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Pseudospherical surfaces with singularities

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Abstract We study a generalization of constant Gauss curvature -1 surfaces in Euclidean 3-space, based on Lorentzian harmonic maps, that we call pseudospherical frontals. We analyse the singularities of these surfaces, dividing them into those of characteristic and non-characteristic type. We give methods for constructing all non-degenerate singularities of both types, as well as many degenerate singularities. We also give a method for solving the singular geometric Cauchy problem: construct a pseudospherical frontal containing a given regular space curve as a non-degenerate singular curve. The solution is unique for most curves, but for some curves there are infinitely many solutions, and this is encoded in the curvature and torsion of the curve.

Keywords Differential geometry · Integrable systems · Loop groups · Pseudospherical surfaces · Constant Gauss curvature · Singularities

Mathematics Subject Classification Primary 53A05 · 53C43; Secondary 53C42

1 Introduction

It is a well-known theorem of Hilbert that there do not exist complete isometric immersions in $\mathbb{R}^3$ of surfaces with constant negative Gauss curvature $K = -1$. These surfaces have nevertheless been much studied since classical times. The integrability condition is the sine-Gordon equation $\phi_{xy} = \sin \phi$, where x and y are unit speed asymptotic coordinates and ϕ is the angle between the asymptotic directions. Most of the literature on these surfaces deals with them via the solutions of this equation, naturally leading to singularities along the curves $\phi = n\pi$ for integers n. A more general approach for pseudospherical surfaces is the formulation in terms of Lorentz harmonic maps. The Gauss map N of a pseudospherical surface is harmonic

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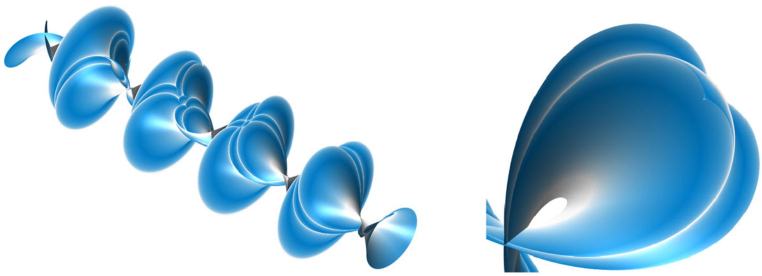


Fig. 1 Pseudospherical surface generated by a Viviani Fig. 8 space curve. The curve has non-vanishing curvature, but $|\tau| = 1$ at four points. This surface is a frontal but not a wave front (Example 4.7)

with respect to the Lorentzian metric induced by the second fundamental form. Conversely, if we restrict to *weakly regular* harmonic maps, i.e. those where the derivatives N_x and N_y with respect to a null coordinate system never vanish, then these maps correspond to solutions of the sine-Gordon equation. The associated surfaces are called weakly regular pseudospherical surfaces, and this has been the standard class of pseudospherical surfaces investigated in the literature.

In this article, we aim to study the natural singularities of pseudospherical surfaces. We will drop the weak regularity assumption, as it serves only to make a connection with the sine-Gordon equation. This connection is not needed in the harmonic map approach. Given a harmonic map $N : S \to \mathbb{S}^2$, from a simply connected Lorentz surface, there is a canonically associated map $f : S \to \mathbb{R}^3$, unique up to a translation, that is pseudospherical wherever it is immersed, and such that $\mathrm{d}f$ is orthogonal to N (see Sect. 2.2). We take such maps f as the definition of a generalized pseudospherical surface.

Abandoning the identification with solutions of the sine-Gordon equation is advantageous for two reasons. In the first place, in order to solve the Cauchy problem along an arbitrary non-characteristic curve, it is necessary to choose asymptotic coordinates (x, y) such that the curve is given by $y = \pm x$. This can always be achieved, at least locally, but not if we require that the coordinate lines are *constant speed*, the choice for which the angle between the coordinate curves is a solution of sine-Gordon. The second advantage is that we are interested in the natural singularities of these surfaces, and for many of these (for example the bifurcating cusp lines in Fig. 1, or the rank zero singularities in Fig. 10), there is no corresponding local solution of the sine-Gordon equation.

We will use a variant of the generalized d'Alembert method given by Toda [19] to study the surfaces. In brief, a loop group lift $\hat{F}$ of a harmonic map is obtained, via integration and a loop group decomposition, from a *potential pair* $(\hat{\chi}, \hat{\psi})$ of loop algebra valued 1-forms along a pair of transverse null coordinate lines. Essentially, the solution is thus given by more or less arbitrary functions of one variable along two characteristic lines, in analogue with the d'Alembert solution of the wave equation. The challenge is to find the potentials that correspond to particular geometric properties, as the geometry is difficult to see in the potentials. To address this problem, in joint work with Svensson [8], we defined special potentials that allow one to solve a geometric Cauchy problem: find a surface that contains a given curve with prescribed surface normal. Here, we generalize these potentials to the case where the curve is required to be a *singular* curve, in place of prescribing the surface normal.

1.1 Main results

A *frontal* is a differentiable map f from a surface M into $\mathbb{R}^3$ that locally has a well-defined unit normal, that is a map N into $\mathbb{S}^2 \subset \mathbb{R}^3$ such that $\mathrm{d}f$ is orthogonal to N. Generalized pseudospherical surfaces, as defined here, are frontals, and we may thus call them *pseudospherical frontals*. If the map $(f, N) : M \to \mathbb{R}^3 \times \mathbb{S}^2$ is everywhere regular, then f is called a *(wave) front*. A pseudospherical frontal is a wave front if and only if it is weakly regular. That is, wave front solutions are exactly those that correspond to solutions of the sine-Gordon equation.

A point p on a frontal f is called a *singular point* if the derivative $\mathrm{d}f$ has rank less than 2 at p, and the local singular locus is called a *singular curve*. The singular point p is *non-degenerate* if the singular curve is locally a regular curve in M. The image in $\mathbb{R}^3$ of a non-degenerate singular curve need not be a regular curve, demonstrated by the case of a swallowtail singularity or a cone singularity (Fig. 3). Below we will divide non-degenerate singular curves into two types, *characteristic* singular curves that are always tangent to a null coordinate direction and *non-characteristic* those that are never tangent to a null direction.

Theorem 4.2 gives the potentials for constructing all non-degenerate non-characteristic singular curves, together with the conditions on the data for cuspidal edges, swallowtails and cone singularities. We then use this to prove Theorem 4.3, which states that given an arbitrary space curve with non-vanishing curvature κ, and torsion $\tau \neq \pm 1$, there is a *unique* pseudospherical wave front that contains this curve as a cuspidal edge. Moreover, the potentials are given by a very simple formula in terms of κ and τ. We use this formula to compute several examples. In fact, the potentials in Theorem 4.2 generate a pseudospherical frontal from an arbitrary pair of functions κ and τ. At a point where κ vanishes, the singular curve is degenerate. At a point where $|\tau| = 1$, the surface is a frontal but not a wave front, and the singular curve is also degenerate. Examples are shown in Figs. 1, 7 and 11.

In Sect. 5, we analyse the problem for *characteristic* singular curves. These singularities are non-generic, but nevertheless of some interest. For example, a *weakly regular* pseudospherical surface (i.e. a wave front) contains a non-degenerate characteristic singular curve if and only if this curve is a straight line segment. For a general frontal, the singular curve, if it is not a straight line, must instead have non-vanishing curvature and constant torsion $\tau = \pm 1$, incidentally the same conditions that are satisfied by asymptotic curves on a *regular* pseudospherical surface. In the characteristic case, the solution is not unique, and there are infinitely many pseudospherical frontals containing a prescribed curve of the allowed type. We give the precise statement and the potentials for all solutions in Theorem 5.1.

We have computed many examples of solutions using a numerical implementation of the generalized d'Alembert method.[1] We have tried to include some representative images throughout the article, as well as further examples illustrating degenerate singularities and surfaces generated from curves with unbounded curvature in Sect. 6. The surfaces are coloured here according to mean curvature, which generally blows up near singularities, showing the singular curves more clearly (Fig. 2).

1.2 Concluding remarks

This work is part of a series investigating how to analyse the singularities arising naturally in integrable systems formulations of geometric problems [4,5,7,9]. The singularities in each case studied arise in a different way. For spacelike and timelike constant mean curvature

[1] Currently available at http://davidbrander.org/software.html.

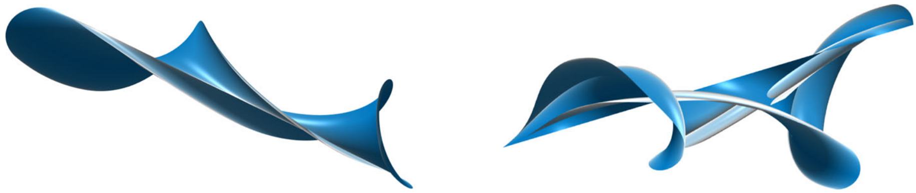
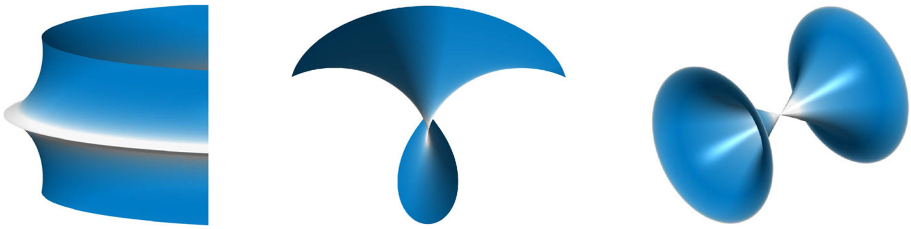


Fig. 2 *Left* One of many pseudospherical fronts that contain a straight line as a singular curve: Theorem 5.1, with $\kappa = 0, \alpha = 1, \beta(t) = t$. *Right* Example 5.4, a higher-order "cuspidal edge", $\kappa = 0, \alpha(t) = t^2, \beta(t) = t$. This surface is not a wave front



$(A, \varepsilon B) = (1, 1)$ $\quad$ $(A, \varepsilon B) = (1+t, -1+t)$ $\quad$ $(A, \varepsilon B) = (1, -1)$

Fig. 3 Non-degenerate singularities: cuspidal edge, swallowtail and cone

surfaces in Minkowski 3-space [4,7,9], singularities are caused by the break down of the Iwasawa and Birkhoff loop group decompositions for non-compact groups. Approaching such points, the direction of the surface normal becomes null, and so the (harmonic) unit normal is not defined. For constant Gauss curvature surfaces in $\mathbb{R}^3$, the loop group decompositions are globally defined, and hence, the unit normal is well defined everywhere, but this does not guarantee that the surface is regular. This is because the unit normal is harmonic with respect to the metric induced by the *second* fundamental form, so the existence of conformal coordinates with respect to this metric does not imply surface regularity. Positive curvature surfaces, studied in [5], differ substantially in treatment from negative curvature surfaces, because the former constitute an elliptic problem corresponding to Riemannian harmonic maps, and the latter case, treated here, is hyperbolic and corresponds to Lorentzian harmonic maps.

We generally consider maps to be in the smooth category. The methods we use involve only integration and loop group decompositions, which preserve smoothness: if real analytic data are given, then the solutions are also real analytic. Our solutions, as frontals, are defined globally, because the Birkhoff decomposition used is shown in [3] to be global. We work with a simply connected (which implies contractible) Lorentz surface S. For non-trivial topologies, this amounts to working on the universal cover. Note, however, that by Kulkarni's theorem [12], there are infinitely many Lorentzian conformal structures on the plane, and not all of these can be realized as conformal submanifolds of the Lorentz plane $\mathbb{R}^{1,1}$. This raises interesting questions for the global theory of pseudospherical frontals.

Andrey Popov [16] proved the existence and uniqueness part of our Theorem 4.3, by using the sine-Gordon equation. The potentials given in Theorem 4.3 improve this result by including solutions for curves where κ vanishes or $|\tau|$ takes the value 1 and by providing a means of easily computing the solutions. Popov concluded that a pseudospherical surface is uniquely determined by a cuspidal edge on its boundary, but this is not strictly accurate: even if we restrict to the class of pseudospherical wave fronts (as he did), there exist cuspidal edges (necessarily straight lines) that are characteristic curves. For such a curve, there are infinitely many different pseudospherical wave fronts that contain it as a cuspidal edge.

An important motivation for studying the singularities of pseudospherical surfaces is to characterize the natural boundaries of the regular surfaces, given that there are no complete immersions. See, e.g. [1,21]. Generalizations that *include* the singular curves as a part of the surface have previously been studied within the framework of weakly regular surfaces. In this article, we construct real analytic pseudospherical frontals (examples 5.4 and 5.5) that are immersed on open dense sets, but have non-degenerate singular curves where the surface is not weakly regular. This demonstrates that the weakly regular framework is not sufficiently general for the task of including even *regular* boundary curves of immersed pseudospherical surfaces. Given this, and the direct relationship between arbitrary Lorentz harmonic maps and globally defined pseudospherical frontals, we conclude that frontals are a more natural candidate for a global theory of pseudospherical surfaces.

2 Generalized pseudospherical surfaces

We first summarize necessary background material on pseudospherical surfaces and the loop group representation. For more references, see, for example [2,13,15].

2.1 Lorentz surfaces and box charts

Any pseudospherical immersion has a natural Lorentz structure induced by the second fundamental form. We therefore outline a little background on Lorentz surfaces from Weinstein [20].

A Lorentz surface $(S, [h])$ is an oriented $\mathbb{C}^\infty$ surface S equipped with a conformal equivalence class of indefinite metrics $[h]$. There is naturally associated an ordered pair of nowhere parallel null direction fields $\mathscr{X}$ and $\mathscr{Y}$. A local *proper null coordinate system* with respect to $[h]$ is a local coordinate chart (x, y) such that ∂_x and ∂_y are parallel to $\mathscr{X}$ and $\mathscr{Y}$, respectively, and $h = 2B\mathrm{d}x\mathrm{d}y$ for some positive function B.

The Lorentzian analogue to a holomorphic chart of a Riemann surface is a box chart. A pair of charts $\phi = (x, y)$, $\hat{\phi} = (\hat{x}, \hat{y})$, on a surface S are $C^\square$-*related* if the orientation and the directions ∂_x and ∂_y are preserved by the transition function, that is $\hat{\phi} \circ \phi^{-1}(x, y) = (f(x), g(y))$ with $f'g' > 0$. A $C^\square$-*atlas* $\mathscr{A}^\square$ is a subatlas of the atlas of S in which all charts are $C^\square$-related. A *box surface* is an ordered pair $(S, \mathscr{A}^\square)$, consisting of a surface and a maximal $C^\square$-atlas, and any element of $\mathscr{A}^\square$ is called a *box chart*.

By Theorem 1 of [20], box surfaces are in one-one correspondence with Lorentz surfaces $(S, [h])$, where $[h]$ is a conformal equivalence class of Lorentz metrics. In particular, given a Lorentz surface (S, h), the set of all proper null coordinate charts is a maximal $C^\square$ atlas on S.

A *grid box* in $\mathbb{R}^2$ is a product of intervals $B = (a, b) \times (c, d)$ where $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$. Since the property of being a grid box is preserved by the transition functions of $C^\square$-related charts, the concept of a grid box is well defined on a Lorentz surface. We call $\phi^{-1}(B)$ a grid box on S if B is a grid box and ϕ is a box chart.

2.2 Lorentz harmonic maps and the associated pseudospherical frontal

Let (S, h) be a simply connected Lorentz surface. Suppose $N : S \to \mathbb{S}^2$ to be a smooth map. Then, N is harmonic if and only if the mixed partial derivative N_{xy} is proportional to N otherwise stated as $N \times N_{xy} = 0$, where (x, y) are any null coordinate system (box chart).

Consider now the system

$$f_x = N \times N_x, \quad f_y = -N \times N_y, \tag{2.1}$$

for a map $f : S \to \mathbb{R}^3$. The compatibility of the system (2.1), i.e. $\partial_y(N \times N_x) = \partial_x(-N \times N_y)$, is equivalent to the equation $N \times N_{xy} = 0$, i.e. to the harmonicity of N.

Definition 2.1 The smooth map $f : S \to \mathbb{R}^3$, unique up to a translation, obtained by integrating the system (2.1) is called the *pseudospherical frontal associated with* N. The map $L = (f, N) : S \to \mathbb{R}^3 \times \mathbb{S}^2$ is called the *Legendrian lift of* f.

Clearly $\mathrm{d}f$ is orthogonal to N, and so f is a frontal. At points where f is an immersion, the Gauss curvature is -1, and the null coordinates are *asymptotic coordinates* for f (see below). Hence the name "pseudospherical frontal".

Conversely, if $\tilde{f} : S \to \mathbb{R}^3$ is a regular constant Gauss curvature -1 surface, where S is simply connected, it is well known that one can find a global asymptotic coordinate system for $\tilde{f}$, and that the unit normal is a harmonic map with respect to the Lorentz structure defined by the second fundamental form. Hence, all standard pseudospherical surfaces are obtained in the above manner from their Gauss maps.

2.3 The extended frame

Let K denote the diagonal subgroup of $SU(2)$ and represent $\mathbb{S}^2$ as the symmetric space $SU(2)/K$, with projection $\pi : SU(2) \to \mathbb{S}^2$ given by $\pi(g) = \mathrm{Ad}_g\, e_3$, where

$$e_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

are an orthonormal basis for $\mathfrak{su}(2)$, with respect to the inner product $\langle X, Y\rangle = -2\,\mathrm{trace}(XY)$. We have the commutators $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$ and $[e_3, e_1] = e_2$, so that the cross-product in $\mathbb{R}^3 = \mathfrak{su}(2)$ is

$$A \times B = [A, B].$$

Let $N : S \to \mathbb{S}^2 = SU(2)/K$ be a harmonic map, as above, and $F : S \to SU(2)$ any lift of N, i.e. a map such that $N = \pi(F) = \mathrm{Ad}_F\, e_3$. We can express the Maurer–Cartan form of F as

$$\alpha := F^{-1}\mathrm{d}F = (U_{\mathfrak{k}} + U_{\mathfrak{p}})\mathrm{d}x + (V_{\mathfrak{k}} + V_{\mathfrak{p}})\mathrm{d}y,$$

where the $\mathfrak{k}$ and $\mathfrak{p}$ components are with respect to the Lie algebra decomposition $\mathfrak{k} = \mathrm{span}\{e_3\}$, $\mathfrak{p} = \mathrm{span}\{e_1, e_2\}$.

Equation (2.1) for the associated pseudospherical frontal can be written

$$f_x = \mathrm{Ad}_F\, U_{\mathfrak{p}}, \quad f_y = -\mathrm{Ad}_F\, V_{\mathfrak{p}},$$

and f is immersed precisely at the points where $U_{\mathfrak{p}}$ and $V_{\mathfrak{p}}$ are linearly independent. At such a point, the first and second fundamental forms are

$$\mathbb{I} = \begin{pmatrix} |U_{\mathfrak{p}}|^2 & |U_{\mathfrak{p}}||V_{\mathfrak{p}}|\cos\phi \\ |U_{\mathfrak{p}}||V_{\mathfrak{p}}|\cos\phi & |V_{\mathfrak{p}}|^2 \end{pmatrix}, \quad \mathbb{II} = \begin{pmatrix} 0 & |U_{\mathfrak{p}}||V_{\mathfrak{p}}|\sin\phi \\ |U_{\mathfrak{p}}||V_{\mathfrak{p}}|\sin\phi & 0 \end{pmatrix},$$

where ϕ is the angle from $U_{\mathfrak{p}}$ to $-V_{\mathfrak{p}}$, and $|\cdot|$ is the standard norm in $\mathbb{R}^3 \equiv \mathfrak{su}(2)$. Thus, x and y are *asymptotic coordinates* for f, and the Gauss curvature is -1.

To characterize the harmonicity of N in terms of F, we differentiate $N = \mathrm{Ad}_F\, e_3$ to obtain

$$\mathrm{Ad}_{F^{-1}}\, N_{xy} = [U_{\mathfrak{p}}, [V_{\mathfrak{p}}, e_3]] + \left[\frac{\partial V_{\mathfrak{p}}}{\partial x} + [U_{\mathfrak{k}}, V_{\mathfrak{p}}], e_3\right].$$

Hence, N_{xy} is proportional to $\mathrm{Ad}_F\, e_3$ if and only if the $\mathfrak{p}$ part of the right hand side vanishes, i.e. if and only if $\left[\partial_x V_{\mathfrak{p}} + [U_{\mathfrak{k}}, V_{\mathfrak{p}}], e_3\right] = 0$, and this holds if and only if

$$\partial_x V_{\mathfrak{p}} + [U_{\mathfrak{k}}, V_{\mathfrak{p}}] = 0. \tag{2.2}$$

If α is the Maurer–Cartan form of a frame F for an arbitrary smooth map $N : S \to \mathbb{S}^2$, we can define

$$\alpha_\lambda := (U_{\mathfrak{k}} + U_{\mathfrak{p}}\lambda)\mathrm{d}x + (V_{\mathfrak{k}} + V_{\mathfrak{p}}\lambda^{-1})\mathrm{d}y,$$

where the parameter λ takes values in $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The basis of the loop group set-up is that the Maurer–Cartan equation

$$\mathrm{d}\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0, \tag{2.3}$$

is satisfied for all λ if and only if Eq. (2.2) holds, if and only if N is harmonic.

Fix some point $p \in S$ with $F(p) = F_0$. We want to retain the *twisted* structure that α_λ already has, namely that diagonal and off-diagonal matrix components are, respectively, even and odd functions of λ. We therefore set

$$F_0^\lambda := \begin{pmatrix} a & \lambda b \\ -\bar{b}\lambda^{-1} & \bar{a} \end{pmatrix}, \quad \text{where} \quad F_0 = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Give that N is harmonic, the Maurer–Cartan equation (2.3) means that, for any value of λ, we can solve the equations

$$(F^\lambda)^{-1}\mathrm{d}F^\lambda = \alpha_\lambda, \quad F^\lambda(p) = F_0^\lambda,$$

to obtain a family of maps $F^\lambda : S \to SL(2, \mathbb{C})$, which take values in $SU(2)$ for real values of λ, and we have an associated family $N^\lambda : S \to \mathbb{S}^2$ of harmonic maps given by

$$N^\lambda := \mathrm{Ad}_{F^\lambda}\, e_3, \quad \text{for } \lambda \in \mathbb{R}^*.$$

Given a fixed basepoint p, the family N^λ is independent of the choice of lift F of N. Any other lift is of the form $\tilde{F} = FD$ where D is a diagonal matrix-valued function, and the extended frame works out to be $\tilde{F}^\lambda = F^\lambda D$, leaving $N^\lambda = \mathrm{Ad}_{F^\lambda}\, e_3$ unchanged. Let us call the family N^λ the *extended harmonic map*, or *the extended unit normal*, and F^λ an extended frame. There is a convenient way to obtain the associated pseudospherical frontal f from F^λ. The *Sym formula* is defined as:

$$\mathscr{S}_\lambda(N^\lambda) := \lambda \frac{\partial F^\lambda}{\partial \lambda}(F^\lambda)^{-1}. \tag{2.4}$$

This formula is independent of the choice of extended frame F^λ (given a fixed basepoint) and hence well defined on N^λ. By computing the derivatives, one verifies:

Lemma 2.2 *For each $\lambda \in \mathbb{R}^*$, the map $f^\lambda : S \to \mathbb{R}^3 = \mathfrak{su}(2)$, given by the* Sym *formula: $f^\lambda = \mathscr{S}_\lambda(N^\lambda)$, is (up to a translation) the unique pseudospherical frontal associated with the harmonic map N^λ.*

The Sym formula was given by A. Sym [18]. A geometric explanation of this formula can be found in [6].

Finally, we remark that the choice of basepoint in the construction of the extended harmonic map N^λ has no geometric significance. Choosing a different basepoint will result in a translation of the surface obtained from the formula $f = \mathscr{S}_1(N^\lambda)$, and this is the same freedom we have in the definition of the associated pseudospherical frontal.

3 Singularities of pseudospherical frontals

For notational convenience, we now use $\hat{X}$ instead of X^λ to denote a family of objects parametrized by λ. For such an object, we also write X for $\hat{X}|_{\lambda=1}$.

Analysis of singularities is local, and so, in this section, we are generally discussing a harmonic map $N : R \to \mathbb{S}^2$, where R is a grid box $I_x \times I_y \subset \mathbb{R}^2$, a product of open intervals. A harmonic map N is called *weakly regular* if the kernel of $\mathrm{d}N$ is everywhere of dimension at most 1 and never contains a nonzero null vector.

Definition 3.1 An *admissible connection* is an integrable family of 1-forms

$$\hat{\alpha} := (U_\mathfrak{k} + U_\mathfrak{p}\lambda)\mathrm{d}x + (V_\mathfrak{k} + V_\mathfrak{p}\lambda^{-1})\mathrm{d}y,$$

on $R := I_x \times I_y$, where $U_\mathfrak{k}$, $V_\mathfrak{k}$ and $U_\mathfrak{p}$, $V_\mathfrak{p}$ take values, respectively, in $\mathfrak{k}$ and $\mathfrak{p}$ in $\mathfrak{su}(2)$. The connection is *weakly regular* at $p \in R$, if both $U_\mathfrak{p}$ and $V_\mathfrak{p}$ are nonzero at p and *regular* if $U_\mathfrak{p}$ are $V_\mathfrak{p}$ are linearly independent at p. The connection is weakly regular or regular if these conditions hold on the whole of R. An *admissible frame* is a family of maps $\hat{F} : R \to SU(2)$ such that $\hat{F}^{-1}\mathrm{d}\hat{F}$ is an admissible connection.

The problem of constructing harmonic maps $R \to \mathbb{S}^2$ is essentially equivalent to that of finding admissible connections. The only freedom in the choice of admissible frame $\hat{F}$ is a gauge $\hat{F} \mapsto \hat{F}D$, where D takes values in the diagonal subgroup $K \subset SU(2)$. Equivalently, $\hat{\alpha} \mapsto D^{-1}\hat{\alpha}D + D^{-1}\mathrm{d}D$. The harmonic map $N = \mathrm{Ad}_F\, e_3$ is (weakly) regular if and only if the admissible connection is.

Lemma 3.2 *Let $\hat{F}$ be an admissible frame, with associated harmonic map $N = \mathrm{Ad}_F\, e_3$ and $f = \mathscr{S}_1(\hat{F})$. The connection $\hat{\alpha} := \hat{F}^{-1}d\hat{F}$ is weakly regular if and only if f is a wave front.*

Proof We have

$$\mathrm{Ad}_{F^{-1}}\, \mathrm{d}f = U_\mathfrak{p}\mathrm{d}x - V_\mathfrak{p}\mathrm{d}y.$$

If $\hat{\alpha}$ is not weakly regular, then at least one of $U_\mathfrak{p}$ and $V_\mathfrak{p}$ is zero at some point. Since the derivatives $\mathrm{d}N$ and $\mathrm{d}f$ are computed in terms of these, the rank of $\mathrm{d}L = (\mathrm{d}f, \mathrm{d}N)$ is at most 1 at this point and f is not a wave front.

Now suppose that $\hat{\alpha}$ is weakly regular. We need to show that $\mathrm{d}L = (\mathrm{d}f, \mathrm{d}N)$ has rank 2. Define $W : R \to \mathbb{S}^1 \subset \mathfrak{p}$ by $W = U_\mathfrak{p}/|U_\mathfrak{p}|$. We can write

$$U_\mathfrak{p} = AW, \qquad V_\mathfrak{p} = -BR_\phi W,$$

where A and B are smooth positive real-valued functions, ϕ is smooth and real valued and R_ϕ denotes the rotation of angle ϕ in the e_1e_2 plane. The connection is regular when ϕ is not an integer multiple of π. Writing $W = R_\gamma e_1$, let us multiply the extended frame $\hat{F}$ on the right by $D = \mathrm{diag}(e^{i\gamma/2}, e^{-i\gamma/2})$. This has no effect on the harmonic map $N = \mathrm{Ad}_F\, e_3$ or the map

$f = \mathscr{S}_1(\hat{F})$. Thus, it is equivalent to consider the admissible connection $D^{-1}\hat{\alpha}D + D^{-1}\mathrm{d}D$, which we now denote by $\hat{\alpha}$. The conclusion is that we can assume that

$$U_\mathfrak{p} = Ae_1, \quad V_\mathfrak{p} = -B(\cos\phi e_1 + \sin\phi e_2).$$

Now

$$\begin{aligned}\mathrm{Ad}_{F^{-1}}\,\mathrm{d}N &= [Ae_1\mathrm{d}x - B(\cos\phi e_1 + \sin\phi e_2)\mathrm{d}y\,,\; e_3]\\ &= -B\sin\phi\mathrm{d}y\, e_1 + (-A\mathrm{d}x + B\cos\phi\mathrm{d}y)\, e_2,\end{aligned}$$

and

$$\mathrm{Ad}_{F^{-1}}\,\mathrm{d}f = (A\mathrm{d}x + B\cos\phi\mathrm{d}y)e_1 + B\sin\phi\mathrm{d}y\, e_2.$$

Since A and B are non-vanishing, it follows that $\mathrm{d}L = (\mathrm{d}f, \mathrm{d}N)$ has rank 2 and f is a wave front. □

3.1 The singular curve for pseudospherical wave fronts

Assume that $\hat{\alpha}$, N and f are as above, and $\hat{\alpha}$ is weakly regular. Using the same choices as in the previous lemma, we have

$$f_x = A\,\mathrm{Ad}_F\, e_1, \quad f_y = B\,\mathrm{Ad}_F(\cos\phi e_1 + \sin\phi e_2), \quad N = \mathrm{Ad}_F(e_1 \times e_2). \tag{3.1}$$

Thus,

$$f_x \times f_y = AB\sin\phi N. \tag{3.2}$$

Since A and B are assumed non-vanishing, the singular set is the set of points $\sin\phi = 0$, i.e. $\phi = k\pi$, for $k \in \mathbb{Z}$. A singular point q on a frontal is non-degenerate if and only if one can write $f_x \times f_y = \mu N$, where $\mu(q) = 0$ and $\mathrm{d}\mu|_q \neq 0$. Here, we have $\mu = AB\sin\phi$ and $\mathrm{d}\mu = \pm AB\mathrm{d}\phi$. Thus, the non-degeneracy condition in our case is

$$\mathrm{d}\phi \neq 0. \tag{3.3}$$

In a neighbourhood of a non-degenerate singular point, the singular set is a regular curve in the coordinate domain, and there is a well-defined 1-dimensional direction field η along the curve called the *null direction* (not to be confused with null coordinate directions!) such that

$$\mathrm{d}f(\eta) = 0.$$

The generic singularities of pseudospherical surfaces were studied by Ishikawa and Machida [10] and shown to be cuspidal edges and swallowtails. For general wave fronts, these singularities can be identified by the following characterization:

Proposition 3.3 ([11]) *Let f be a wave front and q a non-degenerate singular point. Let $\sigma(t)$ be a local parametrization for the singular curve around q, with $\sigma(0) = q$. Then, the image of f in a neighbourhood of q is diffeomorphic to:*

(1) *A cuspidal edge if and only if $\eta(0)$ is not proportional to $\sigma'(0)$;*
(2) *A swallowtail if and only if $\eta(0)$ is proportional to $\sigma'(0)$, and*

$$\frac{d}{dt}(\det(\sigma'(t), \eta(t))\big|_{t=0} \neq 0.$$

In our situation, assuming, for concreteness' sake that the singular curve is given locally by $\phi(x, y) = 0$, we have $\mathrm{d}f = (A\mathrm{d}x + B\mathrm{d}y)\,\mathrm{Ad}_F\, e_1$, and so the null direction is given on this curve by

$$\eta = B\partial_x - A\partial_y.$$

Assume first that the singular curve is not tangent to either ∂_x or ∂_y. In that case, we can, after a change of box coordinates (see, e.g. [8]), assume that our singular curve is locally given by $y = \varepsilon x$, where $\varepsilon = \pm 1$. Note that this special choice of coordinates means that we cannot assume that A and B are constant. Now we have, in the basis ∂_x, ∂_y,

$$\eta(t) = (B(t), -A(t)), \quad \sigma'(t) = (1, \varepsilon), \quad \frac{\mathrm{d}}{\mathrm{d}t}\det(\sigma'(t), \eta(t)) = A'(t) + \varepsilon B'(t).$$

Let us add here that the special case that $A(t) + \varepsilon B(t) \equiv 0$ corresponds to a *cone singularity*, i.e. a non-degenerate singular curve that maps to a single point. This follows from the formula $\mathrm{d}f(\sigma'(t)) = (A(t) + \varepsilon B(t))\,\mathrm{Ad}_F\, e_1$. Constructing pseudospherical wave fronts with cone singularities is discussed by Pinkall [15].

Now consider the case that the singular curve is tangent, at a point p, to one of the coordinate directions ∂_x or ∂_y. Then, it is not proportional to η, because both B and A are nonzero. In this case, by the proposition above, the surface is a cuspidal edge at p. We summarize this as:

Theorem 3.4 *Let f be a pseudospherical wave front. Suppose that q is a non-degenerate singular point. If the singular curve is tangent at q to a null coordinate direction, then the surface is locally diffeomorphic to a cuspidal edge at q. Otherwise, there exist box coordinates (x, y) such that, in a neighbourhood of $q = (0, 0)$, the singular set is parametrized by $(x(t), y(t)) = (t, \varepsilon t)$, and the image of f is diffeomorphic to:*

(1) *A cuspidal edge if $A(0) + \varepsilon B(0) \neq 0$;*
(2) *A swallowtail if $A(0) + \varepsilon B(0) = 0$ and $A'(0) + \varepsilon B'(0) \neq 0$.*
(3) *A cone singularity if $A(t) + \varepsilon B(t) \equiv 0$,*

where $A(t) = |f_x(t, \varepsilon t)|$ and $B(t) = |f_y(t, \varepsilon t)|$.

3.2 Singular curves that are not wave fronts

Let us now consider the case that $\hat{\alpha}$ is *semi-regular* – meaning that the derivative of the associated harmonic map N has rank at least 1 – but not weakly regular. This means that at least one of $U_{\mathfrak{p}}$ and $V_{\mathfrak{p}}$ is nonzero, but the other may vanish. We assume then that $U_{\mathfrak{p}} \neq 0$, the other case being analogous. We can, as before, assume that $U_{\mathfrak{p}} = Ae_1$. After a change of box coordinates, we can take $A = 1$. The angle ϕ is not well defined at points where $V_{\mathfrak{p}}$ vanishes, so we now have:

$$U_{\mathfrak{k}} = u_0 e_3, \quad U_{\mathfrak{p}} = e_1,$$
$$V_{\mathfrak{k}} = v_0 e_3, \quad V_{\mathfrak{p}} = ae_1 + be_2,$$

where u_0, v_0, a and b are real-valued functions. The integrability condition $\mathrm{d}\hat{\alpha} + \hat{\alpha} \wedge \hat{\alpha} = 0$ is equivalent to the following set of equations

$$\frac{\partial u_0}{\partial y} = b, \quad \frac{\partial a}{\partial x} = u_0 b, \quad \frac{\partial b}{\partial x} = -u_0 a, \quad v_0 = 0.$$

Now we have

$$f_x = \mathrm{Ad}_F\, e_1, \quad f_y = \mathrm{Ad}_F(ae_1 + be_2), \quad f_x \times f_y = bN.$$

Thus, the frontal f has a singular point precisely when b vanishes, i.e. the singular set is given by

$$b = 0,$$

and the non-degeneracy condition is $\mathrm{d}b \neq 0$. If a is non-vanishing, then we are at a weakly regular point, already discussed. We therefore consider now a point q at which

$$a(q) = 0, \quad b(q) = 0, \quad \mathrm{d}b|_q \neq 0.$$

We relabel coordinates so that $q = (0, 0)$. The integrability conditions above for a and b give, along the line $y = 0$, the system:

$$\frac{\partial a(x,0)}{\partial x} = u_0(x,0)b(x,0), \quad \frac{\partial b(x,0)}{\partial x} = -u_0(x,0)a(x,0), \quad a(0,0) = b(0,0) = 0,$$

which has the unique local solution

$$a(x,0) = b(x,0) = 0.$$

Hence, assuming the non-degeneracy condition, which is now

$$\partial_y b|_{(x,0)} \neq 0,$$

the singular curve is locally given by

$$y = 0.$$

The other integrability condition becomes $\partial_y u_0 = 0$ along $y = 0$. The null direction is $\eta = \partial_y$, which is transverse to the singular curve, but the singularity is not a standard cuspidal edge because the surface is not a wave front along this curve. We call such a singularity a *higher-order cuspidal edge*, because it is non-degenerate and the image of the singular curve is a regular curve in $\mathbb{R}^3$. A *fold* singularity is of this type.

We have shown that if a pseudospherical surface has a non-degenerate singularity at a point where the surface is not a wave front, then the singular curve at that point is a *characteristic curve*, or null coordinate curve. However, we saw in the previous section that it is also possible for a weakly regular singular curve to be tangent to a characteristic direction.

4 Prescribed non-characteristic singular curves

4.1 The generalized d'Alembert method

A well-known method for producing essentially all admissible frames is the *generalized d'Alembert representation* given by Toda in [19]. Here, is a summary, using definitions and notation as in [8]: let $\mathscr{G} := \Lambda SL(2,\mathbb{C})_{\sigma\rho}$ denote the group of smooth maps $\gamma : \mathbb{S}^1 \to SL(2,\mathbb{C})$ that are fixed by the involutions σ and ρ given by

$$(\sigma\gamma)(\lambda) = \mathrm{Ad}_P\, \gamma(-\lambda), \quad (\rho\gamma)(\lambda) = (\overline{\gamma(\bar{\lambda})}^t)^{-1}.$$

where $P = \mathrm{diag}(-1, 1)$, and λ is the $\mathbb{S}^1$ parameter. All loops considered here extend holomorphically to $\mathbb{C} \setminus \{0\}$, and the reality condition given by ρ means that they take values in $SU(2)$

for real values of the loop parameter λ. We also consider the subgroups $\mathscr{G}^{\pm}$ consisting of loops the Fourier expansions of which are power series in $\lambda^{\pm 1}$. We denote the corresponding Lie algebras by Lie($\mathscr{G}$), Lie($\mathscr{G}^{\pm}$).

Definition 4.1 Let I_x and I_y be two real intervals, with coordinates x and y, respectively. A potential pair $(\hat{\chi}, \hat{\psi})$ is a pair of smooth Lie($\mathscr{G}$)-valued 1-forms on I_x and I_y, respectively, with Fourier expansions in λ as follows:

$$\hat{\chi} = \sum_{j=-\infty}^{1} \chi_j \lambda^j \, \mathrm{d}x, \quad \hat{\psi} = \sum_{j=-1}^{\infty} \psi_j \lambda^j \, \mathrm{d}y.$$

We will call the potential pair *semi-regular* at a point p if at least one of the “leading coefficients” χ_1 and ψ_{-1} is nonzero at p and *regular* if both are nonzero, and the potential pair is called (semi-)regular if the condition holds at every point.

An admissible frame $\hat{F}$ is then obtained by solving $\hat{X}^{-1}\mathrm{d}\hat{X} = \hat{\chi}$, and $\hat{Y}^{-1}\mathrm{d}\hat{Y} = \hat{\psi}$ for $\hat{X}(x)$ and $\hat{Y}(y)$, each with initial condition the identity matrix, thereafter performing, at each (x, y), a *Birkhoff decomposition* (see [3,17]):

$$\hat{X}^{-1}(x)\hat{Y}(y) = \hat{H}_-(x, y)\hat{H}_+(x, y), \quad \text{with} \quad \hat{H}_\pm(x, y) \in \mathscr{G}^{\pm}, \tag{4.1}$$

and finally defining $\hat{F}$ by:

$$\hat{F}(x, y) = \hat{X}(x)\hat{H}_-(x, y). \tag{4.2}$$

The admissible frame is semi-regular if and only if the potential pair is semi-regular and *weakly* regular if and only if the potential pair is regular.

Conversely, any admissible frame $\hat{F}$ is associated with a potential pair $\left(\hat{X}_+^{-1}\mathrm{d}\hat{X}_+, \hat{Y}_-^{-1}\mathrm{d}\hat{Y}_-\right)$, where $\hat{X}_+$ and $\hat{Y}_-$ are obtained by the pair of pointwise *normalized* Birkhoff factorizations

$$\hat{F} = \hat{X}_+\hat{G}_-, \qquad \hat{X}_+(x) \in \mathscr{G}^+, \; \hat{G}_-(x, y) \in \mathscr{G}^-, \quad \hat{X}_+\big|_{\lambda=0} = I,$$
$$\hat{F} = \hat{Y}_-\hat{G}_+, \qquad \hat{Y}_-(y) \in \mathscr{G}^-, \; \hat{G}_+(x, y) \in \mathscr{G}^+, \quad \hat{Y}_-\big|_{\lambda=\infty} = I.$$

Note that the special form of an admissible connection automatically implies that $\hat{X}_+$ and $\hat{Y}_-$ depend only on x and y, respectively. Because of the normalization, these potentials are uniquely determined by $\hat{F}$ and have particularly simple forms:

$$\hat{X}_+^{-1}\mathrm{d}\hat{X}_+ = \begin{pmatrix} 0 & \zeta(x) \\ -\zeta(x) & 0 \end{pmatrix} \lambda \mathrm{d}x, \quad \hat{Y}_-^{-1}\mathrm{d}\hat{Y}_- = \begin{pmatrix} 0 & \xi(y) \\ -\xi(y) & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}y,$$

and are called *normalized potentials*.

4.2 Potentials for non-characteristic singularities

Given the d’Alembert representation just described, a generalized pseudospherical surface is locally determined by an arbitrary pair of (real)-differentiable complex-valued functions $\zeta(x)$ and $\xi(y)$. A generic function $\mathbb{R} \to \mathbb{C}$ is non-vanishing, and so a generic normalized potential pair is regular, and the corresponding pseudospherical surface is a wave front.

Our aim here is to give potentials that produce *prescribed* singular curves. We will consider separately two cases: that the singular set is or is not a characteristic curve, starting with the non-characteristic case. For this, rather than normalized potentials, a better choice is a form of the *boundary potential* pairs, introduced in [8] for the purpose of giving prescribed values of $\hat{F}$ along a non-characteristic curve. We assume that the singular curve is non-degenerate and never parallel to a null curve. Then, we can always find local box coordinates (x, y) such that the curve is given by

$$y = \varepsilon x, \qquad \varepsilon = \pm 1.$$

Suppose given the value for $\hat{F}(x, y)$, along the curve $y = \varepsilon x$. In the coordinates

$$u = \frac{1}{2}(x + \varepsilon y), \quad v = \frac{1}{2}(x - \varepsilon y),$$

the curve is given by $v = 0$, and the value of $\hat{F}$ along the curve is given by

$$\hat{F}_0(u) = \hat{F}(u, 0).$$

Since $\hat{F}$ is assumed to be an admissible frame we have, from Definition 3.1,

$$\hat{F}_0^{-1}\mathrm{d}\hat{F}_0 = (\varepsilon V_{\mathfrak{p}}\lambda^{-1} + U_{\mathfrak{k}} + \varepsilon V_{\mathfrak{k}} + U_{\mathfrak{p}}\lambda)\mathrm{d}u. \tag{4.3}$$

Since the highest and lowest powers of λ appearing are 1 and -1, respectively, this 1-form is valid as either $\hat{\chi}$ or $\hat{\psi}$ or both in a potential pair. Hence, setting

$$\hat{X}(x) = \hat{F}_0(x), \quad \hat{Y}(y) = \hat{F}_0(\varepsilon y)$$

gives a valid potential pair $(\hat{X}^{-1}\mathrm{d}\hat{X}, \hat{Y}^{-1}\mathrm{d}\hat{Y})$, called the *boundary potential pair* relative to the curve $v = 0$. For this potential pair, the Birkhoff decomposition (4.1) is trivial along the curve $v = 0$, since $\hat{X}(v = 0) = \hat{Y}(v = 0)$, and so the admissible frame $\widetilde{F}$ obtained by (4.2) agrees with $\hat{F}$ along this curve. A uniqueness argument using normalized potentials (see [8]) then shows that $\widetilde{F}$ and $\hat{F}$ determine the same harmonic map.

We now want to construct $\hat{F}_0(u)$ along a curve $v = 0$ from geometric data of a pseudospherical frontal f prescribed along the curve. Since the curve is non-characteristic, and assumed non-degenerate, f is necessarily a *wave front* (see Sect. 3.2). From Sect. 2.3, we can assume that we are given box coordinates (x, y) that are asymptotic coordinates for f, the angle ϕ is the oriented angle between f_x and f_y, and the first and second fundamental forms are:

$$I = A^2\mathrm{d}x^2 + 2\cos(\phi)AB\mathrm{d}x\mathrm{d}y + B^2\mathrm{d}y^2, \quad II = 2AB\sin(\phi)\mathrm{d}x\mathrm{d}y,$$

where $A = |f_x|$ and $B = |f_y|$. Using the same frame F as in the proof of Lemma 3.2, defined by (3.1), we have:

$$U_{\mathfrak{k}} = -\phi_x e_3, \quad U_{\mathfrak{p}} = Ae_1, \quad V_{\mathfrak{k}} = 0, \quad V_{\mathfrak{p}} = -B(\cos\phi e_1 + \sin\phi e_2).$$

In the coordinates (u, v) we have $\phi_x = \frac{1}{2}(\phi_u + \phi_v)$. If $v = 0$ is a singular curve, we have $\phi = k\pi$ constant along the curve, so $\phi_u(u, 0) = 0$. Without loss of generality, we take $k = 0$, i.e. $\phi(u, 0) = 0$. The basic data that determine the boundary potential are thus

$$U_{\mathfrak{k}} = -\frac{\phi_v(u, 0)}{2}e_3, \quad U_{\mathfrak{p}} = A(u)e_1, \quad V_{\mathfrak{k}} = 0, \quad V_{\mathfrak{p}} = -B(u)e_1,$$

where $A(u) = |f_x(u, 0)|$ and $B(u) = |f_y(u, 0)|$. Substituting into (4.3) and applying Theorem 3.4, we conclude that all non-degenerate non-characteristic singular curves on pseudospherical frontals are obtained from the following theorem:

Theorem 4.2 *Let J be an open interval, $A, B : J \to (0, \infty)$ and $\beta : J \to \mathbb{R}$ three differentiable functions. Let $\varepsilon = \pm 1$ and set*

$$\hat{\eta} := \left(-\varepsilon B(t)e_1\lambda^{-1} - \frac{\beta(t)}{2}e_3 + A(t)e_1\lambda\right)dt.$$

Consider the potential pair $(\hat{\eta}, \hat{\eta})$ *defined on the intervals* $I_x = J$ *and* $I_y = \varepsilon J$. *Let* $f : I_x \times I_y \to \mathbb{R}^3$ *be the generalized pseudospherical surface obtained from* $(\hat{\eta}, \hat{\eta})$ *via the generalized d'Alembert method. Then,*

(1) *The set* $C := \{y = \varepsilon x\}$ *is a singular set for* f.
(2) C *is non-degenerate at a point* $(x_0, \varepsilon x_0)$ *if and only if* $\beta(x_0) \neq 0$. *In this case,*
   (a) C *is diffeomorphic to a cuspidal edge in a neighbourhood of* $(x_0, \varepsilon x_0)$ *if and only if* $A(x_0) + \varepsilon B(x_0) \neq 0$.
   (b) C *is diffeomorphic to a swallowtail in a neighbourhood of* $(x_0, \varepsilon x_0)$ *if and only if* $A(x_0) + \varepsilon B(x_0) = 0$ *and* $A'(x_0) + \varepsilon B'(x_0) \neq 0$.
   (c) C *is diffeomorphic to a cone singularity if and only if* $A(x) + \varepsilon B(x) \equiv 0$.

Three non-degenerate examples are computed in Fig. 3, all with $\beta(t) = 2$. Some degenerate examples are shown in Fig. 7.

4.3 Prescribed non-characteristic cuspidal edges

Theorem 4.2 gives the boundary potential pair for the generic non-characteristic singularities of pseudospherical surfaces, as well as cones. We now adapt this to produce pseudospherical surfaces with a given curve in $\mathbb{R}^3$ as a singular curve. We treat the case that the curve is regular in $\mathbb{R}^3$, which means that the singular curve, where non-degenerate, must be a cuspidal edge.

The geometric Cauchy problem for *regular* pseudospherical surfaces was studied in [8]. For a non-characteristic curve, there is a unique immersed solution containing a given curve γ and with the surface normal N prescribed along the curve, with a regularity condition $\langle \gamma'(t), N'(t) \rangle \neq 0$. For the non-characteristic *singular* geometric Cauchy problem, we replace the regularity condition with a *singularity* condition, $\langle \gamma'(t), N'(t) \rangle = 0$:

Non-characteristic singular geometric Cauchy data along an open interval J:

(1) A regular curve $\gamma : J \to \mathbb{R}^3$;
(2) A unit vector field $Z : J \to \mathbb{S}^2 \subset \mathbb{R}^3$, satisfying

$$\langle Z(t), \gamma'(t) \rangle = 0, \qquad \langle Z'(t), \gamma'(t) \rangle = 0.$$

(3) Weak regularity condition:

$$|\gamma'(t)| \neq |Z'(t)|.$$

The above conditions are necessarily satisfied along a non-characteristic singular curve on a pseudospherical frontal. We also find that the singular curve is non-degenerate at a point if and only if the curvature κ of the curve γ is nonzero at that point. Adding this assumption then simplifies the above description of the geometric Cauchy data. Suppose that $\gamma(s)$ is parameterized by arc length. Let $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ be the Frenet–Serret frame along the curve. The vector field Z must satisfy: $\langle Z, \mathbf{t} \rangle = 0$ and $\langle Z', \mathbf{t} \rangle = 0$. Differentiating the first equation gives

$$\langle Z', \mathbf{t} \rangle = -\langle Z, \mathbf{t}' \rangle = -\kappa \langle Z, \mathbf{n} \rangle.$$

Hence, the assumptions $\langle Z, \mathbf{t} \rangle = 0$ and $\kappa \neq 0$ imply that $\langle Z, \mathbf{n} \rangle = 0$. It follows that $Z = \pm \mathbf{b}$, where $\mathbf{b}$ is the unit binormal to the curve. Since $\mathbf{b}' = -\tau \mathbf{n}$, where τ is the torsion, the weak regularity condition $|\gamma'| \neq |N'|$ becomes $\tau \neq \pm 1$. To simplify matters, we will also take $\tau > -1$. Hence, for non-degenerate singular curves, the geometric Cauchy data are the curve given in the following result:

Theorem 4.3 *Let $\gamma : J \to \mathbb{R}^3$ be a regular arc-length parameterized curve, with curvature κ and torsion τ satisfying*

$$\kappa(s) \neq 0, \quad \textit{and either} \quad |\tau(s)| < 1, \quad \textit{or} \quad \tau(s) > 1$$

along J. Let $\varepsilon := sign(\tau - 1)$. Then,

(1) *There exists, unique up to a Euclidean motion, a pseudospherical wave front $f(u, v)$, with box coordinates (x, y) and $u = (x + \varepsilon y)/2$, $v = (x - \varepsilon y)/2$, containing γ as a non-characteristic singular curve in the form $f(u, 0) = \gamma(u)$. The singular curve is non-degenerate.*

(2) *The surface f is given by the d'Alembert method, with potential pair $(\hat{\eta}, \hat{\eta})$ on $J \times \varepsilon J$, with*

$$\hat{\eta} = \left(\frac{\tau - 1}{2} e_1 \lambda^{-1} + \kappa e_3 + \frac{\tau + 1}{2} e_1 \lambda \right) ds.$$

(3) *All non-degenerate non-characteristic singular curves of pseudospherical frontals that have a regular image in $\mathbb{R}^3$ are obtained this way.*

Proof By Theorem 4.2, there is a generalized pseudospherical surface generated by any triple of functions A, B and β. The surface is a wave front if and only if both A and B are non-vanishing, which in this case means $\tau \neq \pm 1$. The non-degeneracy condition is $\beta = -2\kappa(t) \neq 0$.

Now suppose the existence of a pseudospherical wave front $f : J \times \varepsilon J \to \mathbb{R}$ with $f(u, 0) = \gamma(u)$ a non-degenerate non-characteristic singular curve. As described above, it follows that the surface normal satisfies $N(u, 0) = \pm \mathbf{b}(u)$. Since we are only looking for the potential up to a Euclidean motion, we can take

$$N(u, 0) = \mathbf{b}(u).$$

Along the singular curve, the vectors f_u, f_v, f_x and f_y are all parallel. As previously, let F be the frame defined at (3.1), so that, on $v = 0$,

$$\mathrm{Ad}_F\, e_1 = \frac{f_x}{|f_x|} = \frac{f_y}{|f_y|}, \quad \mathrm{Ad}_F\, e_3 = N,$$

which is to say that

$$\mathrm{Ad}_F\, e_1 = f_u = \gamma', \quad \mathrm{Ad}_F\, e_2 = \mathbf{n}, \quad \mathrm{Ad}_F\, e_3 = \mathbf{b}.$$

We have already shown in Sect. 4.2 that along $v = 0$

$$F^{-1} F_u = (-\varepsilon B(u) + A(u)) e_1 - \frac{\beta(u)}{2} e_3,$$

where $A(u) = |f_x(u, 0)|$ and $B(u) = |f_y(u, 0)|$. Differentiating $\mathbf{b} = \mathrm{Ad}_F(e_3)$, we have

$$\begin{aligned} \mathbf{b}' &= \mathrm{Ad}_F [F^{-1} F_u, e_3] \\ &= (\varepsilon B - A)\, \mathrm{Ad}_F(e_2), \end{aligned}$$

so that

$$\varepsilon B(u) - A(u) = -\tau(u).$$

We also have $\gamma'(u) = f_u = f_x + \varepsilon f_y$, from which

$$1 = A(u)^2 + 2\varepsilon A(u) B(u) + B(u)^2.$$

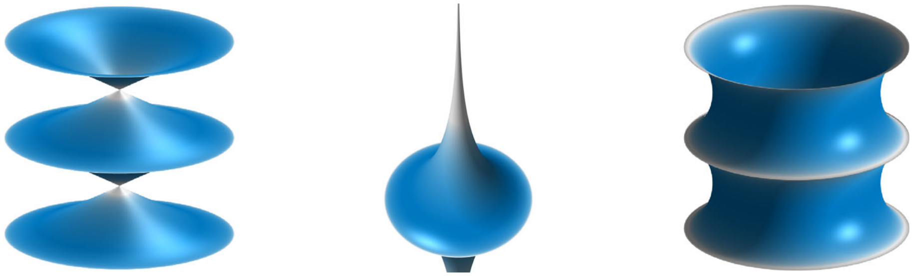


Fig. 4 Example 4.4, $R = 0.5$, $R = 1$ and $R = 1.5$

There are, in general, two solutions for positive A and B, but the surfaces obtained from the corresponding potentials are congruent after interchanging x and y. Hence, we can take the solution:

$$A = \frac{\tau + 1}{2}, \quad \varepsilon B = \frac{1 - \tau}{2}, \quad \varepsilon = \operatorname{sign}(1 - \tau).$$

To find β, we use

$$\begin{aligned} \kappa \mathbf{n} = \gamma'' &= \mathrm{Ad}_F[F^{-1}F_u, e_1] \\ &= -\frac{\beta}{2}\,\mathrm{Ad}_F(e_2), \end{aligned}$$

so $\beta = -2\kappa$. Substituting the expressions for A, B, ε and β into the potential $\hat{\eta}$ of Theorem 4.2 gives the potential in the theorem statement. Since the above data were obtained from an arbitrary solution of the geometric Cauchy problem, this also proves uniqueness, and so items (1) and (2) are proved. Item (3) follows from the fact, already explained, that, for a non-degenerate non-characteristic singular curve the curvature is non-vanishing and the torsion satisfies $|\tau| \neq 1$ (Fig. 4). □

Example 4.4 Circles: take $\gamma(t) = R(\cos t, \sin t, 0)$, where $R > 0$. The arc-length parameter, curvature and torsion are $s = Rt$, $\kappa = 1/R$ and $\tau = 0$. The potential is thus:

$$\hat{\eta} = \left(-\frac{R}{2} e_1 \lambda^{-1} + e_3 + \frac{R}{2} e_1 \lambda \right) \mathrm{d}t,$$

and this gives the well-known pseudospherical surfaces of revolution. The case $R = 1$ is the pseudosphere.

Example 4.5 Helices: taking κ and τ both constant, with $\tau \neq 0$, gives a surface containing a circular helix as a cuspidal edge (Fig. 5). Helical, as well as rotational, constant curvature surfaces, were studied by Minding in [14]. These surfaces are generally periodic in the v direction, which can be seen by considering that the curve is invariant under a 1-parameter family of rigid motions (a screw motion). The surface must also have this symmetry by uniqueness of the solution to the geometric Cauchy problem. Hence, the next singular curve encountered when moving in the v direction is also a circular helix. By the symmetry of the initial data, it follows that every second singular curve is congruent.

As with the case of the circle, there are essentially three types:

(1) Case $\kappa^2 + \tau^2 > 1$: here, there are two sets of helices with the same axis but different radius. The initial curve is on the outer cylinder when $|\tau| > 1$, and the inner when $|\tau| < 1$.

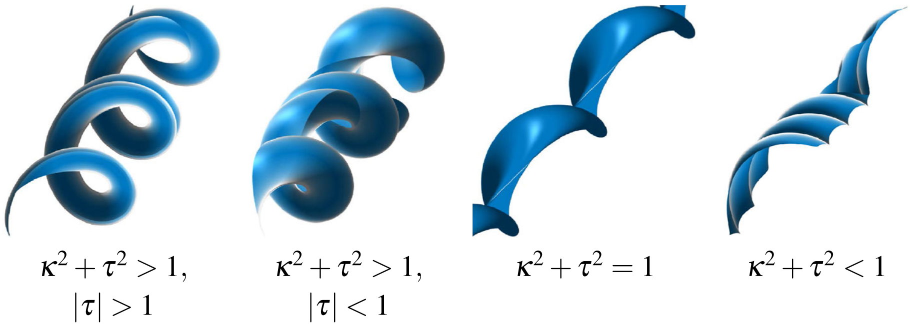


Fig. 5 Examples of helical pseudospherical surfaces

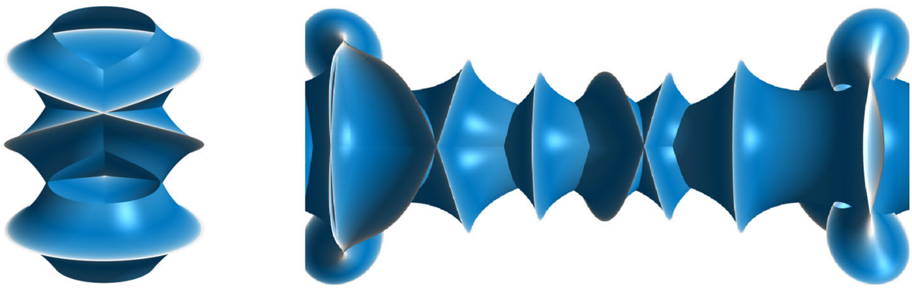


Fig. 6 Example 4.6

(2) Case $\kappa^2 + \tau^2 = 1$: the special case where the inner helices degenerate to a straight line. These are *Dini's surfaces*, which can be parametrized as

$$f(\zeta, \xi) = (a \cos\zeta \sin\xi, a \sin\zeta \sin\xi, a(\cos\xi + \ln(\tan(\xi/2))) + b\xi),$$

where, for the case of constant curvature $K = -1$, we must have $a^2 + b^2 = 1$. The surface has singularities at $\cos(\xi) = 0$, so we can take the helix

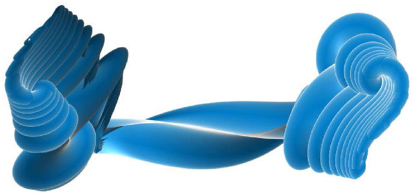
$$\gamma(t) = f(t, \pi/2) = (a \cos t, a \sin t, bt)$$

as the initial curve. We then have $\kappa = |a|$ and $\tau = b$. Hence, Dini's surfaces are given by constant κ and τ, with $\kappa^2 + \tau^2 = 1$.

(3) Case $\kappa^2 + \tau^2 < 1$: here, the inner helix disappears completely, so that all singular curves are congruent.

Example 4.6 The closed curve $\gamma(t) = (\cos(3t), \sin(3t), -\sin(t))$ lies on a round cylinder and has two self-intersections. Computing $\kappa(t) = 3(8\cos^2(t) + 82)^{1/2}(\cos^2(t) + 9)^{3/2}$, $\tau = -12\cos(t)/(4\cos^2(t) + 41)$ and $\mathrm{d}s = \sqrt{\cos^2(t) + 9}\mathrm{d}t$, we see that κ is non-vanishing and $|\tau| < 1$. The surface that contains this curve as a cuspidal edge is shown in Fig. 6.

Example 4.7 Examples with inflections and with τ taking the value 1: Theorem 4.3 is stated for curves with κ non-vanishing and $\tau \neq \pm 1$. However, we can use any functions κ and τ and still obtain a valid potential pair and therefore a pseudospherical frontal. If we take $\kappa \equiv 0$, the solution degenerates to a straight line. If we take $\tau \equiv \pm 1$, the solution degenerates to a helix curve.


$\kappa(t) = t, \ \tau(t) = 1/2.$

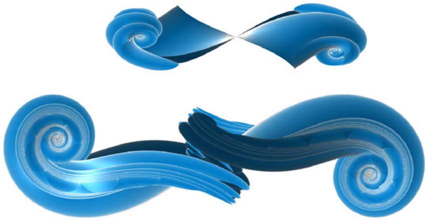

$\kappa(t) = t, \ \tau(t) = 0.$

Fig. 7 Singular curves with inflections

If κ vanishes at just one point, we will get a singular curve that is degenerate at this point, but non-degenerate elsewhere, provided $|\tau| \neq 1$. The most basic example is $\kappa(t) = t$, $\tau(t) = 1/2$, shown in Fig. 7. At the point $(0, 0)$, there are two cuspidal edges crossing each other. For the example $\kappa(t) = t$, $\tau(t) = 0$, the surface appears to have a degenerate cone point. The case $\kappa(t) = t^2$, $\tau(t) = 1/2$ is also computed and shown in Fig. 11. In this case, the singular set is a single curve through the point $(0, 0)$.

If we take $\tau = \pm 1$ at just one point, the surface is not a wave front at this point. This is because the potential pair is $(\hat{\chi}, \hat{\psi})$, where

$$\hat{\chi} = \frac{1}{2}\big((\tau(x) - 1)e_1\lambda^{-1} + 2\kappa(x)e_3 + (\tau(x) + 1)e_1\lambda\big)\,\mathrm{d}x,$$
$$\hat{\psi} = \frac{1}{2}\big((\tau(y) - 1)e_1\lambda^{-1} + 2\kappa(y)e_3 + (\tau(y) + 1)e_1\lambda\big)\,\mathrm{d}y,$$

so exactly one of χ_1 and ψ_{-1} vanishes. The potential pair is semi-regular but not regular. Moreover, the singular curve must be degenerate at this point, because we showed in Sect. 3.2 that if the singular curve is non-degenerate at a point where the surface is not a wave front, then the curve is a characteristic curve on a *neighbourhood* of this point, which is not the case here. The surface shown at Fig. 1 is generated by the Viviani figure-8 space curve $\gamma(t) = 0.3(1 + \cos(t), \sin(t), 2\sin(t/2)$. The torsion takes the values ± 1 twice each, and at each such point another singular curve branches off from the figure eight (Fig. 1, right).

5 Prescribed characteristic singular curves

Now we want to give potentials for non-degenerate *characteristic* singular curves. As expected for a Cauchy problem along a characteristic, we will find that data along a curve do not specify a unique solution: further data must be provided along another, transverse, characteristic curve. Moreover, with our solution, the non-degeneracy is only guaranteed in a neighbourhood of the intersection of these two curves.

As explained in Sect. 3.2, given that the map is semi-regular, we can assume that box coordinates are chosen such that the singular curve is locally given as $\{y = 0\}$, and can choose a local frame satisfying

$$f_x = \mathrm{Ad}_F\, e_1, \quad f_y = \mathrm{Ad}_F(ae_1 + be_2), \quad f_x \times f_y = bN, \quad N = \mathrm{Ad}_F\, e_3,$$

where

$$b(x, 0) = 0, \quad \frac{\partial b}{\partial y}(x, 0) \neq 0.$$

The surface is a wave front at points where $a(x, 0) \neq 0$. The curve $\gamma(x) = f(x, 0)$ is already arc-length parameterized. Hence, differentiating the expression for f_x, we have:

$$f_{xx} = \mathrm{Ad}_F[u_0 e_3 + e_1, e_1] = u_0 \,\mathrm{Ad}_F\, e_2,$$

along $y = 0$. Thus, up to a change of orientation, $u_0(x, 0) = \kappa(x)$, the curvature of γ. Note that if $\kappa(x) \neq 0$ for all x, then the curve has a well-defined normal $\mathbf{n} = \mathrm{Ad}_F\, e_2$, and hence, the binormal is $\mathbf{b} = \mathrm{Ad}_F\, e_3 = N$. We then have

$$-\tau n = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}x} = \mathrm{Ad}_F[\kappa e_3 + e_1, e_3] = -\,\mathrm{Ad}_F\, e_2,$$

from which we conclude that $\tau(x) = 1$ along the whole curve. Although the curve is singular, this is the same property that asymptotic curves (of non-vanishing curvature) have on a regular pseudospherical surface, namely that $\tau = \pm 1$.

Now differentiating the expression $f_x \times f_y = b\,\mathrm{Ad}_F\, e_3$, using $b(x, 0) = b_x(x, 0) = 0$, we also have

$$0 = -\kappa(x) a(x, 0)\,\mathrm{Ad}_F\, e_3.$$

Hence, if the surface is a wave front we must have $\kappa(x) = 0$ for all x. In other words, the only possible non-degenerate characteristic singular curve on a pseudospherical *wave front* is a straight line.

Theorem 5.1 *Let I_x be an open interval containing* 0*, and $\gamma : I_x \to \mathbb{R}^3$ a regular space curve, parameterized by arc length, with either non-vanishing curvature function κ, and constant torsion $\tau = \pm 1$, or with curvature everywhere zero on I_x. Let I_y be an open interval containing* 0*. For every choice of differentiable* 1*-form of type*

$$\hat{\psi} = (\alpha(y) e_1 + \beta(y) e_2)\lambda^{-1} dy.$$

with

$$\beta(0) = 0, \qquad \beta'(0) \neq 0,$$

and

$$\alpha(0) = 0, \quad \text{if} \quad \kappa \not\equiv 0,$$

there corresponds a unique pseudospherical frontal $f : I_x \times I_y \to \mathbb{R}^3$, such that

(1) *f is semi-regular on an open set containing $I_x \times \{0\}$, and*
(2) *$f(x, 0) = \gamma(x)$ is a characteristic singular curve in the surface, non-degenerate on a neighbourhood of* $(0, 0)$.

Up to a Euclidean motion, the surface f is given by the d'Alembert method with potential pair $(\hat{\chi}, \hat{\psi})$ on $I_x \times I_y$, where

$$\hat{\chi} = (\kappa(x) e_3 + \lambda e_1) dx,$$

and all such surfaces f satisfying (a) and (b) are obtained this way.

Proof The 1-forms defined satisfy the requirements for a potential pair, and therefore integrating $\hat{X}^{-1}\mathrm{d}\hat{X} = \hat{\chi}$, and $\hat{Y}^{-1}\mathrm{d}\hat{Y} = \hat{\psi}$, with initial conditions $\hat{X}(0) = I$ and $\hat{Y}(0) = I$, performing a Birkhoff decomposition

$$\hat{X}^{-1}(x)\hat{Y}(y) = \hat{H}_-(x, y)\hat{H}_+(x, y), \quad \hat{H}_\pm(x, y) \in \mathscr{G}^\pm, \quad \hat{H}_-(x, y)\big|_{\lambda=\infty} = I$$

give us an admissible frame $\hat{F} = \hat{X}\hat{H}_- = \hat{Y}\hat{H}_+^{-1}$. We write $O_\pm(\lambda^{\pm k})$ for any convergent Fourier series of the form $\sum_{j=k}^{\infty} a_k \lambda^{\pm j}$. The normalization of $\hat{H}_-$ means that its Fourier expansion is $\hat{H}_- = I + O_-(\lambda^{-1})$, so

$$\hat{F}^{-1}\mathrm{d}\hat{F} = \lambda e_1 \mathrm{d}x + O_-(1).$$

Since the coefficient of λ is $e_1\mathrm{d}x$, we can apply the analysis of Sect. 3.2 to conclude that

$$\hat{F}^{-1}\mathrm{d}\hat{F} = (u_0 e_3 + \lambda e_1)\mathrm{d}x + (a(x,y)e_1 + b(x,y)e_2)\lambda^{-1}\mathrm{d}y.$$

Along the curve $y = 0$, we have $\hat{Y} = I$, and so the unique factor $\hat{H}_-$ in the Birkhoff decomposition above satisfies $\hat{H}_-(x,0) = I$. Thus, $\hat{F}(x,0) = \hat{X}(x)$, and along $y = 0$ we have

$$\hat{F}^{-1}\hat{F}_x = (\kappa(x)e_3 + \lambda e_1).$$

Hence,

$$u_0(x,0) = \kappa(x).$$

To check the non-degeneracy condition on $\partial_y b(x,0)$, we will use the expression $\hat{F} = \hat{Y}\hat{H}_+^{-1}$. Since $\hat{H}_+$ is $\mathscr{G}^+$-valued, we can write

$$\hat{H}_+ = D_0 + O_+(\lambda), \quad D_0 = \mathrm{diag}\left(e^{i\theta/2}, e^{-i\theta/2}\right).$$

We have $\hat{H}_+^{-1}(x,0) = \hat{X}(x)$, and so, along $y = 0$,

$$\frac{\theta_x(x,0)}{2}e_3 + O_+(\lambda) = \hat{H}_+\frac{\partial H_+^{-1}}{\partial x} = \hat{X}^{-1}\frac{\partial \hat{X}}{\partial x} = \kappa(x)e_3 + \lambda e_1,$$

whilst along $x = 0$, we also have $\hat{H}_+(0,y) = I$. Hence,

$$\theta_x(x,0) = 2\kappa(x), \qquad \theta_y(0,y) = 0.$$

From $\hat{F} = \hat{Y}\hat{H}_+^{-1}$, we obtain

$$\hat{F}^{-1}\mathrm{d}\hat{F} = \mathrm{Ad}_{D_0}(\alpha e_1 + \beta e_2)\lambda^{-1} + O_+(\lambda),$$

which gives

$$b(x,y) = \cos(\theta(x,y))\beta(y) + \sin(\theta(x,y))\alpha(y).$$

Differentiating this, using $\beta(0) = 0$:

$$\frac{\partial b}{\partial y}(x,0) = \alpha(0)\theta_y\cos\theta + \beta'(0)\cos\theta + \alpha'(0)\sin\theta.$$

For the case $\kappa(x) \equiv 0$, we have $\theta_x(x,0) = 0$, so θ is constant along $x = 0$, and $\cos(\theta(x,0)) = 1$, $\sin(\theta(x,0) = 0$ by the initial condition at $(0,0)$. Thus,

$$\frac{\partial b}{\partial y}(x,0) = \alpha(0)\theta_y(x,0) + \beta'(0).$$

Since $\theta_y(0,0) = 0$ and $\beta'(0) \neq 0$, it follows that the non-degeneracy condition $b_y(x,0) \neq 0$ is satisfied on an open set containing $(0,0)$. On the other hand, for the case $\kappa \neq 0$, where we take $\alpha(0) = 0$, we have

$$\frac{\partial b}{\partial y}(x,0) = \beta'(0)\cos(\theta(x,0)) + \alpha'(0)\sin(\theta(x,0)).$$

In this case, we use $\cos(\theta(0,0)) = 1$, $\sin(\theta(0,0) = 0$ to again conclude that $b_y(x,0) \neq 0$ is satisfied on an open set containing $(0,0)$.

To see that the singular curve $f(x,0)$, of the solution f, coincides with γ, the discussion preceding the statement of this theorem shows $f(x,0)$ has curvature κ and if κ is non-vanishing, constant torsion $\tau = 1$. Since a curve is determined by its curvature and torsion, we must have, up to a Euclidean motion, $f(x,0) = \gamma(x)$. If κ is everywhere zero, then the curve is just a straight line segment of the same length as I_x, again identical with $\gamma(x)$ up to a Euclidean motion.

For uniqueness given the potential $\hat{\psi}$, it is enough to observe that $\hat{\psi}$ is a normalized potential, with normalization point $(0,0)$, which is uniquely determined by the surface $f : I_x \times I_y \to \mathbb{R}^3$ and the choice of normalization point. Thus, given any surface $\tilde{f}$ satisfying $\tilde{f}(x,0) = \gamma(x)$, we obtain $\hat{\chi}$ from the knowledge of κ and the frame $\hat{\tilde{F}}(x,0)$, and we recover $\hat{\psi}$ from a normalized Birkhoff decomposition of $\hat{\tilde{F}}(x,y)$ as described at the end of Sect. 4.1. Hence, $\tilde{f} = f$. Since $\hat{\psi}$ is the most general normalized potential satisfying the regularity conditions, all possible solutions are obtained this way. □

Remark 5.2 (a) Because $\beta(0) = 0$ and $\beta'(0) \neq 0$, we can, on a neighbourhood of $y = 0$, change y-coordinates to $\tilde{y}(y)$ so that $\beta(y)\mathrm{d}y = \tilde{y}\mathrm{d}\tilde{y}$. In these coordinates, the potential $\hat{\psi}$ is of the form

$$\hat{\psi} = (\tilde{\alpha}(\tilde{y})e_1 + \tilde{y}e_2)\lambda^{-1}\mathrm{d}\tilde{y}.$$

Thus, given κ, the unique solution is determined, on an open set containing the curve, by a single function $\tilde{\alpha}(\tilde{y})$ that is arbitrary if $\kappa \equiv 0$ but in the general case must satisfy $\tilde{\alpha}(0) = 0$

(2) For the case that $\kappa(x) \equiv 0$, adding the assumption $\alpha(0) = 0$ guarantees that the entire singular curve is non-degenerate.

(3) Suppose coordinates are chosen such that $\beta(y) = y$, as just described. Then, if α is an odd function of y, the surface has a *fold* singularity along $y = 0$, i.e f satisfies $f(x,y) = f(x,-y)$. This can be seen from the symmetry $\hat{\psi}(-y) = \hat{\psi}(y)$. Such a singularity, at least if α is analytic, can be "removed" in the sense that one half of the folded surface is part of a regular pseudospherical surface which contains the same curve: writing $\alpha(y) = y(a_1 + a_3y^2 + \cdots)$, and setting $2\tilde{y} = y^2$, we have, for $y > 0$, the expressions $\tilde{\alpha}(\tilde{y})\mathrm{d}\tilde{y} = \alpha(y)\mathrm{d}y = (a_1 + a_3 2\tilde{y} + a_5(2\tilde{y})^2 + \cdots)\mathrm{d}\tilde{y}$ and $y\mathrm{d}y = \mathrm{d}\tilde{y}$. Hence, the surface corresponding to the pair $\hat{\psi} = (\tilde{\alpha}(\tilde{y})e_1 + e_2)\lambda^{-1}\mathrm{d}\tilde{y}$ and $\hat{\chi} = (\kappa(x)e_3 + \lambda e_1)\mathrm{d}x$ is regular on an open set containing the x-axis and agrees with the folded surface on the set $y > 0$.

Of course, the Lorentz structure corresponding to the two surfaces is different here at the line $y = 0$. For a given global Lorentz structure, there is no way to remove this singularity because the vanishing of a 1-form $g(y)\mathrm{d}y$ is well defined with respect to changes of box charts. An example of a folded Amsler surface is shown in Fig. 8.

Example 5.3 Weakly regular characteristic singularities: these are all given by data of the form $\kappa \equiv 0$, $\beta(y) = y$ and an arbitrary choice of α with $\alpha(0) \neq 0$. The singular curve is guaranteed to be non-degenerate in a neighbourhood of $(0,0)$. An example is shown in Fig. 2.

Example 5.4 Straight lines that are not weakly regular: these are given by $\kappa \equiv 0$, $\beta(y) = y$ and any choice of α with $\alpha(0) = 0$. The entire line is a non-degenerate singularity. These

Fig. 8 Non-weakly regular singular curves. *Left* Folded Amsler surface. *Middle* Higher-order cuspidal edge. *Right* Spiral singularity. All have $\beta(y) = y$ (Examples 5.4 and 5.5)

are all higher-order cuspidal edges. See Fig. 8. If α is an odd function, we have a fold. If α is not an odd function, then we cannot "remove" the singular curve as can be done with the fold. For example, for the case $\alpha(y) = y^2$ and $\beta(y) = y$, let S_+ denote the surface generated by $(\hat{\chi}, \hat{\psi})$, for $y > 0$. Then, S_+ does not extend to a pseudospherical wave front over the curve $y = 0$. If it did, because asymptotic directions are well defined on a pseudospherical surface, the surface would be generated by a potential pair $(\hat{\chi}, \tilde{\hat{\psi}})$, where $\hat{\chi}$ is unchanged and the one-form $\tilde{\hat{\psi}}$ agrees with $\hat{\psi}$ on the set $y \geq 0$, but where $\tilde{\hat{\psi}}$ is regular at $y = 0$. In other words, we are looking for a change of coordinates $\tilde{y}(y)$ valid on $y > 0$ such that the 1-form $(y^2, y)\mathrm{d}y = \left(y^2 \frac{\mathrm{d}y}{\mathrm{d}\tilde{y}}, y \frac{\mathrm{d}y}{\mathrm{d}\tilde{y}}\right) \mathrm{d}\tilde{y}$ extends to a regular 1-form at $y = 0$. By definition, this means that both components are smooth and at least one non-vanishing at $y = 0$. If $y \frac{\mathrm{d}y}{\mathrm{d}\tilde{y}}$ is non-vanishing, we can assume that $\tilde{y}$ is chosen so that $y \frac{\mathrm{d}y}{\mathrm{d}\tilde{y}} = 1$, that is $\tilde{y} = y^2/2$, and hence, $(y^2, y)\mathrm{d}y = \left(\sqrt{2\tilde{y}}, 1\right) \mathrm{d}\tilde{y}$, which is not differentiable at $\tilde{y} = 0$. A similar argument shows that coordinates cannot be found such that the first component $y^2\mathrm{d}y$ is nonzero.

Example 5.5 Figure 8 (right) shows a pseudospherical frontal that contains a helix curve. The surface is not a wave front because the singular curve is characteristic and not a straight line. The singularity is non-degenerate in a neighbourhood of (0, 0), but degenerates at some points, which can be seen where it is intersected by other singular curves.

Example 5.6 Weakly regular characteristic singularities: these are all given by data of the form $\kappa \equiv 0$, $\beta(y) = y$ and an arbitrary choice of α with $\alpha(0) \neq 0$. The singular curve is guaranteed to be non-degenerate in a neighbourhood of (0, 0). An example is shown in Fig. 2.

6 Examples and numerics

In this section, we use numerics to give a picture of some degenerate singularities, as well as to show the global appearance of solutions generated by certain types of singular curve.

6.1 Degenerate singularities

In Example 4.7, we saw some degenerate singularities where κ vanishes or $|\tau|$ takes the value 1 on a singular curve generated by Theorem 4.3. Theorem 4.2 is slightly more general, and the condition for a degenerate singularity for the potential $\hat{\eta} = (-B(t)e_1\lambda^{-1} - \beta(t)/2e_3 + A(t)e_1\lambda)\mathrm{d}t$ is that β vanishes. Two examples are shown in Fig. 9. Both are degenerate cone points.

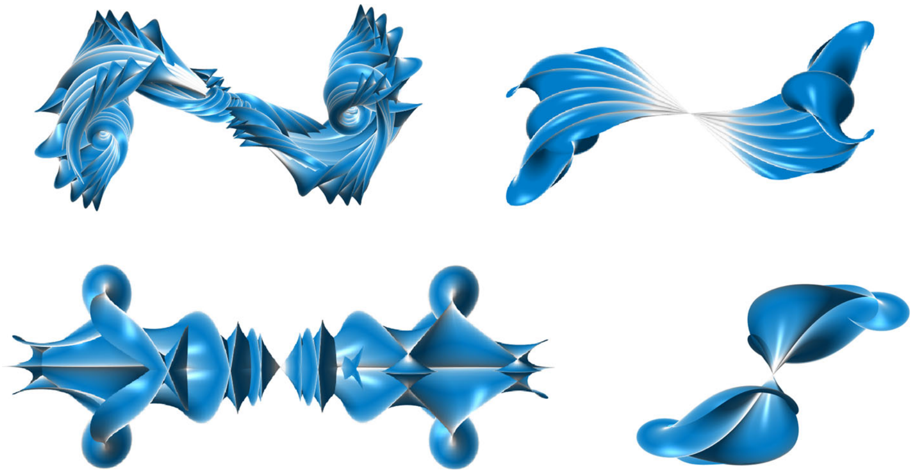


Fig. 9 Degenerate singularities. *Top* $\beta(t) = t$, $A(t) = B(t) = 1$. *Bottom* $\beta(t) = t$, $A(t) = B(t) = -1$

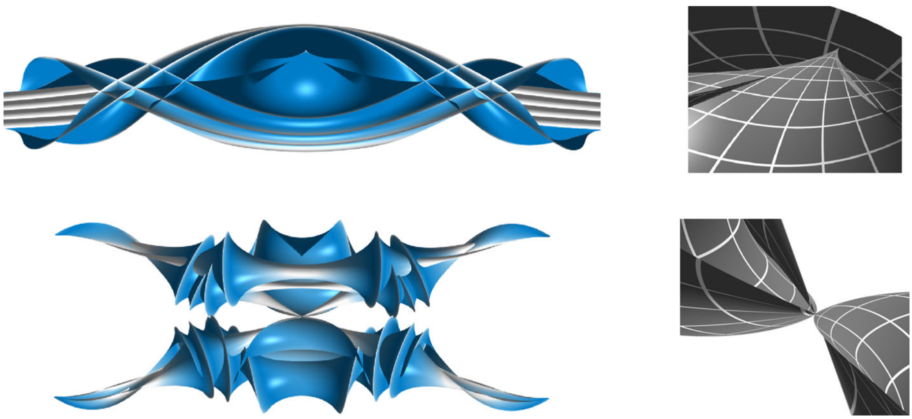


Fig. 10 Rank zero singularities. *Top* $\beta(t) = 1$, $A(t) = B(t) = t$. *Bottom* $\beta(t) = 1$, $A(t) = t$, $B(t) = -t$. (See Sect. 6.2)

6.2 Singularities where the derivative vanishes

Any potential pair $(\hat{\chi}, \hat{\psi})$ with $\chi_1(x_0) = \psi_{-1}(y_0) = 0$ produces a pseudospherical frontal the derivative of which has rank 0 at (x_0, y_0). Examples computed with $\hat{\chi} = \hat{\psi} = (-B(t)e_1\lambda^{-1} - \beta(t)/2e_3 + A(t)e_1\lambda)\mathrm{d}t$, $\beta(t) = 1$ and A and B vanishing are shown in Fig. 10.

6.3 Global properties of solutions

If we consider a surface generated by singular curve data (κ, τ), where $|\kappa(t)| \to \infty$ as $t \to \pm\infty$ and τ is bounded, then the solution becomes concentrated spatially for large

Fig. 11 Pseudospherical surfaces generated from curves with unbounded curvature functions. *Left* $\kappa(t) = 2 - t^2$, $\tau(t) = 0$. *Middle* $\kappa(t) = \exp(t^2)$, $\tau(t) = 0$. *Right* $\kappa(t) = t^2$ and $\tau(t) = 1/2$

(u, v), with a spiral in the u direction and many singularities in the v direction. This means that computing a finite subdomain gives a realistic sense of what the surface looks like, as in Fig. 7. More examples are shown in Fig. 11.

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