{i_l,m} \le b^{n_{i_l} d^m (1+\varepsilon)}.$$

We distinguish below two cases:

Case I $n_{i_\omega} > \rho n_{i_l}(1+\varepsilon)/(1-2\varepsilon)$. Since $\rho \ge 1$, we have $i_\omega > i_l$ and thus there exists a smallest integer v such that $l < v \le \omega$ such that

$$n_{i_v} > \rho n_{i_l} \frac{1+\varepsilon}{1-2\varepsilon}.$$

Consequently, we have $t_{i_v,m} \ge b^{n_{i_v} d^m (1-2\varepsilon)/\rho} > b^{n_{i_l} d^m (1+\varepsilon)} \ge t_{i_l,m}$ and

$$\frac{\log t_{i_v,m}}{\log t_{i_l,m}} \le \frac{\rho n_{i_v}(1+\varepsilon)}{n_{i_l}(1-2\varepsilon)}.$$

By the minimality of v, we obtain

$$n_{i_v} < (\rho+\varepsilon)n_{i_{v-1}} \le (\rho+\varepsilon)\rho n_{i_l} \frac{1+\varepsilon}{1-2\varepsilon},$$

from which we deduce directly

$$1 < \frac{\log r_{\ell,j+1}}{\log r_{\ell,j}} \le \frac{\log t_{i_v,m}}{\log t_{i_l,m}} < \frac{\rho^2(\rho+\varepsilon)(1+\varepsilon)^2}{(1-2\varepsilon)^2}.$$

Case II $n_{i_\omega} \le \rho n_{i_l}(1+\varepsilon)/(1-2\varepsilon)$. Since $n_{i_\omega} < d^\ell \le d n_{i_1}$, we get $d n_{i_1}/n_{i_\omega} > 1$ and we obtain, for all $\varepsilon > 0$ small enough and by our choice of i_3, that

$$\rho n_{i_\omega} \frac{1+\varepsilon}{1-2\varepsilon} < (\rho+\varepsilon) d n_{i_1} \le d n_{i_3},$$

as $\frac{\rho(1+\varepsilon)}{(1-2\varepsilon)(\rho+\varepsilon)}$ converges to 1 when ε tends to 0. Then, we get

$$\frac{\log t_{i_3,m+1}}{\log t_{i_l,m}} \ge \frac{n_{i_3} d(1-2\varepsilon)}{\rho n_{i_l}(1+\varepsilon)} > \frac{n_{i_\omega}}{n_{i_l}} \ge 1.$$

Moreover, from $n_{i_\omega} \le \rho n_{i_l}(1+\varepsilon)/(1-2\varepsilon)$, we obtain also

$$\frac{\log t_{i_3,m+1}}{\log t_{i_l,m}} \le \frac{\rho d n_{i_3}(1+\varepsilon)}{n_{i_l}(1-2\varepsilon)} \le \frac{\rho^2 d n_{i_3}(1+\varepsilon)^2}{n_{i_\omega}(1-2\varepsilon)^2}.$$

Note that $n_{i_\omega} > \frac{d^\ell}{\rho+\varepsilon}$ and $n_{i_3} = n_{i_2+1} < (\rho+\varepsilon)^3(d^{\ell-1}-1)$, hence

$$\frac{d n_{i_3}}{n_{i_\omega}} < (\rho+\varepsilon)^4 \frac{d(d^{\ell-1}-1)}{d^\ell} < (\rho+\varepsilon)^4,$$

and then we deduce

$$1 < \frac{\log r_{\ell,j+1}}{\log r_{\ell,j}} \leq \frac{\log t_{i_3,m+1}}{\log t_{i_l,m}} < \frac{\rho^2(\rho+\varepsilon)^4(1+\varepsilon)^2}{(1-2\varepsilon)^2}.$$

Since $\rho \geq 1$, thus for all integers $j \geq 0$, we have in both cases

$$1 < \frac{\log r_{\ell,j+1}}{\log r_{\ell,j}} < \frac{\rho^2(\rho+\varepsilon)^4(1+\varepsilon)^2}{(1-2\varepsilon)^2}. \tag{3.16}$$

Once again applying Lemma 2.1 with (3.14) and (3.16), we get

$$\mu(f(b)) \leq \frac{(1+\rho)\rho(1+3\varepsilon)}{2(1-3\varepsilon)-1} \cdot \frac{\rho^2(\rho+\varepsilon)^4(1+\varepsilon)^2}{(1-2\varepsilon)^2}.$$

Since ε is positive and can be chosen arbitrarily small, hence we have

$$\mu(f(b)) \leq (1+\rho)\rho^7. \tag{3.17}$$

The bounds (3.15) and (3.17) are obtained under the assumption that $d > \rho$. As noticed previously, we can remove this assumption by replacing d with d^k, where k is the smallest integer such that $d^k > \rho$. In particular, we have $d^k \leq d\rho$. Consequently, we have shown that

$$\mu(f(b)) \leq (1+\rho)\rho^3 d$$

and, under the assumption $\rho < d^{k/3}$,

$$\mu(f(b)) \leq (1+\rho)\rho^7,$$

from which we deduce the desired result by noting that $\min\{d, \rho^4\} = d$ when $\rho \geq d^{k/3}$. In particular, if $\rho = 1$, then $f(b) \leq 2$. But $f(b)$ is transcendental; thus, its irrationality exponent is equal to 2. □

Remark In the statement of Theorem 3.1, if we replace $b = p^w$ by $b = rp^w/s$, where r, s are coprime integers such that $r \neq 0$, $s > 0$, and p does not divide rs, then the same reason (see [34, p. 464]) yields that the p-adic number $f(rp^w/s)$ is transcendental. Moreover, with slight modifications, we can also show that

$$\mu\left(f\left(\frac{rp^w}{s}\right)\right) \leq \frac{w}{2w - \log_p \max\{rp^w, s\}}(1+\rho)\rho^3 \min\{\rho^4, d\},$$

if $\max\{rp^w, s\} < p^{2w}$. The verification is slightly technical, but direct and routine. In the real case, an analogous result was given in [13]. See also Dubickas [16] for the irrationality exponent of the Thue–Morse power series evaluated at the rational number a/b with $a^2 < b$.

If the sequence $(c_j)_{j\geq 0}$ takes only finitely many integer values, then the conditions in Theorem 3.1 can be simplified as follows.

Theorem 3.2 *Let p be a prime number, w a positive integer, and $b = p^w$. Let $(c_m)_{m\geq 0}$ be an integer sequence taking only finitely many values, and set $f(z) = \sum_{j=0}^{+\infty} c_j z^j$. Suppose that there exist an integer $d \geq 2$ and $A(z), B(z), C(z), D(z) \in \mathbb{Z}[z]$ such that $C(b^{d^m}) \neq 0$ for all $m \geq 0$, and*

$$f(z) = \frac{A(z)}{B(z)} + \frac{C(z)}{D(z)} f(z^d).$$

If there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ *such that* $\mathcal{H}_{n_i}(f) \neq 0$ *for all integers* $i \geq 0$ *and* $\limsup_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$, *then the* p*-adic number* $f(b)$ *is transcendental and its irrationality exponent is equal to* 2.

Proof Since the sequence $(c_j)_{j\geq 0}$ is bounded, we can find an integer $\ell > 2$ such that $|c_j| < b^{d^\ell - 1}$, for all integers $j \geq 0$. As in the proof of Theorem 3.1, we know also that $f(z)$ is not rational, and there exist $A_\ell(z)$, $B_\ell(z)$, $C_\ell(z)$, and $D_\ell(z)$ in $\mathbb{Z}[z]$ such that

$$f(z) = \frac{A_\ell(z)}{B_\ell(z)} + \frac{C_\ell(z)}{D_\ell(z)} f(z^{d^\ell}). \tag{3.18}$$

Moreover, $C_\ell(b^{d^m}) \neq 0$ for all integers $m \geq 0$, since $C(b^{d^m}) \neq 0$ for all integers $m \geq 0$. Without loss of generality, we can also suppose that

$$\gcd(A_\ell(z), B_\ell(z)) = 1 \ \text{ and } \ \gcd(C_\ell(z), D_\ell(z)) = 1.$$

We argue by contradiction. Suppose that there is an integer $m \geq 0$ such that $B_\ell(b^{d^m})D_\ell(b^{d^m}) = 0$. Then, we can write

$$B_\ell(z) = (z - b^{d^m})^s E(z), \ D_\ell(z) = (z - b^{d^m})^t F(z),$$

where $E(z), F(z) \in \mathbb{Q}[z]$ are not equal to zero at $z = b^{d^m}$, and $s, t \geq 0$ are integers such that $\max\{s, t\} \geq 1$.

If $s > t$, then from the formula (3.18), we obtain

$$(z - b^{d^m})^t f(z) - \frac{C_\ell(z)}{F(z)} f(z^{d^\ell}) = \frac{A_\ell(z)}{(z - b^{d^m})^{s-t} E(z)}.$$

The left-hand side is regular at $z = b^{d^m}$, while the right-hand side is not, giving us the required contradiction.

If $s \leq t$, then from the formula (3.18), we have

$$(z - b^{d^m})^t f(z) - \frac{(z - b^{d^m})^{t-s} A_\ell(z)}{E(z)} = \frac{C_\ell(z)}{F(z)} f(z^{d^\ell}).$$

Hence, $f(b^{d^{m+\ell}})$ is a rational number. But $(c_j)_{j\geq 0}$ is the sequence of coefficients of this rational number in its base-$b^{d^{m+\ell}}$ expansion, and it is bounded by $b^{d^\ell - 1}$. Thus, the sequence $(c_j)_{j\geq 0}$ is ultimately periodic. This gives again a contradiction since $f(z)$ is not rational.

To conclude, it suffices to apply Theorem 3.1 to the formula (3.18). □

4 Some applications

All the results in the real case presented in [13] have corresponding p-adic versions. In this section, we only summarize the main applications of Theorem 3.1.

We begin with the p-adic analogue of Theorem 6.1 from [13].

Theorem 4.1 *Let* $f(z) \in \mathbb{Z}[[z]]$ *be a power series such that*

$$A(z) + B(z)f(z) + C(z)f(z^2) = 0, \tag{4.1}$$

where $A(z)$, $B(z)$, *and* $C(z)$ *are integer polynomials satisfying one of the following conditions:*

(i) $B(0) \equiv 1,\ C(0) \equiv 0 \pmod 2$,
(ii) $A(0) \equiv 0,\ B(0) \equiv 1,\ C(0) \equiv 1 \pmod 2$.

Let p be a prime number, w a positive integer, and $b = p^w$ such that $B(b^{2^m})C(b^{2^m}) \neq 0$ for all integers $m \geq 0$. If $f(z) \pmod 2$ is not a rational function, then the p-adic number $f(b)$ is transcendental and its irrationality exponent is equal to 2.

Proof Put $F(z) = f(z) \pmod 2 \in \mathbb{F}_2[[z]]$. By the formula (4.1), we obtain

$$A(z) + B(z)F(z) + C(z)F(z)^2 = 0.$$

By Theorem 5.2 in [13] (with conditions (i) and (iii), respectively), the sequence $\mathcal{H}(F)$ of Hankel determinants of F is ultimately periodic over $\mathbb{F}_2$. Since $F(z)$ is not a rational function in $\mathbb{F}_2[[z]]$, there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $\mathcal{H}_{n_i}(F) \neq 0$ for all $i \geq 0$ and $\lim\limits_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. The conclusion comes from Theorem 3.1. □

Theorem 1.1 can be deduced immediately from Theorem 4.1.

Proof of Theorem 1.1 Put $f(z) = \sum\limits_{n\geq 0} t_n z^n$. Then,

$$z - (1 - z^2)f(z) + (1 - z^2)(1 - z)f(z^2) = 0.$$

But the Thue–Morse sequence $(t_n \pmod 2)_{n\geq 0}$ is not ultimately periodic; thus, by Theorem 4.1 (ii), we obtain the desired result. □

For all integers $n \geq 0$ with binary expansion $n = \sum_{j=0}^{k} n_j 2^j$ where $n_j = 0, 1$ $(0 \leq j \leq k)$, put $s_2(n) = \sum_{j=0}^{k} n_j$, called the sum of the binary digits of n. One checks that $s_2(n) \equiv t_n \pmod 2$, for all integers $n \geq 0$.

Inspired by the above proof and also by the recent work [15] of Coons, we obtain the following result.

Theorem 4.2 *Let p be a prime number and w a positive integer. Then, the p-adic number $\sum_{n\geq 0} s_2(n)p^{wn}$ is transcendental, and its irrationality exponent is equal to* 2.

Proof Indeed, if we put $f(z) = \sum_{n\geq 0} s_2(n)z^n$, then

$$z - (1 - z^2)f(z) + (1 - z^2)(1 + z)f(z^2) = 0.$$

To conclude, it suffices to proceed as for the proof of Theorem 1.1. □

Theorem 4.3 *Let $f(z) \in \mathbb{Z}[[z]]$ be the power series defined by*

$$f(z) = \prod_{n\geq 0} \left(1 + uz^{2^n} + 2z^{2^{n+1}} \frac{C(z^{2^n})}{D(z^{2^n})} \right), \tag{4.2}$$

with $u \in \mathbb{Z}$, $C(z), D(z) \in \mathbb{Z}[z]$, and $D(0) = 1$. Let p be a prime number, w a positive integer, and $b = p^w$ such that $D(b^{2^m})f(b^{2^m}) \neq 0$ for all integers $m \geq 0$. If $f(z) \pmod 4$ is not a rational function, then the p-adic number $f(b)$ is transcendental and its irrationality exponent is equal to 2.

Proof We proceed exactly as in the proof of Theorem 2.2 of [13] to show that there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $\mathcal{H}_{n_i}(f) \neq 0$ for all integers $i \geq 0$ and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. We conclude by applying Theorem 3.1. □

Remark Note that Theorem 4.3 cannot be deduced from Theorem 4.1, for $f(z) \pmod 2$ is a rational function.

Theorem 1.1 is also a direct consequence of Theorem 4.3.

Another proof of Theorem 1.1 Put $f(z) = \sum_{n\geq 0}(1-2t_n)z^n$. Then, we have

$$f(z) = \prod_{n\geq 0}(1 - z^{2^n}).$$

Applying Theorem 4.3 with $C(z) = 0$, $D(z) = 1$, and $u = -1$, we obtain that $\mu(f(b)) = 2$. But $f(b) = \frac{1}{1-b} - 2\sum_{n\geq 0} t_n b^n$; hence,

$$\mu\left(\sum_{n\geq 0} t_n b^n\right) = \mu(f(b)) = 2.$$

□

For all integers $\alpha, \beta \geq 0$, define the functions

$$F_{\alpha,\beta}(z) = \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} \frac{z^{2^{n+\alpha}}}{1 + z^{2^{n+\beta}}} = \sum_{n,j\geq 0} (-1)^j z^{(j2^{\beta-\alpha}+1)2^{n+\alpha}-2^\alpha},$$

$$G_{\alpha,\beta}(z) = \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} \frac{z^{2^{n+\alpha}}}{1 - z^{2^{n+\beta}}} = \sum_{n,j\geq 0} z^{(j2^{\beta-\alpha}+1)2^{n+\alpha}-2^\alpha}.$$

It is shown in [13] that $G_{\alpha,\beta}(z)$ is rational if $\beta = \alpha + 1$, and $G_{\alpha,\beta}(z) \pmod 2$ is not a rational function if $\beta \neq \alpha + 1$. The sequence of coefficients of $G_{0,0}(z)$ is usually called the *Gros sequence* [19,22].

For $\beta \neq \alpha + 1$, we have the following result.

Theorem 4.4 *Let $\alpha, \beta \geq 0$ be integers such that $\beta \neq \alpha + 1$. Let p be a prime number, w be a positive integer, and set $b = p^w$. Then, the p-adic numbers $F_{\alpha,\beta}(b)$ and $G_{\alpha,\beta}(b)$ are transcendental and their irrationality exponents are equal to* 2.

Proof From the definition, we know directly that $F_{\alpha,\beta}(z)$ and $G_{\alpha,\beta}(z)$ have integer coefficients in power series expansion. Moreover, we have also

$$-1 + (1 + z^{2^\beta})F_{\alpha,\beta}(z) - z^{2^\alpha}(1 + z^{2^\beta})F_{\alpha,\beta}(z^2) = 0, \tag{4.5}$$

$$-1 + (1 - z^{2^\beta})G_{\alpha,\beta}(z) - z^{2^\alpha}(1 - z^{2^\beta})G_{\alpha,\beta}(z^2) = 0. \tag{4.6}$$

Note that $F_{\alpha,\beta}(z) \pmod 2 = G_{\alpha,\beta}(z) \pmod 2$ is not rational over $\mathbb{F}_2$. To conclude, it suffices to apply Theorem 4.1 (i). □

Recall here that the regular paperfolding sequence $(u_n)_{n\geq 0}$ on $\{0, 1\}$ is defined recursively by $u_{4n} = 1$, $u_{4n+2} = 0$, and $u_{2n+1} = u_n$, for all integers $n \geq 0$. The p-adic regular paperfolding numbers are defined by

$$f_{\mathrm{RPF}}(b) := \sum_{n\geq 0} u_n b^n,$$

where p is a prime number, w a positive integer, and $b = p^w$. Recall also that the regular paperfolding sequence is 4-automatic, but not ultimately periodic (see, for example [4]); thus, all the regular paperfolding p-adic numbers $f_{\mathrm{RPF}}(b)$ are transcendental.

Theorem 4.5 *Let p be a prime number, w a positive integer, and $b = p^w$. Then, the irrationality exponent of the regular paperfolding p-adic number $f_{\mathrm{RPF}}(b)$ is equal to* 2

Proof It suffices to apply Theorem 4.4 to $G_{0,2}(z)$. □

Stern's sequence $(a_n)_{n\geq 0}$ and its twisted version $(b_n)_{n\geq 0}$ (see [5,14,30]) are defined, respectively, by

$$\begin{cases} a_0 = 0, \quad a_1 = 1, \\ a_{2n} = a_n, \quad a_{2n+1} = a_n + a_{n+1} \ (n \geq 1), \end{cases}$$

and

$$\begin{cases} b_0 = 0, \quad b_1 = 1, \\ b_{2n} = -b_n, \quad b_{2n+1} = -(b_n + b_{n+1}) \ (n \geq 1). \end{cases}$$

Put $S(z) = \sum_{n=0}^{\infty} a_{n+1} z^n$ and $T(z) = \sum_{n=0}^{\infty} b_{n+1} z^n$. Our next result gives the exact irrationality exponent of the p-adic (twisted) Stern numbers.

Theorem 4.6 *Let p be a prime number, w a positive integer, and $b = p^w$. Then, the p-adic numbers $S(b)$ and $T(b)$ are transcendental and their irrationality exponents are equal to* 2.

Proof From the above definitions, we obtain (see also [14])

$$S(z) = (1 + z + z^2)S(z^2), \ T(z) = 2 - (1 + z + z^2)T(z^2).$$

Han has recently shown in [21] that for all integers $n \geq 2$, we have

$$\frac{H_n(S)}{2^{n-2}} \equiv \frac{H_n(T)}{2^{n-2}} \equiv \begin{cases} 0, & \text{if } n \equiv 0, 1 \pmod 4, \\ 1, & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

Hence, there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $\mathcal{H}_{n_i}(S)\mathcal{H}_{n_i}(T) \neq 0$ for all integers $i \geq 0$, and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. The rest follows from Theorem 3.1 □.

Theorem 4.7 *Let $f(z) \in \mathbb{Z}[[z]]$ be a power series defined by*

$$f(z) = \prod_{n=0}^{\infty} \frac{C(z^{3^n})}{D(z^{3^n})}, \tag{4.7}$$

with $D(z), C(z) \in \mathbb{Z}[z]$ such that $C(0) = D(0) = 1$. Let p be a prime number, w a positive integer, and $b = p^w$ such that $C(b^{3^m})D(b^{3^m}) \neq 0$ for all integers $m \geq 0$. If $f(z) \pmod 3$ is not rational, then $f(b)$ is transcendental and its irrationality exponent is equal to 2.

Proof Over the finite field $\mathbb{F}_3$, the power series $F(z) = f(z) \pmod 3$ satisfies the quadratic equation $-D(z) + C(z)F(z)^2 = 0$. Consequently, by Theorem 5.2 (iv) in [13], the sequence $\mathcal{H}(F)$ is ultimately periodic. Since $F(z)$ is not a rational function in $\mathbb{F}_3[[z]]$, there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $\mathcal{H}_{n_i}(F) \neq 0$ for all integers $i \geq 0$ and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. The conclusion follows from Theorem 3.1. □

Letting $C(z) = 1 - z$ (resp. $C(z) = 1 \pm z - z^2$) and $D(z) = 1$ in Theorem 4.7, we obtain at once the following corollary. The underlying Hankel determinants are evaluated in [20].

Corollary 4.8 *Let p be a prime number, w a positive integer, and $b = p^w$. Then, the p-adic numbers*

$$\prod_{k\geq 0}(1-b^{3^k}) \quad \text{and} \quad \prod_{k\geq 0}(1\pm b^{3^k}-b^{2\cdot 3^k})$$

are transcendental and their irrationality exponents are equal to 2.

The Cantor sequence $(v_n)_{n\geq 0}$ on $\{0, 1\}$ is defined as follows: For all integers $n \geq 0$, we have $v_n = 1$ if and only if the ternary expansion of n does not contain the digit 1. The Cantor p-adic numbers take the form

$$f_C(b) := \sum_{n\geq 0} v_n b^n,$$

where p is a prime number, w a positive integer, and $b = p^w$. Recall also that the Cantor sequence is 3-automatic, but not ultimately periodic (see, for example, [4]); thus, all the p-adic Cantor numbers $f_C(b)$ are transcendental. Note finally that $f_C(z) = (1+z^2)f_C(z^3)$.

Theorem 4.9 *Let p be a prime number, w a positive integer, and $b = p^w$. Then, the irrationality exponent of the Cantor p-adic number $f_C(b)$ is equal to* 2.

Proof Apply Theorem 4.7 with $C(z) = 1 + z^2$ and $D(z) = 1$. □

We give below further concrete examples of transcendental numbers with irrationality exponent equal to 2.

In [33], Väänänen studied the following two power series

$$L(z) = \sum_{j=0}^{\infty} \frac{z^{2^j}}{\prod_{i=0}^{j-1}(1-z^{2^i})}, \quad M(z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2^j}}{\prod_{i=0}^{j-1}(1-z^{2^i})},$$

which satisfy, respectively, the functional equations

$$z(z-1) + (1-z)L(z) - L(z^2) = 0,$$
$$z(z-1) + (1-z)M(z) + M(z^2) = 0.$$

One can check directly that neither $L(z)$ nor $M(z)$ is rational modulo 2.

Theorem 4.10 *Let p be a prime number, w a positive integer, and $b = p^w$. Then, the p-adic numbers $L(b)$ and $M(b)$ are transcendental and their irrationality exponents are equal to* 2.

Proof It suffices to apply Theorem 4.1 (ii). □

Fu and Han [18] studied the Hankel determinants of the following power series F_5, F_{11}, F_{13}, F_{17A}, and F_{17B}, satisfying the equations

$$\begin{aligned}
F_5(z) &= (1-z-z^2-z^3+z^4)\, F_5(z^5),\\
F_{11}(z) &= (1-z-z^2+z^3-z^4+z^5+z^6+z^7+z^8-z^9-z^{10})\, F_{11}(z^{11}),\\
F_{13}(z) &= (1-z-z^2+z^3-z^4-z^5-z^6-z^7-z^8\\
&\quad +z^9-z^{10}-z^{11}+z^{12})\, F_{13}(z^{13})\\
F_{17A}(z) &= (1-z-z^2+z^3-z^4+z^5+z^6+z^7+z^8+z^9\\
&\quad +z^{10}+z^{11}-z^{12}+z^{13}-z^{14}-z^{15}+z^{16})\, F_{17A}(z^{17}),\\
F_{17B}(z) &= (1-z-z^2-z^3+z^4+z^5-z^6+z^7+z^8+z^9\\
&\quad +z^{11}+z^{12}-z^{13}-z^{14}-z^{15}+z^{16})\, F_{17B}(z^{17})
\end{aligned}$$

and established that they verify the following relations

$$H_n(F_5)/2^{n-1} \equiv H_n(F_{11})/2^{n-1} \equiv H_n(F_{13})/2^{n-1} \equiv 1 \pmod 2,$$
$$H_n(F_{17A})/2^{n-1} \equiv H_n(F_{17B})/2^{n-1} \equiv 1 \pmod 2.$$

All these power series satisfy the conditions of Theorem 3.1; thus, we obtain the following result.

Theorem 4.11 *Let p be a prime number, w a positive integer, and $b = p^w$. Then, all the p-adic numbers $F_5(b)$, $F_{11}(b)$, $F_{13}(b)$, $F_{17A}(b)$, $F_{17B}(b)$ are transcendental and their irrationality exponents are equal to* 2.

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