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New developments on constant angle property in $\mathbb{S}^2 \times \mathbb{R}$

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Abstract In the present paper, we classify curves and surfaces in $\mathbb{S}^2 \times \mathbb{R}$, which make constant angle with a rotational Killing vector field. We obtain the explicit parametrizations of such curves and surfaces, and we find examples in some particular cases. Finally, we give the complete classification of minimal constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$.

Keywords Killing vector field · Homogeneous space · Constant angle surface

Mathematics Subject Classification 53B25

1 Introduction

The study of the properties of curves and surfaces has always represented a major topic in the classical differential geometry. For example, let us recall the problem of Bertrand-Lancret-de Saint Venant, according to which a *generalized helix* denotes a curve of constant slope in the Euclidean 3-space, i.e., the tangent along it makes constant angle with a fixed direction, if and only if the ratio of its curvature and torsion is constant. Later, this classical problem was extended for curves in other ambient spaces, such as real space forms, Sasakian manifolds or Lie groups. Analogously, a *general helix* in a 3-dimensional space form is defined by a curve for which there exists a Killing vector field V along it, with constant length such that the angle between the tangent and V is constant, see [2,13].

Another motivation in the study of the constant angle between a curve and a Killing vector field comes from the theory of *magnetic curves*. For example, on a Sasakian manifold, a normal magnetic curve (parametrized by arc length) corresponding to the magnetic field defined by the fundamental 2-form makes constant angle with the Reeb vector field, which is Killing, see, e.g., [10] and references therein. Passing from curves to surfaces, it is natural

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to ask which surfaces make constant angle with certain directions, whose choice depends on the ambient space. Recall that a surface for which the unit normal makes constant angle with a fixed direction is called a *constant angle surface*.

In the last years many geometers have been studying the constant angle property for surfaces in several ambient spaces and for different choices of the fixed direction, obtaining important classification results. For example, in the product spaces of type $\mathbb{M}^2 \times \mathbb{R}$, where $\mathbb{M}^2$ is a surface of constant Gaussian curvature, taken as model the Euclidean plane $\mathbb{E}^2$, the 2-sphere $\mathbb{S}^2$, or the hyperbolic plane $\mathbb{H}^2$, a standard choice for the fixed direction is the real line $\mathbb{R}$ [4,5,7,17]. Later on, the classification of constant angle surfaces was provided in other ambient spaces, e.g.: in a warped product manifold [8], in the Heisenberg group [11], in Berger spheres [15], in the special linear group [16], in Sol_3 space [14] and in the 2-parameter solvable Lie groups [19]. In these cases, the angle was measured between the surface and a particular vector field for each case. Some results involving constant angle surfaces in higher dimensional spaces are obtained in [6,9,12].

This paper is mainly related with two articles: the first one is [5], which provides the classification of constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ with respect to the $\mathbb{R}-$direction, and the second one is the author's previous joint work [18], where there are studied the curves and the surfaces making constant angle with a rotational Killing vector field in the Euclidean 3-space.

Let us go back and see what happens in the Euclidean space. Recall that the Euclidean 3-space has 6-dimensional isometry group and a basis of Killing vector fields is given by $\{\partial_x,\ \partial_y,\ \partial_z,\ x\partial_y - y\partial_x,\ z\partial_x - x\partial_z,\ z\partial_y - y\partial_z\}$. The first three vector fields may be considered as translation Killing vector fields. The surfaces making constant angle with ∂_z are well known (maybe since Gauss), and they were emphasized recently in [17]. The last three vector fields represent the rotational Killing vector fields in $\mathbb{E}^3$, and the study of the surfaces making constant angle with $V = -y\partial_x + x\partial_y$ was developed in [18], where a complete list of such surfaces was provided. The explicit parametrizations of these surfaces were given in cylindrical coordinates. More precisely, we have the following:

Theorem A [18] *Let M be a surface isometrically immersed in $\mathbb{E}^3\backslash\{Oz\}$ and consider the Killing vector field $V = -y\partial_x + x\partial_y$. Then M makes a constant angle θ with V if and only if it is an open subset of one of the following surfaces, up to vertical translations and rotations around the z-axis:*

(i) either a half-plane with the $z-$axis as boundary (for $\theta = 0$),
(ii) or a rotational surface around the $z-$axis (for $\theta = \frac{\pi}{2}$),
(iii) or a right cylinder over a logarithmic spiral given by

$$F(u, z) = \left(u\cos\theta,\ \log(cu^{-\tan\theta}),\ z\right),\ c \in \mathbb{R}\backslash\{0\},$$

(iv) or, finally, the Dini's surface defined in cylindrical coordinates by

$$F(u, v) = \left(-\frac{\cos\theta\sin(cu)}{c},\ -\frac{cv\tan\theta}{\cos\theta} - \tan\theta\log\left(\tan\left(\frac{cu}{2}\right)\right), v - \frac{\cos\theta\cos(cu)}{c}\right),$$

where c is a nonzero real constant.

At this point, let us return to the surfaces in the product space $\mathbb{S}^2 \times \mathbb{R}$ (see also [5]). In order to fix some notations, let us briefly mention that a vector field V in $\mathbb{S}^2 \times \mathbb{R}$ is Killing if and only if it satisfies the Killing equation:

$$\langle\tilde{\nabla}_Y V, Z\rangle + \langle\tilde{\nabla}_Z V, Y\rangle = 0,$$

for any vector fields Y, Z in $\mathbb{S}^2 \times \mathbb{R}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\mathbb{S}^2 \times \mathbb{R}$.

The homogeneous space $\mathbb{S}^2 \times \mathbb{R}$ has a 4-dimensional isometry group, fact proved first by Cartan [3]. Moreover, the associated Lie algebra admits the following basis of Killing vector fields:

$$\{\partial_t,\ \ -y\partial_x + x\partial_y,\ \ z\partial_x - x\partial_z,\ \ z\partial_y - y\partial_z\}.$$

Here x, y, z denote the coordinates on $\mathbb{R}^3 \supset \mathbb{S}^2$ and t is the global coordinate on $\mathbb{R}$.

The Killing vector field ∂_t represents the translations along $\mathbb{R}$ and the surfaces making constant angle with this direction were classified by Dillen et al. [5], as follows:

Theorem B [5] *A surface M immersed in $\mathbb{S}^2 \times \mathbb{R}$ is a constant angle surface relative to ∂_t if and only if the immersion F is (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$) locally given by $F : M \to \mathbb{S}^2 \times \mathbb{R}$, $F(u, v) = (\cos(u \cos\theta)) f(v) + \sin(u \cos\theta)) f(v) \times f'(v),\ u \sin\theta)$, where $f : I \to \mathbb{S}^2$ is a unit speed curve in $\mathbb{S}^2$ and $\theta \in [0, \pi]$ is the constant angle.*

In this paper, we focus on the rotational Killing vector fields in $\mathbb{S}^2 \times \mathbb{R}$ and, without loss of generality, we choose $V = -y\partial_x + x\partial_y$ due to symmetry reasons. Our aim is to combine ideas from [5] and [18] in order to classify the curves and the surfaces in $\mathbb{S}^2 \times \mathbb{R}$ making constant angle with V. Since the vector field V must be nonzero, we have to remove from the ambient space all the points with coordinates $(0, 0, \pm 1, t)$. Hence, from now on we will assume that the curves and the surfaces lie in $\mathbb{S}^2 \times \mathbb{R} \backslash \{(0, 0, \pm 1, t)\}$, even that we do not mention explicitly this fact.

In the next section, we classify the curves making constant angle with the Killing vector field ∂_t obtaining the geodesics of the ambient space in Theorem 1, and respectively with V, obtaining the explicit parametrizations of the generalized helices in Theorem 2.

The study of the surfaces in $\mathbb{S}^2 \times \mathbb{R}$ making constant angle with V is presented in Sect. 3. An important role in the proof is played by the use of the spherical coordinates as well as by the use of the almost contact metric structure of $\mathbb{S}^2 \times \mathbb{R}$. We obtain the explicit parametrizations included in the classification Theorem 3. The particular case when the constant angle θ defined by the unit normal to the surface and the Killing vector field V is $\frac{\pi}{2}$ is treated in Theorem 4. Moreover, under the extra assumption of minimality, we give a classification result in Theorem 5.

2 Curves in $\mathbb{S}^2 \times \mathbb{R}$

In this section, we study the curves in $\mathbb{S}^2 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$,

$$\gamma : I \subset \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R},\ \gamma(s) = \big(x(s), y(s), z(s), t(s)\big), \tag{1}$$

parametrized by the arc length, which make a constant angle $\theta \in [0, \pi]$ with a Killing vector field.

An appropriate way to work is to choose spherical coordinates on the 2-sphere, and thus, we consider the curve γ given in the coordinates $(\varphi,\ \psi,\ t)$ as:

$$\gamma(s) = \big(\cos\varphi(s)\cos\psi(s),\ \cos\varphi(s)\sin\psi(s),\ \sin\varphi(s),\ t(s)\big), \tag{2}$$

where $\varphi \in (-\pi/2, \pi/2)$ and $\psi \in [0, 2\pi]$.

Let us denote by $\tilde{\nabla}$ the Levi-Civita connection on $\mathbb{S}^2 \times \mathbb{R}$ and by (T, N, B) the Frenet frame field associated to γ, where T, N and B represent the tangent, the principal normal and the binormal vector fields, respectively. The Frenet-Serret equations are given by:

$$\tilde{\nabla}_T T = \boldsymbol{\kappa} N,\ \ \tilde{\nabla}_T N = -\boldsymbol{\kappa} T + \boldsymbol{\tau} B,\ \ \tilde{\nabla}_T B = -\boldsymbol{\tau} N,$$

where $\boldsymbol{\kappa} = |\tilde{\nabla}_T T|$ and $\boldsymbol{\tau}$ are the *curvature* and the *torsion* of γ, respectively.

In the next two subsections, we find the explicit coordinate functions for a curve which makes constant angle with the $\mathbb{R}$ direction and with the Killing vector field V, respectively.

2.1 Constant angle with ∂_t

The curves in $\mathbb{S}^2 \times \mathbb{R}$ making constant angle θ with the Killing vector field ∂_t are classified as follows:

Theorem 1 *A curve γ in $\mathbb{S}^2 \times \mathbb{R}$ makes a constant angle θ with the Killing vector field ∂_t if and only if it is parametrized in $(\varphi, \psi, t)-$coordinates, up to isometries of the ambient space, by:*

$$\varphi(s) = \sin\theta \int^s \cos\alpha(\zeta)\mathrm{d}\zeta, \quad \psi(s) = \sin\theta \int^s \frac{\sin\alpha(\zeta)}{\cos\varphi(\zeta)}\mathrm{d}\zeta, \quad t(s) = s\cos\theta + t_0, \quad t_0 \in \mathbb{R},$$

where α is a smooth function. Moreover, when $\theta \notin \{0, \pi\}$, this curve is a generalized helix on $\mathbb{S}^2 \times \mathbb{R}$, namely the ratio of its curvature $\boldsymbol{\kappa}$ and torsion $\boldsymbol{\tau}$ is constant, $\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}} = \cot\theta$. When $\theta = 0$ or $\theta = \pi$, the curve γ is the geodesic $s \mapsto (p_0, s + t_0)$, where p_0 is an arbitrary point of $\mathbb{S}^2$.

Proof Let γ be given by (1). The property that γ makes constant angle θ with ∂_t may be expressed using the Euclidean scalar product $\langle\ ,\ \rangle$ from $\mathbb{E}^4$ restricted in the points of $\mathbb{S}^2 \times \mathbb{R}$, as: $\cos\theta = \langle\dot{\gamma},\ \partial_t|_\gamma\rangle$, which is equivalent with $\dot{t}(s) = \cos\theta$. Hence, the last component of γ is obtained:

$$t(s) = s\cos\theta + t_0, \quad t_0 \in \mathbb{R}. \tag{3}$$

The condition that γ is parametrized by arc length, $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \sin^2\theta$, may be rewritten, using (2), in terms of φ and ψ as:

$$\dot{\varphi}^2 + \dot{\psi}^2\ \cos^2\varphi = \sin^2\theta. \tag{4}$$

Thus, there exists $\alpha = \alpha(s)$ such that

$$\varphi(s) = \sin\theta \int^s \cos\alpha(\zeta)\mathrm{d}\zeta, \quad \psi(s) = \sin\theta \int^s \frac{\sin\alpha(\zeta)}{\cos\varphi(\zeta)}\mathrm{d}\zeta. \tag{5}$$

Let us observe that if $\theta = 0$ or $\theta = \pi$, then γ is the straight line in a point p_0 of $\mathbb{S}^2$, that is $\gamma : s \mapsto (p_0, s + t_0)$.

If $\theta \notin \{0, \pi\}$ one can prove that the curvature $\boldsymbol{\kappa}$ and the torsion $\boldsymbol{\tau}$ of the curves parametrized by (2) where $\varphi(s)$, $\psi(s)$ and $t(s)$ are given by (5) and (3), respectively, satisfy the relation

$$\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}} = \cot\theta.$$

Hence, these curves are generalized helices on $\mathbb{S}^2 \times \mathbb{R}$.

Finally, when the constant angle is $\theta = \frac{\pi}{2}$, γ is an arbitrary curve on $\mathbb{S}^2$ for any point $t_0 \in \mathbb{R}$.

The converse part follows checking that the generalized helices make a constant angle θ with the Killing vector field ∂_t. □

2.2 Constant angle with $V = -y\partial_x + x\partial_y$

Regarding the other three Killing vector fields, the problem is basically the same for each of them and therefore let us choose

$$V = -y\partial_x + x\partial_y.$$

In the sequel, we give the complete list of curves in $\mathbb{S}^2 \times \mathbb{R}\backslash\{(0, 0, \pm 1, t)\}$ making constant angle with V.

Theorem 2 *A curve γ in $\mathbb{S}^2 \times \mathbb{R}\backslash\{(0, 0, \pm 1, t)\}$ makes a constant angle θ with the Killing vector field $V = -y\partial_x + x\partial_y$ if and only if it is given, in $(\varphi, \psi, t)-$coordinates, up to isometries of the ambient space, by*

$$\varphi(s) = \sin\theta \int^s \cos\omega(\zeta)\mathrm{d}\zeta, \quad \psi(s) = \cos\theta \int^s \frac{\mathrm{d}\zeta}{\cos\varphi(\zeta)}, \quad t(s) = \sin\theta \int^s \sin\omega(\zeta)\mathrm{d}\zeta,$$

where ω is a smooth function on $I \subset \mathbb{R}$.

Proof Since $V|_\gamma = (-y, x, 0, 0)$ and $\sphericalangle\big(\dot{\gamma},\ V|_\gamma\big) = \theta$, we have $\frac{x\dot{y}-\dot{x}y}{\sqrt{x^2+y^2}} = \cos\theta$, which in spherical coordinates is written as:

$$\dot{\psi}\cos\varphi = \cos\theta. \tag{6}$$

As γ is parametrized by arc length and using (6), we get $\dot{\varphi}(s)^2 + \dot{t}(s)^2 = \sin^2\theta$. Hence, there exists a function $\omega(s)$ such that

$$\dot{\varphi}(s) = \sin\theta\cos\omega(s), \quad \dot{t}(s) = \sin\theta\sin\omega(s).$$

Integrating the above expressions and combining with (6), we obtain the expressions of the (φ, ψ, t)-coordinates and thus the direct implication is proved.

The converse part follows by straightforward computations. □

Remark 1 If the function ω involved in the Theorem 2 vanishes identically, then the curve γ lies on the 2−sphere having (in $\mathbb{S}^2 \times \mathbb{R}$) the curvature $\boldsymbol{\kappa} = \cos\theta\cot(s\sin\theta)$ and null torsion, $\boldsymbol{\tau} = 0$.

3 Surfaces making constant angle with V

In this section, we are interested in finding all the *surfaces in $\mathbb{S}^2 \times \mathbb{R}$ which make a constant angle with the rotational Killing vector field $V = -y\partial_x + x\partial_y$.*

First, let us fix the notations.

Let M be a regular surface isometrically immersed in the product space $\mathbb{S}^2 \times \mathbb{R}$ endowed with the metric $\tilde{g} = \bar{g} + dt^2$ and Levi-Civita connection $\tilde{\nabla}$, where $\bar{g}$ denotes the metric on $\mathbb{S}^2$ induced from $\mathbb{R}^3$ with the corresponding Levi-Civita connection $\bar{\nabla}$, and t denotes the global parameter on $\mathbb{R}$.

The Gauss and Weingarten formulas are:

(G) $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$,
(W) $\tilde{\nabla}_X \xi = -AX$,

for every X, Y tangent to M. Here h is a symmetric $(1,2)-$tensor field called the second fundamental form of the surface, and A is a symmetric $(1,1)-$tensor field called the shape operator associated to the unit normal ξ, satisfying $\tilde{g}(h(X,Y),\ \xi) = g(X, AY)$ for any X, Y tangent to M.

On the 2-sphere we use the spherical coordinates (φ, ψ), $\varphi \in (-\pi/2, \pi/2)$, $\psi \in [0, 2\pi]$

$$\begin{cases} x(u,v) = \cos\varphi(u,v)\cos\psi(u,v) \\ y(u,v) = \cos\varphi(u,v)\sin\psi(u,v) \\ z(u,v) = \sin\varphi(u,v), \end{cases}$$

where (x, y, z) are the Cartesian coordinates, and let us denote by $p(x, y, z)$ the position vector on the 2-sphere. Then, the surface in $\mathbb{S}^2 \times \mathbb{R}$ is parametrized by:

$$F : D \subset \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{R} \hookrightarrow \mathbb{R}^4, \quad F(u,v) = (x(u,v),\ y(u,v),\ z(u,v),\ t(u,v)).$$

In spherical coordinates, the rotational Killing vector field $V = (-y, x, 0)$ becomes $V = \partial_\psi$ and the metric on $\mathbb{S}^2$ is a warped metric, $\bar{g} = \cos^2\varphi d\psi^2 + d\varphi^2$. Denoting by J_p the Euclidean rotation of angle $\pi/2$ in the tangent plane $T_p\mathbb{S}^2$, for any $p \in \mathbb{S}^2$, one has

$$J\partial_\psi = \cos\varphi\partial_\varphi, \quad J\partial_\varphi = -\frac{1}{\cos\varphi}\partial_\psi. \tag{7}$$

The Levi-Civita connection $\bar{\nabla}$ associated to the metric $\bar{g}$ is given by the expressions:

$$\bar{\nabla}_{\partial_\psi}\partial_\psi = \cos\varphi\sin\varphi\partial_\varphi, \quad \bar{\nabla}_{\partial_\psi}\partial_\varphi = \bar{\nabla}_{\partial_\varphi}\partial_\psi = -\tan\varphi\partial_\psi, \quad \bar{\nabla}_{\partial_\varphi}\partial_\varphi = 0. \tag{8}$$

Obviously $\bar{\nabla}J = 0$, i.e., J is parallel with respect to $\bar{\nabla}$.

At this point, we set the following geometric objects on $\mathbb{S}^2 \times \mathbb{R}$:

- a 1-form η as the algebraic dual of ∂_t, that is $\eta(X) = \tilde{g}(X, \partial_t)$, equivalently $\eta(\partial_t) = 1$ and $\eta(X) = 0$ for any X tangent to $\mathbb{S}^2$;
- a field of endomorphisms ϕ of the tangent spaces defined by

$$\phi|_{\mathbb{S}^2} = J, \quad \phi\partial_t = 0.$$

It is straightforward that

$$\phi^2 = -I + \eta \otimes \partial_t, \quad \eta \circ \phi = 0. \tag{9}$$

Moreover, it can be easily checked that the metric $\tilde{g}$ is compatible with the triple (ϕ, ∂_t, η), that is

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y).$$

Hence, we have defined an almost contact metric structure $(\phi, \partial_t, \eta, \tilde{g})$ on $\mathbb{S}^2 \times \mathbb{R}$. We will use this structure in the proof of the Theorem 3.

The next result is a classification of surfaces making constant angle with V.

Theorem 3 *A surface in $\mathbb{S}^2 \times \mathbb{R} \backslash \{(0, 0, \pm 1, t)\}$ makes constant angle $\theta \notin \{0, \frac{\pi}{2}\}$ with the Killing vector field $V = -y\partial_x + x\partial_y$ if and only if it is given by one of the following cases:*

(a) a cylinder over a loxodrome of the $2-$sphere;
(b) the parametrization $F : D \subset \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{R} \hookrightarrow \mathbb{R}^4$,

$$F(u,v) = (\cos\varphi(u,v)\cos\psi(u,v),\ \cos\varphi(u,v)\sin\psi(u,v),\ \sin\varphi(u,v),\ t(u,v)),$$

where the coordinate functions are given by:

$$\varphi(u, v) = \mp\arctan\left(\frac{\sqrt{1 - m\sin^2\alpha}}{\sqrt{m}\sin\alpha}\right),$$

$$\psi(u, v) = \pm\left(\frac{\tan\theta}{\sqrt{m}}\,v + \frac{\sin\theta}{\sqrt{m}}\int^{u}\frac{1}{\sin\alpha(\zeta)}\mathrm{d}\zeta\right),$$

$$t(u, v) = v + \cos\theta\int^{u}\sin\alpha(\zeta)\mathrm{d}\zeta,$$

with $\alpha(u) = \pm\mathrm{am}\left(\frac{1}{\sqrt{m}}u\cos\theta + c\middle|m\right)$, $m, c \in \mathbb{R}$. *Here* am *denotes the Jacobi amplitude.*

Proof For an arbitrary vector field $X = X^1\partial_\varphi + X^2\partial_\psi + X^3\partial_t$ in $\mathbb{S}^2 \times \mathbb{R}$, taking into account that $V = \partial_\psi$, and using formulas (7) and (8) we may compute

$$\tilde{\nabla}_X V = \tilde{\nabla}_{X^1\partial_\varphi + X^2\partial_\psi + X^3\partial_t}\partial_\psi = \sin\varphi J(X^1\partial_\varphi + X^2\partial_\psi).$$

Since ϕ coincides with J on $\mathbb{S}^2$ and it vanishes along $\mathbb{R}$ (due to the construction), the previous expression becomes

$$\tilde{\nabla}_X V = \sin\varphi\phi X. \tag{10}$$

Projecting V onto the tangent plane to M one has $V = T + \mu\cos\theta\xi$, where ξ denotes the vector field normal to the surface, T is the tangent part with $\|T\| = \mu\sin\theta$ and $\mu = \|V\| = \cos\varphi \neq 0$, since $V = \partial_\psi$. At this point, we may choose an orthonormal basis $\{e_1, e_2\}$ on the tangent plane to M such that $e_1 = \frac{T}{\|T\|}$ and $e_2 \perp e_1$. We have that $V = \cos\varphi(\sin\theta e_1 + \cos\theta\xi)$. Then, if X is tangent to M, one can compute

$$\tilde{\nabla}_X V = -X(\varphi)\sin\varphi(\sin\theta e_1 + \cos\theta\xi) + \cos\varphi\sin\theta(\nabla_X e_1 + h(X, e_1)) - \cos\varphi\cos\theta AX. \tag{11}$$

Combining (10) and (11) one gets:

$$\begin{aligned} \sin\varphi\phi X = &-X(\varphi)\sin\varphi\sin\theta e_1 + \cos\varphi\sin\theta\nabla_X e_1 - \cos\varphi\cos\theta AX \\ &-X(\varphi)\sin\varphi\cos\theta\xi + \cos\varphi\sin\theta h(X, e_1). \end{aligned} \tag{12}$$

Since $V = \partial_\psi$ and $\langle V, e_2\rangle = 0$, we have that $e_2 \in \mathrm{span}\{\partial_\varphi, \partial_t\}$ and $\phi e_2 \in \mathrm{span}\{\partial_\psi\}$. Hence, it follows that there exists a smooth function $\lambda \in C^\infty(M)$ such that

$$\phi e_2 = \lambda V. \tag{13}$$

Furthermore, $\langle V, \partial_t\rangle = 0$ yields that there exists a smooth function $\rho \in C^\infty(M)$ such that

$$\langle e_1, \partial_t\rangle = \rho\cos\theta, \quad \langle\xi, \partial_t\rangle = -\rho\sin\theta. \tag{14}$$

Applying ϕ in formula (13) and using the almost contact metric structure we obtain

$$-e_2 + \langle e_2, \partial_t\rangle\partial_t = \lambda\phi\partial_\psi.$$

As $\phi\partial_\psi \perp \partial_\psi$, we deduce that $e_2 \in \mathrm{span}\{\partial_\varphi, \partial_t\}$. Using the fact that e_2, ∂_φ and ∂_t are unitary, and denoting by $\alpha \in C^\infty(M)$ the angle between e_2 and ∂_t, we get

$$e_2 = \nu\sin\alpha\partial_\varphi + \cos\alpha\partial_t, \tag{15}$$

where $\nu := \pm 1$. Moreover, the functions λ and α are related by $\lambda = -\nu\frac{\sin\alpha}{\cos\varphi}$.

Next, ϕe_1 and $\phi\xi$ may be expressed as

$$\phi e_1 = \nu \sin\alpha \sin\theta e_2 + B\xi, \quad \phi\xi = -Be_1 + \nu \sin\alpha \cos\theta e_2, \tag{16}$$

for a certain function B. From the fact that $\phi\xi$ is orthogonal to ∂_t it follows $B\rho = \nu \sin\alpha \cos\alpha$. Then, computing $\langle \phi\xi, \phi e_1 \rangle$ in two ways, the functions B and ρ are given by:

$$B = \nu\epsilon \cos\alpha, \quad \rho = \epsilon \sin\alpha. \tag{17}$$

Combining (17) with (13)–(16) and computing

$$\phi^2 e_1 = -e_1 + \langle e_1, \partial_t \rangle \partial_t, \quad \phi^2\xi = -\xi + \langle \xi, \partial_t \rangle \partial_t,$$

we obtain first:

$$\begin{aligned} e_1 &= \frac{\sin\theta}{\cos\varphi}\partial_\psi - \nu\epsilon \cos\alpha \cos\theta \partial_\varphi + \epsilon \sin\alpha \cos\theta \partial_t, \\ \xi &= \frac{\cos\theta}{\cos\varphi}\partial_\psi + \nu\epsilon \cos\alpha \sin\theta \partial_\varphi - \epsilon \sin\alpha \sin\theta \partial_t. \end{aligned} \tag{18}$$

Asking for the basis $\{e_1, e_2, \xi\}$ to have the same orientation as $\{\partial_\psi, \partial_\varphi, \partial_t\}$ one has $\nu\epsilon = -1$. Without loss of generality we take $\nu = -1$ and $\epsilon = 1$.

Thus, the expressions (15) and (18) yield:

$$\begin{aligned} e_1 &= \frac{\sin\theta}{\cos\varphi}\partial_\psi + \cos\alpha \cos\theta \partial_\varphi + \sin\alpha \cos\theta \partial_t, \\ e_2 &= -\sin\alpha \partial_\varphi + \cos\alpha \partial_t, \\ \xi &= \frac{\cos\theta}{\cos\varphi}\partial_\psi - \cos\alpha \sin\theta \partial_\varphi - \sin\alpha \sin\theta \partial_t. \end{aligned} \tag{19}$$

Computing the Lie bracket $[e_1, e_2]$ and using the fact that $\{e_1, e_2\}$ is an involutive system, that is $[e_1, e_2] \in \operatorname{span}\{e_1, e_2\}$, we obtain:

$$e_1(\alpha) + \cos\theta \sin\alpha \tan\varphi = 0. \tag{20}$$

Taking the derivative with respect to e_1 in the above expression and using (19), it follows that the smooth function α on M satisfies the following second-order partial differential equation:

$$e_1 e_1(\alpha) = -\cos^2\theta \sin\alpha \cos\alpha. \tag{21}$$

Our aim now is to define appropriate coordinates on M in order to be able to describe the surface M explicitly.

First, let us choose one of the coordinates on the surface as $e_1 = \frac{\partial}{\partial u}$. Second, we have to find the other coordinate, let us call it v, such that $e_2 = a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v}$, for a suitable choice of functions a and b. We get on the one hand $e_2(\varphi) = a(u, v)\varphi_u + b(u, v)\varphi_v$, and on the other hand, from (19), $e_2(\varphi) = -\sin\alpha$. We may take

$$a(u, v) = -\frac{\tan\alpha}{\cos\theta}, \tag{22}$$

where $\cos\theta \neq 0$ since $\theta \neq \frac{\pi}{2}$ and we assume $\cos\alpha \neq 0$. The remained cases will be studied separately. This fact immediately implies that α, and consequently φ, do not depend on v. Combining (20) and (21) it follows that there exists a positive real constant p such that

$$e_1(\alpha)^2 = \cos^2\theta(p^2 - \sin^2\alpha).$$

We obtain

$$e_1(\alpha) = \pm \frac{\cos\theta}{\sqrt{m}} \sqrt{1 - m\sin^2\alpha}, \tag{23}$$

where $m := \frac{1}{p^2}$. Next, we may determine the expression of α from the theorem,

$$\alpha(u) = \pm \mathrm{am}\left(\frac{1}{\sqrt{m}}\, u\cos\theta + c \middle| m\right), \tag{24}$$

where am denotes the *Jacobi amplitude* and m, c are some integration constants. For more details on elliptic functions see for example [1].

We have already known

$$[e_1, e_2] = \sin\alpha\tan\varphi e_1 - \cos\theta e_2(\alpha) e_2.$$

Computing now the same Lie bracket in terms of u and v, we get the following partial differential equation

$$\frac{b_u}{b} = \tan\alpha\,\alpha',$$

from which we find the solution

$$b(u, v) = \frac{1}{\cos\alpha}. \tag{25}$$

Using (22) and (25) it follows

$$e_2 = -\frac{\tan\alpha}{\cos\theta}\frac{\partial}{\partial u} + \frac{1}{\cos\alpha}\frac{\partial}{\partial v}. \tag{26}$$

We may easily check that the Lie bracket of ∂_u and ∂_v vanishes, $[\partial_u, \partial_v] = 0$, thus we made the right choice of coordinates.

From (20), when $\sin\alpha \neq 0$, we have

$$\tan\varphi = -\frac{e_1(\alpha)}{\cos\theta\sin\alpha},$$

and combining it with (23) we find

$$\tan\varphi = \mp\frac{\sqrt{1 - m\sin^2\alpha}}{\sqrt{m}\sin\alpha}, \tag{27}$$

obtaining the expression of the $\varphi-$coordinate from the theorem.

Using the expressions of e_1 and e_2, on one side given by (19) and on the other side written in terms of local coordinates u and v, together with (27), we get

$$\psi_u = \pm\frac{\sin\theta}{\sqrt{m}\sin\alpha}, \quad \psi_v = \pm\frac{\tan\theta}{\sqrt{m}}.$$

Thus ψ is also obtained. Finally, from (19) and (26), the $t-$coordinate satisfies

$$t_u = e_1(t) = \cos\theta\sin\alpha, \quad t_v = 1.$$

Hence, case (b) of the theorem is proved.

Let us study now the two particular cases we excluded during the proof.

$\underline{\cos\alpha = 0.}$ Similar reasoning yields to the conclusion that the surfaces for which the angle α between e_2 and ∂_t is $\frac{\pi}{2}$ are given by: an open part of the 2-sphere, $\mathbb{S}^2 \times \{t_0\}$ (when also

$\theta = \frac{\pi}{2}$), or an open part of the product manifold $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ denotes a meridian of the 2-sphere (when $\theta = 0$), these cases being excluded for the constant angle $\theta \notin \{0, \frac{\pi}{2}\}$.

$\underline{\sin\alpha = 0.}$ When $\alpha = 0$, namely the angle between e_2 and ∂_t vanishes, t is a local coordinate on M. We get that these surfaces are cylinders given by $\mathcal{C} \times \mathbb{R}$, where $\mathcal{C}$ is a loxodrome on the 2-sphere, completing case (a) of the direct implication. In particular, regarding the Equator $\mathcal{E}$ as a degenerated loxodrome, the surface $\mathcal{E} \times \mathbb{R}$ should be excluded from the theorem since it corresponds to the constant angle $\theta = \frac{\pi}{2}$.

Conversely, by straightforward computations one may show that the surfaces parametrized as in the theorem make constant angle with the rotational Killing vector field V. $\square$

Let us see now two particular cases for the constant angle θ.

If θ vanishes, then the surface is given by $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ denotes a meridian on $\mathbb{S}^2$.

If $\theta = \frac{\pi}{2}$, we have the following classification result:

Theorem 4 *A surface M isometrically immersed in the product space $\mathbb{S}^2 \times \mathbb{R} \backslash \{(0, 0, \pm 1, t)\}$ makes constant angle $\theta = \frac{\pi}{2}$ with the Killing vector field $V = -y\partial_x + x\partial_y$ if and only if it is given, up to isometries of the ambient space, by: $F : D \subset \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{R} \hookrightarrow \mathbb{R}^4$,*

$$F(\psi, u) = (\cos\varphi(u)\cos\psi,\ \cos\varphi(u)\sin\psi,\ \sin\varphi(u),\ t(u)),$$

where

$$\varphi(u) = -\int^{u} \sin\alpha(\zeta)\mathrm{d}\zeta, \quad t(u) = \int^{u} \cos\alpha(\zeta)\mathrm{d}\zeta,$$

and $\alpha : I \subset \mathbb{R} \to \mathbb{R}$ denotes a smooth function.

Proof Since $\theta = \frac{\pi}{2}$ denotes the angle between $V = \partial_\psi$ and the unit normal ξ, it follows that V is tangent to the surface M, thus ψ is a coordinate on the surface. The circles of radii $r = \cos\varphi_0$ corresponding to $\varphi = \varphi_0$ are included in the surface. Moreover, we have $t = t_0$, $z = \sin\varphi_0$, meaning that we obtain a rotational surface. More precisely, if R_ψ is the rotation of angle ψ in the plane (x, y) of $\mathbb{R}^4$, that is

$$R_\psi : \mathbb{R}^4 \to \mathbb{R}^4, \quad (x, y, z, t) \mapsto (x\cos\psi - y\sin\psi, x\sin\psi + y\cos\psi, z, t),$$

then M is obtained by rotating the unit speed curve γ, namely

$$F(\psi, u) = R_\psi(\gamma(u)), \quad \text{where} \quad \gamma(u) = (\cos\varphi(u), 0, \sin\varphi(u), t(u)).$$

Note that γ lies on a 2-cylinder $x^2 + z^2 = 1$ in the 3-dimensional Euclidean space $(x, 0, z, t)$.

In this particular case ($\theta = \frac{\pi}{2}$), we may consider that the tangent plane T_pM (in $p \in M$) is spanned by

$$\begin{aligned} e_1 &= \frac{1}{\cos\varphi}\partial_\psi, \\ e_2 &= -\sin\alpha\,\partial_\varphi + \cos\alpha\,\partial_t, \end{aligned} \tag{28}$$

for a certain function α. See also (19). Subsequently, the normal to M is given by

$$\xi = -\cos\alpha\,\partial_\varphi - \sin\alpha\,\partial_t. \tag{29}$$

The fact that the Lie bracket $[e_1, e_2] \in \text{span}\{e_1, e_2\}$ yields the compatibility condition

$$e_1(\alpha) = 0,$$

namely the function α does not depend on the coordinate ψ on the surface.

Since $[\partial_\psi, e_2] = 0$, it follows that one can choose a second coordinate on M, call it u, such that $e_2 = \partial_u$. Hence, $\varphi = \varphi(u)$ and the metric on M can be expressed as $g = \cos^2\varphi(u)d\psi^2 + du^2$.

At this moment, the conclusion follows immediately since $\varphi'(u) = e_2(\varphi) = -\sin\alpha$, $e_1(t) = 0$ and $t_u = e_2(t) = \cos\alpha$.

Conversely, one may easily check by straightforward computations that the parametrization of the surface given in the theorem represents a surface for which the unit normal makes constant angle $\theta = \frac{\pi}{2}$ with the Killing vector field V. □

Remark 2 Two examples which arise in Theorem 4 are given by:

- $\mathcal{E} \times \mathbb{R}$, when $\varphi(u) = 0$ (and also $\alpha(u) = 0$);
- $\mathbb{S}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$, when $\alpha(u) = \frac{\pi}{2}$.

In the study of minimality for surfaces making constant angle with the Killing vector field V, we compute first the mean curvature of M. Taking the tangent and normal parts in formula (12) first for $X = e_1$ and then for $X = e_2$ one gets the Levi-Civita connection on M

$$\nabla_{e_1}e_1 = -\tan\varphi\sin\alpha e_2,\ \nabla_{e_1}e_2 = \tan\varphi\sin\alpha e_1,\ \nabla_{e_2}e_1 = \delta\cot\theta e_2,\ \nabla_{e_2}e_2 = -\delta\cot\theta e_1, \tag{30}$$

and the scalar second fundamental form

$$h(e_1, e_1) = -\sin\theta\tan\varphi\cos\alpha, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \delta,$$

where $\delta = \sin\theta e_2(\alpha)$.

Thus, the mean curvature is given by

$$H = \frac{\sin\theta}{2}\Big(e_2(\alpha) - \tan\varphi\cos\alpha\Big). \tag{31}$$

We may formulate the following classification result:

Theorem 5 *Minimal surfaces isometrically immersed in $\mathbb{S}^2 \times \mathbb{R}\backslash\{(0, 0, \pm 1, t)\}$ which make constant angle with the Killing vector field $V = -y\partial_x + x\partial_y$ are given by:*

(a) $\mathbb{S}^1 \times \mathbb{R}$*, where* $\mathbb{S}^1$ *denotes a meridian on the* $2-$*sphere,*
(b) $\mathbb{S}^2 \times \{t_0\}$*,* $t_0 \in \mathbb{R}$*,*
(c) $\mathcal{E} \times \mathbb{R}$*, where* $\mathcal{E}$ *denotes the Equator of the* $2-$*sphere,*
(d) the rotational surface parametrized as $F : D \subset \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{R} \hookrightarrow \mathbb{R}^4$,

$$F(\psi, u) = (\cos\varphi(u)\cos\psi,\ \cos\varphi(u)\sin\psi,\ \sin\varphi(u),\ t(u)),$$

with $\varphi(u) = \arctan\left(\dfrac{c\cos u}{\sqrt{1 + c^2\sin^2 u}}\right)$ *and* $t(u) = \mathrm{EllipticF}(u \mid -c^2)$,
where $\mathrm{EllipticF}(u \mid -c^2)$ *denotes the elliptic integral of first kind.*

Proof Firstly, the minimality condition in (31) implies the case when $\sin\theta = 0$. Namely, when the constant angle θ vanishes, we already mentioned that the surfaces are given by $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ denotes a meridian on $\mathbb{S}^2$, as in case (a) of the theorem.

Secondly, the condition $H = 0$ in (31) becomes:

$$e_2(\alpha) = \cos\alpha\tan\varphi. \tag{32}$$

Using the expressions (23), (30) and (32), we compute the Lie bracket $[e_1, e_2](\alpha)$ in two ways obtaining the compatibility condition:

$$\cos\theta(\cos 2\alpha + 2\sin^2\varphi) = 0. \tag{33}$$

We distinguish the following cases.

- In the most general situation, when $\cos\theta \neq 0$, i.e., $\theta \neq \frac{\pi}{2}$ and $\sin\alpha \neq 0$, $\cos\alpha \neq 0$, we get from (33): $\cos 2\alpha + 2\sin^2\varphi = 0$. Applying e_1 to this expression we find

$$4\cos\theta\cos\alpha\sin\varphi(\sin^2\alpha + \cos^2\varphi) = 0,$$

  which leads to a contradiction, as follows. Firstly, if $\sin\varphi = 0$ we get that also $\alpha = 0$ and the surface is given by $\mathcal{E} \times \mathbb{R}$, where $\mathcal{E}$ denotes the Equator, corresponding to the constant angle $\theta = \frac{\pi}{2}$, which is excluded in this case. Secondly, $\sin^2\alpha + \cos^2\varphi = 0$ leads to another contradiction since $\cos\varphi \neq 0$.
  Moreover, also the cases $\cos\alpha = 0$, $\sin\alpha = 0$ when $\theta \neq \frac{\pi}{2}$ yield contradictions.
- Let us study the minimality in the case $\theta = \frac{\pi}{2}$. Since we already know from Theorem 4 that the local coordinates on the surface were chosen (ψ, u), and $e_2 = \partial_u$, we get that $e_1(\alpha) = 0$ and $e_1(\varphi) = 0$ involving that α and φ depend only on u. Moreover, from

$$e_2(\alpha) = \cos\alpha\tan\varphi, \quad e_2(\varphi) = -\sin\alpha,$$

  we should determine the functions α and φ satisfying the following system of ordinary differential equations:

$$\begin{cases} \alpha'(u) = \cos\alpha(u)\tan\varphi(u) \\ \varphi'(u) = -\sin\alpha(u). \end{cases} \tag{34}$$

Two particular solutions are obtained for $\cos\alpha = 0$, when the surface is given by $\mathbb{S}^2 \times \{t_0\}$, which is totally geodesic, thus minimal, fulfilling case (b) of the theorem; $\sin\alpha = 0$ implies also $\varphi = 0$, and the surface is given by $\mathcal{E} \times \mathbb{R}$, proving case (c) of the theorem.

Going back to the general situation, taking a second derivative with respect to u in the first relation of (34), we get the following second-order ordinary differential equation

$$\alpha''(u) + 2\tan\alpha(u)\alpha'(u)^2 + \sin\alpha(u)\cos\alpha(u) = 0. \tag{35}$$

In order to solve this ODE for $\cos\alpha \neq 0$ and $\sin\alpha \neq 0$, we denote $f(u) := \frac{\alpha'(u)}{\sin\alpha(u)\cos\alpha(u)}$. Then, (35) becomes

$$f'(u) + f(u)^2 + 1 = 0,$$

which has the general solution $f(u) = \cot(u + c_0)$, $c_0 \in \mathbb{R}$. Without loss of generality, we may assume $c_0 = 0$ (eventually after a translation in the u coordinate), and it follows that

$$\cot u = \frac{\alpha'(u)}{\sin\alpha(u)\cos\alpha(u)},$$

which implies that

$$\left(\frac{\tan\alpha}{\sin u}\right)' = 0,$$

and we get the solution for (35)

$$\alpha(u) = \arctan(c\sin u), \quad c \in \mathbb{R}, \quad u \in (0, \pi/2).$$

From the first relation in (34) we easily find

$$\varphi(u) = \arctan\left(\frac{c\cos u}{\sqrt{1 + c^2\sin^2 u}}\right).$$

Finally, the coordinate t along $\mathbb{R}-$direction is obtained from the condition $t'(u) = \cos\alpha(u)$. For more details on elliptic functions see e.g. [1].

Conversely, one can check that the surfaces parametrized in the theorem are minimal and they make constant angle with the Killing vector field V. □

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